SPECTRAL INVARIANCE OF BESOV-BESSEL SUBALGEBRAS

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Abstract. Using principles of the theory of smoothness spaces we give systematic constructions of scales of inverse-closed subalgebras of a given Banach algebra with the action of a d-parameter automorphism group. In particular we obtain the inverse-closedness of Besov algebras, Bessel potential algebras and approximation algebras of polynomial order in their defining algebra. By a proper choice of the group action these general results can be applied to algebras of infinite matrices and yield inverse-closed subalgebras of matrices with off-diagonal decay of polynomial order. Besides alternative proofs of known results we obtain new classes of inverse-closed subalgebras of matrices with off-diagonal decay.

This work is a continuation and extension of results presented in [20].

1. Introduction

We aim at systematic constructions of inverse-closed subalgebras of a given Banach algebra $\mathcal{A}$. Recall that a subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called inverse-closed in $\mathcal{A}$, if

$$b \in \mathcal{B} \text{ invertible in } \mathcal{A} \text{ implies } b^{-1} \in \mathcal{B}\,. \tag{1}$$

Many equivalent notions for this relation are used in the literature, e.g.,one says that $\mathcal{B}$ is a *spectral subalgebra* of $\mathcal{A}$ [30], or $\mathcal{B}$ is *spectrally invariant* in $\mathcal{A}$, see [19] for a collection of synonyms.

A prototypical result is Wiener's Lemma, which states precisely that the Wiener algebra of trigonometric series with absolutely convergent coefficients is inverse-closed in the algebra of continuous functions on the torus. Another example is the algebra $C^m(X)$ of m times continuously differentiable functions on a closed interval X, which is inverse-closed in $C(X)$ by the iterated quotient rule (note that the algebra property of $C^m(X)$ follows from the iterated product rule).

Our original interest was the construction of Banach algebras of matrices with off-diagonal decay that are inverse-closed in $\mathcal{B}(\ell^2)$, the bounded operators in ℓ^2, see [20]. A key result is Jaffard's theorem.

Theorem ([25])**.** *If the entries of the matrix A satisfy $|A(k,l)| \leq C|k-l|^{-r}$ for some $C > 0$ and $r > 0$, and A is invertible in $\mathcal{B}(\ell^2)$, then $|A^{-1}(k,l)| \leq C'|k-l|^{-r}$ for some $C' > 0$.*

In other words Jaffard's theorem states that the Banach algebra $\mathcal{C}_r^\infty$, consisting of matrices A with finite norm $\|A\|_{\mathcal{C}_r^\infty} = \sup_{k,l}|A(k,l)|(1+|k-l|^r)$, is inverse-closed in $\mathcal{B}(\ell^2)$. Generalizations and variants of this theorem have been obtained in, e.g., [5, 7, 21, 22, 25].

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Inverse-closed subalgebras are often related to the concept of smoothness, an elementary example is again the algebra $C^m(X)$. In more generality, the domain of a densely defined, closed and symmetric derivation on the C^* algebra $\mathcal{A}$ was shown to be inverse-closed in $\mathcal{A}$ by Bratteli and Robinson [11], an extension of this result to general Banach algebras was given in [26]. Related concepts of smoothness that define inverse-closed subalgebras are the *differentials norms* [10], the D_p algebras [26, 27], or the *Leibniz seminorms* [36].

Although it might be not immediately obvious the examples of matrices with off-diagonal decay fit into this picture, as off-diagonal decay can be decribed by smoothness conditions. In particular, the formal commutator

$$\delta(A) = [X, A]\,,$$

$X = \operatorname{Diag}((k)_{k\in\mathbb{Z}})$, is a derivation on $\mathcal{B}(\ell^2)$, and its domain defines an algebra of matrices with off-diagonal decay that is inverse-closed in $\mathcal{B}(\ell^2)$ [20, 3.4].

The proof of the spectral invariance of $\mathcal{B} \subseteq \mathcal{A}$ makes often detailed use of some specific properties of the involved algebras. The standard proof for Wiener's Lemma is a prime example of an application of the Gelfand theory. Proofs of the spectral invariance of Banach algebras of matrices involve the theorem of Bochner-Philips [5, 6], interpolation arguments [21, 40, 41] or commutator estimates [25]. As it turns out, all of these proof methods use some related concepts of smoothness [28].

In a previous publication [20] we obtained systematic constructions of inverse-closed subalgebras of a given *Banach algebra* with additional smoothness conditions. In particular it was shown in [20] that

(1) the domain of a (not necessaryly densely defined) closed derivation of a *symmetric* Banach algebra $\mathcal{A}$ is inverse-closed in $\mathcal{A}$,
(2) the subalgebra $C(A)$ of continuous elements of a Banach algebra $\mathcal{A}$ with d-parameter automorphism group Ψ and the associated Hölder-Zygmund spaces $\Lambda_r^\infty(\mathcal{A})$ are inverse-closed in $\mathcal{A}$,
(3) the approximation spaces of polynomial order of a symmetric Banach algebra $\mathcal{A}$ are inverse-closed in $\mathcal{A}$, where the approximating subspaces are adapted to the algebra multiplication (see Section 2.4).

Applied to subalgebras of $\mathcal{B}(\ell^2)$ the theory yields scales of inverse-closed subalgebras of matrices with off-diagonal decay, including the Banach algebras of matrices used in the literature cited above, but also new classes of inverse-closed subalgebras of matrices with off-diagonal decay constructed by approximation with banded matrices.

In this article we generalize the approach (2). In its essence this approach is based on the product and quotient rules of real analysis. The identities

$$\begin{aligned}\Delta_t(fg) &= T_t f\,\Delta_t g + \Delta_t f g\,,\\ \Delta_t(1/f) &= -\frac{\Delta_t f}{T_t f\cdot f}\,,\end{aligned} \tag{2}$$

where $T_t f(x) = f(x-t)$ is the translation operator, imply that the smoothness of f (resp. g) is preserved by the product fg, and by the reciprocal $1/f$. To give an example, the membership of the continuous function f in the Hölder-Zygmund space $\Lambda_r^\infty(L^\infty(\mathbb{R}))$ for $0 < r < 1$ is defined by the condition $\|\Delta_t f\|_\infty \le C|t|^r$ for some $C > 0$, and Equation (2) implies that $\|\Delta_t(1/f)\|_\infty \le C'|t|^r$ for a constant $C' > 0$.

In [20] we have (amongst other things) adapted this approach to more general Banach algebras. If translation is replaced by the action of a d-parameter automorphism group on the Banach algebra $\mathcal{A}$, the noncommutative form of the product and quotient rule (2) impliy that the noncommutative Hölder-Zygmund space $\Lambda_r^\infty(\mathcal{A})$ is an inverse-closed subalgebra of $\mathcal{A}$ [20, 3.21].

In this article we extend this approach to cover Besov and Bessel potential spaces. In more detail, the organization of the paper is as follows.

After introducing notation we describe some classes of matrices that will serve as examples for the theory to be developed, and we define the approximation spaces needed. In Section 3 we introduce smoothness for a Banach algebra $\mathcal{A}$ by the action of a d-parameter automorphism group, and review basic properties. Besov spaces are defined in Section 3.2, and it is proved that they form inverse-closed subalgebras of $\mathcal{A}$. A useful property not only for simplifying several proofs but also of conceptual interest is a reiteration theorem for Besov spaces (Theorem 3.7). In [2] a similar theorem was proved by interpolation methods for Besov spaces of operators. An examination of the details of this proof shows that it uses similar ingredients (and is of similar complexity) as ours, which is valid in the more general setting of a Banach algebra with automorphism group. Instead of interpolation theory the proof given here uses estimates for the moduli of smoothness. In Section 3.3 we identify Besov spaces as approximation spaces and obtain a Littlewood-Paley-decomposition of them.

For the discussion of Bessel potential spaces $\mathcal{P}_r(\mathcal{A})$ in Section 3.4 we introduce the concept of C_w-continuity [2, 3] in order to cover relevant examples of Banach algebras of matrices with off-diagonal decay. A description of Bessel potential spaces by hypersingular integrals is used to prove the algebra properties and the inverse-closedness of $\mathcal{P}_r(\mathcal{A})$ in $\mathcal{A}$.

Again, we illustrate the abstract concepts by constructing subalgebras of matrices with off-diagonal decay. The membership of a matrix A in a Besov or Bessel subalgebra of $\mathcal{B}(\ell^2)$ is then equivalent to a form of off-diagonal decay, so this application of the general theory describes scales of inverse-closed subalgebras of matrices with off-diagonal decay.

Long proofs have been moved to the appendices.

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2. Resources

2.1. **Notation.** The *d-dimensional torus* is $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let $\mathbb{C}^d_* = \mathbb{C}^d \setminus \{0\}$, and $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$. The symbol $\lfloor x \rfloor$ denotes the greatest integer smaller or equal to the real number x.

A multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ is a d-tuple of nonnegative integers. We set $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, and $D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x)$ is the partial derivative. The degree of x^α is $|\alpha| = \sum_{j=1}^d \alpha_j$, and $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for $j = 1, \dots, d$. More generally, $|x|_p = \left(\sum_{k=1}^d |x(k)|^p\right)^{1/p}$ denotes the p-norm on $\mathbb{C}^d$.

Positive constants will be denoted by C, C',C_1,c, etc., The same symbol might denote different constants in each equation. If f and g are positive functions, $f \asymp g$ means that $C_1 f \le g \le C_2 f$. We sometimes use the notation $f \lesssim g$ ($f \gtrsim g$) to express that there is a constant $C > 0$ such that $f \le Cg$ ($f \ge Cg$).

The standard basis of $\ell^p(\mathbb{Z}^d)$ is $e_k = (\delta_{jk})_{j\in\mathbb{Z}^d}$, $\langle x, y\rangle = \sum_{k\in\mathbb{Z}^d} x(k)y(k)$ is the standard dual pairing between $\ell^p(\mathbb{Z}^d)$ and its dual $\ell^{p'}(\mathbb{Z}^d)$. If $p = 2$, we define the scalar product as $\langle x, y\rangle = \sum_{k\in\mathbb{Z}^d} x(k)\overline{y(k)}$. This should not lead to confusion.

A submultiplicative weight on on $\mathbb{Z}^d$ is a positive function $v : \mathbb{Z}^d \to \mathbb{R}$ such that $v(0) = 1$ and $v(x + y) \le v(x)v(y)$ for $x, y \in \mathbb{Z}^d$. The standard polynomial weights are $v_r(x) = (1 + |x|)^r$ for $r \ge 0$. The weighted spaces $\ell^p_w(\mathbb{Z}^d)$ are defined by the norm $\|x\|_{\ell^p_w(\mathbb{Z}^d)} = \|xw\|_{\ell^p(\mathbb{Z}^d)}$. If $w = v_r$ we will simply write $\|x\|_{\ell^p_r(\mathbb{Z}^d)}$.

The Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ is denoted by $\mathscr{S}(\mathbb{R}^d)$. The Fourier transform of $f \in \mathscr{S}(\mathbb{R}^d)$ is $\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\omega\cdot x}\,dx$. This definition is extended by duality to $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions. The same symbols are also used for the Fourier transform on $\mathbb{Z}^d$ and $\mathbb{T}^d$.

The continuous embedding of the normed space X into the normed space Y is denoted as $X \hookrightarrow Y$. The operator norm of a bounded linear mapping $A\colon X \to Y$ is $\|A\|_{X\to Y}$. In the special case of operators $A\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ we write $\|A\|_{\mathcal{B}(\ell^2(\mathbb{Z}^d))} = \|A\|_{\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)}$ or simply $\|A\|_{\mathcal{B}(\ell^2)}$.

We will consider Banach spaces with equivalent norms as equal.

2.2. **Inverse closed subalgebras of Banach algebras.** All Banach algebras are assumed to be *unital*. To verify that a Banach space $\mathcal{A}$ with norm $\|\ \|_{\mathcal{A}}$ is a Banach algebra it is sufficient to prove that $\|ab\|_{\mathcal{A}} \le C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{A}}$ for some constant C. The expression $\|a\|'_{\mathcal{A}} = \sup_{\|b\|_{\mathcal{A}}=1}\|ab\|_{\mathcal{A}}$ is an equivalent norm on $\mathcal{A}$ and satisfies $\|ab\|'_{\mathcal{A}} \le \|a\|'_{\mathcal{A}}\|b\|'_{\mathcal{A}}$.

Definition 2.1 (Inverse-closedness)**.** If $\mathcal{A} \subseteq \mathcal{B}$ are Banach algebras with common multiplication and identity, we call $\mathcal{A}$ *inverse-closed* in $\mathcal{B}$, if

$$a \in \mathcal{A} \text{ and } a^{-1} \in \mathcal{B} \quad \text{implies} \quad a^{-1} \in \mathcal{A}. \tag{3}$$

Inverse-closedness is equivalent to *spectral invariance*. This means that the spectrum $\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C}\colon a-\lambda \text{ not invertible in } \mathcal{A}\}$ of an element $a \in \mathcal{A}$ satisfies

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a), \qquad \text{for all } a \in \mathcal{A}.$$

The relation of inverse-closedness is transitive: If $\mathcal{A}$ is inverse-closed in $\mathcal{B}$ and $\mathcal{B}$ is inverse-closed in $\mathcal{C}$, then $\mathcal{A}$ is inverse-closed in $\mathcal{C}$.

Remark. Spectral invariance is a generalization of *Wiener's Lemma*, which states precisely that the *Wiener algebra* $\mathcal{F}\ell^1(\mathbb{Z}^d)$ of absolutely convergent Fourier series is inverse-closed in $C(\mathbb{T}^d)$. See [19] for a concise overview of the importance of the concept of inverse-closedness.

2.3. **Examples of Matrix Algebras.** To describe the most common forms of off-diagonal decay, let us fix some notation. An infinite matrix A over $\mathbb{Z}^d$ is a function $A : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$. The m-th side diagonal of A is the matrix $\hat{A}(m)$ with entries

$$\hat{A}(m)(k,l) = \begin{cases} A(k,l), & k-l = m, \\ 0, & \text{otherwise.} \end{cases}$$

A matrix A is *banded* with bandwidth N, if

$$A = \sum_{|m|_\infty \le N} \hat{A}(m).$$

Let $1 < p \le \infty$, $r > d(1-1/p)$, or $p = 1$, and $r \ge 0$. The space $\mathcal{C}^p_r$ consists of all matrices A with finite norm

$$\|A\|_{\mathcal{C}^p_r} = \left(\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|A(l,l-k)|^p(1+|k|)^{rp}\right)^{1/p}$$

with the standard change for $p = \infty$.

The following special cases have obtained particular interest. The *Jaffard algebra* $\mathcal{C}^\infty_r$ consists of the matrices A for which $|A(k,l)| \le C(1+|k-l|)^{-r}$, so the norm of $\mathcal{C}^\infty_r$ describes polynomial decay off the diagonal.

The *algebra of convolution-dominated matrices* $\mathcal{C}^1_r, r \ge 0$, (sometimes called the Baskakov-Gohberg-Sjöstrand algebra) consists of all matrices A, for which there is a $h \in \ell^1_r(\mathbb{Z}^d)$ such that $|Ax(k)| \le h * |x|(k)$, where $|x|$ denotes the vector with components $(|x(k)|)_{k\in\mathbb{Z}^d}$.

If $1 < p \leq \infty$ and $r > d(1-1/p)$ or $p = 1$ and $r \geq 0$ the *Schur algebra* $\mathcal{S}_r^p$ is defined by the norm

$$\|A\|_{\mathcal{S}_r^p} = \max\Big\{ \sup_{k\in\mathbb{Z}^d} \big(\sum_{l\in\mathbb{Z}^d} |A(k,l)|^p v_r(k-l)^p\big)^{1/p}, \sup_{l\in\mathbb{Z}^d} \big(\sum_{k\in\mathbb{Z}^d} |A(k,l)|^p v_r(k-l)^p\big)^{1/p} \Big\}$$

with the standard change for $p = \infty$.

Remarks. (1) The scales $\mathcal{S}_r^p$ and $\mathcal{C}_r^p$ are identical at the endpoint $p = \infty$, i.e. $\mathcal{S}_r^\infty = \mathcal{C}_r^\infty$. (2) It follows immediately from the definitions that $\mathcal{C}_r^p \hookrightarrow \mathcal{S}_r^p$.

We note that the norms above depend only on the absolute values of the matrix entries. Precisely, a matrix norm on $\mathcal{A}$ is called *solid*, if $B \in \mathcal{A}$ and $|A(k,l)| \leq |B(k,l)|$ for all k,l implies $A \in \mathcal{A}$ and $\|A\|_\mathcal{A} \leq \|B\|_\mathcal{A}$. In particular, for a solid norm we have $\|\,|A|\|_\mathcal{A} = \|A\|_\mathcal{A}$, where $|A|$ is the matrix with entries $|A|(k,l) = |A(k,l)|$ for $k,l \in \mathbb{Z}^d$.

The following result summarizes the main properties of the matrix classes $\mathcal{C}_r^p$ and $\mathcal{S}_r^p$. See [5, 21, 25, 40] for proofs.

Proposition 2.2. *Assume that $\mathcal{A}$ is one of the matrix classes $\mathcal{C}_r^p$ or $\mathcal{S}_r^p$ for $r > d(1-1/p)$, if $p > 1$, and $r \geq 0$, if $p = 1$. Then $\mathcal{A}$ is a solid Banach $*$-algebra with respect to matrix multiplication and taking adjoints as the involution. Every $\mathcal{A}$ is continuously embedded into the algebra $\mathcal{B}(\ell^p(\mathbb{Z}^d))$ of bounded operators on $\ell^p(\mathbb{Z}^d)$ for $1 \leq p \leq \infty$. With the exception of $\mathcal{S}_0^1$ [42] every class $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^p(\mathbb{Z}^d))$, $1 \leq p \leq \infty$. In particular, $\mathcal{A}$ is symmetric.*

In the sequel we will construct algebras that are inverse-closed in one of the standard algebras $\mathcal{C}_r^p, \mathcal{S}_r^p$, and are therefore inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$ by Proposition 2.2.

We generalize the definitions above.

Definition 2.3. A *matrix algebra* $\mathcal{A}$ (over $\mathbb{Z}^d$) is a Banach algebra of matrices that is continuously embedded in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$.

We drop the reference to the index set $\mathbb{Z}^d$ whenever possible.

Lemma 2.4. *If $\mathcal{A}$ is a matrix algebra, the selection of matrix elements is continuous.*

Proof. $|A(k,l)| = |\langle Ae_k, e_l\rangle| \leq \|A\|_{\mathcal{B}(\ell^2)} \leq C\|A\|_\mathcal{A}$. □

2.4. **Approximation Spaces and Algebras.** Let the index set Λ be either $\mathbb{R}_0^+$ or $\mathbb{N}_0$. An *approximation scheme* on the Banach algebra $\mathcal{A}$ is a family $(X_\sigma)_{\sigma\in\Lambda}$ of closed subspaces of $\mathcal{A}$ that satisfy $X_0 = \{0\}$, $X_\sigma \subseteq X_\tau$ for $\sigma \leq \tau$, and $X_\sigma \cdot X_\tau \subseteq X_{\sigma+\tau}$, $\sigma, \tau \in \Lambda$. If $\mathcal{A}$ possesses an involution, we further assume that $\mathbf{1} \in X_1$ and $X_\sigma = X_\sigma^*$ for all $\sigma \in \Lambda$. The *σ-th approximation error* of $a \in \mathcal{A}$ by X_σ is $E_\sigma(a) = \inf_{x\in X_\sigma} \|a - x\|_\mathcal{A}$. We define approximation spaces $\mathcal{E}_r^p(\mathcal{A})$ by the norm

$$\|a\|_{\mathcal{E}_r^p}^p = \sum_{k=0}^\infty E_k(a)^p (k+1)^{rp} \frac{1}{k+1}, \text{ for } \Lambda = \mathbb{N}_0\,, \tag{4}$$

for $1 \leq p < \infty$ with the obvious change for $p = \infty$. If $\Lambda = \mathbb{R}_0^+$ an equivalent norm is $\|a\|_{\mathcal{E}_r^p}^p = \int_0^\infty E_\sigma(a)^p(\sigma+1)^{rp} \frac{d\,\sigma}{\sigma+1}$. Algebra properties of approximation spaces of approximation spaces are discussed in [1, 20]. In particular, in [20] the following result is proved.

Proposition 2.5. *If $\mathcal{A}$ is a symmetric Banach algebra and $(X_\sigma)_{\sigma\in\Lambda}$ an approximation scheme, then $\mathcal{E}_r^p(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

If $\mathcal{A}$ is a matrix algebra and $\mathcal{T}_N = \mathcal{T}_N(\mathcal{A})$ denotes the set of matrices in $\mathcal{A}$ with bandwidth smaller than N,

$$\mathcal{T}_N = \{A \in \mathcal{A} \colon A = \sum_{|k|_\infty < N} \hat{A}(k)\}$$

then the sequence $(\mathcal{T}_N)_{N\geq 0}$ is an approximation scheme for $\mathcal{A}$. The closure of all banded matrices in $\mathcal{A}$ is the space of *band-dominated matrices* in $\mathcal{A}$ [34, 35].

In [20] we obtained the following constructive description of $\mathcal{E}_r^\infty(\mathcal{C}_0^1)$: *The approximation space $\mathcal{E}_r^\infty(\mathcal{C}_0^1)$ consists of all matrices A satisfying*

$$\|\hat{A}(0)\|_{\mathcal{B}(\ell^2)} < \infty, \quad 2^{rk} \sum_{2^k \leq |l| < 2^{k+1}} \|\hat{A}(l)\|_{\mathcal{B}(\ell^2)} = 2^{rk} \sum_{2^k \leq |l| < 2^{k+1}} \sup_{m \in \mathbb{Z}^d} |A(m, m-l)| \leq C$$

for all $k \geq 0$. Theorem 3.14 is a more general result of this type.

3. Smoothness in Banach Algebras

An important observation in [20] was that the off-diagonal decay of matrices can be described by *smoothness properties*, using derivations and the action of the automorphism group $\chi_t(A) = \sum \hat{A}(k)e^{2\pi i k t}$. In our treatment we focused on Hölder-Zygmund spaces and on spaces of m times differentiable elements. Now we extend our research and cover the more general Besov and Bessel potential spaces. We also establish the isomorphism between Besov spaces and approximation spaces of polynomial order. In all cases we obtain results on the inverse-closedness of the smoothness spaces in their defining algebra.

It turns out that the investigations can be carried out with no additional effort for Banach algebras with the bounded action of a d-parameter automorphism group. Here we obtain new methods for the construction of scales of inverse-closed subalgebras.

3.1. **Automorphism Groups and Continuity.** Let $\mathcal{A}$ be a Banach algebra. A (d-parameter) *automorphism group* acting on $\mathcal{A}$ is a set of Banach algebra automorphisms $\Psi = \{\psi_t\}_{t\in\mathbb{R}^d}$ of $\mathcal{A}$ that satisfy the group properties

$$\psi_s \psi_t = \psi_{s+t} \quad \text{for all} \quad s, t \in \mathbb{R}^d.$$

If $\mathcal{A}$ is a $*$-algebra we assume that Ψ consists of $*$-automorphisms. In order to simplify some proofs, we assume that Ψ is *uniformly bounded*, that is,

$$M_\Psi = \sup_{t\in\mathbb{R}^d} \|\psi_t\|_{\mathcal{A}\to\mathcal{A}} < \infty\,.$$

The abstract theory works for more general group actions [2, 23, 43].

An element $a \in \mathcal{A}$ is (strongly) *continuous*, if

$$\|\psi_t(a) - a\|_{\mathcal{A}} \to 0 \text{ for } t \to 0. \tag{5}$$

The continuous elements of $\mathcal{A}$ are denoted by $C(\mathcal{A})$.

For $t \in \mathbb{R}^d \setminus \{0\}$ the *generator* δ_t is

$$\delta_t(a) = \lim_{h\to 0} \frac{\psi_{ht}(a) - a}{h} \tag{6}$$

The domain $\mathcal{D}(\delta_t, \mathcal{A})$ of δ_t is the set of all $a \in \mathcal{A}$ for which this limit exists. The *canonical generators* of Ψ are $(\delta_{e_k})_{1\leq k\leq d}$, and Ψ is the *automorphism group generated by* $(\delta_{e_k})_{1\leq k\leq d}$. If $\alpha \in \mathbb{N}_0^d$ is a multi-index, then $\delta^\alpha = \delta_{e_1}^{\alpha_1} \cdots \delta_{e_d}^{\alpha_d}$. In [20, 3.15] precise conditions are given, under which condition two derivations commute. In particular, this is true for all cases that will be encountered in this text. Each generator δ_t is a closed derivation, that is $\delta_t(ab) = a\delta_t(b) + \delta_t(a)b$ for all $a, b \in \mathcal{D}(\delta_t, \mathcal{A})$, and the operator δ_t is a closed operator on its domain. If $\mathcal{A}$ is a Banach $*$-algebra, then δ_t is a $*$-derivation [12].

Proposition 3.1 ([20, 3.4])**.** *If $\mathcal{A}$ is symmetric, and δ is a closed $*$-derivation then $\mathcal{D}(\delta_t, \mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

We call the action of Ψ periodic if $\psi_{t+e_k} = \psi_t$ for all $1 \leq k \leq d$ and all $t \in \mathbb{R}^d$.

Definition 3.2. A matrix algebra $\mathcal{A}$ is called *homogeneous* [16, 17], if

$$\chi_t \colon A \mapsto M_t A M_{-t},\ \chi_t(A)(k,l) = e^{2\pi i (k-l)\cdot t} A(k,l) \text{ for all } k,l \in \mathbb{Z}^d \text{ and } t \in \mathbb{R}^d$$

define uniformly bounded mappings on $\mathcal{A}$, where $M_t, t \in \mathbb{R}^d$, is the *modulation operator* $M_t x(k) = e^{2\pi i k\cdot t} x(k)$, $k \in \mathbb{Z}^d$.

Clearly $\chi = \{\chi_t\}_{t\in\mathbb{R}^d}$ defines an automorphism group on $\mathcal{A}$. The algebra of bounded operators on ℓ^2 is a homogeneous matrix algebra, and so are all solid matrix algebras.

In the literature on group actions it is often assumed that Ψ is strongly continuous on all of $\mathcal{A}$, i.e. $\mathcal{A} = C(\mathcal{A})$. This is no longer true for most matrix algebras, and in general $C(\mathcal{A})$ is a closed and inverse-closed subalgebra of $\mathcal{A}$ [20, 3.14].

In [20] the spaces $C(\mathcal{A})$ have been identified for the algebras $\mathcal{C}^1_r, \mathcal{C}^\infty_r, \mathcal{S}^1_r$. We use the opportunity to state the full result for the algebras $\mathcal{C}^p_r$ and $\mathcal{S}^p_r$.

Proposition 3.3. *If $\mathcal{A}$ is one of the algebras $\mathcal{C}^\infty_r, \mathcal{S}^p_r, \mathcal{B}(\ell^2(\mathbb{Z}^d))$ for $r \geq 0$, then $C(\mathcal{A}) \neq \mathcal{A}$.*

$$C(\mathcal{C}^p_r) = \mathcal{C}^p_r, \quad 1 \leq p < \infty\,,$$

$$C(\mathcal{C}^\infty_r) = \{A \in \mathcal{C}^\infty_r \colon \lim_{|k|_\infty \to\infty} \|\hat{A}(k)\|_{\mathcal{C}^\infty_r} = \lim_{|k|_\infty\to\infty} \|\hat{A}(k)\|_{\mathcal{B}(\ell^2)} (1+|k|)^r = 0\}$$

$$\begin{aligned} C(\mathcal{S}^p_r) = \{A \in \mathcal{S}^p_r \colon &\lim_{N\to\infty} \sup_{k\in\mathbb{Z}^d} \sum_{|s|_\infty > N} |A(k,k-s)|^p (1+|s|)^{rp} = 0 \textit{ and} \\ &\lim_{N\to\infty} \sup_{k\in\mathbb{Z}^d} \sum_{|s|_\infty > N} |A(k-s,k)|^p (1+|s|)^{rp} = 0\}\,. \end{aligned}$$

The method of proof is as in [20].

3.2. **Besov Spaces.** The theory of vector valued Besov spaces is well established [9, 13, 31, 44]. The main results of this section are the algebra properties of vector-valued Besov spaces derived from a given Banach algebra $\mathcal{A}$. Though possibly known, we were not able to find any references, so full proofs of the results are included.

Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ. Define the kth difference operator as $\Delta^k_t = (\psi_t - \mathrm{id})^k$, $t \in \mathbb{R}^d$. For a step size $h > 0$, the *modulus of continuity* of $a \in \mathcal{A}$ is $\omega_h(a) = \omega^1_h(a) = \sup_{|t|\leq h} \|\Delta_t a\|_{\mathcal{A}}$. If $k > 1$, the kth *modulus of smoothness* of a is $\omega^k_h(a, \mathcal{A}) = \omega^k_h(a) = \sup_{|t|\leq h} \|\Delta^k_t a\|_{\mathcal{A}}$.

Definition 3.4. Let $1 \leq p \leq \infty$, $r > 0$, $l = \lfloor r \rfloor + 1$. The (vector valued) *Besov space* $\Lambda^p_r(\mathcal{A})$ consists of all $a \in \mathcal{A}$ for which the seminorm

$$|a|_{\Lambda^p_r(\mathcal{A})} = \begin{cases} \left(\int_{\mathbb{R}^+} (h^{-r}\omega^l_h(a))^p \frac{dh}{h}\right)^{1/p}, & 1 \leq p < \infty \\ \|a\|_{\mathcal{A}} + \sup_{h>0} h^{-r} \omega^l_h(a)\,, & p = \infty \end{cases}$$

is finite. The parameter r is the smoothness parameter. The Besov norm is $\|a\|_{\Lambda^p_r(\mathcal{A})} = \|a\|_{\mathcal{A}} + |a|_{\Lambda^p_r(\mathcal{A})}$.

Actually, replacing l by any integer $k > \lfloor r \rfloor$ in the preceding definition yields an equivalent norm for $\Lambda^p_r(\mathcal{A})$. In addition, we will need the following norm equivalences.

$$\begin{aligned} \|a\|_{\Lambda^p_r(\mathcal{A})} &\asymp \|a\|_{\mathcal{A}} + \Big(\int_{\mathbb{R}^d} (|t|^{-r} \|\Delta^k_t a\|_{\mathcal{A}})^p \frac{dt}{|t|^d} \Big)^{1/p} \\ &\asymp \|a\|_{\mathcal{A}} + \Big(\sum_{l=0}^{\infty} \big(2^{rl} \omega^k_{2^{-l}}(a)\big)^p \Big)^{1/p} . \end{aligned} \tag{7}$$

If $l \in \mathbb{N}_0$, and $l \leq r$, these norms are further equivalent to

$$\|a\|_{\mathcal{A}} + \sum_{|\alpha|=l} \|\delta^\alpha(a)\|_{\Lambda^p_{r-l}(\mathcal{A})} .$$

The Besov spaces $\Lambda^p_r(\mathcal{A})$ are Banach spaces for all $1 \leq p \leq \infty$ and $r > 0$. If $1 \leq p, q \leq \infty$ and $0 < r < s$, then $\Lambda^p_s(\mathcal{A}) \hookrightarrow \Lambda^q_r(\mathcal{A})$. If $p < q$ then $\Lambda^p_r(\mathcal{A}) \hookrightarrow \Lambda^q_r(\mathcal{A})$. See, e.g., [9, 13, 31, 44] for these and other basic properties.

Lemma 3.5. *If $\mathcal{A}$ is a Banach algebra with (bounded) automorphism group Ψ, then Ψ is a bounded automorphism group on $\Lambda^p_r(\mathcal{A})$ for every $1 \leq p \leq \infty$, and $r > 0$.*

Proof. Assume that $p < \infty$. If $a \in \Lambda^p_r(\mathcal{A})$, $k > \lfloor r \rfloor$ and $s \in \mathbb{R}^d$, then for every $s > 0$

$$\|\psi_s a\|_{\Lambda^p_r(\mathcal{A})} = \|\psi_s a\|_{\mathcal{A}} + \Big(\int_{\mathbb{R}^d} (|t|^{-r} \|\Delta^k_t \psi_s a\|_{\mathcal{A}})^p \frac{dt}{|t|^d} \Big)^{1/p} \leq M_\Psi \|a\|_{\Lambda^p_r(\mathcal{A})} ,$$

since $\delta^k_t \psi_s = \psi_s \delta^k_t$, so ψ_s is bounded on $\Lambda^p_r(\mathcal{A})$. The proof for $p = \infty$ is similar. □

Proposition 3.6 ([13, 3.1.5, 3.4.3])**.** *If $k > \lfloor r \rfloor$ then*

$$\begin{aligned} &C(\Lambda^p_r(\mathcal{A})) = \Lambda^p_r(\mathcal{A}), \quad 1 \leq p < \infty , \\ &C(\Lambda^\infty_r(\mathcal{A})) = \lambda^\infty_r(\mathcal{A}) = \{a \in \mathcal{A} \colon \lim_{h \to 0} h^{-r} \omega^k_h(a) = 0\} . \end{aligned}$$

Does the iteration of the construction of Besov spaces yield refined smoothness spaces?

Theorem 3.7 (Reiteration theorem)**.** *If $1 \leq p, q \leq \infty$ and $r, s > 0$ then*

$$\Lambda^q_s(\Lambda^p_r(\mathcal{A})) = \Lambda^q_{r+s}(\mathcal{A}) . \tag{8}$$

A proof of is in appendix A.

Remarks. A proof of the reiteration formula for the Banach algebra of bounded operators on a Banach space $\mathcal{X}$ and the automorphism group ψ obtained by conjugation with an automorphism group on $\mathcal{X}$ has been given in[2], using interpolation theory.

We think the reiteration formula is of some conceptual interest. Note that the classical notion of Besov spaces on $\mathbb{R}^d$ does not even allow to formulate the result. We use (8) to simplify proofs of approximation results.

The main result of this section treats the algebra properties of Besov spaces.

Theorem 3.8. *Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ. For all parameters $1 \leq p \leq \infty$ and $r > 0$, the Besov space $\Lambda^p_r(\mathcal{A})$ is a Banach subalgebra of $\mathcal{A}$. Moreover, $\Lambda^p_r(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

Proof. We treat the case $r < 1$ first. To show that $\Lambda^p_r(\mathcal{A})$ is a Banach algebra we use the identity

$$\Delta_t(ab) = \psi_t(a)\Delta_t(b) + \Delta_t(a)b . \tag{9}$$

Taking norms we obtain

$$\begin{aligned}\|\Delta_t(ab)\|_{\mathcal{A}} &\le \|\psi_t(a)\|_{\mathcal{A}}\|\Delta_t(b)\|_{\mathcal{A}} + \|\Delta_t(a)\|_{\mathcal{A}}\|b\|_{\mathcal{A}} \\ &\le M_\Psi\|a\|_{\mathcal{A}}\|\Delta_t(b)\|_{\mathcal{A}} + \|\Delta_t(a)\|_{\mathcal{A}}\|b\|_{\mathcal{A}}\,.\end{aligned}$$

This implies a similar relation for the Besov-seminorms, namely,

$$|ab|_{\Lambda^p_r(\mathcal{A})} \le M_\Psi\|a\|_{\mathcal{A}}|b|_{\Lambda^p_r(\mathcal{A})} + \|b\|_{\mathcal{A}}|a|_{\Lambda^p_r(\mathcal{A})}.$$

So

$$\|ab\|_{\Lambda^p_r(\mathcal{A})} = \|ab\|_{\mathcal{A}} + |ab|_{\Lambda^p_r(\mathcal{A})} \le C\|a\|_{\Lambda^p_r(\mathcal{A})}\|b\|_{\Lambda^p_r(\mathcal{A})}\,,$$

and the assertion follows.

To show that $\Lambda^p_r(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ we assume that $a \in \Lambda^p_r(\mathcal{A})$ is invertible in $\mathcal{A}$. It is sufficient to verify that $|a^{-1}|_{\Lambda^p_r(\mathcal{A})}$ is finite. By a straightforward computation we obtain

$$\Delta_t(a^{-1}) = -\psi_t(a^{-1})\,\Delta_t(a)\,a^{-1}\,. \tag{10}$$

This implies that a^{-1} has a finite $\Lambda^p_r(\mathcal{A})$-norm.

In the general case we can use the reiteration theorem (Theorem 3.7) and the transitivity of inverse-closedness, and prove the statement by induction. Assume that the statement is proved for all smoothness parameters smaller than $s > 0$. As $\Lambda^p_r(\mathcal{A}) = \Lambda^p_{r-s}(\Lambda^p_s(\mathcal{A}))$ for $r > s$, the preceding argument yields $\Lambda^p_{r-s}(\Lambda^p_s(\mathcal{A}))$ is inverse-closed in $\Lambda^p_s(\mathcal{A})$ for $s < r < s+1$. As $\Lambda^p_s(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ by hypotheses, the theorem is proved. □

3.3. Characterization of Besov Spaces as Approximation Spaces.

As in the case of function spaces, $\Lambda^p_r(\mathcal{A})$ can be characterized by approximation properties. This was carried out for $\Lambda^\infty_r(\mathcal{A})$ in [20], so our treatment is very brief.

We focus on the approximation of a Banach algebra $\mathcal{A}$ with automorphism group Ψ by smooth elements.

Definition 3.9 (Bernstein inequality)**.** An element $a \in \mathcal{A}$ is *σ-bandlimited* for $\sigma > 0$, if there is a constant C such that for every multi-index α

$$\|\delta^\alpha(a)\|_{\mathcal{A}} \le C(2\pi\sigma)^{|\alpha|}\,. \tag{11}$$

An element is *bandlimited*, if it is σ-bandlimited for some $\sigma > 0$.

If $\mathcal{A}$ is a Banach algebra with automorphism group Ψ, then,

$$X_0 = \{0\}, \quad X_\sigma = \{a \in \mathcal{A}\colon a \text{ is } \sigma\text{-bandlimited}\}, \quad \sigma > 0$$

is an approximation scheme for $\mathcal{A}$ [20, Lemma 5.8]. From now on we use this approximation scheme without further notice.

In particular, if $\mathcal{A}$ is ahomogeneous matrix algebra, we obtain the following characterization of bandlimited elements

Proposition 3.10 ([20, 5.7])**.** *A matrix A is banded with bandwidth N in the homogeneous matrix algebra $\mathcal{A}$, if and only if it is N-bandlimited with respect to the group action $\{\chi_t\}$.*

Theorem 3.11 (Jackson Bernstein Theorem)**.** *Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ, and assume that $r > 0$ and $1 \le p \le \infty$. If $\{X_\sigma : \sigma \ge 0\}$ is the approximation scheme of bandlimited elements, then*

$$\Lambda^p_r(\mathcal{A}) = \mathcal{E}^p_r(\mathcal{A})\,. \tag{12}$$

The proof is in appendix B.

Littlewood-Paley Decomposition. The norms of Besov spaces are not easily computable. An equivalent explicit norm for these spaces can be obtained by means of a Littlewood-Paley decomposition.

First we need some technical preparation: If $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $a \in C(\mathcal{A})$, the *action of* μ on a is defined by

$$\mu * a = \int_{\mathbb{R}^d} \psi_{-t}(a) d\mu(t). \tag{13}$$

This action is a generalization of the usual convolution and satisfies similar properties:

$$\|\mu * a\|_{\mathcal{A}} \leq M_\Psi \|\mu\|_{M(\mathbb{R}^d)} \|a\|_{\mathcal{A}} .$$

If $f \in C_c^\infty(\mathbb{R}^d)$ then

$$\delta^\alpha(f * a) = D^\alpha f * a \in C(\mathcal{A}) \tag{14}$$

for every multi-index α. See [13] for details and proofs.

In particular, if the group action is periodic, the action of μ on a is

$$\mu * a = \int_{\mathbb{T}^d} \psi_{-t}(a)\, d\mu(t) = \sum_{k \in \mathbb{Z}^d} \mathcal{F}(\mu)(k) \hat{a}(k)\,, \tag{15}$$

where $\hat{a}(k) = \int_{\mathbb{T}^d} \psi_{-t}(a) e^{2\pi i k \cdot t}\, dt$ is the k-th Fourier coefficient of a and the sum converges in the C1-sense.
Now assume that $\varphi \in \mathscr{S}(\mathbb{R}^d)$ with $\operatorname{supp} \hat{\varphi} \subseteq \{\omega \in \mathbb{R}^d \colon 2^{-1} \leq |\omega|_\infty \leq 2\}$, $\hat{\varphi}(\omega) > 0$ for $2^{-1} < |\omega|_\infty < 2$, and $\sum_{k \in \mathbb{Z}} \hat{\varphi}(2^{-k}\omega) = 1$ for all $\omega \in \mathbb{R}^d \setminus \{0\}$. Set $\hat{\varphi}_k(\omega) = \hat{\varphi}(2^{-k}\omega), k \in \mathbb{N}_0$, so $\varphi_k(x) = 2^{kd}\varphi_0(2^k x)$, and let $\hat{\varphi}_{-1} = 1 - \sum_{k=0}^\infty \hat{\varphi}_k$. Then $\{\hat{\varphi}_k\}_{k \geq -1}$ is a dyadic partition of unity.

Proposition 3.12. *Let $\{\hat{\varphi}_k\}_{k \geq -1}$ be a dyadic partition of unity, and $1 \leq p \leq \infty$, $r > 0$. An element $a \in \mathcal{A}$ is in $\Lambda_r^p(\mathcal{A})$, if and only if*

$$\left(\sum_{k=-1}^\infty 2^{rkp} \|\varphi_k * a\|_{\mathcal{A}}^p \right)^{1/p} < \infty\,. \tag{16}$$

The expression (16) defines an equivalent norm on $\Lambda_r^p(\mathcal{A})$. Moreover the Littlewood-Paley decomposition $\sum_{k=0}^\infty \varphi_k * a$ converges to a in the norm of $\mathcal{A}$.

The special case $p = \infty$ was proved in [20] with a weak type argument. This approach does not work for $p < \infty$, so we adapt a proof in [9], see Appendix C.

Approximation of Polynomial Order in Homogeneous Matrix Spaces.

Lemma 3.13. *If $\{\varphi_k\}_{k \geq -1}$ is a dyadic partition of unity and the action of Ψ on $\mathcal{A}$ is periodic, then for $a \in C(\mathcal{A})$,*

$$\varphi_k * a = \sum_{\lfloor 2^{k-1} \rfloor \leq |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l) \hat{a}(l)\,. \tag{17}$$

Proof. Let $\varphi_k^\Pi(t) = \sum_{l \in \mathbb{Z}^d} \varphi_k(t + l)$ denote the periodization of φ_k. Then

$$\varphi_k^\Pi(t) = \sum_{\lfloor 2^{k-1} \rfloor \leq |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l) e^{2\pi i l \cdot t}$$

by Poisson's summation formula. Equation (17) now follows, combining (15) with

$$\varphi_k * a = \int_{\mathbb{R}^d} \psi_{-t}(a) \varphi_k(t)\, dt = \int_{\mathbb{T}^d} \psi_{-t}(a) \varphi_k^\Pi(t)\, dt \qquad \square$$

Equation (17) allows us to obtain a characterization of the approximation spaces for homogeneous matrix algebras by the Littlewood-Paley decomposition of its elements.

Proposition 3.14. *Let $\mathcal{A}$ be a homogeneous matrix algebra, $r > 0$, and $\Phi = \{\varphi_k\}_{k\geq -1}$ a dyadic partition of unity. Then the norm on the approximation space $\mathcal{E}_r^p(\mathcal{A}) = \Lambda_r^p(\mathcal{A})$ is equivalent to*

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Big(\sum_{k=0}^{\infty} 2^{kpr} \Big\| \sum_{\lfloor 2^{k-1} \rfloor \leq |l|_\infty < 2^{k+1}} \hat{\varphi}_k(l)\hat{A}(l) \Big\|_{\mathcal{A}}^p \Big)^{1/p} . \tag{18}$$

If $\mathcal{A}$ is solid, then

$$\|A\|_{\mathcal{E}_r^p(\mathcal{A})} \asymp \Big(\sum_{k=-1}^{\infty} 2^{kpr} \Big\| \sum_{\lfloor 2^{k} \rfloor \leq |l|_\infty < 2^{k+1}} \hat{A}(l) \Big\|_{\mathcal{A}}^p \Big)^{1/p} . \tag{19}$$

Remark. If the matrix algebra $\mathcal{A}$ is solid, similar results can be obtained for approximation spaces of non-polynomial order [28].

Proof. The results for general homogeneous matrix algebras follow from the Jackson Bernstein Theorem (Theorem 3.11) and the Littlewood-Paley decomposition. We still have to prove the norm equivalence (19). Set $C_k = \|\sum_{2^k \leq |l| < 2^{k+1}} \hat{A}(l)\|_{\mathcal{A}}$. The solidity of $\mathcal{A}$ implies that, for $k \geq -1$,

$$B_k = \|\varphi_k * A\|_{\mathcal{A}} \leq \| \sum_{2^{k-1} \leq |l|_\infty < 2^{k+1}} \hat{A}(l)\|_{\mathcal{A}} = C_{k-1} + C_k$$

On the other hand, since $\phi_{k-1} + \phi_k + \phi_{k+1} \equiv 1$ on $\{\xi \colon 2^{k-1} \leq |\xi|_2 \leq 2^{k+1}\}$, we obtain $C_k \leq B_{k-1} + B_k + B_{k+1}$. So $\|A\|^p_{\mathcal{E}_r^p(\mathcal{A})} \asymp \sum_{k=0}^{\infty} 2^{kpr} B_k \asymp \sum_{k=-1}^{\infty} 2^{kpr} C_k$, and this is (19). □

We apply the preceding results to the example of $\mathcal{C}_r^p$ (see Section 2.3 for the definition). We obtain

$$\mathcal{E}_s^q(\mathcal{C}_r^p) = \mathcal{E}_{s+r}^q(\mathcal{C}_0^p) ,$$

$$\|A\|_{\mathcal{E}_r^q(\mathcal{C}_s^p)} \asymp \Big(\sum_{j=0}^{\infty} 2^{jq(r+s)} \big(\sum_{\lfloor 2^{j-1} \rfloor \leq |k|_\infty < 2^j} \|A[k]\|^p_{\ell^\infty(\mathbb{Z}^d)} \big)^{q/p} \Big)^{1/q} \asymp \|A\|_{\mathcal{E}_{r+s}^q(\mathcal{C}_0^p)} .$$

In particular,

$$\mathcal{E}_s^p(\mathcal{C}_r^p) = \mathcal{C}_{r+s}^p .$$

If $p \neq q$ these norms define new classes of inverse-closed subalgebras of $\mathcal{B}(\ell^2)$ with a form of off-diagonal decay suited to approximation with banded matrices.

These results should be compared to the definition of *discrete Besov spaces* [32].

3.4. **Bessel Potential Spaces.** Bessel potentials allow us to define an analogue of *polynomial* weights in a Banach algebra with an automorphism group. For homogeneous matrix algebras the Bessel potential spaces are weighted algebras.

We define the Bessel kernel $\mathcal{G}_r$ by its Fourier transform,

$$\mathcal{F}\mathcal{G}_r(\omega) = (1 + |2\pi\omega|_2^2)^{-r/2} , \quad r > 0.$$

C_w groups. In analogy to the case of real functions we would like to define an element of the Bessel potential space $\mathcal{P}_r(\mathcal{A})$ as an element of the form $a = \mathcal{G}_r * y$ for some $y \in \mathcal{A}$. However, the action $\mathcal{G}_r * y$ is defined only for $y \in C(\mathcal{A})$. Using a weaker form of continuity for the action of the automorphism group we can extend the convolution "$*$" to the whole algebra for all examples of matrix algebras in Section 2.3.

Definition 3.15 ([4, 12])**.** Let $\mathcal{A}$ be a Banach algebra with automorphism group Ψ. For $a \in \mathcal{A}$, $a' \in \mathcal{A}'$ define $G_{a',a}(t) = \langle a', \psi_t(a)\rangle$. Assume that

$$\mathcal{A}'_\Psi = \{a' \in \mathcal{A}' \colon G_{a',a} \text{ is continuous for all } a \in \mathcal{A}\}$$

is a *norm fundamental subspace* of $\mathcal{A}'$, that is

$$\|a\|_{\mathcal{A}} = \sup\{|\langle a', a\rangle| \colon a' \in \mathcal{A}'_\Psi,\ \|a'\|_{\mathcal{A}'} \leq 1\}$$

for all $a \in \mathcal{A}$. Assume that $\mathcal{A}$ is equipped with the weak topology $\sigma(\mathcal{A}, \mathcal{A}'_\Psi)$ with respect to the functionals in $\mathcal{A}'_\Psi$, and
(20)
the convex hull of every $\sigma(\mathcal{A}, \mathcal{A}'_\Psi)$-compact set has $\sigma(\mathcal{A}, \mathcal{A}'_\Psi)$-compact closure.

In this case we call Ψ a C_w group and denote $\mathcal{A}$ with the $\sigma(\mathcal{A}, \mathcal{A}'_\Psi)$ topology by $\mathcal{A}_w$, if necessary.

Condition (20) ensures the existence of the "convolution integral" (13) as a Pettis integral, see below. If $\sigma(\mathcal{A}, \mathcal{A}'_\Psi)$ is *quasi-complete*, i.e., bounded Cauchy nets converge, then condition (20) is automatically satisfied [29].

Example 3.16.

(1) If $\mathcal{A}'_\Psi$ is the predual of $\mathcal{A}$ (in particular, if $\mathcal{A}$ is a von Neumann algebra) the quasi-completeness is a consequence of the Banach-Alaoglu theorem.
(2) If $\mathcal{A}$ is a Banach function space in the sense of [8], and $\mathcal{A}'_\Psi$ is a norm fundamental order ideal of the *Koethe dual* $\mathcal{A}^\sim$, then it is known that $(\mathcal{A}, \sigma(\mathcal{A}, \mathcal{A}'_\Psi))$ is quasi-complete [8, 1.5.2].
(3) If $\mathcal{A} = C(\mathcal{A})$ then $\mathcal{A}'_\Psi = \mathcal{A}'$. It is well-known that ψ_t is strongly continuous at $a \in \mathcal{A}$, if and only if it is continuous with respect to the $\sigma(\mathcal{A}, \mathcal{A}')$-topology [13, 24], so in this case $\mathcal{A}_w = \mathcal{A}$. In this case the condition (20) is a consequence of the Krein-Smulian theorem.

Remarks. If the group action is uniformly bounded the space $\mathcal{A}'_\Psi$ is a norm-closed subspace of $\mathcal{A}'$. Indeed, if $a'_k \in \mathcal{A}'_\Psi$ and $a'_k \to a'$ in norm, then

$$\begin{aligned}\lim_{t\to 0} G_{a',a}(t) - G_{a',a}(0) &= \lim_{t\to 0}\langle a', \psi_t(a) - a\rangle = \lim_{t\to 0}\lim_k \langle a'_k, \psi_t(a) - a\rangle \\ &= \lim_k \lim_{t\to 0}\langle a'_k, \psi_t(a) - a\rangle = 0\,.\end{aligned}$$

We do not have general conditions when the action of χ on a homogeneous matrix algebra is a C_w-group. For the specific examples of matrix algebras introduced in Section 2.3 we can prove that χ is a C_w group.

Example 3.17. Recall that $\mathcal{B}(\ell^2)$ is the dual of the trace class operators $\mathcal{B}_1$, $\mathcal{B}(\ell^2) = (\mathcal{B}_1)'$ and the finite rank operators are dense in $\mathcal{B}_1$. Adapting a continuity argument from [16] we verify that $(\mathcal{B}(\ell^2))'_\chi \supseteq \mathcal{B}_1$. Indeed, for $x, y \in \ell^2(\mathbb{Z}^d)$ and the rank one operator $(x \otimes y)z = \langle z, y\rangle x$ we obtain

$$\begin{aligned}G_{x\otimes y,A}(t) - G_{x\otimes y,A}(t) &= \mathrm{tr}((x\otimes y)\chi_t(A)) - \mathrm{tr}((x \otimes y)A) = \langle x, (\chi_t(A) - A)y\rangle \\ &= \langle x, M_t A M_{-t}(y - M_t y)\rangle\end{aligned}$$

As $\lim_{t\to 0}\|z - M_t z\|_{\ell^2(\mathbb{Z}^d)} = 0$ for every $z \in \ell^2(\mathbb{Z}^d)$, it follows that $G_{x\otimes y,A}$ is continuous. So, if A' is a finite rank operator then $G_{A',A}$ is continuous. As $\mathcal{A}'_\Psi$ is norm closed in $\mathcal{A}'$, and the finite rank operators are dense in $\mathcal{B}_1$ we obtain the continuity of $G_{A',A}$ for all $A' \in \mathcal{B}_1$. We have shown that the space $\mathcal{A}'_\Psi$ contains $\mathcal{B}_1$. This implies that $\mathcal{A}'_\Psi$ is norm fundamental, and so χ is a C_w-group on $\mathcal{B}(\ell^2)$.

Example 3.18. If $\mathcal{A} = \mathcal{S}^p_r$ we can argue as follows: Let $\ell^{\infty,p}_{m_r}(\mathbb{Z}^{2d})$ the mixed norm space on $\mathbb{Z}^{2d}$ with

$$\|(x(k,l))_{k,l\in\mathbb{Z}^d}\|_{\ell^{\infty,p}_{m_r}} = \sup_{k\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d}|x(k,l)|^p(1+|k-l|)^{rp}\Big)^{1/p}$$

and define $(jx)(k,l) = x(l,k)$. Then we obtain the isometric isomorphism

$$\mathcal{S}^p_r \cong \ell^{\infty,p}_{m_r}(\mathbb{Z}^{2d}) \cap j\big(\ell^{\infty,p}_{m_r}(\mathbb{Z}^{2d})\big)\,.$$

From [15, Lemma 1.12] (and using standard facts about sequence spaces, e.g [29, 30.3] we conclude that

$$\big(\ell^{\infty,p}_{m_r}(\mathbb{Z}^{2d}) \cap j\big(\ell^{\infty,p}_{m_r}(\mathbb{Z}^{2d})\big)\big)^{\sim} \cong \ell^{1,p'}_{m_{-r}}(\mathbb{Z}^{2d}) + j\big(\ell^{1,p'}_{m_{-r}}(\mathbb{Z}^{2d})\big)$$

It is routine to verify that $G_{A',A}$ is continuous for $A' \in \ell^{1,p'}_{m_{-r}}(\mathbb{Z}^{2d}) + j\big(\ell^{1,p'}_{m_{-r}}(\mathbb{Z}^{2d})\big)$ and $A \in \mathcal{S}^p_r$, so Example 3.16 (2) verifies that $\mathcal{S}^p_r$ is a C_w-group.

We need the concept of C_w-groups not only to extend the action of a measure defined in (13) to the whole of $\mathcal{A}$, but also to give a weak type description of this action.

Proposition 3.19 ([3, 1.2])**.** *If Ψ is a C_w- group for the Banach algebra $\mathcal{A}$, then for each $\mu \in \mathcal{M}(\mathbb{R}^d)$ and each $a \in \mathcal{A}$ there is an element, denoted as $\mu * a \in \mathcal{A}$, such that*

$$\langle a', \mu * a\rangle = \int_{\mathbb{R}^d}\langle a', \psi_{-t}(a)\rangle\, d\mu(t)$$

for all $a' \in \mathcal{A}'_\Psi$. As usual we write

$$\mu * a = \int_{\mathbb{R}^d}\psi_{-t}(a)\, d\mu(t)\,. \tag{21}$$

We obtain the norm inequality

$$\|\mu * a\| \le M_\Psi \|a\|_{\mathcal{A}}\|\mu\|_{\mathcal{M}(\mathbb{R}^d)}\,.$$

Remarks. Clearly in special cases the existence of the integral (21) can be verified directly. In particular, if $\mathcal{A} = C(\mathcal{A})$ the integral exists in the sense of Bochner.

The following result is straightforward.

Proposition 3.20. *If Ψ is a C_w- group for the Banach algebra $\mathcal{A}$, and $G_{a',a}(t) = \langle a', \psi_t(a)\rangle$ for $a' \in \mathcal{A}'_\Psi$, $a \in \mathcal{A}$, then*

$$\|a\|_{\mathcal{A}} \asymp \sup\{\|G_{a',a}\|_\infty \colon a' \in \mathcal{A}'_\Psi, \|a'\|_{\mathcal{A}'} \le 1\}\,.$$

Moreover,

$$G_{a',\mu * a} = \mu * G_{a',a}\,. \tag{22}$$

Before defining Bessel potential spaces we list properties of the Bessel kernel that will be needed in the sequel.

Lemma 3.21 ([38, V.5])**.**

(1) $\mathcal{G}_r \in \Lambda^\infty_r(L^1(\mathbb{R}^d))$, $\|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)} = 1$,
(2) $\mathcal{G}_r * \mathcal{G}_s = \mathcal{G}_{r+s}$ *for all* $r, s > 0$,
(3) $\mathcal{G}_r * \mathscr{S} = \{\mathcal{G}_r * \varphi \colon \varphi \in \mathscr{S}\} = \mathscr{S}$.

Definition 3.22. Let $\mathcal{A}$ be a Banach space and Ψ a C_w- group acting on $\mathcal{A}$ (this includes the case $A = C(\mathcal{A})$). The *Bessel potential space* of order $r > 0$ is

$$\mathcal{P}_r(\mathcal{A}) = \mathcal{G}_r * \mathcal{A} = \{a \in \mathcal{A} \colon a = \mathcal{G}_r * y \text{ for some } y \in \mathcal{A}\}$$

with the norm

$$\|\mathcal{G}_r * y\|_{\mathcal{P}_r(\mathcal{A})} = \|y\|_{\mathcal{A}}.$$

We have to verify that the definition of the norm on $\mathcal{P}_r(\mathcal{A})$ is consistent, that is, we show that the convolution with $\mathcal{G}_r$ is injective on $\mathcal{A}$. We use a weak type argument.

Let $y \in \mathcal{A}$ with $\mathcal{G}_r * y = 0$. This is equivalent to

$$G_{a',\mathcal{G}_r * y}(t) = \mathcal{G}_r * G_{a',y}(t) = 0$$

for all $t \in \mathbb{R}^d$ and all $a' \in \mathcal{A}'_\Psi$. Now we proceed as in [38, V.3.3]. We choose a test function $\varphi \in \mathscr{S}$ and obtain

$$\int_{\mathbb{R}^d} (\mathcal{G}_r * G_{a',y})(t)\varphi(t)\, dt = \int_{\mathbb{R}^d} G_{a',y}(t)(\mathcal{G}_r * \varphi)(t)\, dt = 0\,.$$

By Lemma 3.21 (3) the convolution with $\mathcal{G}_r$ is surjective on $\mathscr{S}$, and so it follows that $G_{a',y} = 0$ for all $a' \in \mathcal{A}'_\Psi$, that is, $y = 0$.

An immediate consequence of Definition 3.22 is the embedding $\mathcal{P}_r(\mathcal{A}) \hookrightarrow \mathcal{A}$. Indeed, if $a \in \mathcal{P}_r(\mathcal{A})$, then $a = \mathcal{G}_r * y$ for a $y \in \mathcal{A}$, and

$$\|a\|_{\mathcal{A}} \le \|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)}\|y\|_{\mathcal{A}} = \|\mathcal{G}_r\|_{L^1(\mathbb{R}^d)}\|a\|_{\mathcal{P}_r(\mathcal{A})}. \tag{23}$$

As $\mathcal{G}_r * \mathcal{G}_s = \mathcal{G}_{r+s}$ for $r, s > 0$ we obtain a useful reiteration property for the Bessel potential spaces.

Proposition 3.23. *If $\mathcal{A}$ is a Banach algebra and Ψ a C_w-automorphism group on $\mathcal{A}$, then for all $r, s > 0$*

$$\mathcal{P}_r(\mathcal{P}_s(\mathcal{A})) = \mathcal{P}_{r+s}(\mathcal{A})\,.$$

Proof. We have to verify that Ψ is a C_w-automorphism group on $\mathcal{P}_r(\mathcal{A})$. For this we show that the dual pairing defined by

$$\langle a', \mathcal{G}_r * y\rangle_{\mathcal{A}'_\Psi \times \mathcal{P}_r(\mathcal{A})} = \langle a', y\rangle_{\mathcal{A}' \times \mathcal{A}}$$

yields a norm-fundamental subspace of $\mathcal{P}_r(\mathcal{A})'$. As

$$|\langle a', \mathcal{G}_r * y\rangle_{\mathcal{A}'_\Psi \times \mathcal{P}_r(\mathcal{A})}| \le \|a'\|_{\mathcal{A}'}\|y\|_{\mathcal{A}} = \|a'\|_{\mathcal{A}'}\|\mathcal{G}_r * y\|_{\mathcal{P}_r(\mathcal{A})}$$

the mapping $z \mapsto \langle a', z\rangle_{\mathcal{A}'_\Psi \times \mathcal{P}_r(\mathcal{A})}$ is continuous for every $a' \in \mathcal{A}'_\Psi$, so $\mathcal{P}_r(\mathcal{A})'_\Psi \supset \mathcal{A}'_\Psi$. Moreover, a straightforward computation shows that $\|a'\|_{\mathcal{P}_r(\mathcal{A})'} = \|a'\|_{\mathcal{A}'}$. By definition $t \mapsto \langle a', \psi_t z\rangle_{\mathcal{A}'_\Psi \times \mathcal{P}_r(\mathcal{A})}$ is continuous for each $a' \in \mathcal{A}'_\Psi$ and each $z \in \mathcal{P}_r(\mathcal{A})$. Finally, $\mathcal{A}'_\Psi$ is norm fundamental, as we have for $z = \mathcal{G}_r * y$

$$\begin{aligned}
&\sup\{|\langle a', y\rangle_{\mathcal{A}'_\Psi \times \mathcal{P}_r(\mathcal{A})}|\colon a' \in \mathcal{A}'_\Psi, \|a'\|_{\mathcal{P}_r(\mathcal{A})'} \le 1\}\\
=&\sup\{|\langle a', y\rangle_{\mathcal{A}' \times \mathcal{A}}|\colon a' \in \mathcal{A}'_\Psi, \|a'\|_{\mathcal{A}'} \le 1\}\\
=&\|y\|_{\mathcal{A}} = \|z\|_{\mathcal{P}_r(\mathcal{A})}
\end{aligned}$$

□

3.4.1. *Characterization by Hypersingular Integrals.*

Lemma 3.24. *If $a \in \mathcal{P}_r(\mathcal{A})$, then $\|a\|_{\mathcal{P}_r(\mathcal{A})} \asymp \sup_{\|a'\|_{\mathcal{A}'} \le 1}\|G_{a',a}\|_{\mathcal{P}_r(L^\infty)}$, where the dual pairing in $G_{a',a}$ is the one of $\mathcal{A}'_\Psi \times \mathcal{A}$.*

Proof. Let $a = \mathcal{G}_r * y$. Then

$$\begin{aligned}
\|a\|_{\mathcal{P}_r(\mathcal{A})} =& \|y\|_{\mathcal{A}} \asymp \sup_{\|a'\|_{\mathcal{A}'} \le 1}\|G_{a',y}\|_\infty\\
&= \sup_{\|a'\|_{\mathcal{A}'} \le 1}\|\mathcal{G}_r * G_{a',y}\|_{\mathcal{P}_r(L^\infty)} = \sup_{\|a'\|_{\mathcal{A}'} \le 1}\|G_{a',\mathcal{G}_r * y}\|_{\mathcal{P}_r(L^\infty)}\,.
\end{aligned}$$

□

We state a special case of a result by Wheeden [46] (see also [37],[38, V.6.10]).

Theorem 3.25. *Let $0 < r < 2$. A function f is an element of $\mathcal{P}_r(L^\infty(\mathbb{R}^d))$ if and only if $f \in L^\infty(\mathbb{R}^d)$ and*

$$\sup_{\epsilon>0}\Big\|\int_{|t|_2\geq\epsilon} |t|_2^{-r}\Delta_t(f)\frac{dt}{|t|_2^d}\Big\|_{L^\infty(\mathbb{R}^d)} < \infty. \tag{24}$$

If (24) holds,

$$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{\epsilon>0}\Big\|\int_{|t|_2\geq\epsilon} |t|_2^{-r}\Delta_t(f)\frac{dt}{|t|_2^d}\Big\|_{L^\infty(\mathbb{R}^d)} < \infty \tag{25}$$

defines an equivalent norm on $\mathcal{P}_r(L^\infty(\mathbb{R}^d))$.

Combining Lemma 3.24 with Theorem 3.25 we obtain the first statement of the following theorem.

Theorem 3.26. *Let $\mathcal{A}$ be a Banach algebra and Ψ a C_w-automorphism group acting on it. For $0 < r < 2$ the norm $\|a\|_{\mathcal{P}_r(\mathcal{A})}$ is equivalent to*

$$\|a\|_{\mathcal{A}} + \sup_{\epsilon>0}\Big\|\int_{|t|_2\geq\epsilon} |t|_2^{-r}\Delta_t(a)\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}. \tag{26}$$

This norm is further equivalent to

$$\|a\|_{\mathcal{A}} + \sup_{\epsilon>0}\Big\|\int_{\epsilon\leq|t|_2\leq 1} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}.$$

Proof. We only show the second statement. As

$$\begin{aligned}
\Big\|\int_{\epsilon\leq|t|_2} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} &\leq \Big\|\int_{\epsilon\leq|t|_2\leq 1} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} + \Big\|\int_{|t|_2\geq 1} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}\\
&\leq \Big\|\int_{\epsilon\leq|t|_2\leq 1} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} + (1+M_\Psi)\|a\|_{\mathcal{A}}\int_{|t|_2\geq 1}|t|_2^{-r}\frac{dt}{|t|_2^d}\\
&\leq C\Big(\|a\|_{\mathcal{A}} + \Big\|\int_{\epsilon\leq|t|_2\leq 1} \frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}\Big),
\end{aligned}$$

the proof of the other inequality works in a similar way. □

Next we compare Bessel potential spaces with Besov spaces.

Proposition 3.27. *If $\mathcal{A}$ is Banach algebra with C_w-automorphism group Ψ, then*

$$\Lambda_r^1(\mathcal{A}) \hookrightarrow \mathcal{P}_r(\mathcal{A}) \hookrightarrow \Lambda_r^\infty(\mathcal{A}) \qquad \text{if } r > 0.$$

Proof. For the proof of the embedding $\mathcal{P}_r(\mathcal{A}) \hookrightarrow \Lambda_r^\infty(\mathcal{A})$ let $a \in \mathcal{P}_r(\mathcal{A})$ with $a = \mathcal{G}_r * y$, $y \in \mathcal{A}$. The seminorm $|a|_{\Lambda_r^\infty(\mathcal{A})}$ can be estimated for $k > \lfloor r \rfloor$ as

$$\begin{aligned}
|a|_{\Lambda_r^\infty(\mathcal{A})} &= \sup_{|t|\neq 0}\frac{\|\Delta_t^k(\mathcal{G}_r * y)\|_{\mathcal{A}}}{|t|^r} = \sup_{|t|\neq 0}\Big\|\frac{\Delta_t^k(\mathcal{G}_r)}{|t|^r} * y\Big\|_{\mathcal{A}}\\
&\leq \sup_{|t|\neq 0}\Big\|\frac{\Delta_t^k(\mathcal{G}_r)}{|t|^r}\Big\|_{L^1(\mathbb{R}^d)}\|y\|_{\mathcal{A}} = \|\mathcal{G}_r\|_{\Lambda_r^\infty(L^1)}\|a\|_{\mathcal{P}_r(\mathcal{A})},
\end{aligned}$$

and this is the desired embedding. We still have to verify the first inclusion. Assume first that $0 < r < 1$. By Theorem 3.26, for an $a \in \mathcal{P}_r(\mathcal{A})$

$$\begin{aligned}
\|a\|_{\mathcal{P}_r(\mathcal{A})} &\asymp \|a\|_{\mathcal{A}} + \sup_{\epsilon>0}\Big\|\int_{|t|_2\geq\epsilon}|t|_2^{-r}\Delta_t(a)\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}\\
&\leq \|a\|_{\mathcal{A}} + \int_{\mathbb{R}^d}|t|_2^{-r}\|\Delta_t(a)\|_{\mathcal{A}}\frac{dt}{|t|_2^d}\\
&= \|a\|_{\Lambda_r^1(\mathcal{A})}.
\end{aligned}$$

In the general case we proceed by induction. Assume that the statement is true for all positive values up to $s > 0$, and $s < r < s + 1$. Then

$$\Lambda^1_r(\mathcal{A}) = \Lambda^1_{r-s}(\Lambda^1_s(\mathcal{A})) \subseteq \mathcal{P}_{r-s}(\Lambda^1_s(\mathcal{A})) \subseteq \mathcal{P}_{r-s}(\mathcal{P}_s(\mathcal{A})) = \mathcal{P}_r(\mathcal{A}),$$

where we have used the reiteration theorems for the Bessel and the Besov spaces (Theorem 3.7). □

Another application of the reiteration theorem and the representation of the norm of $\mathcal{P}_r(\mathcal{A})$ by the hypersingular integral (26) shows how Besov spaces and Bessel potential spaces interact.

Proposition 3.28. *If $\mathcal{A}$ is a Banach algebra with C_w-automorphism group Ψ, then for all $r, s > 0$ and $1 \le p \le \infty$*

$$\mathcal{P}_r(\Lambda^p_s(\mathcal{A})) = \Lambda^p_s(\mathcal{P}_r(\mathcal{A})) = \Lambda^p_{r+s}(\mathcal{A}). \tag{27}$$

Proof. Again, we need to know first that Ψ is a C_w-automorphism group on $\Lambda^p_r(\mathcal{A})$. If $p < \infty$ then $C(\Lambda^p_r(\mathcal{A})) = \Lambda^p_r(\mathcal{A})$ by Proposition 3.6. If $p = \infty$ the assertion follows from

$$\|a\|_{\Lambda^\infty_r(\mathcal{A})} = \sup_{|t|\neq 0} \sup\{\langle a', \frac{\Delta^k_t(a)}{|t|^r}\rangle : a' \in \mathcal{A}'_\Psi, \|a'\|_{\mathcal{A}'} \le 1\}.$$

The details are similar to the proof of the analogue statement in Proposition 3.23 and are left to the reader.

Using the reiteration theorems for Bessel potential spaces and Besov spaces, it suffices to prove the proposition only for $0 < r, s < 1$. We show first that $\mathcal{P}_r(\Lambda^p_s(\mathcal{A})) \hookrightarrow \Lambda^p_s(\mathcal{P}_r(\mathcal{A}))$. Assume that $a \in \mathcal{P}_r(\Lambda^p_s(\mathcal{A}))$, so $a = \mathcal{G}_r * y$ with $y \in \Lambda^p_s(\mathcal{A})$. We obtain the following estimate.

$$\begin{aligned}
\|a\|^p_{\Lambda^p_s(\mathcal{P}_r(\mathcal{A}))} &= \int_{\mathbb{R}^d} \frac{\|\Delta_t(a)\|^p_{\mathcal{P}_r(\mathcal{A})}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \int_{\mathbb{R}^d} \frac{\|\Delta_t(\mathcal{G}_r * y)\|^p_{\mathcal{P}_r(\mathcal{A})}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \int_{\mathbb{R}^d} \frac{\|\mathcal{G}_r * \Delta_t(y)\|^p_{\mathcal{P}_r(\mathcal{A})}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \int_{\mathbb{R}^d} \frac{\|\Delta_t(y)\|^p_{\mathcal{A}}}{|t|^{sp}} \frac{dt}{|t|^d} \\
&= \|y\|^p_{\Lambda^p_s(\mathcal{A})} = \|\mathcal{G}_r * y\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{A}))}.
\end{aligned}$$

Now let $a = \mathcal{G}_r * y \in \mathcal{P}_r(\Lambda^p_s(\mathcal{A}))$. Then

$$\begin{aligned}
\|a\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{A}))} &= \|\mathcal{G}_r * y\|^p_{\mathcal{P}_r(\Lambda^p_s(\mathcal{A}))} = \|y\|^p_{\Lambda^p_s(\mathcal{A})} = \int_{\mathbb{R}^d} \frac{\|\Delta_t(y)\|^p_{\mathcal{A}}}{|t|^{sp}} \frac{dt}{|t|^d_2} \\
&= \int_{\mathbb{R}^d} \frac{\|\Delta_t(\mathcal{G}_r * y)\|^p_{\mathcal{P}_r(\mathcal{A})}}{|t|^{sp}} \frac{dt}{|t|^d_2} \\
&= \|\mathcal{G}_r * y\|_{\Lambda^p_s(\mathcal{P}_r(\mathcal{A}))} = \|a\|_{\Lambda^p_s(\mathcal{P}_r(\mathcal{A}))}.
\end{aligned}$$

Consequently $\mathcal{P}_r(\Lambda^p_s(\mathcal{A})) = \Lambda^p_s(\mathcal{P}_r(\mathcal{A}))$. Finally, Proposition 3.27 implies that

$$\Lambda^p_s(\Lambda^1_r(\mathcal{A})) \hookrightarrow \Lambda^p_s(\mathcal{P}_r(\mathcal{A})) \hookrightarrow \Lambda^p_s(\Lambda^\infty_r(\mathcal{A})),$$

and the first and last space in this chain equal $\Lambda^p_{r+s}(\mathcal{A})$ by the reiteration theorem for Besov spaces (Theorem 3.7). □

Algebra Properties. The characterization of Bessel potential spaces by a hypersingular integral yields the Banach algebra properties of $\mathcal{P}_r(\mathcal{A})$.

Theorem 3.29. *If $\mathcal{A}$ is a Banach algebra with C_w-group Ψ, then the Bessel potential space $\mathcal{P}_r(\mathcal{A})$ is a Banach subalgebra of $\mathcal{A}$ for every $r > 0$. Moreover, $\mathcal{P}_r(\mathcal{A})$ is inverse-closed in $\mathcal{A}$.*

For functions in $\mathcal{P}_r(L^\infty(\mathbb{R}^d)$ this result is in Strichartz [39].

Proof. We treat the case $r < 1$ first. Let $a, b \in \mathcal{P}_r(\mathcal{A})$. Using

$$\Delta_t(ab) = \Delta_t(a)\Delta_t(b) + a\Delta_t(b) + \Delta_t(a)b$$

we obtain

$$\begin{aligned}\Big\|\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(ab)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} \le & \Big\|\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a)\Delta_t(b)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} \\ &+\Big\|a\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(b)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} + \Big\|\Big(\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big)b\Big\|_{\mathcal{A}}.\end{aligned} \tag{28}$$

The second and third term of the expression on the right hand side of the inequality are dominated by

$$\|a\|_{\mathcal{A}}\|b\|_{\mathcal{P}_r(\mathcal{A})} + \|a\|_{\mathcal{P}_r(\mathcal{A})}\|b\|_{\mathcal{A}} \lesssim \|a\|_{\mathcal{P}_r(\mathcal{A})}\|b\|_{\mathcal{P}_r(\mathcal{A})}\,.$$

For the estimation of the first term in (28) we use the embedding $\mathcal{P}_r(\mathcal{A}) \hookrightarrow \Lambda_r^\infty(\mathcal{A})$ (Proposition 3.27), so $\|\Delta_t a\|_{\mathcal{A}} \lesssim |t|_2^r\|a\|_{\mathcal{P}_r(\mathcal{A})}$, with a similar estimate for b. Therefore

$$\Big\|\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a)\Delta_t(b)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} \lesssim \|a\|_{\mathcal{P}_r(\mathcal{A})}\|b\|_{\mathcal{P}_r(\mathcal{A})}\int_{0\le|t|_2\le 1}|t|_2^r\frac{dt}{|t|_2^d} \le C_r\|a\|_{\mathcal{P}_r(\mathcal{A})}\|b\|_{\mathcal{P}_r(\mathcal{A})}\,,$$

and C_r does not depend on ϵ. Combining the estimates we have proved that

$$\|ab\|_{\mathcal{P}_r(\mathcal{A})} \lesssim \|a\|_{\mathcal{P}_r(\mathcal{A})}\|b\|_{\mathcal{P}_r(\mathcal{A})}.$$

For the verification of the inverse-closedness of $\mathcal{P}_r(\mathcal{A})$ in $\mathcal{A}$ we use a similar argument: Expand the identity ((10)) to obtain

$$\Delta_t(a^{-1}) = -\Delta_t(a^{-1})\Delta_t(a)a^{-1} - a^{-1}\Delta_t(a)a^{-1}.$$

So

$$\begin{aligned}\Big\|\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a^{-1})}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} \le & \Big\|\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a^{-1})\Delta_t(a)a^{-1}}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} \\ &+\Big\|\int_{\epsilon\le|t|_2\le 1}\frac{a^{-1}\Delta_t(a)a^{-1}}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}}.\end{aligned} \tag{29}$$

As $a \in \Lambda_r^\infty(\mathcal{A})$, we know that $\|\Delta_t(a)\|_{\mathcal{A}} \lesssim |t|_2^r\|a\|_{\Lambda_r^\infty(\mathcal{A})}$, and, as $\Lambda_r^\infty(\mathcal{A})$ is inverse-closed in $\mathcal{A}$, $\|\Delta_t(a^{-1})\|_{\mathcal{A}} \lesssim |t|_2^r\|a^{-1}\|_{\Lambda_r^\infty(\mathcal{A})}$.

The first term on the right hand side of (29) can be dominated by

$$\int_{\epsilon\le|t|_2\le 1}\frac{\|\Delta_t(a^{-1})\|_{\mathcal{A}}\|\Delta_t a\|_{\mathcal{A}}\|a^{-1}\|_{\mathcal{A}}}{|t|_2^r}\frac{dt}{|t|_2^d} \lesssim \|a^{-1}\|_{\Lambda_r^\infty(\mathcal{A})}\|a\|_{\Lambda_r^\infty(\mathcal{A})}\|a\|_{\mathcal{A}}$$

The second term can be estimated as

$$\begin{aligned}\Big\|\int_{\epsilon\le|t|_2\le 1}\frac{a^{-1}\Delta_t(a)a^{-1}}{|t|_2^r}\frac{dt}{|t|_2^d}\Big\|_{\mathcal{A}} =&\Big\|a^{-1}\Big(\int_{\epsilon\le|t|_2\le 1}\frac{\Delta_t(a)}{|t|_2^r}\frac{dt}{|t|_2^d}\Big)a^{-1}\Big\|_{\mathcal{A}} \\ \lesssim&\|a^{-1}\|_{\mathcal{A}}^2\|a\|_{\mathcal{P}_r(\mathcal{A})}.\end{aligned}$$

As $\Lambda_r^\infty(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ we obtain the inverse-closedness of $\mathcal{P}_r(\mathcal{A})$ in $\mathcal{A}$.

If $r \geq 1$ we can proceed by induction. Assume that we have already proved that $\mathcal{P}_s(\mathcal{A})$ is inverse-closed in $\mathcal{A}$, and $s < r < s+1$. By what we have just proved $\mathcal{P}_r(\mathcal{A}) = \mathcal{P}_{r-s}(\mathcal{P}_s(\mathcal{A}))$ is inverse-closed in $\mathcal{P}_s(\mathcal{A})$. As $\mathcal{P}_s(\mathcal{A})$ is inverse-closed in $\mathcal{A}$ by hypotheses we are done. □

3.4.2. *Application to Weighted Matrix Algebras.* We call $v_r^*(k) = (1+|2\pi k|_2^2)^{r/2}$ for $r > 0$ the *Bessel weight* of order r. If $\mathcal{A}$ is a Banach space of matrices, we say that a matrix A is in the weighted matrix space $\mathcal{A}_{v_r}$, where v_r is the standard polynomial weight $v_r(k) = (1+|k|)^r$, if the matrix with entries $A(k,l)v_r(k-l)$ is in $\mathcal{A}$. The norm in $\mathcal{A}_{v_r}$ is $\|A\|_{\mathcal{A}_{v_r}} = \|(A(k,l)v_r(k-l))_{k,l\in\mathbb{Z}^d}\|_{\mathcal{A}}$. In a similar way we introduce $\mathcal{A}_{v_r^*}$.

Proposition 3.30. *If $\mathcal{A}$ is a homogeneous matrix algebra, and χ is a C_w- group on $\mathcal{A}$, then*

$$\mathcal{A}_{v_r^*} = \mathcal{P}_r(\mathcal{A})\,.$$

Proof. By definition A is in $\mathcal{P}_r(\mathcal{A})$, if there is a $A_0 \in \mathcal{A}$ such that $A = \mathcal{G}_r * A_0$. This is equivalent to

$$\hat{A}(k) = (1+|2\pi k|^2)^{-r/2}\hat{A}_0(k)\,,$$

or $\hat{A}_0(k) = (1+|2\pi k|^2)^{r/2}\hat{A}(k)$, and therefore

$$\|A\|_{\mathcal{P}_r(\mathcal{A})} = \|A_0\|_{\mathcal{A}} = \|A\|_{\mathcal{A}_{v_r^*}}\,,$$

i.e., $A \in \mathcal{A}_{v_r^*}$. □

Proposition 3.31. *If $\mathcal{A}$ is a homogeneous matrix algebra, χ is a C_w- group on $\mathcal{A}$, and v_r^*, $r > 0$, is a Bessel weight, then $\mathcal{A}_{v_r^*} = \mathcal{P}_r(\mathcal{A})$ is a matrix algebra. This algebra is inverse-closed in $\mathcal{A}$.*

Proof. This is an application of Theorem 3.29. □

Proposition 3.31 applies in particular to the weighted subalgebras of $\mathcal{B}(\ell^2)$.

For solid matrix algebras the standard polynomial weights v_r can be taken instead of v_r^*.

Corollary 3.32. *If $\mathcal{A}$ is a solid matrix algebra, and χ is a C_w- group on $\mathcal{A}$, then $\mathcal{A}_{v_r}$ is an inverse-closed subalgebra of $\mathcal{A}$.*

We state the results of Proposition 3.27 and Proposition 3.28 for weighted matrix algebras.

Proposition 3.33. *If $\mathcal{A}$ is a homogeneous matrix algebra, and $r, s > 0$, then*

$$\Lambda_r^1(\mathcal{A}) \hookrightarrow \mathcal{A}_{v_r^*} \hookrightarrow \Lambda_\infty^r(\mathcal{A}),$$
$$\Lambda_r^p(\mathcal{A}_{v_s^*}) = \mathcal{E}_r^p(\mathcal{A}_{v_s^*}) = (\Lambda_r^p(\mathcal{A}))_{v_s^*} = \Lambda_{r+s}^p(\mathcal{A}) = \mathcal{E}_{r+s}^p(\mathcal{A}).$$

Example 3.34. For the Schur algebras $\mathcal{S}_r^p$ we obtain

$$\mathcal{E}_s^q(\mathcal{S}_r^p) = \mathcal{E}_{s+r}^q(\mathcal{S}_0^p)\,.$$

Appendix A. Proof of the Reiteration theorem

We give a proof of Theorem 3.7. We need some properties of the moduli of smoothness.

Lemma A.1. *If $l, k \in \mathbb{N}$, $l \geq k$, $t \in \mathbb{R}^d$ and $h > 0$, then*

(1) $\|\Delta_t^l(x)\|_{\mathcal{X}} \leq (M_\Psi+1)^k \|\Delta_t^{l-k}(x)\|_{\mathcal{X}}$ *and* $\omega_h^l(x) \leq (M_\Psi+1)^k \omega_h^{l-k}(x)\,,$

(2)

$$\omega_t^k(x) \asymp \sup_{\substack{|h_j|\leq t\\ 1\leq j\leq k}} \|(\prod_{j=1}^k \Delta_{h_j})x\|_{\mathcal{X}}.$$

The proof of (1) is an easy calculation in complete analogy to the corresponding properties of the moduli of smoothness for functions. See, e.g., [18]. Item 2 is proved in [8, 5.4.11].

Proof of the Reiteration theorem. We assume first that x is in $\Lambda^q_{s+r}(\mathcal{A})$ and estimate $\|x\|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))}$. As $\|x\|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))} = \|x\|_{\Lambda^p_r(\mathcal{A})} + |x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))}$ and the inclusion relations of Besov spaces imply that $\|x\|_{\Lambda^p_r(\mathcal{A})} \leq C\|x\|_{\Lambda^q_{r+s}(\mathcal{A})}$, it suffices to estimate $|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))}$.

Assume that $\lfloor r \rfloor < m$ and $\lfloor s \rfloor < n$, $m, n \in \mathbb{N}$. Using the norm equivalences in (7) we can write

$$|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))} \asymp \left\{ \int_{\mathbb{R}^+} \left[h^{-s}\omega_h^{n+m}(x, \Lambda^p_r(\mathcal{A}))\right]^q \frac{dh}{h} \right\}^{1/q} = \|h^{-s}\omega_h^{n+m}(x, \Lambda^p_r(\mathcal{A}))\|_{L^q_*}\,, \tag{30}$$

where $\|f\|_{L^q_*} = \left(\int_0^\infty f(t)^q \frac{dt}{t}\right)^{1/q}$. An estimate of the modulus of smoothness is

$$\begin{aligned} \omega_h^{n+m}(x, \Lambda^p_r(\mathcal{A})) &= \sup_{|u|\leq h} \|\Delta_u^{n+m}x\|_{\Lambda^p_r(\mathcal{A})} \\ &\leq \sup_{|u|\leq h} \|\Delta_u^{n+m}x\|_{\mathcal{A}} + \sup_{|u|\leq h} |\Delta_u^{n+m}x|_{\Lambda^p_r(\mathcal{A})} \\ &\lesssim \sup_{|u|\leq h} \|\Delta_u^{n+m}x\|_{\mathcal{A}} + \sup_{|u|\leq h} |\Delta_u^{n+m}x|_{\Lambda^1_r(\mathcal{A})}\,, \end{aligned} \tag{31}$$

where the last inequality uses the embedding $\Lambda^1_r(\mathcal{A}) \hookrightarrow \Lambda^p_r(\mathcal{A})$ for $p \geq 1$.

Inserting this estimate into (30) we obtain

$$|x|_{\Lambda^q_s(\Lambda^p_r(\mathcal{A}))} \lesssim |x|_{\Lambda^q_s(\mathcal{A})} + \|h^{-s} \sup_{|u|\leq h} |\Delta_u^{n+m}x|_{\Lambda^1_r(\mathcal{A})}\|_{L^q_*}\,. \tag{32}$$

With $\phi(v,u) = \|\Delta_v^{n+m}\Delta_u^{n+m}x\|_{\mathcal{A}}$ the L^q_*-norm in (32) can be rewritten as

$$\begin{aligned} &\|h^{-s} \sup_{|u|\leq h} \int_{\mathbb{R}^+} t^{-r} \sup_{|v|\leq t} \phi(v,u)\frac{dt}{t}\|_{L^q_*} \\ &\leq \|h^{-s} \sup_{|u|\leq h} \int_0^h t^{-r} \sup_{|v|\leq t} \phi(v,u)\frac{dt}{t}\|_{L^q_*} + \|h^{-s} \sup_{|u|\leq h} \int_h^\infty t^{-r} \sup_{|v|\leq t} \phi(v,u)\frac{dt}{t}\|_{L^q_*} \end{aligned}$$

$=: \mathrm{I} + \mathrm{II}.$

We can estimate the first term further using Hardy's inequality.

$$\begin{aligned} \mathrm{I}^q &= \int_0^\infty h^{-sq} \sup_{|u|\leq h}\left[\int_0^h \sup_{|v|\leq t} t^{-r}\phi(v,u)\frac{dt}{t}\right]^q \frac{dh}{h} \\ &\leq \int_0^\infty h^{-sq}\left[\int_0^h \sup_{|v|,|u|\leq h} t^{-r}\phi(v,u)\frac{dt}{t}\right]^q \frac{dh}{h} \\ &\overset{(*)}{\lesssim} \int_0^\infty \left(t^{-(r+s)} \sup_{|v|,|u|\leq t} \phi(v,u)\right)^q \frac{dt}{t} \\ &\overset{(**)}{\lesssim} \int_0^\infty \left(t^{-(r+s)}\omega_t^{2(n+m)}(x,\mathcal{A})\right)^q \frac{dt}{t} = |x|^q_{\Lambda^q_{r+s}(\mathcal{A})}\,, \end{aligned}$$

$(*)$ by Hardy's inequality, and $(**)$ using Lemma A.1(2). For the second term we use (1) of Lemma A.1 to get

$$\phi(v,u) = \|\Delta_v^{n+m}\Delta_u^{n+m}x\|_{\mathcal{A}} \lesssim \|\Delta_u^{n+m}x\|_{\mathcal{A}}.$$

Then $\sup_{|v|\le t}\phi(u,v)$ is independent of t, and

$$\begin{aligned}\mathrm{II}^q &\lesssim \int_0^\infty \left(h^{-s}\sup_{|u|\le h}\int_h^\infty t^{-r}\|\Delta_u^{n+m}x\|_{\mathcal{A}}\frac{dt}{t}\right)^q\frac{dh}{h}\\ &= \int_0^\infty h^{-(r+s)q}\sup_{|u|\le h}\|\Delta_u^{n+m}x\|_{\mathcal{A}}^q\frac{dh}{h} = |x|^q_{\Lambda^q_{r+s}(\mathcal{A})}.\end{aligned} \tag{33}$$

I and II together give the desired estimate.

For the converse assume that $x\in\Lambda^q_s(\Lambda^p_r(\mathcal{A}))$. Then,

$$\begin{aligned}|x|^q_{\Lambda^q_{r+s}(\mathcal{A})} &\asymp \int_{\mathbb{R}^d}\left(|t|^{-(r+s)}\|\Delta_t^{n+m}x\|_{\mathcal{A}}\right)^q\frac{dt}{|t|^d}\\ &\asymp \int_{\mathbb{R}^d}|t|^{-rq}\left(\int_{|\eta|\ge|t|}|\eta|^{-sp}\|\Delta_t^{n+m}x\|^p_{\mathcal{A}}\frac{d\eta}{|\eta|^d}\right)^{q/p}\frac{dt}{|t|_2^d},\end{aligned} \tag{34}$$

where we have used $|t|^{-s}\asymp\left(\int_{|\eta|\ge|t|}|\eta|^{-sp}\frac{d\eta}{|\eta|_2^d}\right)^{1/p}$ for the last equivalence. As $\|\Delta_t^{n+m}x\|_{\mathcal{A}}\le\sup_{|v|\le|\eta|}\|\Delta_v^m\Delta_t^n\|_{\mathcal{A}}$ for $|\eta|\ge|t|$, we can dominate the right hand side of (34) by

$$\begin{aligned}&\int_{\mathbb{R}^d}|t|^{-rq}\sup_{|u|\le|t|}\left(\int_{|\eta|\ge|t|}|\eta|^{-sp}\sup_{|v|\le|\eta|}\|\Delta_v^n\Delta_u^m x\|^p_{\mathcal{A}}\frac{d\eta}{|\eta|^d}\right)^{q/p}\frac{dt}{|t|^d}\\ &\le \int_{\mathbb{R}^d}|t|^{-rq}\sup_{|u|\le|t|}\left(\int_{\mathbb{R}^d}|\eta|^{-sp}\sup_{|v|\le|\eta|}\|\Delta_v^n\Delta_u^m x\|^p_{\mathcal{A}}\frac{d\eta}{|\eta|^d}\right)^{q/p}\frac{dt}{|t|^d}\\ &\le \int_{\mathbb{R}^d}|t|^{-rq}\sup_{|u|\le|t|}(|\Delta_u^m x|_{\Lambda^p_s(\mathcal{A})})^q\frac{dt}{|t|^d}\\ &\le |x|^q_{\Lambda^q_r(\Lambda^p_s(\mathcal{A}))}.\end{aligned}$$

□

Appendix B. Jackson Bernstein Theorem

Proposition B.1 ([20, 5.12])**.** *Let $a\in\mathcal{A}$ and $\sigma>0$.*

(1) *There is a σ-bandlimited element $a_\sigma\in C(\mathcal{A})$ such that*

$$\|a-a_\sigma\|_{\mathcal{A}}\le C\omega_{1/\sigma}(a)$$

with C independent of σ and a.

(2) *If $\delta^\alpha(a)\in C(\mathcal{A})$, for all multi-indices α with $|\alpha|=k$ then there exists a σ-bandlimited element $a_\sigma\in\mathcal{A}$ such that*

$$\|a-a_\sigma\|_{\mathcal{A}}\le C\sigma^{-k}\sum_{|\alpha|=k}\omega^2_{1/\sigma}(\delta^\alpha a).$$

Corollary B.2. *If $a\in\Lambda^p_r(\mathcal{A})$ for $r>0$, then $a\in\mathcal{E}^p_r(\mathcal{A})$.*

Proof. We use the integral version of the norm for an approximation space in (4) and assume that $1\le p<\infty$. The proof for $p=\infty$ is simpler and done in [20].

Assume first that $0<r<1$. Then, by Proposition B.1(1),

$$\int_1^\infty\left(E_\sigma(a)\sigma^r\right)^p\frac{d\sigma}{\sigma}\le C\int_0^1\left(\omega_\tau(a)\tau^{-r}\right)^p\frac{d\tau}{\tau}\le C|a|^p_{\Lambda^p_r(\mathcal{A})},$$

and so the approximation norm is dominated by the Besov norm.

Likewise, if $r=k+\eta, 0<\eta\le 1$, and $k\in\mathbb{N}$, Proposition B.1(2) yields

$$\int_1^\infty\left(E_\sigma(a)\sigma^r\right)^p\frac{d\sigma}{\sigma}\le C\sum_{|\alpha|=k}\int_0^1\left(\omega^2_\tau(\delta^\alpha(a))\tau^{-\eta}\right)^p\frac{d\tau}{\tau}$$

and again $\|a\|_{\mathcal{E}^p_r(\mathcal{A})}$ is dominated by the Besov norm. □

Before proving the converse implication in Theorem 3.11, i.e., the Bernstein-type result, we need a mean-value property of automorphism groups.

Lemma B.3 ([20, 5.15]). *If a is σ-bandlimited, then*

$$\|\Delta_t a\|_{\mathcal{A}} \le C\sigma\,|t|\,\|a\|_{\mathcal{A}}\,. \tag{35}$$

Proposition B.4. *Let $a \in \mathcal{A}$, and $r > 0$, $1 \le p \le \infty$. If $a \in \mathcal{E}^p_r(\mathcal{A})$, then $a \in \Lambda^p_r(\mathcal{A})$.*

Proof. We adapt a standard proof [14] and verify the statement for $p < \infty$. If $a \in \mathcal{E}^p_r(\mathcal{A})$, the representation theorem of approximation theory (see, e.g [33, 3.1]) implies that

$$a = \sum_{k=0}^{\infty} a_k, \quad \text{with } a_k \in X_{2^k} \quad \text{and } \sum_{k=0}^{\infty} 2^{krp}\|a_k\|^p_{\mathcal{A}} < \infty\,, \tag{36}$$

where $(X_\sigma)_{\sigma\ge 0}$ is the approximation scheme of bandlimited elements, and

$$\|a\|_{\mathcal{E}^p_r(\mathcal{A})} \asymp \Big(\sum_{k=0}^{\infty} 2^{krp}\|a_k\|^p_{\mathcal{A}}\Big)^{1/p}\,,$$

where the infimum is taken over all admissible representations as in (36). An application of Hölders inequality shows that $\sum_{k=0}^{\infty} a_k$ is convergent in $\mathcal{A}$. Note that (36) implies that $\|a_k\|_{\mathcal{A}} \le C2^{-kr}$for all $k \in \mathbb{N}_0$.

We assume first that $0 < r < 1$. We need an estimate for the norm of $\Delta_t a$.

$$\begin{aligned}
\|\Delta_t a\|_{\mathcal{A}} &\le \sum_{k=0}^{M}\|\Delta_t a_k\|_{\mathcal{A}} + \sum_{k=M+1}^{\infty}\|\Delta_t a_k\|_{\mathcal{A}} \\
&\le \sum_{k=0}^{M}\|\Delta_t a_k\|_{\mathcal{A}} + (M_\Psi + 1)\sum_{k=M+1}^{\infty}\|a_k\|_{\mathcal{A}}\,,
\end{aligned} \tag{37}$$

where the value of M will be chosen later.

Lemma B.3 implies that

$$\|\Delta_t a_k\|_{\mathcal{A}} \le C2^k|t|\,\|a_k\|_{\mathcal{A}}$$

for all $k \in \mathbb{N}$. Substituting back into (37) yields

$$\|\Delta_t a\|_{\mathcal{A}} \le C\Big(\sum_{k=0}^{M} 2^k|t|\|a_k\|_{\mathcal{A}} + \sum_{k=M+1}^{\infty}\|a_k\|_{\mathcal{A}}\Big)\,. \tag{38}$$

We use this relation for the estimation of the Besov seminorm.

$$\begin{aligned}
|a|_{\Lambda^p_r(\mathcal{A})} &\asymp \left(\sum_{l=0}^{\infty}\big(2^{lr}\omega_{2^{-l}}(a)\big)^p\right)^{1/p} \\
&\lesssim \left(\sum_{l=0}^{\infty} 2^{lrp}\Big(\sum_{k=0}^{M} 2^k 2^{-l}\|a_k\|_{\mathcal{A}} + \sum_{k=M+1}^{\infty}\|a_k\|_{\mathcal{A}}\Big)^p\right)^{1/p}.
\end{aligned}$$

We split this expression into two parts and assume that $M = l$ in the inner sums.

$$|a|_{\Lambda^p_r(\mathcal{A})} \lesssim \left(\sum_{l=0}^{\infty} 2^{l(r-1)p}\Big(\sum_{k=0}^{l} 2^k\|a_k\|_{\mathcal{A}}\Big)^p\right)^{1/p} + \left(\sum_{l=0}^{\infty} 2^{lrp}\Big(\sum_{k=l+1}^{\infty}\|a_k\|_{\mathcal{A}}\Big)^p\right)^{1/p}.$$

We apply Hardy's inequalities to both terms on the right hand side and obtain

$$\begin{aligned}|a|_{\Lambda^p_r(\mathcal{A})} &\lesssim \Big(\sum_{l=0}^{\infty} 2^{l(r-1)p} 2^{lp} \|a_l\|_{\mathcal{A}}^p\Big)^{1/p} + \Big(\sum_{l=0}^{\infty} 2^{lrp} \|a_l\|_{\mathcal{A}}^p\Big)^{1/p} \\ &= 2\Big(\sum_{l=0}^{\infty} 2^{lrp} \|a_l\|_{\mathcal{A}}^p\Big)^{1/p} .\end{aligned}$$

As the representations $a = \sum_{k=0}^{\infty} a_k$ were arbitrary we conclude that $|a|_{\Lambda^p_r(\mathcal{A})} \lesssim \|a\|_{\mathcal{E}^p_r(\mathcal{A})}$, using again the representation theorem. Next we consider the case $r = m + \eta$ for $m \in \mathbb{N}_0$ and $0 < \eta < 1$. The Bernstein inequality implies that

$$\|\delta^\alpha(a_k)\|_{\mathcal{A}} \le C(2\pi 2^k)^{|\alpha|} \|a_k\|_{\mathcal{A}}$$

for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^d$. Consequently $\sum_{k=0}^{\infty} \delta^\alpha a_k$ converges in $\mathcal{A}$ for all α with $|\alpha| \le m$ and its sum must be $\delta^\alpha(a)$, as each δ_j is closed on $\mathcal{D}(\delta^\alpha)$. We now apply the above estimates $\delta^\alpha(a)$ instead of a and deduce that $\delta^\alpha(a)$ must be in $\Lambda^p_\eta(\mathcal{A})$ for $|\alpha| \le k$. Thus $a \in \Lambda^p_r(\mathcal{A})$.

If r is an integer, then we have to use second order differences and a corresponding version of the mean value theorem. The argument is almost the same as above (see [45] for details in the scalar case). □

Combining Propositions B.2 and B.4, we have completed the proof of Theorem 3.11.

Appendix C. Littlewood-Paley decomposition

Proof of Proposition 3.12. We include the derivation of the relevant results to keep the presentation self-contained. We follow [9], but we use approximation arguments where feasible.

We use some obvious facts of the dyadic partition of unity $(\varphi_k)_{k\ge -1}$. By definition, $\operatorname{supp}\hat{\varphi}_k = 2^k \operatorname{supp}\hat{\varphi} \subseteq \{\omega \colon 2^{k-1} \le |\omega|_\infty \le 2^{k+1}\}$ for $k \ge 0$, and $\operatorname{supp}\varphi_{-1} \subseteq \{\omega \colon |\omega|_\infty \le 1\}$. As the intersection of $\operatorname{supp}(\hat{\varphi}_k)$ with $\operatorname{supp}(\hat{\varphi}_l)$ is nonempty only for $l \in \{k-1, k, k+1\}$ we obtain that $\varphi_k = \varphi_k * (\varphi_{k-1} + \varphi_k + \varphi_{k+1})$ if $k \ge 0$, and $\varphi_{-1} = \varphi_{-1} * (\varphi_{-1} + \varphi_0)$.

Assume first that (16) holds. Then $\|\varphi_k * a\|_{\mathcal{A}} \le C 2^{-rk}$, and so $\sum_{k=-1}^{\infty} \varphi_k * a$ is norm convergent in $\mathcal{A}$. A standard weak type argument shows that the limit is actually a.

For $a \in \Lambda^p_r(\mathcal{A})$ and $m > \lfloor r \rfloor$ we use $\|a\|_{\Lambda^p_r(\mathcal{A})} \asymp \|a\|_{\mathcal{A}} + \big(\sum_{k=0}^{\infty} (2^{rk}\omega^m_{2^{-k}}(a))^p\big)^{1/p}$. As $\|\Delta^m_t(\varphi_k * a)\|_{\mathcal{A}} \le C_m \|\varphi_k * a\|_{\mathcal{A}}$ by Lemma A.1 (1), and $\|\Delta^m_t(\varphi_k * a)\|_{\mathcal{A}} \le C'|t|^m 2^{mk} \|\varphi_k * a\|_{\mathcal{A}}$ by repeated application of Lemma B.3 we conclude that

$$\|\Delta^m_t(\varphi_k * a)\|_{\mathcal{A}} \le C_1 \min(1, |t|^m 2^{mk}) \|\varphi_k * a\|_{\mathcal{A}} . \tag{39}$$

As an immediate consequence we obtain

$$\omega^m_{|t|}(a) \le C_1 \sum_{k=-1}^{\infty} \min(1, t^m 2^{mk}) \|\varphi_k * a\|_{\mathcal{A}} , \tag{40}$$

and so

$$2^{rj}\omega^m_{2^{-j}}(a) \le C_1 \sum_{k=1}^{\infty} 2^{(j-k)r} 2^{kr} \min(1, 2^{-(j-k)m}) \|\varphi_k * a\|_{\mathcal{A}} . \tag{41}$$

The right hand side of this relation can be written as a convolution. If we set $u(l) = \min(1, 2^{-lm}) 2^{lr}$ for $l \in \mathbb{Z}$, and $v(l) = 2^{lr}\|\varphi_l * a\|_{\mathcal{A}}$ if $l > -1$ and 0 else, then u and v are sequences in $\ell^1(\mathbb{Z})$, and the right hand side of (41) is just $(u * v)(j)$.

So $\|\big(2^{rj}\omega^m_{2^{-j}}(a)\big)_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} \le C\|u\|_{\ell^1(\mathbb{Z})}\|v\|_{\ell^p(\mathbb{Z})}$, and this means that

$$\|a\|_{\Lambda^p_r(\mathcal{A})} \le C\Big(\sum_{k=-1}^{\infty} 2^{rkp}\|\varphi_k * a\|^p_{\mathcal{A}}\Big)^{1/p}, \tag{42}$$

so (16) implies that $a \in \Lambda^p_r(\mathcal{A})$.

For the other inequality we use $\|a\|_{\Lambda^p_r(\mathcal{A})} \asymp \|a\|_{\mathcal{A}} + \sum_{|\alpha|=m}\|\delta^\alpha(a)\|_{\Lambda^p_{r-m}(\mathcal{A})}$ with $m < r \le m+1$.

First we show that

$$\|\varphi_k * a\|_{\mathcal{A}} \le C2^{-mk}\|\varphi_k * \delta^\alpha(a)\|_{\mathcal{A}}\,, \quad m = |\alpha| \tag{43}$$

and

$$\|\varphi_k * \delta^\alpha(a)\|_{\mathcal{A}} \le C\omega^2_{2^{-k}}(\delta^\alpha a). \tag{44}$$

For the proof of these relations choose an even function $\Phi \in S(\mathbb{R}^d)$ such that $\hat{\Phi} \equiv 1$ on $\operatorname{supp}\hat{\varphi}_0$, and $\hat{\Phi} \equiv 0$ in a neighbourhood of 0. Set $\Phi_k(t) = 2^{kd}\Phi(2^k t)$, then $\|\Phi_k\|_1 = \|\Phi\|_1$ and $\Phi_k * \varphi_k = \varphi_k$. The function $\hat{\eta}^{(\alpha)}\colon \omega \to (2\pi i\omega)^{-\alpha}\hat{\Phi}(\omega)$ is an element of $\mathscr{S}$. Again, if we set $\eta^{(\alpha)}_k(t) = 2^{kd}\eta^{(\alpha)}(2^k t)$, then $\|\eta^{(\alpha)}_k\|_1 = \|\eta^{(\alpha)}\|_1$. Then $\hat{\Phi}_k(\omega) = \hat{\Phi}(2^{-k}\omega) = 2^{-k|\alpha|}(2\pi i\omega)^\alpha\hat{\eta}^{(\alpha)}_k(w)$, and so, assuming that $|\alpha| = m$, we obtain $\hat{\varphi}_k(\omega) = 2^{-km}\hat{\eta}^{(\alpha)}_k(\omega)(2\pi i\omega)^\alpha\hat{\varphi}_k(\omega)$ for all $\omega \in \mathbb{R}^d$, which implies

$$\varphi_k * a = 2^{-k\,m}\eta^{(\alpha)}_k * \delta^\alpha(\varphi_k * a) = 2^{-k\,m}\eta^{(\alpha)}_k * \varphi_k * \delta^\alpha(a),$$

the last equality by (14). Now (43) follows immediately.

For the proof of (44) set $y = \delta^\alpha(a)$ and $y_k = \varphi_k * y = \Phi_k * \varphi_k * y = \Phi_k * y_k$. We obtain

$$\begin{aligned}\varphi_k * y = \Phi_k * y_k &= \int_{\mathbb{R}^d}\Phi_k(t)\psi_{-t}(y_k)\,dt\\ &= \tfrac{1}{2}\int_{\mathbb{R}^d}\Phi_k(t)\big\{\psi_{-t}(y_k) - 2y_k + \psi_t(y_k)\big\}\,dt = \tfrac{1}{2}\int_{\mathbb{R}^d}\Phi_k(t)\psi_{-t}\Delta^2_t(y_k)\,dt\,,\end{aligned}$$

as $\int_{\mathbb{R}^d}\Phi_k = 0$ and $\Phi_k(-t) = \Phi_k(t)$. Changing variables we obtain

$$\varphi_k * y = \tfrac{1}{2}\int_{\mathbb{R}^d}\Phi(u)\psi_{-2^{-k}u}\Delta^2_{2^{-k}u}(y_k)\,dt = \tfrac{1}{2}\int_{\mathbb{R}^d}\Phi(u)\psi_{-2^{-k}u}\big(\varphi_k * \Delta^2_{2^{-k}u}(y)\big)\,dt.$$

Taking norms we get

$$\begin{aligned}\|\varphi_k * y\|_{\mathcal{A}} &\le \tfrac{M_\psi}{2}\int_{\mathbb{R}^d}|\Phi(u)|\|\varphi_k\|_1\omega^2_{2^{-k}|u|}(y)\,dt.\\ &\le \tfrac{M_\psi}{2}\|\varphi_0\|_1\int_{\mathbb{R}^d}|\Phi(u)|(1+|u|^2)\omega^2_{2^{-k}}(y)\,dt\\ &\le C\omega^2_{2^{-k}}(y)\,,\end{aligned}$$

where the estimate for $\omega^2_{2^{-k}|u|}(y)$ follows from Lemma A.1. This is what we wanted to show.

The proof of the reverse inclusion now follows by putting (43) and (44) together.

$$2^{rk}\|\varphi_k * a\|_{\mathcal{A}} \le C2^{(r-m)k}\|\varphi_k * \delta^\alpha(a)\|_{\mathcal{A}} \le C2^{(r-m)k}\omega^2_{2^{-k}}(\delta^\alpha(a))\,,$$

and so

$$\begin{aligned}\sum_{k=-1}^{\infty} 2^{rpk}\|\varphi_k * a\|^p_{\mathcal{A}} &\le C\big(\|a\|^p_{\mathcal{A}} + \sum_{k=0}^{\infty} 2^{(r-m)pk}\omega^2_{2^{-k}}(\delta^\alpha(a))^p\big)\\ &\le C'(\|a\|^p_{\mathcal{A}} + |\delta^\alpha(a)|^p_{\Lambda^p_{r-m}(\mathcal{A})}) \le C"\|a\|^p_{\Lambda^p_{r-m}(\mathcal{A})}\,.\end{aligned} \tag{45}$$

We have shown that $a \in \Lambda^p_r(\mathcal{A})$ implies (16). The norm equivalence follows from (42) and (45). $\square$

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