COFREE COMPOSITIONS OF COALGEBRAS

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Abstract. We develop the notion of the composition of two coalgebras, which arises naturally in higher category theory and in the theory of species. We prove that the composition of two cofree coalgebras is again cofree, and we give sufficient conditions that ensure the composition is a one-sided Hopf algebra. We show these conditions are satisfied when one coalgebra is a graded Hopf operad $\mathcal{D}$ and the other is a connected graded coalgebra with coalgebra map to $\mathcal{D}$. We conclude by computing the primitive elements for compositions of coalgebras built on the vertices of multiplihedra, composihedra, and hypercubes.

Introduction

The Malvenuto-Reutenauer Hopf algebra of ordered trees [12, 2] and the Loday-Ronco Hopf algebra of planar binary trees [11, 3] are cofree as coalgebras and are connected by cellular maps from the permutahedra to the associahedra. Closely related polytopes include Stasheff's multiplihedra [16] and the composihedra [7], and it is natural to study to what extent Hopf structures may be placed on these objects. The map from permutahedra to associahedra factors through the multiplihedra, and in [8] we used this factorization to place Hopf structures on bi-leveled trees, which correspond to vertices of multiplihedra.

The multiplihedra form an operad module over the associahedra, and this leads to the concept of painted trees, which also correspond to the vertices of the multiplihedra. Moreover, expressing the Hopf structures of [8] in terms of painted trees relates these Hopf structures to the operad module structure. Abstracting this structure leads to the general notion of a composition of coalgebras, which is a functorial construction of a graded coalgebra $\mathcal{D}\circ\mathcal{C}$ from graded coalgebras $\mathcal{C}$ and $\mathcal{D}$. We define this composition in Section 2 and show that it preserves cofreeness. In Section 3, we suppose that $\mathcal{D}$ is a Hopf algebra and give sufficient conditions for the compositions of coalgebras $\mathcal{D}\circ\mathcal{C}$ and $\mathcal{C}\circ\mathcal{D}$ to be one-sided Hopf algebras. These also guarantee that these compositions are Hopf modules and comodule algebras over $\mathcal{D}$.

The definition of the composition of coalgebras is familiar from the theory of operads. In general, a (nonsymmetric) operad is a monoid in the category of graded sets, with product given by composition (also known as the substitution product). In Section 3 we show that an operad $\mathcal{D}$ in the category of connected graded coalgebras is automatically a Hopf algebra. Those familiar with the theory of species will also recognize our

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construction. The coincidence is explained in [1, Appendix B]: species and operads are one-and-the-same.

We conclude in Sections 4, 5, and 6 with a detailed look at several compositions of coalgebras that enrich the understanding of well-known objects from category theory and algebraic topology. In particular, we prove that the (one sided) Hopf algebra of simplices in [9] is cofree as a coalgebra.

1. Preliminaries

We work over a fixed field $\mathbb{K}$ of characteristic zero. For a graded vector space $V = \bigoplus_n V_n$, we write $|v| = n$ and say v has *degree* n if $v \in V_n$.

1.1. **Hopf algebras and cofree coalgebras.**

A bialgebra H is a unital associative algebra equipped with two algebra maps: a coproduct homomorphism $\Delta\colon H \to H \otimes H$ that is coassociative and a counit homomorphism $\varepsilon\colon H \to \mathbb{K}$ which plays the role of the identity for Δ. See [13] for more details. A graded bialgebra $H = (\bigoplus_{n\geq 0} H_n, \cdot, \Delta, \varepsilon)$ is *connected* if $H_0 = \mathbb{K}$. In this case, a result of Takeuchi [17, Lemma 14] guarantees the existence of an antipode map for H, making it a Hopf algebra.

We recall Sweedler's coproduct notation for later use. A coalgebra $\mathcal{C}$ is a vector space $\mathcal{C}$ equipped with a coproduct Δ and counit ε. Given $c \in \mathcal{C}$, the coproduct $\Delta(c)$ is written $\sum_{(c)} c' \otimes c''$. Coassociativity means that

$$\sum_{(c),(c')} (c')' \otimes (c')'' \otimes c'' \;=\; \sum_{(c),(c'')} c' \otimes (c'')' \otimes (c'')'' \;=\; \sum_{(c)} c' \otimes c'' \otimes c''' \,,$$

and the counit condition means that $\sum_{(c)} \varepsilon(c')c'' = \sum_{(c)} c'\varepsilon(c'') = c$.

The *cofree coalgebra* on a vector space V has underlying vector space $\mathsf{C}(V) := \bigoplus_{n\geq 0} V^{\otimes n}$. Its counit is the projection $\varepsilon\colon \mathsf{C}(V) \to \mathbb{K} = V^{\otimes 0}$. Its coproduct is the *deconcatenation coproduct*: writing "$\backslash$" for the tensor product in $V^{\otimes n}$, we have

$$\Delta(c_1\backslash \cdots \backslash c_n) \;=\; \sum_{i=0}^{n} (c_1\backslash \cdots \backslash c_i) \otimes (c_{i+1}\backslash \cdots \backslash c_n)\,.$$

Observe that V is exactly the set of primitive elements of $\mathsf{C}(V)$. A coalgebra $\mathcal{C}$ is *cofreely cogenerated* by a subspace $V \subset \mathcal{C}$ if $\mathcal{C} \simeq \mathsf{C}(V)$ as coalgebras. Necessarily, V is the space of primitive elements of $\mathcal{C}$. Many of the coalgebras and Hopf algebras arising in combinatorics are cofree. We recall a few key examples.

1.2. **Cofree Hopf algebras on trees.**

We describe three cofree Hopf algebras built on rooted planar binary trees: *ordered trees* $\mathfrak{S}_n$, *binary trees* $\mathcal{Y}_n$, and *(left) combs* $\mathfrak{C}_n$ on n internal nodes. Let $\mathfrak{S}_{\bullet}$ denote the union $\bigcup_{n\geq 0} \mathfrak{S}_n$ and define $\mathcal{Y}_{\bullet}$ and $\mathfrak{C}_{\bullet}$ similarly.

1.2.1. *Constructions on trees.* The nodes of a tree $t \in \mathcal{Y}_n$ are a poset (with root maximal) whose Hasse diagram is the internal edges of t. An *ordered tree* $w = w(t)$ is a linear extension of this node poset of t that we indicate by placing a permutation in the gaps between its leaves. Ordered trees are in bijection with the permutations of n. The map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$ forgets the total ordering of the nodes of an ordered tree $w(t)$ and gives the underlying tree t. The map $\kappa\colon \mathcal{Y}_n \to \mathfrak{C}_n$ shifts all nodes of a tree t to the right branch from the root. We let $\mathfrak{S}_0 = \mathcal{Y}_0 = \mathfrak{C}_0 = \mathsf{I}$. Note that $|\mathfrak{C}_n| = 1$ for all $n \geq 0$.

Figure 1 gives some examples from $\mathfrak{S}_\bullet$, $\mathcal{Y}_\bullet$, and $\mathfrak{C}_\bullet$ and indicates the natural maps τ and κ between them. See [8] for more details.

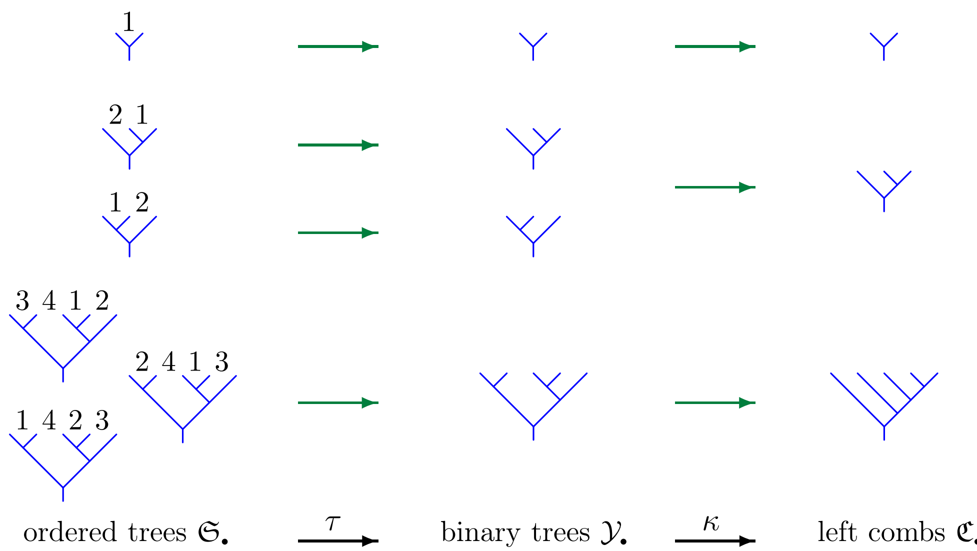
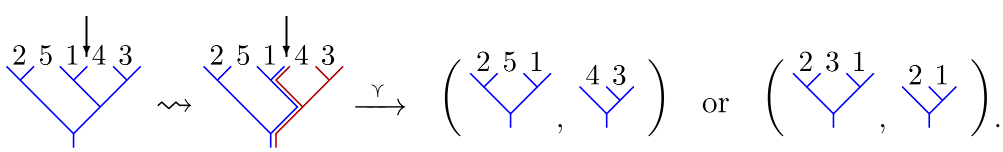


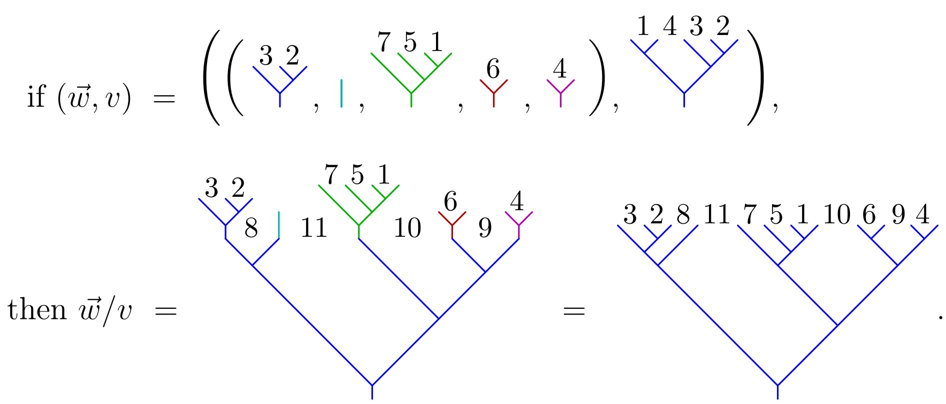
Figure 1. Maps between binary trees.

Splitting an ordered tree w along the path from a leaf to the root yields an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,



Write $w \xrightarrow{\curlyvee} (w_0, w_1)$ when the ordered forest (w_0, w_1) (or pair of ordered trees) is obtained by splitting w. (Context will determine how to interpret the result.)

We may *graft* an ordered forest $\vec{w} = (w_0, \ldots, w_n)$ onto an ordered tree $v \in \mathfrak{S}_n$, obtaining the tree $\vec{w}/v$ as follows. First increase each label of v so that its nodes are greater than the nodes of $\vec{w}$, and then graft tree w_i onto the i^{th} leaf of v. For example,



The notions of splitting and grafting make sense for trees in $\mathcal{Y}_{\bullet}$ (simply forget the labels on the nodes). They also work for $\mathfrak{C}_{\bullet}$, if after grafting a forest of combs onto the leaves of a comb, κ is applied to the resulting planar binary tree to get a new comb.

1.2.2. *Three cofree Hopf algebras.* Let $\mathfrak{S}Sym := \bigoplus_{n\geq 0}\mathfrak{S}Sym_n$ be the graded vector space whose n^{th} graded piece has basis $\{F_w \mid w \in \mathfrak{S}_n\}$. Define $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$ similarly. The set maps τ and κ induce vector space maps $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$, $\boldsymbol{\tau}(F_w) = F_{\tau(w)}$ and $\boldsymbol{\kappa}(F_t) = F_{\kappa(t)}$. Fix $\mathfrak{X} \in \{\mathfrak{S}, \mathcal{Y}, \mathfrak{C}\}$. For $w \in \mathfrak{X}_{\bullet}$ and $v \in \mathfrak{X}_n$, set

$$F_w \cdot F_v \ := \sum_{w \overset{\curlyvee}{\to} (w_0,\dots,w_n)} F_{(w_0,\dots,w_n)/v}\,,$$

the sum over all ordered forests obtained by splitting w at a multiset of n leaves, and

$$\Delta(F_w) \ := \sum_{w \overset{\curlyvee}{\to} (w_0,w_1)} F_{w_0} \otimes F_{w_1}\,,$$

the sum over pairs of trees obtained by splitting w at one leaf. The counit ε is projection onto the 0^{th} graded piece, which is spanned by the unit element $1 = F_{|}$ for the multiplication.

Proposition 1.1. *For $(\Delta, \cdot, \varepsilon)$ above, $\mathfrak{S}Sym$ is the Malvenuto–Reutenauer Hopf algebra of permutations, $\mathcal{Y}Sym$ is the Loday–Ronco Hopf algebra of planar binary trees, and $\mathfrak{C}Sym$ is the divided power Hopf algebra. Moreover, $\mathfrak{S}Sym \xrightarrow{\boldsymbol{\tau}} \mathcal{Y}Sym$ and $\mathcal{Y}Sym \xrightarrow{\boldsymbol{\kappa}} \mathfrak{C}Sym$ are surjective Hopf algebra maps.* □

The part of the proposition involving $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ is found in [2, 3]; the part involving $\mathfrak{C}Sym$ is straightforward and we leave it to the reader.

Remark 1.2. Typically [13, Example 5.6.8], the divided power Hopf algebra is defined to be $\mathbb{K}[x] := \text{span}\{x^{(n)} \mid n \geq 0\}$, with basis vectors $x^{(n)}$ satisfying $x^{(m)} \cdot x^{(n)} = \binom{m+n}{n} x^{(m+n)}$, $1 = x^{(0)}$, $\Delta(x^{(n)}) = \sum_{i+j=n} x^{(i)} \otimes x^{(j)}$, and $\varepsilon(x^{(n)}) = 0$ for $n > 0$. An isomorphism between $\mathbb{K}[x]$ and $\mathfrak{C}Sym$ is given by $x^{(n)} \mapsto F_{c_n}$, where c_n is the unique comb in $\mathfrak{C}_n$.

The following result is important for what follows.

Proposition 1.3. *The Hopf algebras $\mathfrak{S}Sym$, $\mathcal{Y}Sym$, and $\mathfrak{C}Sym$ are cofreely cogenerated by their primitive elements.* □

The result for $\mathfrak{C}Sym$ is easy. Proposition 1.3 is proven for $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in [2] and [3] by performing a change of basis—from the *fundamental basis* F_w to the *monomial basis* M_w—by means of Möbius inversion in a poset structure placed on $\mathfrak{S}_{\bullet}$ and $\mathcal{Y}_{\bullet}$. We revisit this in Section 4.3.

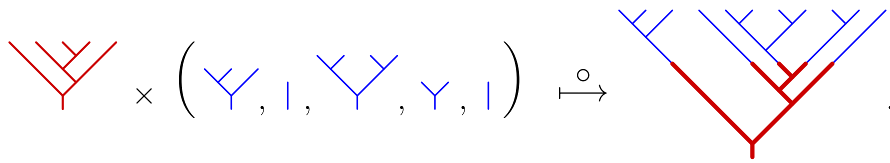
2. Cofree Compositions of Coalgebras

2.1. **Cofree composition of coalgebras.** Let $\mathcal{C}$ and $\mathcal{D}$ be graded coalgebras. We form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

$$\mathcal{D} \circ \mathcal{C} \ := \ \bigoplus_{n \geq 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes(n+1)} . \tag{2.1}$$

We write $\mathcal{E} = \bigoplus_{n\geq 0} \mathcal{E}_{(n)}$, where $\mathcal{E}_{(n)} = \mathcal{D}_n \otimes \mathcal{C}^{\otimes(n+1)}$. This gives a coarse coalgebra grading of $\mathcal{E}$ by *$\mathcal{D}$-degree*. There is a finer grading of $\mathcal{E}$ by *total degree*, in which a decomposable tensor $c_0 \otimes \cdots \otimes c_n \otimes d$ (with $d \in \mathcal{D}_n$) has total degree $|c_0| + \cdots + |c_n| + |d|$. Write $\mathcal{E}_n$ for the linear span of elements of total degree n.

Example 2.1. This composition is motivated by a grafting construction on trees. Let $d \times (c_0, \ldots, c_n) \in \mathcal{Y}_n \times \big(\mathcal{Y}_{\bullet}{}^{n+1}\big)$. Define $\circ$ by attaching the forest $(c_0, \ldots, c_n)$ to the leaves of d while remembering d,



This is a new type of tree (*painted trees* in Section 4). Applying this construction to the indices of basis elements of $\mathcal{C}$ and $\mathcal{D}$ and extending by multilinearity gives $\mathcal{D} \circ \mathcal{C}$.

Motivated by this example, we represent an decomposable tensor in $\mathcal{D} \circ \mathcal{C}$ as

$$d \circ (c_0 \bullet \cdots \bullet c_n) \qquad \text{or} \qquad \frac{c_0 \bullet \cdots \bullet c_n}{d}$$

to compactify notation.

2.1.1. *The coalgebra $\mathcal{D} \circ \mathcal{C}$.* We define the *compositional coproduct* Δ for $\mathcal{D} \circ \mathcal{C}$ on indecomposable tensors and extend multilinearly: if $|d| = n$, put

$$\Delta\left(\frac{c_0 \bullet \cdots \bullet c_n}{d}\right) \ = \ \sum_{i=0}^{n} \sum_{\substack{(d) \\ |d'|=i}} \sum_{(c_i)} \frac{c_0 \bullet \cdots \bullet c_{i-1} \bullet c_i'}{d'} \otimes \frac{c_i'' \bullet c_{i+1} \bullet \cdots \bullet c_n}{d''} , \tag{2.2}$$

where the coproducts in $\mathcal{C}$ and $\mathcal{D}$ are expressed using Sweedler notation.

The *counit* $\varepsilon : \mathcal{D} \circ \mathcal{C} \to \mathbb{K}$ is given by $\varepsilon(d \circ (c_0 \bullet \cdots \bullet c_n)) = \varepsilon_{\mathcal{D}}(d) \cdot \prod_j \varepsilon_{\mathcal{C}}(c_j)$. Hence, it is zero off of $\mathcal{D}_0 \otimes \mathcal{C}_0$.

Remark 2.2. The reader may check that for the painted trees of Example 2.1, if $c_0, \ldots, c_n$ and d are elements of the F-basis of $\mathcal{Y}Sym$, then $d \circ (c_0 \bullet \cdots \bullet c_n)$ represents a painted tree t and $\Delta(d \circ (c_0 \bullet \cdots \bullet c_n))$ is the sum over all splittings $t \xrightarrow{\curlyvee} (t', t'')$ of t into a pair of painted trees.

Theorem 2.3. *$(\mathcal{D} \circ \mathcal{C}, \Delta, \varepsilon)$ is a coalgebra. This composition is functorial, if $\varphi \colon \mathcal{C} \to \mathcal{C}'$ and $\psi \colon \mathcal{D} \to \mathcal{D}'$ are morphisms of graded coalgebras, then*

$$\frac{c_0 \bullet \cdots \bullet c_n}{d} \longmapsto \frac{\varphi(c_0) \bullet \cdots \bullet \varphi(c_n)}{\psi(d)}$$

defines a morphism of graded coalgebras $\varphi \circ \psi \colon \mathcal{D} \circ \mathcal{C} \to \mathcal{D}' \circ \mathcal{C}'$.

Proof. Let $\mathbb{1}$ be the identity map. Fix $e := d\circ(c_0{\bullet}\cdots{\bullet}c_n) \in (\mathcal{D}\circ\mathcal{C})_{(n)}$. From (2.2), we have

$$(\Delta\otimes\mathbb{1})\Delta(e) \;=\; \sum_{i=0}^{n}\sum_{j=0}^{i-1}\sum_{\substack{(d),(d')\\ |d'|=i,|(d')'|=j}}\sum_{(c_i),(c_j)} \frac{c_0{\bullet}\cdots{\bullet}c_j'}{(d')'}\otimes\frac{c_j''{\bullet}\cdots{\bullet}c_i'}{(d')''}\otimes\frac{c_i''{\bullet}\cdots{\bullet}c_n}{d''}$$
$$+\sum_{i=0}^{n}\sum_{\substack{(d),(d')\\ |d'|=i,|(d')''|=0}}\sum_{(c_i),(c_i')} \frac{c_0{\bullet}\cdots{\bullet}(c_i')'}{(d')'}\otimes\frac{(c_i')''}{(d')''}\otimes\frac{c_i''{\bullet}\cdots{\bullet}c_n}{d''}\,.$$

Using coassociativity, this becomes

$$\sum_{i=0}^{n}\sum_{j=0}^{i-1}\sum_{\substack{(d)\\ |d'|=i,|d''|=j}}\sum_{(c_i),(c_j)} \frac{c_0{\bullet}\cdots{\bullet}c_j'}{d'}\otimes\frac{c_j''{\bullet}\cdots{\bullet}c_i'}{d''}\otimes\frac{c_i''{\bullet}\cdots{\bullet}c_n}{d'''}$$
$$+\sum_{i=0}^{n}\sum_{\substack{(d)\\ |d'|=i,|d''|=0}}\sum_{(c_i)} \frac{c_0{\bullet}\cdots{\bullet}c_i'}{d'}\otimes\frac{c_i''}{d''}\otimes\frac{c_i'''{\bullet}\cdots{\bullet}c_n}{d'''}\,.$$

Simplification of $(\mathbb{1}\otimes\Delta)\Delta(e)$ reaches the same expression, proving coassociativity.

For the counital condition, we have

$$(\varepsilon\otimes\mathbb{1})\Delta(e) \;=\; \sum_{i=0}^{n}\sum_{\substack{(d)\\ |d'|=i}}\sum_{(c_i)} \varepsilon\left(\frac{c_0{\bullet}\cdots{\bullet}c_{i-1}{\bullet}c_i'}{d'}\right)\frac{c_i''{\bullet}c_{i+1}{\bullet}\cdots{\bullet}c_n}{d''}$$
$$=\; \sum_{\substack{(d)\\ |d'|=0}}\sum_{(c_0)} \varepsilon\left(\frac{c_0'}{d'}\right)\frac{c_0''{\bullet}c_1{\bullet}\cdots{\bullet}c_n}{d''}\,,$$

since $\varepsilon_{\mathcal{D}}(d')=0$ unless $|d'|=0$. Continuing, this becomes

$$\sum_{\substack{(d)\\ |d'|=0}}\sum_{(c_0)} \frac{\varepsilon(c_0')c_0''{\bullet}c_1{\bullet}\cdots{\bullet}c_n}{\varepsilon(d')d''} \;=\; e\,,$$

by the counital conditions in $\mathcal{C}$ and $\mathcal{D}$. The identity $(\mathbb{1}\otimes\varepsilon)\Delta(e)=e$ is similarly verified, proving the counital condition for $\mathcal{D}\circ\mathcal{C}$. Lastly, the functoriality is clear. □

2.1.2. *The cofree coalgebra $\mathcal{D}\circ\mathcal{C}$.* Suppose that $\mathcal{C}$ and $\mathcal{D}$ are graded, connected, and cofree. Then $\mathcal{C}=\mathsf{C}(P_{\mathcal{C}})$, where $P_{\mathcal{C}}\subset\mathcal{C}$ is its space of primitive elements. Likewise, $\mathcal{D}=\mathsf{C}(P_{\mathcal{D}})$. As in Section 1.1, we use "\" for internal tensor products.

Theorem 2.4. *If $\mathcal{C}$ and $\mathcal{D}$ are cofree coalgebras then $\mathcal{D}\circ\mathcal{C}$ is also a cofree coalgebra. Its space of primitive elements is spanned by indecomposible tensors of the form*

$$\frac{1{\bullet}c_1{\bullet}\cdots{\bullet}c_{n-1}{\bullet}1}{\delta} \qquad \textit{and} \qquad \frac{\gamma}{1} \tag{2.3}$$

where $\gamma, c_i\in\mathcal{C}$ and $\delta\in\mathcal{D}_n$ with γ,δ primitive.

Proof. Let $\mathcal{E} = \mathcal{D}\circ\mathcal{C}$ and let $P_{\mathcal{E}}$ denote the vector space spanned by the vectors in (2.3). We compare the compositional coproduct Δ to the deconcatenation coproduct Δ_{C} on the space $\mathsf{C}(P_{\mathcal{E}})$. We define a vector space isomorphism $\varphi\colon \mathcal{E} \to \mathsf{C}(P_{\mathcal{E}})$ and check that $\Delta_{\mathsf{C}}\,\varphi(e) = (\varphi\otimes\varphi)\Delta(e)$ for all $e \in \mathcal{E}$.

Let $e = d\circ(c_0\bullet\cdots\bullet c_n)$. Define φ recursively as follows:

- If $d = 1$ and $c_0 = c_0'\backslash c_0''$, put $\varphi\left(\dfrac{c_0}{1}\right) = \varphi\left(\dfrac{c_0'}{1}\right)\Big\backslash\varphi\left(\dfrac{c_0''}{1}\right)$.
- If $|c_0| > 0$, put $\varphi(e) = \varphi\left(\dfrac{c_0}{1}\right)\Big\backslash\varphi\left(\dfrac{1\bullet c_1\bullet\cdots\bullet c_n}{d}\right)$.
- If $|c_n| > 0$, put $\varphi(e) = \varphi\left(\dfrac{c_0\bullet\cdots\bullet c_{n-1}\bullet 1}{d}\right)\Big\backslash\varphi\left(\dfrac{c_n}{1}\right)$.
- If $d = d'\backslash d''$ with $|d'| = i$, then put $\varphi(e) = \varphi\left(\dfrac{c_0\bullet\cdots\bullet c_i}{d'}\right)\Big\backslash\varphi\left(\dfrac{1\bullet c_{i+1}\bullet\cdots\bullet c_n}{d''}\right)$.

We illustrate φ with an example from $\mathcal{E}_{(5)}$:

$$\frac{a'\backslash a''\bullet b\bullet c\bullet u'\backslash u''\bullet v\bullet w}{d'\backslash d''} \quad\overset{\varphi}{\longmapsto}\quad \frac{a'}{1}\Big\backslash\frac{a''}{1}\Big\backslash\frac{1\bullet b\bullet c\bullet 1}{d'}\Big\backslash\frac{u'}{1}\Big\backslash\frac{u''}{1}\Big\backslash\frac{1\bullet v\bullet 1}{d''}\Big\backslash\frac{w}{1}\,.$$

Here $|d'| = 3$ and all variables belong to $P_{\mathcal{C}}\cup P_{\mathcal{D}}$.

To see that φ is a coalgebra map, notice that locations to deconcatenate $\varphi(e)$,

$$t_1\backslash\cdots\backslash t_N \;\longmapsto\; t_1\backslash\cdots\backslash t_i\otimes t_{i_1}\backslash\cdots\backslash t_N\,,$$

are in bijection with pairs of locations: a place to deconcatenate d and a place to deconcatenate an accompanying c_i. These are exactly the choices governing (2.2), given that d and each c_i belong to tensor powers of $P_{\mathcal{D}}$ and $P_{\mathcal{C}}$, respectively. □

2.2. **Examples of cofree compositions of coalgebras.** The graded Hopf algebras of ordered trees $\mathfrak{S}Sym$, planar trees $\mathcal{Y}Sym$, and divided powers $\mathfrak{C}Sym$ are all cofree, and so their compositions are cofree. We have the surjective Hopf algebra maps

$$\mathfrak{S}Sym \;\longrightarrow\; \mathcal{Y}Sym \;\longrightarrow\; \mathfrak{C}Sym$$

giving a commutative diagram of nine cofree coalgebras (Figure 2), as the composition $\circ$ is functorial. In Section 3, we use operads to further analyze eight of these nine (all except $\mathfrak{S}Sym\circ\mathfrak{S}Sym$). We show that these eight are one-sided Hopf algebras. The algebra $\mathcal{P}Sym$ of painted trees appears in the center of this 3×3 grid. We discuss $\mathcal{P}Sym$ further in Section 4, the algebra $\mathcal{Y}Sym\circ\mathfrak{C}Sym$ in Section 5, and the algebra $\mathfrak{C}Sym\circ\mathfrak{C}Sym$ in Section 6.

2.3. **Enumeration.** We enumerate the graded dimension of many examples from Section 2.2. Set $\mathcal{E} := \mathcal{D}\circ\mathcal{C}$ and let C_n and E_n be the dimensions of $\mathcal{C}_n$ and $\mathcal{E}_n$, respectively.

Theorem 2.5. *When $\mathcal{D}_n$ has a basis indexed by combs with n internal nodes we have the recursion*

$$E_0 \;=\; 1\,,\quad \text{and for}\;\; n>0,\quad E_n \;=\; C_n \;+\; \sum_{i=0}^{n-1} C_i E_{n-i-1}\,.$$

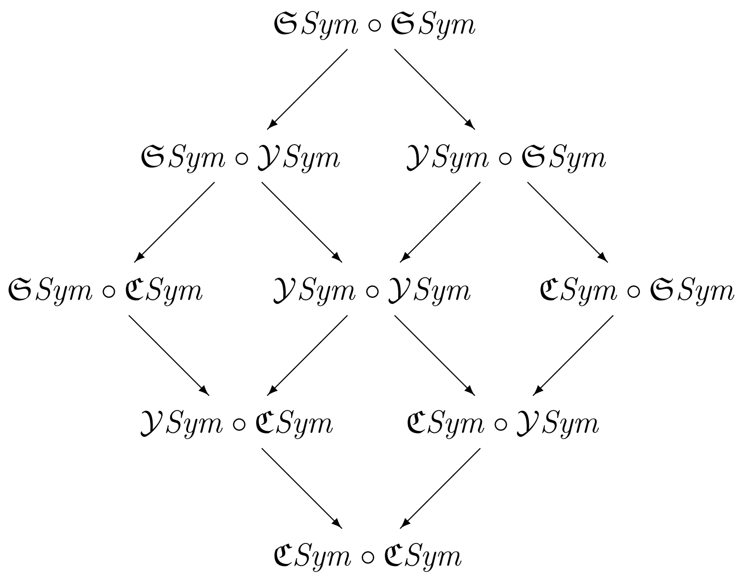


FIGURE 2. A commutative diagram of cofree compositions of coalgebras.

Proof. The first term in the expression for E_n counts elements in $\mathcal{E}_n$ of the form $| \circ c$. Removing the root node of d from $d \circ (c_0 \cdot \cdots \cdot c_k)$ gives a pair $| \circ (c_0)$ and $d' \circ (c_1 \cdot \cdots \cdot c_k)$ with $c_0 \in \mathcal{C}_i$, whose dimensions are enumerated by the terms $C_i E_{n-i-1}$ of the sum. □

For combs over a comb, $E_n = 2^n$. For trees over a comb, E_n are the Catalan numbers. For permutations over a comb, we have the recursion

$$E_0 \;=\; 1\,, \quad \text{and for } \; n > 0, \quad E_n \;=\; n! + \sum_{i=0}^{n-1} i! E_{n-i-1}\,,$$

which begins $1, 2, 5, 15, 54, 235, \ldots$, and is sequence A051295 in the On-line Encyclopedia of Integer Sequences (OEIS) [14]. (This is the invert transform [4] of the factorial numbers.)

Theorem 2.6. *When $\mathcal{D}_n$ has a basis indexed by $\mathcal{Y}_n$, then we have the recursion*

$$E_0 \;=\; 1\,, \quad \textit{and for } \; n > 0, \quad E_n \;=\; C_n + \sum_{i=0}^{n-1} E_i E_{n-i-1}\,.$$

Proof. Again, the first term in the expression for E_n is the number of basis elements of $\mathcal{C}_n$, since each of these trees is grafted on to the unit element of $\mathcal{D}$. The sum accounts for the possible pairs of trees obtained from removing root nodes in $\mathcal{D}$. In this case, each subtree from the root is another tree in $\mathcal{E}$. □

For example, combs over a tree are enumerated by the binary transform of the Catalan numbers [7]. Trees over a tree are enumerated by the Catalan transform of the Catalan numbers [6]. Permutations over a tree are enumerated by the recursion

$$E_0 \;=\; 1\,, \quad \text{and for } \; n > 0, \quad E_n \;=\; n! + \sum_{i=0}^{n-1} E_i E_{n-i-1}\,,$$

which begins $1, 2, 6, 22, 92, 428, \ldots$ and is not a recognized sequence in the OEIS [14].

For $\mathcal{E} = \mathfrak{S}Sym \circ \mathcal{C}$, we do not have a recursion, but do have a formula from direct inspection of the possible trees $d \circ (c_0 \cdot \cdots \cdot c_k)$ with $|d| = k$ (since $|\mathfrak{S}_k| = k!$)

$$E_n \;=\; \sum_{k=0}^{n} k! \sum_{(\gamma_0,\ldots,\gamma_k)} C_{\gamma_0} \cdots C_{\gamma_k}\,,$$

the sum over all weak compositions $\gamma = (\gamma_0, \ldots, \gamma_k)$ of $n-k$ into $k+1$ parts ($\gamma_i \geq 0$). Since the number of such weak compositions is $\binom{(n-k)+(k+1)-1}{(k+1)-1} = \binom{n}{k}$, when $\mathcal{C} = \mathfrak{C}Sym$ so that $C_n = 1$, this formula becomes

$$E_n \;=\; \sum_{k=0}^{n} k!\tbinom{n}{k} \;=\; \sum_{k=0}^{n} n!/k!\,,$$

which is sequence A000522 in the OEIS [14].

3. Composition of Coalgebras and Hopf Modules

We give conditions ensuring that a composition of coalgebras is a one-sided Hopf algebra, interpret these in the language of operads, and then investigate which compositions of Section 2.2 are one-sided Hopf algebras.

3.1. **Module coalgebras.**

Let $\mathcal{D}$ be a connected graded Hopf algebra with product $m_{\mathcal{D}}$, coproduct $\Delta_{\mathcal{D}}$, and unit element $1_{\mathcal{D}}$.

A map $f : \mathcal{E} \to \mathcal{D}$ of graded coalgebras is a *connection* on $\mathcal{D}$ if $\mathcal{E}$ is a $\mathcal{D}$–module coalgebra, f is a map of $\mathcal{D}$-module coalgebras, and $\mathcal{E}$ is connected. That is, $\mathcal{E}$ is an associative (left or right) $\mathcal{D}$-module whose action (denoted $\star$) commutes with the coproducts, so that $\Delta_{\mathcal{E}}(e \star d) = \Delta_{\mathcal{E}}(e) \star \Delta_{\mathcal{D}}(d)$, for $e \in \mathcal{E}$ and $d \in \mathcal{D}$, *and* the coalgebra map f is also a module map, so that for $e \in \mathcal{E}$ and $d \in \mathcal{D}$ we have

$$(f \otimes f)\,\Delta_{\mathcal{E}}(e) \;=\; \Delta_{\mathcal{D}}\, f(e) \qquad \text{and} \qquad f(e \star d) \;=\; m_{\mathcal{D}}\,(f(e) \otimes d)\,.$$

We may sometimes use subscripts (f_l or f_r) on a connection f to indicate that the action is a left- or right-module action.

Theorem 3.1. *If $\mathcal{E}$ is a connection on $\mathcal{D}$, then $\mathcal{E}$ is also a Hopf module and a comodule algebra over $\mathcal{D}$. It is also a one-sided Hopf algebra with one-sided unit $1_{\mathcal{E}} := f^{-1}(1_{\mathcal{D}})$ and antipode.*

Proof. Suppose $\mathcal{E}$ is a right $\mathcal{D}$-module. Define the product $m_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ via the $\mathcal{D}$-action: $m_{\mathcal{E}} := \star \circ (1 \otimes f)$. The one-sided unit is $1_{\mathcal{E}}$. Then $\Delta_{\mathcal{E}}$ is an algebra map. Indeed, for $e, e' \in \mathcal{E}$, we have

$$\Delta_{\mathcal{E}}(e \cdot e') \;=\; \Delta_{\mathcal{E}}(e \star f(e')) \;=\; \Delta_{\mathcal{E}} e \star \Delta_{\mathcal{D}} f(e') \;=\; \Delta_{\mathcal{E}} e \star (f \otimes f)(\Delta_{\mathcal{E}} e') \;=\; \Delta_{\mathcal{E}} e \cdot \Delta_{\mathcal{E}} e'\,.$$

As usual, $\varepsilon_{\mathcal{E}}$ is just projection onto $\mathcal{E}_0$. The unit $1_{\mathcal{E}}$ is one-sided, since

$$e \cdot 1_{\mathcal{E}} \;=\; e \star f(1_{\mathcal{E}}) \;=\; e \star f(f^{-1}(1_{\mathcal{D}})) \;=\; e \star 1_{\mathcal{D}} \;=\; e\,,$$

but $1_{\mathcal{E}} \cdot e = 1_{\mathcal{E}} \star f(e)$ is not necessarily equal to e. As $\mathcal{E}$ is a graded bialgebra, the antipode S may be defined recursively to satisfy $m_{\mathcal{E}}(S \otimes \mathbb{1})\Delta_{\mathcal{E}} = \varepsilon_{\mathcal{E}}$, see 4.2. (If instead $\mathcal{E}$ is a left $\mathcal{D}$-module, then it has a left-sided unit and right-sided antipode.)

Define $\rho\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{D}$ by $\rho := (1 \otimes f)\,\Delta_{\mathcal{E}}$. This gives a coaction so that $\mathcal{E}$ is a Hopf module and a comodule algebra over $\mathcal{D}$. □

3.2. **Operads and operad modules.** Composition of coalgebras is the same product used to define operads internal to a symmetric monoidal category [1, Appendix B]. A *monoid* in a category with a product $\bullet$ is an object $\mathcal{D}$ with a morphism $\gamma\colon \mathcal{D} \bullet \mathcal{D} \to \mathcal{D}$ that is associative. An *operad* is a monoid in the category of graded sets with an analog of the composition product $\circ$ defined in Section 2.1.

The category of connected graded coalgebras and coalgebra maps is a symmetric monoidal category with the composition $\circ$ of coalgebras. A *graded Hopf operad* $\mathcal{D}$ is a monoid in this category. That is, $\mathcal{D}$ has associative composition maps $\gamma\colon \mathcal{D} \circ \mathcal{D} \to \mathcal{D}$ obeying

$$\Delta_{\mathcal{D}}\gamma(a) \;=\; (\gamma \otimes \gamma)\,(\Delta_{\mathcal{D}\circ\mathcal{D}}(a)) \quad \text{for all} \;\; a \in \mathcal{D} \circ \mathcal{D}\,.$$

By Theorem 3.4, $\mathcal{D}$ is a Hopf algebra; this explains our nomenclature.

A *graded Hopf operad module* $\mathcal{E}$ is an operad module (left or right) over $\mathcal{D}$ and a graded coassociative coalgebra whose module action is compatible with its coproduct. Write $\mu_l : \mathcal{D} \circ \mathcal{E} \to \mathcal{E}$ and $\mu_r : \mathcal{E} \circ \mathcal{D} \to \mathcal{E}$ for the left and right actions, which obey, e.g.,

$$\Delta_{\mathcal{E}}\mu_r(b) \;=\; (\mu_r \otimes \mu_r)\Delta_{\mathcal{E}\circ\mathcal{D}} b \qquad \text{for all } b \in \mathcal{E} \circ \mathcal{D}\,.$$

Example 3.2. $\mathcal{Y}Sym$ is an operad in the category of vector spaces. The action of γ on $F_t \circ (F_{t_0} \cdot \cdots \cdot F_{t_n})$ grafts the indexing trees $t_0, \ldots, t_n$ onto the tree t and, unlike in Example 2.1, forgets which nodes of the resulting tree came from t. This is associative in the appropriate sense. The same action γ makes $\mathcal{Y}Sym$ an operad in the category of connected graded coalgebras, and thus a graded Hopf operad. Finally, operads are operad modules over themselves, so $\mathcal{Y}Sym$ is also graded Hopf operad module.

Remark 3.3. This notion differs from that of Getzler and Jones [10], who defined a Hopf operad $\mathcal{D}$ to be an operad where each component $\mathcal{D}_n$ is a coalgebra.

Theorem 3.4. *A graded Hopf operad $\mathcal{D}$ is also a Hopf algebra with product*

$$a \cdot b \;:=\; \gamma(b \otimes \Delta^{(n)} a) \tag{3.1}$$

where $b \in \mathcal{D}_n$ and $\Delta^{(n)}$ is the iterated coproduct from $\mathcal{D}$ to $\mathcal{D}^{\otimes(n+1)}$.

Remark 3.5. If we swap the roles of a and b on the right-hand side of (3.1), we also obtain a Hopf algera, for H^{op} is a Hopf algebra whenever H is one. Our choice agrees with the description (Section 1.2) of products in $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$.

Before we prove Theorem 3.4, we restate an old result in the language of operads.

Proposition 3.6. *The well-known Hopf algebra structures of $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$ are induced by their structure as graded Hopf operads.*

Proof. The operad structure on $\mathcal{Y}Sym$ is the operad of planar, rooted, binary trees, where composition γ is grafting. The operad structure on $\mathfrak{C}Sym$ is the terminal operad, which has a single element in each component. Representing the single element of degree n as a comb of n leaves, the composition γ becomes grafting and combing all branches of the result.

We check that these compositions γ are coalgebra maps. For $\mathcal{D} = \mathcal{Y}Sym$, the coproduct $\Delta_{\mathcal{D}\circ\mathcal{D}}$ in the F-basis is the sum over possible splittings of the composite trees. Then splitting an element of $e \in \mathcal{D}\circ\mathcal{D}$ and grafting both resulting trees (via $\gamma\otimes\gamma$) yields the same result as first grafting ($e \to \gamma(e)$), then splitting the resulting tree. When $\mathcal{D} = \mathfrak{C}Sym$, virtually the same analysis holds, with the proviso that graftings are always followed by combing all branches to the right.

Finally, we note that the product in $\mathcal{Y}Sym$ in terms of the F-basis is simply $a\cdot b = \gamma(b\otimes\Delta^{(|b|)}a)$. The same holds for $\mathfrak{C}Sym$, again with the proviso that γ is grafting, followed by combing. □

Proof of Theorem 3.4. We have $\gamma(1\otimes 1) = 1$ and $\gamma(b\otimes 1^{\otimes|b|+1}) = b$ by construction, since $\mathcal{D}$ is connected. Thus $1 = 1_{\mathcal{D}}$ is the unit in $\mathcal{D}$.

The image of $\mathbb{1}\otimes\Delta^{(n)}$ lies in $\mathcal{D}\circ\mathcal{D}$. As γ is a map of graded coalgebras, $\Delta(a\cdot b) = \Delta a\cdot\Delta b$. Indeed, for $b\in\mathcal{D}$ homogeneous,

$$\begin{aligned}\Delta(a\cdot b) \;=\; \Delta(\gamma(b\otimes\Delta^{(|b|)}a)) = \;& (\gamma\otimes\gamma)(\Delta_{\mathcal{D}\circ\mathcal{D}}(b\otimes\Delta^{(|b|)}a))\\ =\;& (\gamma\otimes\gamma)((\Delta b\otimes\Delta^{(|b|)}\Delta a)) \;=\; \Delta a\cdot\Delta b\,.\end{aligned}$$

Associativity of the product follows, since for b, c homogeneous elements of $\mathcal{D}$, we have

$$\begin{aligned} a\cdot(b\cdot c) \;=\; a\cdot\gamma(c\otimes\Delta^{(|c|)}b) \;&=\; \gamma\left(\gamma(c\otimes\Delta^{(|c|)}b)\otimes\Delta^{(|b|+|c|)}a\right)\\ &=\; \gamma\left(c\otimes\gamma^{\otimes(|c|+1)}(\Delta^{(|c|)}b\otimes\Delta^{(|b|+|c|)}a)\right) \qquad (3.2)\\ &=\; \gamma\left(c\otimes(\Delta^{(|c|)}a\cdot\Delta^{(|c|)}b)\right)\\ &=\; \gamma(c\otimes\Delta^{(|c|)}(a\cdot b)) \;=\; (a\cdot b)\cdot c\,. \qquad (3.3)\end{aligned}$$

Here, (3.2) is by the associativity of composition γ in an operad, where we assume the isomorphism $\mathcal{D}\circ(\mathcal{D}\circ\mathcal{D})\cong(\mathcal{D}\circ\mathcal{D})\circ\mathcal{D}$. The step (3.3) follows as $\mathcal{D}$ is a bialgebra ($\Delta^{(n)}$ is an algebra map since $\Delta = \Delta^{(1)}$ is one). □

Lemma 3.7. *If $\mathcal{C}$ is a graded coalgebra and $\mathcal{D}$ is a graded Hopf operad, then $\mathcal{D}\circ\mathcal{C}$ is a (left) graded Hopf operad module and $\mathcal{C}\circ\mathcal{D}$ is a (right) graded Hopf operad module.*

Proof. We grade $\mathcal{D}\circ\mathcal{C}$ and $\mathcal{C}\circ\mathcal{D}$ by total degree. An operad module of vector spaces is a sequence of vector spaces acted upon by the operad. The action $\mu_l : \mathcal{D}\circ(\mathcal{D}\circ\mathcal{C}) \to (\mathcal{D}\circ\mathcal{C})$ is given by

$$\mu_l\left(d\otimes\frac{c_{0_0}\bullet\cdots\bullet c_{i_0}}{d_0}\otimes\cdots\otimes\frac{c_{0_n}\bullet\cdots\bullet c_{i_n}}{d_n}\right) \;=\; \frac{c_{0_0}\bullet\cdots\bullet c_{i_n}}{\gamma(d\otimes d_0\otimes\cdots\otimes d_n)}\,.$$

Associativity of γ implies that this action is associative. The action $\mu_r : (\mathcal{C}\circ\mathcal{D})\circ\mathcal{D} \to (\mathcal{C}\circ\mathcal{D})$ is given by

$$\begin{aligned}&\mu_r\left(\frac{d_0\bullet\cdots\bullet d_m}{c}\otimes d_{0_0}\otimes\cdots\otimes d_{j_m}\right)\\ &\qquad\qquad = \frac{\gamma(d_0\otimes d_{0_0}\otimes\cdots\otimes d_{j_0})\bullet\cdots\bullet\gamma(d_m\otimes d_{0_m}\otimes\cdots\otimes d_{j_m})}{c}\,.\end{aligned}$$

Associativity of γ implies that this action is associative as well. We leave the reader to check that $\Delta\mu_l = (\mu_l\otimes\mu_l)\Delta$ and $\Delta\mu_r = (\mu_r\otimes\mu_r)\Delta$. □

Lemma 3.8. *A graded Hopf operad module $\mathcal{E}$ over a graded Hopf operad $\mathcal{D}$ is also a module coalgebra for the Hopf algebra $\mathcal{D}$.*

Proof. Fix $e \in \mathcal{E}$ and $d \in \mathcal{D}$ to be homogeneous elements. If $\mathcal{E}$ is a right operad module over $\mathcal{D}$ then define a left action by $d \star e := \mu_r(e \otimes \Delta^{(|e|)} d)$. If $\mathcal{E}$ is a left operad module over $\mathcal{D}$ then $e \star d := \mu_l(d \otimes \Delta^{(|d|)} e)$ defines a right action.

Checking that either case defines an associative action and a module coalgebra uses the same reasoning as for the proof of Theorem 3.4, with μ replacing γ. □

Theorem 3.9. *Given a coalgebra map $\lambda \colon \mathcal{C} \to \mathcal{D}$ from a connected graded coalgebra $\mathcal{C}$ to a graded Hopf operad $\mathcal{D}$, the maps $f_r = \gamma \circ (\mathbb{1} \circ \lambda) \colon \mathcal{D} \circ \mathcal{C} \to \mathcal{D}$ and $f_l = \gamma \circ (\lambda \circ \mathbb{1}) \colon \mathcal{C} \circ \mathcal{D} \to \mathcal{D}$ give connections on $\mathcal{D}$.*

Proof. By Theorem 3.4 and Lemmas 3.7 and 3.8, $\mathcal{D} \circ \mathcal{C}$ and $\mathcal{C} \circ \mathcal{D}$ are connected graded module coalgebras over $\mathcal{D}$. We need only show that the maps f_r and f_l are coalgebra maps and module maps. In terms of decomposable tensors, the maps take the form,

$$f_r\left(\frac{c_0 \bullet \cdots \bullet c_n}{d}\right) := \gamma\left(\frac{\lambda(c_0) \bullet \cdots \bullet \lambda(c_n)}{d}\right) \quad \text{and} \quad f_l\left(\frac{d_0 \bullet \cdots \bullet d_n}{c}\right) := \gamma\left(\frac{d_0 \bullet \cdots \bullet d_n}{\lambda(c)}\right).$$

These are coalgebra maps since both λ and γ are coalgebra maps. The associativity of γ implies that f_r and f_l are maps of right and left $\mathcal{D}$-modules, respectively. □

3.3. **Examples of module coalgebra connections.** Eight of the nine compositions of coalgebras from Section 2.2 are connections on one or both of the factors $\mathcal{C}$ and $\mathcal{D}$.

Theorem 3.10. *For $\mathcal{C} \in \{\mathfrak{S}Sym, \mathcal{Y}Sym, \mathfrak{C}Sym\}$, the coalgebra compositions $\mathcal{C} \circ \mathfrak{C}Sym$ and $\mathfrak{C}Sym \circ \mathcal{C}$ are connections on $\mathfrak{C}Sym$. For $\mathcal{C} \in \{\mathfrak{S}Sym, \mathcal{Y}Sym, \mathfrak{C}Sym\}$, the coalgebra compositions $\mathcal{C} \circ \mathcal{Y}Sym$ and $\mathcal{Y}Sym \circ \mathcal{C}$ are connections on $\mathcal{Y}Sym$.*

Proof. By Theorem 3.9 and Proposition 3.6, we need only show the existence of coalgebra maps from $\mathcal{C}$ to $\mathcal{D}$, for $\mathcal{C} \in \{\mathfrak{S}Sym, \mathcal{Y}Sym, \mathfrak{C}Sym\}$ and $\mathcal{D} \in \{\mathcal{Y}Sym, \mathfrak{C}Sym\}$.

For $\mathcal{D} = \mathfrak{C}Sym$, the maps $\boldsymbol{\kappa\tau}$, $\boldsymbol{\kappa}$, and $\mathbb{1}$ are all coalgebra maps to $\mathfrak{C}Sym$ (Proposition 1.1). For $\mathcal{D} = \mathcal{Y}Sym$, the maps $\boldsymbol{\tau}$ and $\mathbb{1}$ are coalgebra maps to $\mathcal{Y}Sym$. Lastly, combs are binary trees, and the induced inclusion map $\mathfrak{C}Sym \hookrightarrow \mathcal{Y}Sym$ is a coalgebra map. □

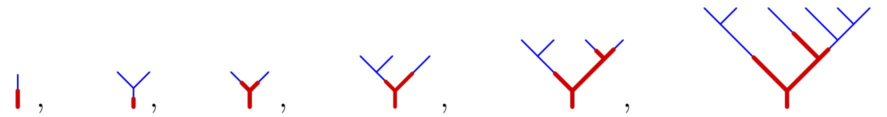
Note that in particular, $\mathcal{Y}Sym \circ \mathfrak{C}Sym$ is a connection on both $\mathfrak{C}Sym$ and $\mathcal{Y}Sym$. This yields two distinct one-sided Hopf algebra structures on $\mathcal{Y}Sym \circ \mathfrak{C}Sym$. Likewise, $\mathcal{Y}Sym \circ \mathcal{Y}Sym$ is a connection on $\mathcal{Y}Sym$ in two distinct ways (again leading to two distinct one-sided Hopf structures). In the remaining sections, we discuss three of the compositions of Section 2.2 which have appeared previously.

4. A Hopf Algebra of Painted Trees

Our motivating example is the self-composition $\mathcal{P}Sym := \mathcal{Y}Sym \circ \mathcal{Y}Sym$. Elements of the fundamental basis of $\mathcal{P}Sym$ are $F_p = d \circ (c_0 \bullet \cdots \bullet c_{|d|})$, where $c_0, \ldots, c_{|d|}$ and d are elements of the fundamental basis of $\mathcal{Y}Sym$. The indexing trees of $c_1, \ldots, c_{|d|}$ and d may be combined to form painted trees as in Example 2.1. We describe the topological origin of painted trees and their relation to the multiplihedron, and we relate the Hopf structures of $\mathcal{P}Sym$ to the Hopf structures of $\mathcal{M}Sym$ developed in [8].

4.1. **Painted binary trees in topology.** A *painted binary tree* is a planar binary tree t, together with a (possibly empty) upper order ideal of its node poset. We indicate this ideal by painting on top of a representation of t. For clarity, we stop our painting in the middle of edges. Here are a few simple examples,

(4.1)



An *A_n-space* is a topological H-space with a weakly associative multiplication of points [15]. (Products are represented by planar binary trees as these distinguish between possible choices of associations.) Maps between A_n-spaces preserve the multiplicative structure only up to homotopy. Stasheff [15] described these maps combinatorially using cell complexes called multiplihedra, while Boardman and Vogt [5], used spaces of painted trees. Both the spaces of trees and the cell complexes are homeomorphic to convex polytope realizations of the multiplihedra as shown in [6].

If $f\colon (X,\bullet) \to (Y,*)$ is a map of A_n-spaces, then the different ways to multiply and map n points of X are represented by painted trees. Unpainted nodes are multiplications in X, painted nodes are multiplications in Y, and the beginning of the painting indicates that f is applied to a given point in X. See Figure 3.

$$f(a) * \big(f(b \bullet c) * f(d)\big) \longleftrightarrow$$

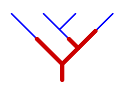
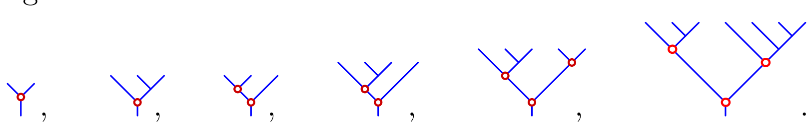


FIGURE 3. A_n-maps between H-spaces correspond to painted binary trees.

Figure 5 shows the three-dimensional multiplihedron with its vertices labeled by painted trees having three internal nodes. This picture of the multiplihedron also shows that the vertices are the elements of a lattice whose Hasse diagram is the one-skeleton of the polytope in the view shown. See [8] for an explicit description of the covering relations in terms of *bi-leveled trees*.

4.2. **Painted trees as bi-leveled trees.** A *bi-leveled tree* is a planar binary tree t together with an order ideal T of its node poset which contains the leftmost node, but none of its children. We display bi-leveled trees corresponding to the painted trees of (4.1), circling the nodes in T.



Bi-leveled trees having $n{+}1$ internal nodes are in bijection with painted trees having n internal nodes, the bijection being given by pruning: Remove the leftmost branch and node from a bi-leveled tree to get a tree whose order ideal is the order ideal of the bi-leveled tree, minus the leftmost node. For an illustration of this and the inverse mapping, see Figure 4.

Let $\mathcal{M}_n$ be the set of bi-leveled trees with n internal nodes. In [8] we developed several algebraic structures on the graded vector space $\mathcal{M}Sym$ with basis F_b indexed by bi-leveled trees b, graded by the number of internal nodes of b. We also placed a

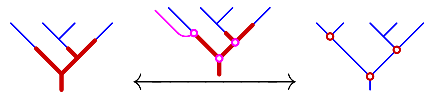


FIGURE 4. Painted trees correspond to bi-leveled trees.

$\mathcal{Y}Sym$–Hopf module structure on $\mathcal{M}Sym_+$, the positively graded part of $\mathcal{M}Sym$. We revisit this structure in Section 4.4.

4.3. **The coalgebra of painted trees.** Let $\mathcal{P}_n$ be the poset of painted trees on n internal nodes, with partial order inherited from the identification with bi-leveled trees $\mathcal{M}_{n+1}$. We show $\mathcal{P}_3$ in Figure 5. We refer to [8] for a description of the order on $\mathcal{M}_{n+1}$.

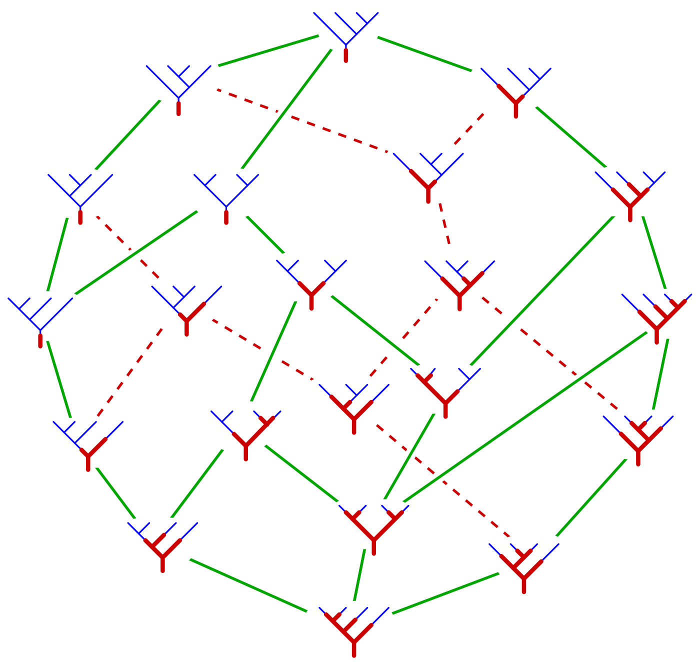


FIGURE 5. The one-skeleton of the three-dimemsional multiplihedron, $\mathcal{M}_4$.

(Note that the map from $\mathcal{P}_\bullet$ to $\mathcal{M}_\bullet$ actually lands in $\mathcal{M}_+$, which consists of the trees in $\mathcal{M}_\bullet$ with one or more nodes.)

We reproduce the compositional coproduct defined in Section 2.1.

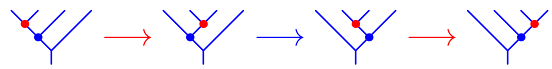
Definition 4.1 (Coproduct on $\mathcal{P}Sym$)**.** Given a painted tree p, define the coproduct in the fundamental basis $\left\{F_p \mid p \in \mathcal{P}_\bullet\right\}$ by

$$\Delta(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0,p_1)} F_{p_0} \otimes F_{p_1},$$

where the painting in p is preserved in the splitting $p \xrightarrow{\curlyvee} (p_0, p_1)$.

The counit ε is projection onto $\mathcal{P}Sym_0$, which is spanned by $F_{|}$.

Theorem 2.4 describes the primitive elements of $\mathcal{P}Sym = \mathcal{Y}Sym \circ \mathcal{Y}Sym$ in terms of the primitive elements of $\mathcal{Y}Sym$. We recall the description of primitive elements of $\mathcal{Y}Sym$ as given in [3]. The set of trees $\mathcal{Y}_n$ forms a poset whose covering relation is obtained by moving a child node of a given node from the left to the right branch above the given node. Thus



is an increasing chain in $\mathcal{Y}_3$ (the moving vertices are marked with dots).

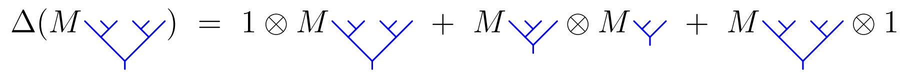
Let μ be the Möbius function of $\mathcal{Y}_n$ which is defined by $\mu(t,s) = 0$ unless $t \leq s$,

$$\mu(t,t) \;=\; 1\,, \qquad \text{and} \qquad \mu(t,r) \;=\; -\sum_{t\leq s<r} \mu(t,s)\,.$$

We define a new basis for $\mathcal{Y}Sym$ using the Möbius function. For $t \in \mathcal{Y}_n$, set

$$M_t \;:=\; \sum_{t\leq s} \mu(t,s) F_s\,.$$

Then the coproduct for $\mathcal{Y}Sym$ with respect to this M-basis is still given by splitting of trees, but only at leaves emanating directly from the right branch above the root:



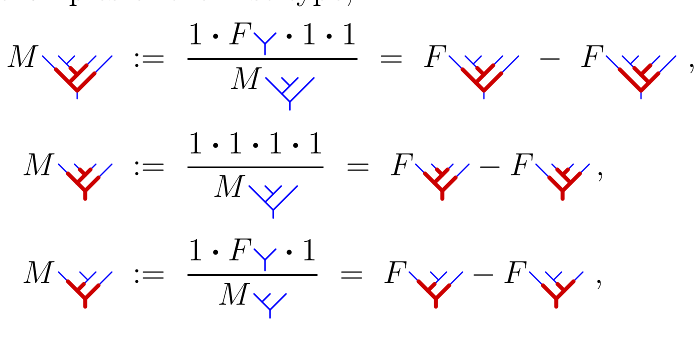
A tree $t \in \mathcal{Y}_n$ is *progressive* if it has no branching along the right branch above the root node. A consequence of the description of the coproduct in this M-basis is Corollary 5.3 of [3] that the set $\{M_t \mid t \text{ is progressive}\}$ is a linear basis for the space of primitive elements in $\mathcal{Y}Sym$.

Thus according to Theorem 2.4 the cogenerating primitives in $\mathcal{P}Sym$ are of two types:

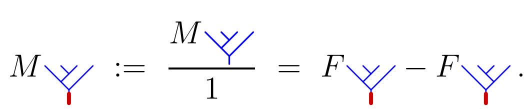
$$\frac{1{\bullet}c_1{\bullet}\cdots{\bullet}c_{n-1}{\bullet}1}{M_t} \qquad \text{and} \qquad \frac{M_t}{1}\,,$$

where t is a progressive tree.

Here are some examples of the first type,



and one of the second type,



The primitives can be described in terms of Möbius inversion on certain subintervals of the multiplihedra lattice. For the first type, the subintervals are those with a fixed unpainted forest of the form $(|, t, \ldots, s, |)$. For the second type, the subinterval consists of those trees whose painted part is trivial, . Each subinterval of the first type is isomorphic to $\mathcal{Y}_m$ for some $m \leq n$, and the second subinterval is isomorphic to $\mathcal{Y}_n$. Figure 6 shows the multiplihedron lattice $\mathcal{P}_3$, with these subintervals highlighted.

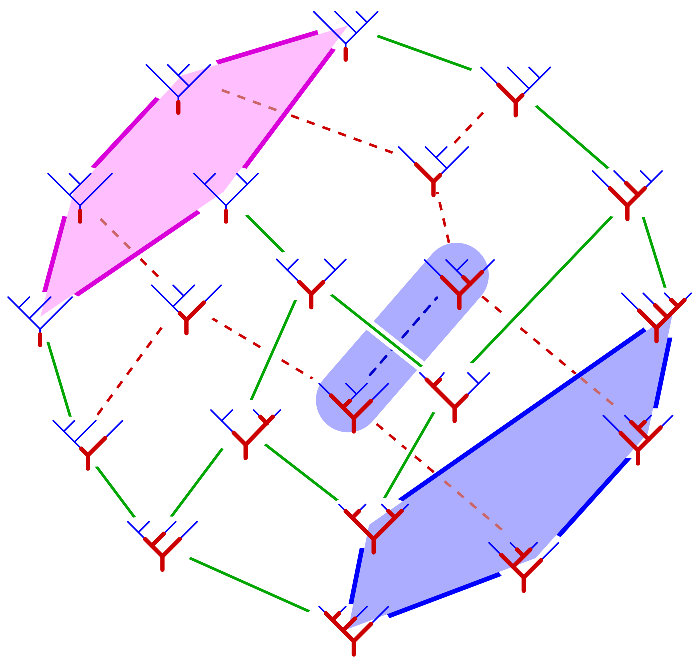


FIGURE 6. The multiplihedron lattice $\mathcal{M}_4$ showing the three subintervals that yield primitives via Möbius inversion.

4.4. **Hopf structures on painted trees.** As determined in the proof of Theorem 3.10, the identity map $\mathbb{1} : \mathcal{Y}Sym \to \mathcal{Y}Sym$ yields a connection $f_r \colon \mathcal{P}Sym \to \mathcal{Y}Sym$. In particular (Theorem 3.1), $\mathcal{P}Sym$ is a one-sided Hopf algebra, a $\mathcal{Y}Sym$–Hopf module, and a $\mathcal{Y}Sym$–comodule algebra. We discuss these structures, and relate them to structures placed on $\mathcal{M}Sym$ in [8].

Let p, q be painted trees with $|q| = n$. In terms of the F-basis, f_r simply forgets the painting level, e.g., $f_r(F) = F$. Thus Theorem 3.1 describes the product $F_p \cdot F_q$ in $\mathcal{P}Sym$ as

$$F_p \cdot F_q \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0,p_1,\ldots,p_n)} F_{(p_0,p_1,\ldots,p_n)/q^+}\,,$$

where the painting in p is preserved in the splitting $(p_0, p_1, \ldots, p_n)$, and q^+ signifies that q is painted completely before grafting. Here is an example of the product,

$$F \cdot F \;=\; F \;+\; F \;+\; F \;+\; F\,.$$

The painted tree with 0 nodes is only a right unit: for all $q \in \mathcal{P}_{\bullet}$,

$$F \cdot F_q \;=\; F_{q^+} \qquad \text{and} \qquad F_q \cdot F \;=\; F_q\,.$$

Although the antipode is guaranteed to exist, we include a proof for purpose of exposition.

Theorem 4.2. *There are unit and antipode maps $\eta\colon \mathbb{K} \to \mathcal{P}Sym$ and $S\colon \mathcal{P}Sym \to \mathcal{P}Sym$ making $\mathcal{P}Sym$ a one-sided Hopf algebra.*

Proof. We just observed that $\eta\colon 1 \mapsto F_{|}$ is a right unit for $\mathcal{P}Sym$. We verify that a *left antipode* exists. That is, there exists a map $S\colon \mathcal{P}Sym \to \mathcal{P}Sym$ such that $S(F_{|}) = F_{|}$, and for $p \in \mathcal{P}_+$, we have

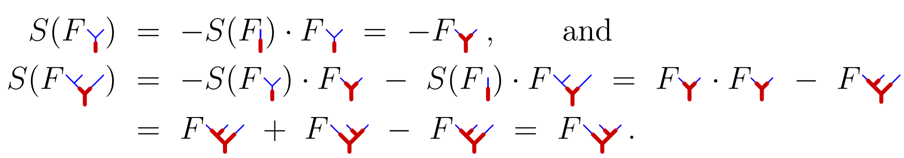
$$\sum_{p \xrightarrow{\curlyvee} (p_0,p_1)} S(F_{p_0}) \cdot F_{p_1} \;=\; 0\,. \tag{4.2}$$

Since $\mathcal{P}Sym$ is graded, and $|p| = |p_0| + |p_1|$ whenever $p \xrightarrow{\curlyvee} (p_0,p_1)$, we may recursively construct S using induction on $|p|$. First, set $S(F_{|}) = F_{|}$. Then, given any painted tree p, the only term involving $S(F_q)$ in (4.2) with $|q| = |p|$ is $S(F_p)\cdot F_{|} = S(F_p)$, and so we may solve (4.2) for $S(F_p)$ to obtain

$$S(F_p) \;:=\; -\sum_{\substack{p \xrightarrow{\curlyvee} (p_0,p_1)\\ |p_0|,|p_1|>0}} S(F_{p_0}) \cdot F_{p_1} \;-\; S(F_{|}) \cdot F_p\,,$$

expressing $S(F_p)$ in terms of previously defined values $S(F_q)$. □

For example,



Remark 4.3. One may be tempted to artificially adjoin a true unit e to $\mathcal{P}Sym$, but this only pushes the problem to the antipode map: $S(F_{|})$ cannot be defined if $\eta(1) = e$.

The $\mathcal{Y}Sym$-Hopf module structure on $\mathcal{P}Sym$ of Theorem 3.1 has coaction,

$$\rho(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0,p_1)} F_{p_0} \otimes F_{f(p_1)}\,,$$

where the painting in p is preserved in the first half of the splitting (p_0,p_1), and forgotten in the second half.

Under the bijection between $\mathcal{P}_{\bullet}$ and $\mathcal{M}_+$ that grows an extra node as in Figure 4, the splittings and graftings on $\mathcal{P}Sym$ map to the restricted splittings $\xrightarrow{\curlyvee_+}$ and graftings defined in [8, Section 4.1]. Moreover, we can split and graft before or after the bijection to achieve the same results. These facts allow the following corollary.

Corollary 4.4. *The $\mathcal{Y}Sym$ action and coaction defined in* [8, Section 4.1] *make $\mathcal{M}Sym_+$ into a Hopf module isomorphic to the Hopf module $\mathcal{P}Sym$.* □

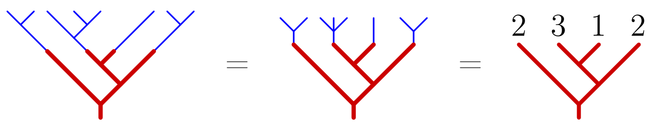
The *coinvariants* of a Hopf module $\rho\colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{D}$ are elements $e \in \mathcal{E}$ such that $\rho(e) = e \otimes 1$. The coinvariants for the action of Corollary 4.4 were described explicitly in [8, Corollary 4.5]. In contrast to the discussion in Section 4.3, Möbius inversion in the entire lattice $\mathcal{M}_{\bullet}$ helps to find the coinvariants.

5. A Hopf Algebra of Weighted Trees

The composition of coalgebras $\mathcal{Y}Sym \circ \mathfrak{C}Sym$ has fundamental basis indexed by forests of combs attached to binary trees, which we will call weighted trees. By the first statement of Theorem 3.10, it has a connection on $\mathfrak{C}Sym$ that gives it the structure of a one-sided Hopf algebra. We examine this Hopf algebra in more detail.

5.1. **Weighted trees in topology.** In a forest of combs attached to a binary tree, the combs may be replaced by corollae or by a positive *weight* counting the number of leaves in the comb. These all give *weighted trees*.

(5.1)



Let $\mathcal{CK}_n$ denote the weighted trees with weights summing to $n{+}1$. These index the vertices of the n-dimensional *composihedron*, $\mathcal{CK}(n{+}1)$ [7]. This sequence of polytopes parameterizes homotopy maps between strictly associative and homotopy associative H-spaces. Figure 7 gives a picture of the composihedron $\mathcal{CK}_3$. For small values of n,

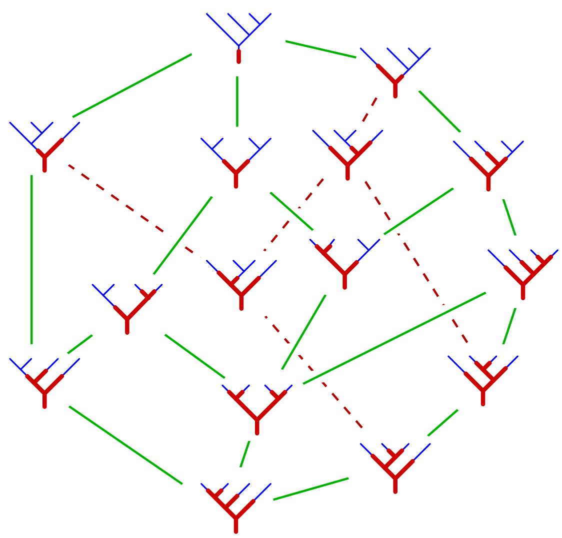


FIGURE 7. The one-skeleton of the three-dimensional composihedron.

the composihedra $\mathcal{CK}(n)$ also appear as the commuting diagrams in enriched bicategories [7]. These diagrams appear in the definition of pseudomonoids [1, Appendix C].

5.2. **A Hopf algebra of weighted trees.** We describe the key definitions of Section 2.1 and Section 3 for *$\mathcal{CK}Sym$* $:= \mathcal{Y}Sym \circ \mathfrak{C}Sym$. In the fundamental basis $\{F_p \mid p \in \mathcal{CK}_\bullet\}$ of $\mathcal{CK}Sym$, the coproduct is

$$\Delta(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0, p_1)} F_{p_0} \otimes F_{p_1}\,,$$

where the painting in $p \in \mathcal{CK}_\bullet$ is preserved in the splitting $p \xrightarrow{\curlyvee} (p_0, p_1)$. The counit ε is projection onto $\mathcal{CK}Sym_0$, which is spanned by $F_{|}$. Here is an example in terms of weighted trees,

$$\Delta(F_{2\,1\,2}) \;=\; F_{1} \otimes F_{2\,1\,2} \;+\; F_{2} \otimes F_{1\,1\,2} \;+\; F_{2\,1} \otimes F_{1\,2} \;+\; F_{2\,1\,1} \otimes F_{2} \;+\; F_{2\,1\,2} \otimes F_{1}\,.$$

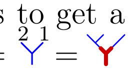
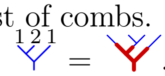
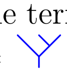
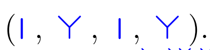
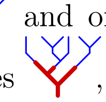
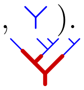
The primitive elements of $\mathcal{CK}Sym = \mathcal{Y}Sym \circ \mathfrak{C}Sym$ have the form

$$F_{2} \;=\; \frac{F_{\curlyvee}}{1} \qquad \text{and} \qquad \frac{1 \bullet c_1 \bullet \cdots \bullet c_{n-1} \bullet 1}{M_t}\,,$$

where t is a progressive tree with n nodes and $c_1, \ldots, c_{n-1}$ are any elements of $\mathfrak{C}Sym$. In terms of weighted trees, the indices of the second type are weighted progressive trees with weights of 1 on their leftmost and rightmost leaves.

Let $f_l \colon \mathcal{CK}Sym \to \mathfrak{C}Sym$ be the connection given by Theorem 3.9 (built from the coalgebra map $\boldsymbol{\kappa}$). Then Theorem 3.1 gives the product

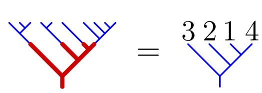
$$F_p \cdot F_q \;:=\; f_l(F_p) \star F_q\,, \qquad \text{where } p, q \in \mathcal{CK}_\bullet\,.$$

In terms of the F-basis, f_l acts on indices, sending a weighted tree p to the unique comb $f_l(p)$ with the same number of nodes as p. The action $\star$ in the F-basis is given as follows: split $f_l(p)$ in all ways to make a forest of $|q|+1$ combs; graft each splitting onto the leaves of the forest of combs in q; comb the resulting forest of trees to get a new forest of combs. We illustrate one term in the product. Suppose that $p = {}^{2\,1}$ =  and $q = {}^{1\,2\,1}$ = . Then $f_l(p)$ =  and one way to split $f_l(p)$ gives the forest . Grafting this onto q gives , which after combing the forest yields the term  $= {}^{1\,3\,2}$ in the product $p \cdot q$. Doing this for the other nine splittings of $f_l(p)$ gives

$$F_{2\,1} \cdot F_{1\,2\,1} \;=\; F_{3\,2\,1} \;+\; 3F_{1\,4\,1} \;+\; F_{1\,2\,3} \;+\; 2F_{2\,3\,1} \;+\; F_{2\,2\,2} \;+\; 2F_{1\,3\,2}.$$

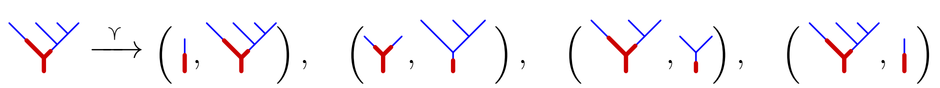
6. Composition trees and the Hopf algebra of simplices

The simplest composition of Section 2.2 is $\mathfrak{C}Sym \circ \mathfrak{C}Sym$. As shown in Section 2.3, the graded component of total degree n has dimension 2^n, indexed by trees with n interior nodes obtained by grafting a forest of combs to the leaves of a comb (which is painted). Analogous to (5.1), these are weighted combs. As these are in bijection with compositions of $n+1$, we refer to them as *composition trees*.



$$= \; {}^{3\,2\,1\,4} \; = \; (3,2,1,4)\,.$$

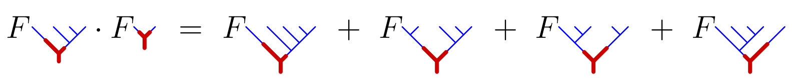
6.1. **Hopf algebra structures on composition trees.** The coproduct may again be described via splitting. Since the composition tree $(1,3)$ has the four splittings

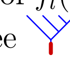
(6.1) 

we have $\Delta(F_{1,3}) = F_1 \otimes F_{1,3} + F_{1,1} \otimes F_3 + F_{1,2} \otimes F_2 + F_{1,3} \otimes F_1$.

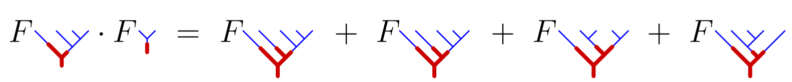
The identity map on $\mathfrak{C}Sym$ gives two connections $\mathfrak{C}Sym \circ \mathfrak{C}Sym \to \mathfrak{C}Sym$ (using either f_l or f_r from Theorem 3.9). This gives two new one-sided Hopf algebra structures on compositions.

6.1.1. *Hopf structure induced by f_l.* Let p, q be composition trees and consider the product $F_p \cdot F_q \ := \ f_l(F_p) \star F_q$. At the level of indices in the F-basis, the connection f_l sends the composition tree p to the unique comb $f_l(p)$ with the same number of vertices as p. The action $f_l(F_p) \star F_q$ may be described as follows: split the comb $f_l(p)$ into a forest of $|q|{+}1$ combs in all possible ways; graft each splitting onto the leaves of the forest in q; comb the resulting forest of trees to get a new forest of combs. For example, $F_{1,3} \cdot F_{1,1} = F_{1,4} + F_{2,3} + F_{3,2} + F_{4,1}$, or alternatively,

(6.2) 

This may be seen by unpainting and grafting the splittings (6.1) onto the tree . Likewise, $F_{1,3} \cdot F_2 = 4F_4$, for no matter which of the four splittings of $f_l(1,3)$ is chosen, the grafting onto  and subsequent combing will yield the same tree .

6.1.2. *Hopf structure induced by f_r.* Let p, q be composition trees and consider the product $F_p \cdot F_q \ := \ F_p \star f_r(F_q)$. At the level of indices in the F-basis, the connection f_r sends a composition tree q to the unique comb $f_r(q)$ with $|q|$ vertices. The action $F_p \star f_r(F_q)$ may be described as follows: first paint the comb $f_r(q)$; next split the composition tree p into a forest of $|q|{+}1$ composition trees in all possible ways; finally, graft each forest onto the leaves of the painted tree $f_r(q)$ and comb the resulting painted subtree (which comes from the nodes of q and the painted nodes of p). For example, $F_{1,3} \cdot F_2 = 2F_{1,1,3} + F_{1,2,2} + F_{1,3,1}$, or alternatively,

(6.3) 

This may be seen by grafting the splittings (6.1) onto the tree $f_r($$) =$ .

6.2. **Composition trees in topology.** A one-sided Hopf algebra ΔSym was defined in [9, Section 7.3] whose nth graded piece had a basis indexed by the faces of the $(n{-}1)$-dimensional simplex. We recount the product and coproduct introduced there. (The notation $\Delta\widetilde{Sym}$ was used for this algebra in [9] to distinguish it from an algebra based only on the vertices of the simplex.) Faces of the $(n{-}1)$-dimensional simplex correspond to subsets S of $[n] := \{1, \ldots, n\}$, so this is a Hopf algebra whose nth graded piece also has dimension 2^n, with fundamental basis $F^{[n]}_{\mathsf{S}}$.

An ordered decomposition $n = p + q$ gives a splitting of $[n]$ into two pieces $[p]$ and $\iota_p([q]) := \{p{+}1, \ldots, n\}$. Any subset $\mathsf{S} \subseteq [n]$ gives a pair of subsets $\mathsf{S}' \subseteq [p]$ and $\mathsf{S}'' \subseteq [q]$,

$$\mathsf{S}' \ := \ \mathsf{S} \cap [p] \qquad \text{and} \qquad \mathsf{S}'' \ := \ \iota_p^{-1}(\mathsf{S} \cap \{p{+}1, \ldots, n\}) \,.$$

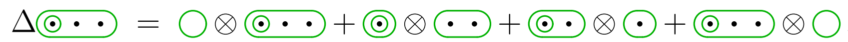
Then the coproduct is

$$\Delta(F^{[n]}_{\mathsf{S}}) \ = \ \sum_{p+q=n} F^{[p]}_{\mathsf{S}'} \otimes F^{[q]}_{\mathsf{S}''} \,.$$

For example, the coproduct on the basis element corresponding to $\{1\} \subseteq [3]$ is

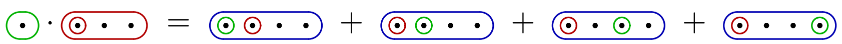
$$\Delta\big(F^{[3]}_{\{1\}}\big) \ = \ F^{\emptyset}_{\emptyset} \otimes F^{[3]}_{\{1\}} + F^{[1]}_{\{1\}} \otimes F^{[2]}_{\emptyset} + F^{[2]}_{\{1\}} \otimes F^{[1]}_{\emptyset} + F^{[3]}_{\{1\}} \otimes F^{\emptyset}_{\emptyset} \,.$$

This was motivated by constructions based on certain tubings of graphs. In terms of tubings on an edgeless graph with three nodes, the coproduct takes the form

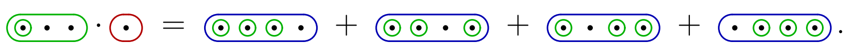
(6.4) 

We leave it to the reader to make the identification (or see [9]).

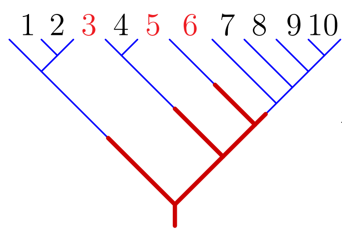
The product $F_{\mathsf{S}}^{[p]} \cdot F_{\mathsf{T}}^{[q]}$ has one term for each shuffle of $[p]$ with $\iota_p([q])$. The corresponding subset $\mathsf{R} \subseteq [p{+}q]$ is the image of $[p]$ in the shuffle (not just S), together with the image of T. For example,

(6.5) 

and



Let α be the bijection between subsets $\mathsf{S} = \{a, b, \ldots, c, d\} \subseteq [n]$ and compositions $\alpha(\mathsf{S}) = (a, b{-}a, \ldots, d-c, n{+}1{-}d)$ of $n{+}1$. Numbering the nodes of a composition tree $1, \ldots, n$ from left to right, the subset of $[n]$ corresponding to the tree is comprised of the colored nodes.



$$\longleftrightarrow \quad \{3, 5, 6\} \longleftrightarrow (3, 2, 1, 4)\,.$$

Applying this bijection to the indices of their fundamental bases gives a linear isomorphism $\boldsymbol{\alpha}\colon \Delta Sym \xrightarrow{\simeq} \mathfrak{C}Sym \circ \mathfrak{C}Sym$. Comparing the definitions above, this is clearly and isomorphism of coalgebras. Compare (6.1) and (6.4). If we use the second product on $\mathfrak{C}Sym \circ \mathfrak{C}Sym$ (induced by the connection f_r), then $\boldsymbol{\alpha}$ is nearly an isomorphism of the algebra, which can be seen by comparing the examples (6.3) and (6.5). In fact, from the definitions given above, it is an *anti-isomorphism* ($\boldsymbol{\alpha}(p \cdot q) = \boldsymbol{\alpha}(q) \cdot \boldsymbol{\alpha}(p)$) of one-sided algebras. We may summarize this discussion as follows.

Theorem 6.1. *The map* $\boldsymbol{\alpha}\colon \Delta Sym \to (\mathfrak{C}Sym \circ \mathfrak{C}Sym, f_r)^{\mathrm{op}}$ *is an isomorphism of one-sided Hopf algebras (with left-sided unit and right-sided antipode).*

Corollary 6.2. *The one-sided Hopf algebra of simplices introduced in* [9] *is cofree as a coalgebra.*

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