Almost commuting self-adjoint matrices — the real and self-dual cases

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Abstract

We show that a pair of almost commuting self-adjoint, symmetric matrices are close to commuting self-adjoint, symmetric matrices (in a uniform way). Moreover we prove that the same holds with self-dual in place of symmetric. Since a symmetric, self-adjoint matrix is real, the former gives a real version of Huaxin Lin's famous theorem on almost commuting matrices. There are applications to physics of Lin's original theorem and both new cases. The self-dual case applies specifically to systems that respect time reversal. Along the way we develop some theory for semiprojective real C^*-algebras.

1 Introduction

In 1997 Lin proved an important theorem about almost commuting matrices [18]. Nowadays it is know as Lin's theorem. Loosely speaking it states that two almost commuting self-adjoint matrices are close to commuting self-adjoint matrices, and in a way that is uniform over all dimensions. Formally it says:

Theorem. (Lin) *For all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ the following holds: Whenever $A, B \in M_n(\mathbb{C})$ are two self-adjoint matrices such that $\|AB - BA\| < \delta$ there exists self-adjoint matrices $A', B' \in M_n(\mathbb{C})$ such that $A'B' = B'A'$ and*

$$\|A - A'\|, \|B - B'\| < \varepsilon.$$

Lin proved this result on complex matrices using C^*-algebra techniques, with an eye on corollaries in classification of C^*-algebras, KK-theory and the extensions of C^*-algebras (e.g [17]). However the theorem itself does not mention C^*-algebra, and it seems to have siblings and applications outside of C^*-algebra theory.

A famous algorithm, developed by Cardoso and Souloumiac for use in blind source separation [5], is "Joint Approximate Diagonalization" (JADE). This algorithm takes two (or more) matrices, either real or complex, and finds a change of basis to make both matrices approximately diagonal. This is closely related to the problem of finding a small perturbation of an almost commuting pair of matrices to a commuting pair. Of course there are various interpretations of "small perturbation" and of "approximately diagonal" and JADE is only claiming to minimize off-diagonal parts, not promising small off-diagonal parts. Nevertheless, we feel this is a connection to be explored.

Hastings discovered a connection between Lin's theorem and finite systems in condensed matter physics [11]. The versions of Lin's theorem that we prove here involve an additional symmetry beyond being self-adjoint. This type of symmetry, which we call a reflection, is needed when working with systems in condensed matter physics that have time-reversal symmetry [13, 14, 21, 24, 25]. The use of reflections (or equivalently, generalized conjugations) in physics is certainly not restricted to condensed matter physics. For example, this sort of symmetry arises in Connes' derivation of the standard model, section 2 of [6].

We have two main theorems, which we state as one.

Theorem 1. *For all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ the following holds: Whenever A, B are two n-by-n, self-adjoint, real (resp. self-dual) matrices such that $\|AB - BA\| < \delta$ there exists n-by-n, self-adjoint, real (resp. self-dual) matrices A', B' such that $A'B' = B'A'$ and*

$$\|A - A'\|, \|B - B'\| < \varepsilon.$$

There are essentially three known ways to prove (the complex case of) Lin's theorem: Lin's original proof; the Friis-Rørdam proof ([9]) that utilizes semiprojectivity results and generalizes the result to work in C^*-algebras of "low topological dimension"; Hastings' quantitative proof ([12]) that is valid only in the matrix case but gives a relation ship between ε and δ. We modeled our proof on that of that of Friis and Rørdam, and so had to develop some theory of semiprojectivity of real C^*-algebras. We made this choice since this was the most natural proof for us, and since we felt that some of the real C^*-algebra techniques we would study would have independent interest. Our proof is limited to the matrix case, although the semiprojectivity results are for semiprojectivity with respect to general real C^*-algebras.

2 Real C^*-algebras

2.1 Two types of real C^*-algebras

In the past, there have been two ways to talk about real C^*-algebras. There have been real C^*-algebras (that is, with lowercase r) and Real C^*-algebras (with uppercase R). For general background on real/Real C^*-algebras, see [10] and [16]. Real and real C^*-algebras are different objects, and even though they are closely related the similar names cause confusion (especially in verbal communication). We are not the first to feel this way. See, for example, [22, page 698] regarding Atiyah's [1] use of Real and real as distinct terms. Adding to the confusion is that fact the Real C^*-algebras have $\mathbb{C}$ as their scalar field. In fairness to Atiyah we should mention that the category of spaces sits nicely inside the category of Real spaces, thus reducing potential confusion. This, however, is not true for noncommutative C^*-algebra. To minimize confusion we suggest new names.

First we describe a class of algebras with scalar field $\mathbb{R}$.

Definition 2.1. *Given a real Banach $*$-algebra we let $A_{\mathbb{C}}$ be the set of formal sums $a_1 \dotplus i \cdot a_2$, $a_1, a_2 \in A$. Letting $a_1, a_2, b_1, b_2 \in A$ and $\alpha, \beta \in \mathbb{R}$ we define algebraic operations on $A_{\mathbb{C}}$ by:*

$$\begin{aligned}
(a_1 \dotplus i \cdot a_2) + (b_1 \dotplus i \cdot b_2) &= (a_1 + b_1) \dotplus i \cdot (a_2 + b_2), \\
(a_1 \dotplus i \cdot a_2)(b_1 \dotplus i \cdot b_2) &= (a_1 b_1 - a_2 b_2) \dotplus i \cdot (a_2 b_1 + a_1 b_2), \\
(a_1 \dotplus i \cdot a_2)^* &= a_1^* \dotplus i \cdot (-a_2^*), \\
(\alpha + \beta i)(a_1 \dotplus i \cdot a_2) &= (\alpha a_1 - \beta a_2) \dotplus i \cdot (\alpha a_2 + \beta a_1).
\end{aligned}$$

With those operations $A_{\mathbb{C}}$ is a complex $*$-algebra. We call it the complexification of A.

Definition 2.2. *A real Banach $*$-algebra A is called an R^*-algebra if the there exist a norm on $A_{\mathbb{C}}$ such that $A_{\mathbb{C}}$ becomes a C^*-algebra, and the norm on $A_{\mathbb{C}}$ extends the norm on A.*

Remark 2.3. An R^*-algebra is known in the literature as a real C^*-algebra [23].

We have the obvious morphisms, and with those we have a category.

Definition 2.4. *A map $\phi\colon A \to B$ between two R^*-algebras is called an R^*-homomorphism if it is $\mathbb{R}$-linear, multiplicative and $*$-preserving.*

Definition 2.5. *Denote by $\mathbf{R}^*$ the category with objects all R^*-algebras and morphisms all R^*-homomorphisms. Denote by $\mathbf{R}^*_{\mathbf{1}}$ the category of unital R^*-algebras and morphisms.*

We will also define a class of algebras that is seemingly closer to C^*-algebras. The motivation for this is that the real matrices can be described as those where $A^* = A^T$. We define something similar to the transpose in a more general setting.

Definition 2.6. *Let A be a C^*-algebra. A linear and $*$-preserving map $\tau\colon A \to A$ such that $\tau(ab) = \tau(b)\tau(a)$, and $\tau(\tau(a)) = a$ for all $a, b \in A$ is called a reflection on A.*

Remark 2.7. A reflection is just an isomorphism between A and its opposite, with the range changed. Thus it is automatically norm preserving and continuous. Furthermore, the 0 element in A must be mapped to 0 by τ, if A has a unit it too must be mapped to it self by τ, and for any $a \in A$ the spectrum of a equals that of $\tau(a)$

Definition 2.8. *A $C^{*,\tau}$-algebra is a pair (A, τ) where A is a C^*-algebra and τ is a reflection of A. We will often write $\tau(a)$ as a^τ.*

Similar to how the letter d is almost always used to represent a generic metric, we will write (A, τ) when we do not know anything special about τ.

Remark 2.9. The Real C^*-algebras correspond to $C^{*,\tau}$-algebras.

We also have morphisms between $C^{*,\tau}$-algebras, and so we also get a category.

Definition 2.10. *By a $C^{*,\tau}$-homomorphism (or $*$-τ-homomorphism) we mean a map $\phi\colon (A, \tau) \to (B, \tau)$ such that ϕ is a $*$-homomorphism from A to B and $\phi(a^\tau) = \phi(a)^\tau$ for all $a \in A$.*

Definition 2.11. *Let $\mathbf{C}^{*,\tau}$ be the category with objects all $C^{*,\tau}$-algebras and morphisms all $*$-τ-homomorphisms. Let $\mathbf{C}_\mathbf{1}^{*,\tau}$ be the category of unital $C^{*,\tau}$-algebras and morphisms.*

2.2 Connections between $\mathbf{R}^*$ and $\mathbf{C}^{*,\tau}$

We will now consider the close relationship between R^*-algebras and $C^{*,\tau}$-algebras. We have a notion of real elements inside a $C^{*,\tau}$-algebra.

Definition 2.12. *Given $a \in (A, \tau)$ we let $\Re_\tau(a) = (a + a^{*\tau})/2$.*

We will say that a is a real element or is in the real part of (A, τ) if $\Re_\tau(a) = a$. This happens precisely when $a^* = a^\tau$.

Lemma 2.13. *If $a \in (A, \tau)$ then*

$$a = \Re_\tau(a) - i\Re_\tau(ia).$$

Lemma 2.14. *If $a \in (A, \tau)$ and we can write $a = a_1 + ia_2$ with a_1 and a_2 in the real part of A then $a_1 = \Re_\tau(a)$ and $a_2 = \Re_\tau(-ia)$.*

We use this newfound knowledge to show that inside all $C^{*,\tau}$-algebras lives an R^*-algebra.

Proposition 2.15. *If (A, τ) is a $C^{*,\tau}$-algebra then $\{a \in A \mid a^* = a^\tau\}$ is an R^*-algebra.*

Proof. Let $A_0 = \{a \in A \mid a^* = a^\tau\}$. The map from A to $(A_0)_\mathbb{C}$ sending $a \in A$ to $\Re_\tau(a) \dotplus i \cdot \Re_\tau(-ia)$ is an R^*-isomorphism. □

We now define a functor from $\mathbf{R}^*$ to $\mathbf{C}^{*,\tau}$.

Definition 2.16. *Define $\Re\colon \mathbf{C}^{*,\tau} \to \mathbf{R}^*$ on objects by*

$$\Re((A,\tau)) = \{a \in A \mid a^* = a^\tau\},$$

and if $\phi\colon (A,\tau) \to (B,\tau)$ we let

$$\Re(\phi) = \phi|_{\Re(A,\tau)},$$

where we co-restrict the right hand side to $\Re((B,\tau))$.

We also wish to have a functor from $\mathbf{R}^*$ to $\mathbf{C}^{*,\tau}$.

Lemma 2.17. *If A is an R^*-algebra then $\bar{*}\colon A_\mathbb{C} \to A_\mathbb{C}$ given by*

$$(a_1 \dotplus i \cdot a_2)^{\bar{*}} = a_1^* \dotplus i \cdot a_2^*,$$

is a reflection on $A_\mathbb{C}$. Furthermore $\Re(A_\mathbb{C}, \bar{*}) \cong A$.

Definition 2.18. *Define $\overline{\bigstar}$ to be the functor from $\mathbf{R}^*$ to $\mathbf{C}^{*,\tau}$ that maps R^*-algebras A to $(A_\mathbb{C}, \bar{*})$ and R^*-homomorphism $\phi\colon A \to B$ to $\overline{\bigstar}(\phi)\colon (A_\mathbb{C}, \bar{*}) \to (B_\mathbb{C}, \bar{*})$ given by*

$$\overline{\bigstar}(\phi)(a_1 \dotplus i \cdot a_2) = \phi(a_1) \dotplus i \cdot \phi(a_2).$$

It is not obvious that $\overline{\bigstar}$ is a functor, but on the other hand it is not hard to prove.

Remark 2.19. The functor $\overline{\bigstar}$ maps surjections to surjections and injections to injections.

It can shown that our two functors are almost inverses, that is if A is an R^*-algebra and (B,τ) is a $C^{*,\tau}$-algebra, then

$$\overline{\bigstar}(\Re(B,\tau)) \cong (B,\tau), \quad \text{and} \quad \Re(\overline{\bigstar}(A)) \cong A.$$

In fact it is know that they both yield categorical equivalences. As such a lot of the study of R^*-algebras can be done using $C^{*,\tau}$ algebras. That is the approach we will take through out this paper. The reasoning behind this choice is that the $C^{*,\tau}$-algebras lets us utilize a lot of our C^*-algebra knowledge. Hence there is less reproving of theorems.

2.3 Two examples

Example 2.20. We modeled a reflection on the transpose so of course it is a reflection, and $\Re(M_n(\mathbb{C}), T) = M_n(\mathbb{R})$.

There is another reflection on $M_{2n}(\mathbb{C})$. If $A \in M_{2n}(\mathbb{C})$ we let A_{ij} be the $n \times n$ blocks and define

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{\sharp} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}^T & A_{11}^T \end{pmatrix}.$$

This is a reflection, and $\Re(M_{2n}(\mathbb{C}), \sharp) = M_n(\mathbb{H})$, where $\mathbb{H}$ is the quaternions. This is an imporant operation in physics, as is dicussed in the survey [27] of applications of random matrices in physics.

Example 2.21. Consider the C^*-algebra of continuous complex-valued functions on the circle, i.e. $C(S^1)$. Since $C(S^1)$ is abelian a reflection is just a an order-two isomorphism. Hence any reflection will come from an order-two homeomorphism of the circle. From [7] we glean that there are only three such maps (up to conjugation), namely:

1. $z \mapsto -z$,
2. $z \mapsto \overline{z}$, and,
3. $z \mapsto z$.

Each gives rise to a $C^{*,\tau}$ algebra by defining for instance $f^\tau(z) = f(-z)$. The real parts will be

1. $\{f \in C(S^1, \mathbb{C}) \mid \overline{f(z)} = f(-z) \text{ for all } z \in S^1\}$,
2. $\{f \in C(S^1, \mathbb{C}) \mid \overline{f(z)} = f(\overline{z}) \text{ for all } z \in S^1\}$, and,
3. $\{f \in C(S^1, \mathbb{C}) \mid \overline{f(z)} = f(z)\} \cong C(S^1, \mathbb{R})$.

As there are two essentially distinct reflections on $\mathbf{M}_{2n}(\mathbb{C})$ and three on $C(S^1)$, we immediately find six replacements in the real case for

$$U_{2n}(A) \cong \hom\left(C(S^1), \mathbf{M}_{2n}(A)\right).$$

We are therefore unsurprised to find that $K_1(A)$ gets replaced by six odd K-groups, counting degrees 1, 3, 5 and 7 in KO and degrees 1 and 3 in self-conjugate K-theory [1, 4].

2.4 Ideals in and operations on

We wish to study ideals in $C^{*,\tau}$ algebras. In C^*-algebras the ideals are precisely the kernels of $*$-homomorphisms. The kernel of $C^{*,\tau}$-homomorphism will be self-τ (that is, if $x \in \ker\phi$ then $x^\tau \in \ker\phi$), but there are C^*-ideals that need not be self-τ. We wish to eliminate those ideals, and so we give the following definition.

Definition 2.22. *Let (A, τ) be a $C^{*,\tau}$-algebra. We say that $I \subseteq A$ is an ideal in (A, τ) if I is a C^*-ideal in A and I is self-τ. We will sometimes write $I \triangleleft_\tau A$ or $I \triangleleft (A, \tau)$.*

With ideals at hand, we can define quotients.

Lemma 2.23. *If $I \triangleleft (A, \tau)$ then $(I, \tau|_I)$ is a $C^{*,\tau}$-algebra. Let $\pi\colon A \to A/I$ be the C^* quotient map. The map $\pi(a)^\tau \mapsto \pi(a^\tau)$ defines a reflection on A/I. Thus A/I is naturally a $C^{*,\tau}$-algebra and π is $C^{*,\tau}$-homomorphism.*

We note that we now have obtained what we wanted: The $C^{*,\tau}$ ideals are precisely the kernels of the $C^{*,\tau}$-homomorphisms.

The following lemma and theorem tells us that we have direct sums and pullbacks in the category $\mathbf{C}^{*,\tau}$.

Lemma 2.24. *Given two $C^{*,\tau}$-algebras (A, τ) and (B, σ) the map $\tau \oplus \sigma\colon A \oplus B \to A \oplus B$ will be a reflection.*

Theorem 2.25. *Suppose $\varphi_1\colon (A_1, \tau) \to (C, \tau)$ and $\varphi_2\colon (A_1, \tau) \to (C, \tau)$ are $*$-τ-homomorphisms, and form the pull-back C^*-algebra*

$$A_1 \oplus_C A_2 = \left\{ (a_1, a_2) \in A_1 \oplus A_2 \,\middle|\, \varphi_1(a_1) = \varphi_2(a_2) \right\}.$$

This becomes a $C^{*,\tau}$-algebra with

$$(a_1, a_2)^\tau = (a_1^\tau, a_2^\tau)$$

and it gives us the pull-back of the given $C^{*,\tau}$-algebras, where we are using the restricted projection maps $\pi_j\colon (A_1, \tau) \oplus_{(C,\tau)} (A_2, \tau) \to (A_j, \tau)$.

Proof. We need to check some axioms, but all are clear. Given $\psi_j\colon (D, \tau) \to (A_j, \tau)$ with $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$, we know from the underlying $*$-homomorphisms that we have a unique $*$-homomorphisms

$$\psi\colon D \to A_1 \oplus_C A_2$$

for which $\pi_j \circ \psi = \psi_j$. It is defined by

$$\psi(d) = (\psi_1(d), \psi_2(d))$$

and

$$\psi(d^\tau) = (\psi_1(d^\tau), \psi_2(d^\tau)) = (\psi_1(d)^\tau, \psi_2(d)^\tau) = (\psi_1(d), \psi_2(d))^\tau = \psi(d)^\tau.$$

□

We can also define what it means to unitize a $C^{*,\tau}$-algebra.

Lemma 2.26. *Let (A, τ) be a $C^{*,\tau}$ algebra. The formula*

$$(a + \lambda\mathbb{1})^\sigma = a^\tau + \lambda\mathbb{1}, \quad a \in A, \lambda \in \mathbb{C},$$

defines a reflection on $\tilde{A}$. Thus $(\tilde{A}, \sigma)$ is a $C^{*,\tau}$-algebra. And it is the only way to unitize (A, τ) while preserving the reflection on A.

Proof. Let $a, b \in A$ and let $\lambda, \mu \in \mathbb{C}$. We must check that σ is linear, anti-multiplicative and $*$-preserving. The only thing that is not immediately obvious is that σ is anti-multiplicative. To see that we compute:

$$\begin{aligned} ((a+\lambda\mathbb{1})(b+\mu\mathbb{1}))^\sigma &= ((ab+\mu a+\lambda b)+(\lambda\mu)\mathbb{1})^\sigma \\ &= (ab+\mu a+\lambda b)^\tau+(\lambda\mu)\mathbb{1} \\ &= b^\tau a^\tau+\mu a^\tau+\lambda b^\tau+\lambda\mu\mathbb{1} \\ &= (b^\tau+\mu\mathbb{1})(a^\tau+\lambda\mathbb{1}) \\ &= (b+\mu\mathbb{1})^\sigma(a+\lambda)^\sigma. \end{aligned}$$

Since any reflection must preserve the unit σ is defined in the only possible way. □

Definition 2.27. *If (A, τ) is a $C^{*,\tau}$-algebra we will also denote by τ the extension of τ to $\tilde{A}$ given in lemma 2.26 (this should cause no confusion, as the lemma shows this extension is unique). The $C^{*,\tau}$-algebra $(\tilde{A}, \tau)$ we denoted by $\widetilde{(A,\tau)}$ or $(A,\tau)^\sim$, and call the unitization of (A, τ).*

Example 2.28. We will compute the unitization of the $C^{*,\tau}$-algebra $C_0((0,1),\mathrm{id})$. Since $C_0((0,1))^\sim \cong C(S^1)$, we have that $(C_0((0,1),\mathrm{id}))^\sim \cong C(S^1,\tau)$, where τ is a reflection that extends id. Since the unit is always self-τ and everything in $C_0((0,1),\mathrm{id})$ is self-id, we have that all elements of $C_0((0,1),\mathrm{id})^\sim$ are self-τ. Thus we have $C_0((0,1),\mathrm{id})^\sim \cong C(S^1,\mathrm{id})$.

3 (Semi) Projective real C^*-algebras

The definition of a semiprojective C^*-algebra that we use today was given by Blackadar in [2]. We will modify that definition so we can use it for $C^{*,\tau}$-algebras. The theory of semiprojective C^*-algebras is well developed, for good resources on the subject see [3], [20], and the references therein. In what follows we try to develop some theory of semiprojective $C^{*,\tau}$-algebras. To do so we borrow proof ideas from across the field of semiprojective C^*-algebras without further references.

3.1 Definitions

We give the obvious definitions of projectivity and semiprojectivity in the categories $\mathbf{C}^{*,\tau}$ and $\mathbf{C}_{\mathbf{1}}^{*,\tau}$.

Definition 3.1. *Let $\mathbf{C}$ be one of the categories $\mathbf{C}^{*,\tau}$ or $\mathbf{C}_{\mathbf{1}}^{*,\tau}$. An object A in $\mathbf{C}$ is said to be projective, if whenever J is an ideal in B, another object in $\mathbf{C}$, and we have a morphism $\phi\colon A \to B/J$ in $\mathbf{C}$, we can find a morphism $\psi\colon A \to B$ in $\mathbf{C}$ such that $\pi \circ \psi = \phi$, where π it the quotient map from B to B/J.*

Definition 3.2. *Let $\mathbf{C}$ be one of the categories $\mathbf{C}^{*,\tau}$ or $\mathbf{C}_{\mathbf{1}}^{*,\tau}$. An object A in $\mathbf{C}$ is said to be semiprojective, if whenever $J_1 \subseteq J_2 \subseteq \cdots$ is an increasing sequence of ideals in B, another object in $\mathbf{C}$, and we have a morphism $\phi\colon A \to B/J$, $J = \overline{\cup_n J_n}$, in $\mathbf{C}$, we can find an $m \in \mathbb{N}$ and morphism $\psi\colon A \to B/J_m$ in $\mathbf{C}$ such that $\pi_{m,\infty} \circ \psi = \phi$, where $\pi_{m,\infty}$ it the quotient map from B/J_m to B/J.*

Notation. Whenever we have a $C^{*,\tau}$-algebra B containing an increasing sequence of τ-invariant ideals $J_1 \subseteq J_2 \subseteq \cdots$ we denote the quotient maps as follows:

$$\begin{aligned}
\pi_n &\colon B \twoheadrightarrow B/J_n,\\
\pi_{n,m} &\colon B/J_n \twoheadrightarrow B/J_m,\\
\pi_{m,\infty} &\colon B/J_m \twoheadrightarrow B/J,\\
\pi_\infty &\colon B \twoheadrightarrow B/J,
\end{aligned}$$

where $n < m$ are natural numbers and $J = \overline{\cup_n J_n}$.

Of course one could just as easily define semiprojective R^*-algebras. Studying how the functors $\Re$ and $\overline{\bigstar}$ behave with respect to ideals and lifting problems, the following two propositions can be proved. For reasons of brevity we have chosen not to include proofs of these propositions.

Proposition 3.3. *If A, B are R^*-algebras, J is an ideal in B, and $\phi\colon A \to B/J$ is an R^*-homomorphism, then we can find an R^*-homomorphism $\psi\colon A \to B$ such that $\pi \circ \psi = \phi$ if and only if we can find a $*$-τ-homomorphism $\chi\colon \overline{\bigstar}(A) \to \overline{\bigstar}(B)$ such that $\overline{\bigstar}(\pi) \circ \psi = \overline{\bigstar}(\phi)$.*

Proposition 3.4. *If A is an R^*-algebra then A is (semi-) projective if and only if $\overline{\bigstar}(A)$ is. If (B,τ) is a $C^{*,\tau}$-algebra then (B,τ) is (semi-) projective if and only if $\Re(B,\tau)$ is.*

Just as in the C^*-case we can somewhat simplify the task of proving semiprojectivity.

Proposition 3.5. *To show that a $C^{*,\tau}$-algebra (B,τ) is semiprojective it suffices to solve lifting problems*

$$\begin{tikzcd}
& (B,\tau) \arrow[d, two heads] \\
& (B/J_n,\tau) \arrow[d, two heads] \\
(A,\tau) \arrow[ur, dashed] \arrow[r, "\phi"'] & (B/J,\tau)
\end{tikzcd}$$

where ϕ is either injective, surjective or both.

Proof. This is well know in the C^*-case [20], and is no harder in the $C^{*,\tau}$-case. To get injective we replace ϕ with $\phi \oplus \mathrm{id}\colon (A,\tau) \to (B/J \oplus A, \tau \oplus \tau)$. To get surjective we focus on the image of ϕ. □

Functional calculus is indispensable when working with lifting problems. The following lemma tells some of the story about $C^{*,\tau}$-algebras and functional calculus.

Lemma 3.6. *Suppose a is normal element in a $C^{*,\tau}$-algebra (A,τ). If f is a continuous function from $\sigma(a)$ to $\mathbb{C}$ then $f(a)^\tau = f(a^\tau)$.*

If $b \in (A,\tau)$ is a normal and self-τ element and $\sigma(b) \subseteq X \subseteq \mathbb{C}$ then the C^*-homomorphism $\phi\colon C_0(X) \to A$ given by $f \mapsto f(b)$ is a $C^{*,\tau}$-homomorphism from $C(X,\mathrm{id})$ to (A,τ).

Proof. We remind the reader that $\sigma(a) = \sigma(a^\tau)$. Since τ is linear we have $p(a)^\tau = p(a^\tau)$, for any polynomial p. By continuity of τ we now get $f(a)^\tau = f(a^\tau)$ for any function $f \in C(\sigma(a))$.

For any function $f \in C_0(X)$ we have

$$f(b)^\tau = f(b^\tau) = f(b) = (f \circ \mathrm{id})(b).$$

□

With that lemma at our disposal, we can give some basic examples of (semi-) projective $C^{*,\tau}$-algebras.

Example 3.7. We will show that the $C^{*,\tau}$-algebra $C_0((0,1], \mathrm{id})$ is projective. Suppose we are given the following lifting problem:

$$\begin{tikzcd} & (B,\tau) \arrow[d, two heads, "\pi"] \\ C_0((0,1],\mathrm{id}) \arrow[r, "\phi"'] & (B/J,\tau) \end{tikzcd}$$

Let $h = \phi(t \mapsto t)$. Then h is a self-τ positive contraction. Let x be a positive contractive lift of h, and let $k = (x + x^\tau)/2$. Then k is a self-τ, positive contraction, and $\pi(x) = h$. By Lemma 3.6 the map $f \mapsto f(k)$ is a $C^{*,\tau}$-homomorphism. It is a lift of ϕ by standard C^*-theory.

Example 3.8. We will show that the $C^{*,\tau}$-algebra $(\mathbb{C}, \mathrm{id})$ is semiprojective. Suppose we are given the following lifting problem:

$$\begin{tikzcd} & (B,\tau) \arrow[d, two heads, "\pi_n"'] \\ & (B/J_n,\tau) \arrow[d, two heads, "\pi_{n,\infty}"'] \\ (A,\tau) \arrow[r, "\phi"'] & (B/J,\tau) \end{tikzcd}$$

Let $p = \phi(1)$. Then p is a self-τ projection. Let $y \in (B, \tau)$ be any self-adjoint lift of p. If we let $x = (y+y^\tau)/2$ then x is a self-τ and self-adjoint lift of p. Since $\pi_n(x^2 - x) \to 0$ as $n \to \infty$ we can find some $m \in \mathbb{N}$ such that $1/2 \notin \sigma(\pi_m(x))$. Now let f be the function that is 0 on $(-\infty; 1/2)$ and 1 on $(1/2; \infty)$. Then $q = f(\pi_m(x))$ is a projection and a lift of p. Since x is self-τ q will be self-τ. We can now define a $C^{*,\tau}$ homomorphism from $\mathbb{C}$ to B/J_m by $\lambda \mapsto \lambda q$ (Lemma 3.6). It is a lift of ϕ.

3.2 Closure results

3.2.1 Unitizing

We aim to get the $C^{*,\tau}$ equivalent of C^* result that A is semiprojective if and only if $\tilde{A}$ is. First we show that if A is unital it suffices to solve unital lifting problems.

Lemma 3.9. *A unital $C^{*,\tau}$-algebra is semiprojective in $\mathbf{C}^{*,\tau}$ if and only if it is semiprojective in $\mathbf{C}_1^{*,\tau}$.*

Proof. Let (A, τ) be a unital $C^{*,\tau}$-algebra.

The proof that (A, τ) semiprojective in $\mathbf{C}^{*,\tau}$ implies that it is semiprojective in $\mathbf{C}_1^{*,\tau}$ is precisely the same as in the C^*-case.

Suppose that (A,τ) is semiprojective in $\mathbf{C}_{\mathbf{1}}^{*,\tau}$. Let (B,τ) be a $C^{*,\tau}$-algebra containing an increasing sequence of $C^{*,\tau}$ ideals $I_1 \subseteq I_2 \subseteq \cdots$, let $I = \overline{\cup_n I_n}$, and let $\phi\colon (A,\tau) \to (B/I,\tau)$ be a $C^{*,\tau}$-homomorphism. Put $p = \phi(1_A)$. Then p is a self-τ projection in $(B/J,\tau)$. Since $(\mathbb{C},\mathrm{id})$ is semiprojective, we can find some $n_0 \in \mathbb{N}$ and self-τ projection $q \in B/J_{n_0}$ such that $\pi_{n_0,\infty}(q) = p$. For each $n \geq n_0$ define $q_n = \pi_{n_0,n}(q)$. Since all the q_n are self-τ all the corners $q_n(B/J_n)q_n$ are self-τ. Hence for all $n \geq n_0$ we have that $q_n(B/J_n)q_n \cong (qBq)/(qJ_nq)$ and that by restricting the τ's we get the following commutative diagram of $C^{*,\tau}$-algebras:

$$\begin{tikzcd}
 & (q_{n_0}(B/J_{n_0})q_{n_0},\tau) \arrow[r, hook] \arrow[d] & (B/J_{n_0},\tau) \arrow[d] \\
 & (q_n(B/J_n)q_n,\tau) \arrow[r, hook] \arrow[d] & (B/J_n,\tau) \arrow[d] \\
(A,\tau) \arrow[r, "\phi"'] & (p(B/J)p,\tau) \arrow[r, hook] & (B/J,\tau)
\end{tikzcd}$$

In the two left most columns there are only unital maps and algebras, so since (A,τ) is semiprojective in the unital category, we can find a lift for some $n \geq n_0$. This lifting combines with the inclusion $q_n(B/J_n)q_n \hookrightarrow B/J_n$ to show that (A,τ) is semiprojective. □

The lemma is a stepping stone towards a goal, but it also has its own applications.

Example 3.10. The $C^{*,\tau}$-algebras $C(S^1,\mathrm{id}), C(S^1, z \mapsto \overline{z})$ and $C(S^1, z \mapsto -z)$ are all semiprojective. We will only show the first one, but the remaining proofs are similar. By Lemma 3.9 it suffices to solve lifting problems of the form:

$$\begin{tikzcd}
 & (B,\tau) \arrow[d, "\pi_n"] \\
 & (B/J_n,\tau) \arrow[d, "\pi_{n,\infty}"] \\
(C(S^1),\mathrm{id}) \arrow[r, "\phi"'] & (B/J,\tau)
\end{tikzcd}$$

where everything is unital. Let $u = \phi(z \mapsto z)$. Then u is a self-τ unitary. Let y be any self-τ lift of u. We can find an m such that $x = \pi_m(x)$ satisfies that xx^* and x^*x are invertible. Now define $v = x(x^*x)^{-1/2}$. The v is a unitary lift of u and, by Lemma 3.6 and a standard functional calculus trick,

$$v^\tau = ((x^*x)^{-1/2})^\tau x^\tau = ((x^*x)^\tau)^{-1/2}x = (xx^*)^{-1/2}x = x(x^*x)^{-1/2} = v.$$

There is C^*-homomorphism from $C(S^1)$ to B/J_m given by $\psi(f) = f(v)$. Since v is self-τ and every element in $C(S^1,\mathrm{id})$ is self-τ, this is actually a $C^{*,\tau}$ homomorphism from $C(S^1,\mathrm{id})$ to $(B/J_m,\tau)$. Because v is a lift of u, ψ is a lift of ϕ.

Lemma 3.11. *A $C^{*,\tau}$-algebra (A,τ) is semiprojective if and only if $\widetilde{(A,\tau)}$ is semiprojective in the unital $C^{*,\tau}$ category.*

Corollary 3.12. *A $C^{*,\tau}$-algebra is semiprojective if and only if its unitization is.*

Example 3.13. Since $(C((0,1),\mathrm{id})^{\sim} \cong C(S^1,\mathrm{id})$ and the latter is semiprojective, $C((0;1),\mathrm{id})$ is semiprojective.

3.2.2 Direct sums

In this section we aim to show the following.

Proposition 3.14. *If $(A,\tau),(B,\sigma)$ are separable semiprojective $C^{*,\tau}$-algebras, then $(A\oplus B,\tau\oplus\sigma)$ is a semiprojective $C^{*,\tau}$-algebra.*

Before we can do that however, we need to set up some theory.

Lemma 3.15. *The relations $0\le h,k\le 1, h=h^\tau, k=k^\tau, hk=0$ are liftable.*

Proof. Suppose we are given a τ-invariant ideal J in a $C^{*,\tau}$-algebra B, and suppose $h,k\in B/J$ satisfy the relations. Let $a=h-k$. Then a is a a self-τ self-adjoint contraction. Thus we can lift it to a self-adjoint self-τ contraction in B, $\hat a$ say. Define $f\colon\mathbb{R}\to\mathbb{R}$ by $f(x)=(x+|x|)/2$. Then we know from C^*-algebra theory that $f(\hat a)$ is a positive contractive lift of h, that $f(-\hat a)$ is a positive contractive lift of k, and that $f(\hat a)f(-\hat a)=0$. Lemma 3.6 tells us that $f(\hat a)$ and $f(-\hat a)$ are self-τ. □

Lemma 3.16. *Let (B,τ) be a $C^{*,\tau}$-algebra. If $h\in(B,\tau)$ is strictly positive in B then so is h^τ. Hence $\Re_\tau(b)$ is strictly positive.*

Proof. Let $\phi\colon B\to\mathbb{C}$ be a linear positive functional. Then we have, writing τ as a function,

$$\phi(h^\tau)=\phi(\tau(h))=(\phi\circ\tau)(h).$$

Since τ is linear and maps positive elements to positive elements $\phi\circ\tau$ is a positive linear functional. But then if ϕ is non-zero we have

$$\phi(h^\tau)=(\phi\circ\tau)(h)>0.$$

□

Corollary 3.17. *If (B,τ) is a separable $C^{*,\tau}$-algebra then it contains a self-τ positive element h such that $\overline{hBh}=B$.*

A discussion of hereditary subalgebras in the context of real C^*-algebras is to be found in [26].

We are now ready to prove Proposition 3.14.

Proof of Proposition 3.14. Since both (A, τ) and (B, σ) are separable we can use Corollary 3.17 to find $h \in A$ and $k \in B$, positive contractions such that $h^\tau = h$, $k^\sigma = k$, $\overline{hAh} = A$ and $\overline{kBk} = B$. Suppose we are given a $C^{*,\tau}$-algebra (D, τ) containing an increasing sequence of τ-invariant ideals $J_1 \subseteq J_2 \subseteq \cdots$ and a $C^{*,\tau}$-homomorphism

$$\phi\colon (A \oplus B, \tau \oplus \sigma) \to (D/J, \tau),$$

where $J = \overline{\cup_n J_n}$. Let $\hat{h} = \phi((h, 0))$ and $\hat{k} = \phi((k, 0))$. Since $\hat{h}$ and $\hat{k}$ are orthogonal positive contractions we can, by Lemma 3.15, find positive orthogonal contractive lifts $\tilde{h}$, $\tilde{k}$ of them in B. For each $n \in \mathbb{N} \cup \{\infty\}$ let $h_n = \pi_n(\tilde{h})$, $k_n = \pi_n(\tilde{k})$, $A_n = \overline{h_n(D/J_n)h_n}$, and $B_n = \overline{k_n(D/J_n)k_n}$. For each $n \in \mathbb{N} \cup \{\infty\}$ the map $\gamma_n = \tau|_{A_n} \oplus \tau|_{B_n}$ is a reflection since h_n and k_n are self-τ. Observe that we have

$$\overline{\hat{h}(D/J)\hat{h}} = \overline{h_\infty(D/J)h_\infty} = A_\infty \quad \text{and} \quad \overline{\hat{k}(D/J)\hat{k}} = \overline{k_\infty(D/J)k_\infty)} = B_\infty.$$

Define for each $n \in \mathbb{N} \cup \{\infty\}$ a map

$$\alpha_n\colon (A_n \oplus B_n, \gamma_n) \to (D/J_n, \tau),$$

by $\alpha((x, y)) = x + y$. It will be an $C^{*,\tau}$-homomorphism since $h_n k_n = 0$. Noticing that

$$\pi(\overline{\tilde{h}D\tilde{h}}) = A_\infty \quad \text{and} \quad \pi(\overline{\tilde{k}D\tilde{k}}) = B_\infty,$$

we see there must be a $C^{*,\tau}$-homomorphism

$$\psi\colon (A \oplus B, \tau \oplus \sigma) \to (A_\infty \oplus B_\infty, \gamma),$$

such that $\phi\colon \alpha_\infty \circ \psi$. Hence we get the following commutative diagram for all $n \in \mathbb{N}$

$$\begin{tikzcd}
 & (A_n \oplus B_n, \gamma_n) \arrow[r, "\alpha_n"] \arrow[d] & (D/J_n, \tau) \arrow[d] \\
(A \oplus B, \tau \oplus \sigma) \arrow[r, "\psi"] \arrow[rr, bend right, "\phi"'] & (A_\infty \oplus B_\infty, \gamma_\infty) \arrow[r, "\alpha_\infty"] & (D/J, \tau)
\end{tikzcd}$$

Since γ is a direct sum of two reflections, we can use the semiprojective of (A, τ) and (B, σ), one at a time, to show that $(A \oplus B, \tau \oplus \sigma)$ is semiprojective. $\square$

Remark 3.18. We observe that we only used (A, τ) and (B, τ) separable to get strictly positive real elements h, k. So we might as well have assumed that A and B were σ-unital. Lemma 3.16 tells us that whether we define (A, τ) to be σ-unital when A is or when (A, τ) contains a strictly positive real element, we get the same class of algebras.

The knowledge we have accumulated so far lets us take a small step towards showing that if X is a finite one-dimensional CW-complex then $C(X, \mathrm{id})$ is semiprojective.

Proposition 3.19. *If X is a wedge of circles (a bouquet) then $C(X, \mathrm{id})$ is semiprojective.*

Proof. By assumption

$$C(X, \mathrm{id}) \cong \left(\bigoplus_{i=1}^{n} C_0((0,1), \mathrm{id}) \right)^{\sim},$$

for some $n \in \mathbb{N}$. By Proposition 3.14 $\bigoplus_{i=1}^{n} C_0((0,1), \mathrm{id})$ is semiprojective since each summand is. So by Corollary 3.11 $C(X, \mathrm{id})$ is semiprojective. □

The above proposition will later be the basis step of an induction proof.

Remark 3.20. If X is a wedge of two circles, then we can put a reflection on $C(X)$ by mapping one circle to the other. This reflection is not a direct sum of two reflections on the circle. Hence showing that the $C^{*,\tau}$-algebra it defines is semiprojective requires different techniques than the ones we have just used.

4 Multiplier algebras

In this section we will study multiplier and corona algebras of $C^{*,\tau}$-algebras. The idea is that we already have multiplier algebras at our disposal. So the main body of work lies in showing that we can extend a reflection on A to a reflection on $M(A)$.

4.1 A reflection on $M(A)$

The following theorem is in [15]. We present it here with a few more details.

Theorem 4.1. *Suppose (A, τ) is a $C^{*,\tau}$-algebra. There is an operation τ on $M(A)$ defined by*

$$m^\tau a = (a^\tau m)^\tau , \quad \text{and} \quad am^\tau = (ma^\tau)^\tau$$

for a in A and m in $M(A)$, and $(M(A), \tau)$ is a $C^{*,\tau}$-algebra, and the C^*-inclusion

$$\iota\colon A \to M(A),$$

is also a $C^{*,\tau}$-homomorphism.

Proof. Consider for a moment a fixed m in $M(A)$. Define $L\colon A \to A$ and $R\colon A \to A$ by

$$L(a) = (a^\tau m)^\tau \quad \text{and} \quad R(a) = (ma^\tau)^\tau .$$

For all a and b in A,

$$L(ab) = (b^\tau a^\tau m)^\tau = (a^\tau m)^\tau b = L(a)b,$$

$$R(ab) = (mb^\tau a^\tau)^\tau = a\,(mb^\tau)^\tau = aR(b),$$

and

$$R(a)b = (ma^\tau)^\tau b = (b^\tau ma^\tau)^\tau = a\,(b^\tau m)^\tau = aL(b),$$

so (L, M) is an element of $M(A)$, which we denote m^τ. Notice m^τ is specified within all multipliers by either one of the formulas

$$m^\tau a = (a^\tau m)^\tau , \quad \text{or} \quad am^\tau = (ma^\tau)^\tau .$$

We claim that the operation defined above, on all multipliers $m \mapsto m^\tau$, makes $M(A)$ a $C^{*,\tau}$-algebra. For any multiplier m, and any a in A,

$$m^{\tau\tau} a = (a^\tau m^\tau)^\tau = (ma^{\tau\tau})^{\tau\tau} = ma,$$

so $\tau \circ \tau = \mathrm{id}$. For n in $M(A)$ and α in $\mathbb{C}$,

$$(\alpha m + n)^\tau a = (a^\tau(\alpha m + n))^\tau = \alpha m^\tau a + n^\tau a = (\alpha m^\tau + n^\tau)a,$$

$$(mn)^\tau a = (a^\tau mn)^\tau = ((m^\tau a)^\tau n)^\tau = n^\tau(m^\tau a) = (n^\tau m^\tau)a,$$

and
$$(m^*)^\tau a = (a^\tau m^*)^\tau = (m a^{*\tau})^{*\tau} = (a^* m^\tau)^* = (m^\tau)^* a,$$
which means τ commutes with $*$, is anti-multiplicative, and $\mathbb{C}$-linear.

If a is in A, them for any other b in A,
$$\iota(a)^\tau b = (b^\tau \iota(a))^\tau = (b^\tau a)^\tau = a^\tau b = \iota(a^\tau) b,$$
so $\iota(a)^\tau = \iota(a^\tau)$. □

Lemma 4.2. *Suppose A is a $C^{*,\tau}$-subalgebra of B, where (B, τ) is a given $C^{*,\tau}$-algebra. The idealizer*
$$I(A : B) = \left\{ b \in B \,\middle|\, bA + Ab \subseteq A \right\},$$
is self-τ, and so a $C^{*,\tau}$-subalgebra of B containing A as a self-τ ideal.

Proof. Suppose b is in the idealizer and a is in A. Then $a^\tau \in A$ and so
$$(b^\tau a)^\tau = a^\tau b \in A \implies b^\tau a \in A,$$
and
$$(a b^\tau)^\tau = b a^\tau \in A \implies a b^\tau \in A,$$
proving b^τ is also in the idealizer. □

Theorem 4.3. *Suppose (B, τ) is a $C^{*,\tau}$-algebra and $A \triangleleft (B, \tau)$ is a self-τ ideal. The unique $*$-homomorphism $\theta \colon B \to M(A)$ for which $\theta(a) = \iota(a)$ for all a in A, is automatically a $*$-τ-homomorphism.*

Proof. We know $\theta(b)a = ba$ defines the only possible $*$-homomorphism from B to $M(A)$ satisfying $\theta(a) = \iota(a)$. For b in B and a in A we compute
$$\theta(b)^\tau a = (a^\tau \theta(b))^\tau = (\theta(a^\tau b))^\tau = \theta((a^\tau b)^\tau) = \theta(b^\tau a) = \theta(b^\tau) a,$$
which proves θ is τ-preserving. □

Lemma 4.4. *If $\varphi : (A, \tau) \to (B, \tau)$ is a proper $*$-τ-homomorphism between σ-unital $C^{*,\tau}$-algebras, then the unique $*$-homomorphism $\widehat{\varphi} : M(A) \to M(B)$ that extends φ is actually a $*$-τ-homomorphism.*

Proof. The fact that φ is proper tells us $B = \varphi(A)B = B\varphi(A)$. The defining formulas for $\widehat{\varphi}$ are
$$\widehat{\varphi}(m)\varphi(a)b = \varphi(ma)b$$
and
$$b\varphi(a)\widehat{\varphi}(m) = b\varphi(am).$$

Therefore

$$\begin{aligned}\widehat{\varphi}(m)^{\tau}\varphi(a)b &= ((\varphi(a)b)^{\tau}\widehat{\varphi}(m))^{\tau}\\ &= (b^{\tau}\varphi(a^{\tau})\widehat{\varphi}(m))^{\tau}\\ &= (b^{\tau}\varphi(a^{\tau}m))^{\tau}\\ &= \varphi(m^{\tau}a)b\\ &= \widehat{\varphi}(m^{\tau})\varphi(a)b.\end{aligned}$$

□

We get "multiplier realization" for free.

Theorem 4.5. *Let C(E) denote the corona of a σ-unital $C^{*,\tau}$-algebra (E,τ), and let D and N be separable $C^{*,\tau}$-subalgebras of $C(E)$. Suppose*

$$A \subseteq C(E) \cap D' \cap N^{\perp},$$

is a σ-unital $C^{*,\tau}$-subalgebra. Then the $*$-τ-homomorphism

$$\theta\colon I(A\colon C(E) \cap D' \cap N^{\perp}) \to M(A)$$

is onto.

Proof. We know that θ is onto, by Corollary 3.2 of [8]. All we are asserting here is that this map is now a morphism in the category of $C^{*,\tau}$-algebras. □

4.2 Corona extendible morphisms

Definition 4.6. *We say a morphism of $C^{*,\tau}$-algebras $\gamma\colon (A,\tau) \to (B,\tau)$ is* corona extendible *if, for every $*$-τ-homomorphism $\varphi\colon A \to C(E)$ with E a σ-unital $C^{*,\tau}$-algebra, there exists a $*$-τ-homomorphism $\widehat{\varphi}\colon A \to C(E)$ so that $\widehat{\varphi} \circ \gamma = \varphi$.*

Theorem 4.7. *Suppose $0 \to A \to X \to P \to 0$ is a short-exact sequence of σ-unital $C^{*,\tau}$-algebras If P is projective then the inclusion $A \to X$ is corona extendible. Moreover, the unitization of this map $\widetilde{A} \to \widetilde{X}$ is also corona extendible.*

Proof. Except for the $*$-τ-homomorphism claim, this is Theorem 3.4 of [8] combined with the usual universal property of a split extension, as in Theorem 7.3.6 of [20]. We summarize those proofs and verify that various maps can be selected to be $*$-τ-homomorphisms.

Since P is projective, the exact sequence has a splitting by a $*$-τ-homomorphism $\lambda\colon P \to X$. We assume we are given a $*$-τ-homomorphism $\varphi : A \to C(E)$ with E being σ-unital. As in the proof of Theorem 3.4 of [8], we have the commutative diagram, ignoring for now ψ_0,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & P \arrow[r] \arrow[l, bend right, "\lambda"'] \arrow[dddl, bend left, "\psi_0"] & 0 \\
& A \arrow[r] \arrow[d] & M(A) \arrow[d] & & \\
& \varphi(A) \arrow[r] \arrow[d, equal] & M(\varphi(A)) & & \\
& \varphi(A) \arrow[r, hook] & I(\varphi(A):C(E)) \arrow[u, two heads] & &
\end{tikzcd}$$

where the map $A \to \varphi(A)$ is the co-restriction of φ making it onto. The essential fact that the arrow up from the idealizer to the multiplier algebra is both surjective and a $*$-τ-homomorphism is Theorem 4.5. The map from B to $M(A)$ in the top square is a $*$-τ-homomorphism by Theorem 4.3. The map from $M(A)$ to $M(\varphi(A))$ in the middle square is a $*$-τ-homomorphism by Lemma 4.4. We use the projectivity, in the $*$-τ-sense, of P to get a $*$-τ-homomorphism ψ_0 making the diagram commute.

Following ψ_0 by the inclusion into the corona algebra give us a $*$-τ-homomorphism $\psi\colon P \to C(E)$ such that

$$\psi(p)\varphi(a) = \varphi(\mu(p)a)$$

for all p in P and a in A. This induces a $*$-homomorphism

$$\Psi\colon X \to C(E)$$

extending φ by

$$\Psi(a + \lambda(p)) = \varphi(a) + \psi(p)$$

which is evidently a $*$-τ-homomorphism.

To get the last claim, we must use more of the power of Theorem 4.5. We are given $\widetilde{A} \to C(E)$ which we regard as a $*$-τ-homomorphism $\varphi : A \to C(E)$ together with a projection p in $C(E)$ such that $p\varphi(a) = \varphi(a)$ for all a in A. We can replace $I(\varphi(A) : C(E))$ in the big diagram by

$$I(\varphi(A) : C(E)) \cap (1 - p)^{\perp}.$$

We still have the needed surjectivity onto $M(\varphi(A))$ and end up with $\Psi : B \to C(E)$ with the property $p\Psi(b) = \Psi(b)$ for all b in B. $\square$

Corollary 4.8. *Suppose X is a compact metrizable space and $Y \subseteq X$ is a closed subset of X homeomorphic to the closed interval $[0;1]$. Let X_1 be the quotient of X obtained by collapsing Y to a point. The inclusion $C(X_1, \mathrm{id}) \hookrightarrow C(X, \mathrm{id})$ of abelian $C^{*,\tau}$-algebras is corona extendible.*

Proof. Let y_0 denote the point in Y associated to 0 in $[0;1]$. Let y_* be the point in X_1 that is the image of Y in the quotient map. We have an exact sequence

$$0 \longrightarrow C_0(X_1 \setminus \{y_*\}) \longrightarrow C(X \setminus \{y_0\}) \longrightarrow C_0(0;1] \longrightarrow 0$$

where all C^*-algebras are equipped with the trivial τ operation. Thus we are done by Theorem 4.7 and Example 3.7. $\square$

4.3 Corona (semi-) projective

Just as in the C^*-case, the work of showing semiprojectivity can be reduced using corona algebras. Most of the proof of the following two theorems can be copied from the proof of [20, Theorem 14.1.7] if one only remembers to change category. The only change is that we have not studied the Calkin algebra in a $C^{*,\tau}$ setting. To avoid using that, use the corona algebra of $\bigoplus_{n=1}^{\infty} \widetilde{(A, \tau)}$.

Theorem 4.9. *Suppose A is a separable $C^{*,\tau}$-algebra. The following are equivalent:*

1. *A is projective;*
2. *we can solve the lifting problem for A whenever ρ is the quotient map $M(E) \to C(E)$ for a separable $C^{*,\tau}$-algebra E and φ is injective;*
3. *we can solve the lifting problem for A whenever ρ is the quotient map $M(E) \to C(E)$ for a separable $C^{*,\tau}$-algebra E;*
4. *we can solve the lifting problem for A whenever ρ is the quotient map $B \to B/I$ for a separable $C^{*,\tau}$-algebra B and closed τ-closed ideal I.*

Theorem 4.10. *Suppose A is a separable $C^{*,\tau}$-algebra. The following are equivalent:*

1. *A is semiprojective;*
2. *we can solve the partial lifting problem for A whenever $B = M(E)$ for a separable $C^{*,\tau}$-algebra E and $\overline{\bigcup E_k} = E$ for some chain of τ-invariant ideals of E and φ is injective;*
3. *we can solve the partial lifting problem for A whenever $B = M(E)$ for a separable $C^{*,\tau}$-algebra E and $\overline{\bigcup E_k} = E$ for some chain of τ-invariant ideals of E;*
4. *we can solve the partial lifting problem for A whenever B is separable.*

Theorem 4.11. *Suppose $0 \to I \to A \to B \to 0$ is an exact sequence of separable $C^{*,\tau}$-algebras. If A and B are projective then I is projective.*

Proof. We need only lift morphisms of the form $I \to C(E)$. These extend to morphisms $B \to C(E)$, and those morphisms lift. $\square$

Theorem 4.12. *Suppose* $0 \to I \to A \to B \to 0$ *is an exact sequence of separable* $C^{*,\tau}$*-algebras. If* A *is semiprojective and* B *is projective then* I *is semiprojective.*

5 Functions on graphs

In this section we show semiprojectivity of continuous functions on finite one-dimensional CW-complexes with the trivial reflection. The proof follows the ideas put forth in [19]. In that paper semiprojectivity of "dimension drop graphs" is shown. Since we have a specific goal in mind, we have chosen to drop the matrix algebras.

Theorem 5.1. *If X is a finite one-dimensional CW complex, then $C(X, \mathrm{id})$ is a semiprojective $C^{*,\tau}$-algebra.*

Proof. Since semiprojectivity is closed under direct sums, we can assume that X is connected. We will do the proof by induction on the number of vertices in X.

The case where X has only one vertex is Proposition 3.19.

Suppose now any one-dimensional CW complex with k vertices gives rise to a semiprojective $C^{*,\tau}$-algebra. Let X be a one-dimensional CW complex with $k+1$ vertices. Fix two vertices, v_1 and v_2 say. Let $\tilde{X}$ be a topological copy of X. Denote the copies of v_1 and v_2 and $\tilde{X}$ by w_1 and w_2 respectively. Choose a continuous function $h_0 \colon \tilde{X} \to [-1;2]$ such that $h_0^{-1}([-1;0])$ consists of the union of closed subintervals, containing w_1, of each of the edges adjacent to w_1, and such that $h_0^{-1}([1,2])$ consists of the same for the edges adjacent to w_2, and also $h_0^{-1}(\{-1\}) = \{w_1\}$ and $h_0^{-1}(\{2\}) = \{w_2\}$. We will identify X with the quotient of $\tilde{X}$ obtained by collapsing $h_0^{-1}([-1;0])$ to one point and $h_0^{-1}([1;2])$ to another. Let $\gamma_X \colon \tilde{X} \twoheadrightarrow X$ be the quotient map. Collapsing v_1 and v_2 to one point we obtain a space, Y say. Let $\eta \colon X \to Y$ be the quotient map. Collapsing w_1 and w_2 in $\tilde{X}$ we get a space $\tilde{Y}$, call the quotient map $\tilde{\eta}$. And we can collapse arcs in $\tilde{Y}$ to obtain Y, with quotient map γ_Y say. Thus we have a nice commuting square of quotient maps

$$\begin{tikzcd} \tilde{X} \arrow[r, two heads, "\gamma_X"] \arrow[d, two heads, "\tilde{\eta}"'] & X \arrow[d, two heads, "\eta"] \\ \tilde{Y} \arrow[r, two heads, "\gamma_Y"'] & Y \end{tikzcd}$$

We will view $C(X, \mathrm{id})$, $C(\tilde{Y}, \mathrm{id})$ and $C(Y, \mathrm{id})$ as sub-algebras of $C(\tilde{X}, \mathrm{id})$ using the following identifications:

$$\begin{aligned} C(X, \mathrm{id}) &\cong \left\{ f \in C(\tilde{X}, \mathrm{id}) \mid \begin{array}{ll} f(x) = f(w_1) & \text{if } h_0(x) \le 0 \\ f(x) = f(w_2) & \text{if } h_0(x) \ge 1 \end{array} \right\}, \\ C(\tilde{Y}, \mathrm{id}) &\cong \{ f \in C(\tilde{X}, \mathrm{id}) \mid f(w_1) = f(w_2) \}, \\ C(Y, \mathrm{id}) &\cong \{ f \in C(\tilde{X}, \mathrm{id}) \mid f(x) = f(w_1) \text{ if } h_0(x) \le 0 \text{ or } h_0(x) \ge 1 \}. \end{aligned}$$

Define $h_1 \colon \tilde{X} \to [0,1]$ by

$$h_1(x) = \begin{cases} 0, & h_0(x) \le 0 \\ h_0(x), & 0 \le h_0(x) \le 1 \\ 1, & 1 \le h_0(x) \end{cases} .$$

Note that h_1 and $C(Y, \mathrm{id})$ generate $C(X, \mathrm{id})$.

Suppose now that we are given a $C^{*,\tau}$-algebra (E, τ) containing an increasing sequence of τ-invariant ideals $E_1 \subseteq E_2 \subseteq \cdots$ such that $\overline{\cup_n E_n} = E$, and an injective $C^{*,\tau}$-homomorphism $\phi\colon C(X, \mathrm{id}) \to (C(E), \tau)$. Putting some of the quotient maps and ϕ into one diagram, we have the following.

$$\begin{tikzcd}
C(\tilde{Y}, \mathrm{id}) \arrow[d, "(\tilde{\eta})_*"'] & & \\
C(\tilde{X}, \mathrm{id}) & C(X, \mathrm{id}) \arrow[l, "\gamma_{X*}"'] \arrow[r, "\phi"] & (C(E), \tau)
\end{tikzcd}$$

where $-_*$ denotes the induced maps. Using Corollary 4.8 repeatedly we get a $C^{*,\tau}$-homomorphism $\hat{\phi}\colon C(\tilde{X}, \mathrm{id}) \to (C(E), \mathrm{id})$ such that $\phi = \hat{\phi} \circ \gamma_{X*}$. Using that $\tilde{Y}$ is a one-dimensional CW complex with one vertex less than X we get that $C(\tilde{Y}, \mathrm{id})$ is semiprojective, so we can find an $n \in \mathbb{N}$ and a $C^{*,\tau}$-homomorphism $\psi\colon C(\tilde{Y}, \mathrm{id}) \to (M(E)/E_n, \tau)$ such that $\pi_{n,\infty} \circ \psi = \hat{\phi} \circ (\tilde{\eta})_*$. All in all we have the following commutative diagram

$$\begin{tikzcd}
C(\tilde{Y}, \mathrm{id}) \arrow[rr, "\psi"] \arrow[d, "(\tilde{\eta})_*"'] & & (M(E)/E_n, \tau) \arrow[d, "\pi_{n,\infty}"] \\
C(\tilde{X}, \mathrm{id}) \arrow[rr, bend right, "\hat{\phi}"'] & C(X, \mathrm{id}) \arrow[l, "(\gamma_X)_*"'] \arrow[r, "\phi"] & (C(E), \tau)
\end{tikzcd}$$

We will now find a positive contractive self-τ lift of $\hat{\phi}(h_1)$ in $(M(E)/E_n, \tau)$ that commutes with $(\psi \circ (\gamma_Y)_*)(C(Y, \mathrm{id}))$. Since $\hat{\phi}(h_1)$ is positive and contractive, we can find a positive and contractive lift. Averaging this lift with τ of it, we get a self-τ positive contractive lift of $\hat{\phi}(h_1)$. Let us call it H. Define functions $l, m, k\colon [-1; 2] \to [0; 1]$ by

$$\begin{aligned}
l(t) &= \begin{cases} 0, & -1 \leq t \leq 0, \\ t, & 0 \leq t \leq 1, \\ 2 - t, & 1 \leq t \leq 2 \end{cases}, \\
m(t) &= \begin{cases} -t, & -1 \leq t \leq 0, \\ 0, & 0 \leq t \leq 1, \\ t - 1, & 1 \leq t \leq 2 \end{cases}, \\
k(t) &= \begin{cases} 0, & -1 \leq t \leq 0, \\ t, & 0 \leq t \leq 1, \\ 1, & 1 \leq t \leq 2 \end{cases}.
\end{aligned}$$

Observe that $l + mk = k$, that $k \circ h_0 = h_1$, and that $l \circ h_0$ and $m \circ h_0$ both are in $C(\tilde{Y}, \mathrm{id})$. Hence we can define

$$\tilde{H} = \psi(l \circ h_0) + \psi((m \circ h_0)^{1/2}) H \psi((m \circ h_0)^{1/2}).$$

Since all the functions are real valued we get that $\tilde{H}$ is self-adjoint. Since every thing else is self-τ so is $\tilde{H}$. It is a lift of $\hat{\phi}(h_1)$ since

$$\begin{aligned}\pi_{n,\infty}(\tilde{H}) &= \hat{\phi}(l \circ h_0) + \hat{\phi}((m \circ h_0)^{1/2})\hat{\phi}(h_1)\hat{\phi}((m \circ h_0)^{1/2}) \\ &= \hat{\phi}((l \circ h_0) + (m \circ h_0)h_1) = \hat{\phi}((l \circ h_0) + (m \circ h_0)(k \circ h_0)) \\ &= \hat{\phi}((l + mk) \circ h_0) = \hat{\phi}(k \circ h_0) = \hat{\phi}(h_1).\end{aligned}$$

By functional calculus can replace $\tilde{H}$ with $\hat{H} = k(\tilde{H})$ to obtain a positive contractive lift of $\hat{\phi}(h_1)$. By Lemma 3.6 $\hat{H}$ is self-τ. To show that this lifts commutes with $(\psi \circ (\gamma_Y)_*)(C(Y, \mathrm{id}))$ it suffices to show that $\tilde{H}$ does. Let $f \in C(Y, \mathrm{id})$. Then $f(m \circ h_0) = 0$ so we must have

$$\psi((\gamma_Y)_*(f))\psi(m \circ h_0) = 0.$$

Hence $\psi((\gamma_Y)_*(f))$ commutes with $\tilde{H}$.

Let $D = C(Y \times [0, 1])$. We have shown that given a $C^{*,\tau}$-homomorphism $\phi\colon C(X, \mathrm{id}) \to (C(E), \tau)$ we can find an $n_0 \in \mathbb{N}$ and a C^*-homomorphism $\chi\colon D \to M(E)/E_{n_0}$ such that the following diagram commutes

$$\begin{tikzcd} D \arrow[r, "\chi"] \arrow[d, "\beta"'] & M(E)/E_{n_0} \arrow[d, "\pi_{n_0,\infty}"] \\ C(X, \mathrm{id}) \arrow[r, "\phi"'] & C(E) \end{tikzcd}$$

Where β denotes that map induced by sending $C(Y, \mathbb{R})$ (inside D) to $C(Y, \mathbb{R})$ (inside $C(X, \mathbb{R})$) and h to h_1. Since $\hat{H}$ is self-τ Lemma 3.6 gives that χ is actually τ-preserving. Hence we can view the above diagram as being a commutative diagram in the $\mathbf{C}^{*,\tau}$ category.

For each $n \geq n_0$ define $\chi_n = \pi_{n_0,n} \circ \chi$ and let $D_n = D/\ker \chi_n$. Then if $n_0 \leq n \leq m$ we have a surjection $D_n \twoheadrightarrow D_m$. Since $D = C(Y \times [0; 1])$ and each D_n, $n \geq n_0$, is a quotient of D, there must be spaces Y_n, $n \geq n_0$, such that $D_n \cong C(Y_n)$. Thus we have an inductive system

$$D \twoheadrightarrow C(Y_{n_0}) \twoheadrightarrow C(Y_{n_0+1}) \twoheadrightarrow C(Y_{n_0+2}) \twoheadrightarrow \cdots$$

Call the bonding maps $\delta_{k,l}$. This is an inductive system in the category of C^*-algebras, so we can compute the limit as

$$D/\ker(\pi_{n_0,\infty} \circ \chi) \cong (\pi_{n_0,\infty} \circ \chi)(D) = (\phi \circ \beta)(D) = \phi(C(X)) \cong C(X).$$

Since X is an ANR we can find an $m \geq n_0$ and a C^*-homomorphism $\lambda\colon C(X) \to C(Y_m)$ such that $\delta_{m,\infty} \circ \lambda = \mathrm{id}$. Clearly λ and $\delta_{m,\infty}$ are $C^{*,\tau}$-homomorphisms if we equip all the commutative algebras with the identity reflection.

Consider the the following commutative diagram.

$$\begin{tikzcd}
(D, \mathrm{id}) \arrow[r, "\chi"] \arrow[d, two heads, "\delta_n"'] & (M(E)/E_{n_0}, \tau) \arrow[d, two heads, "\pi_{n_0,n}"] \\
(C(Y_n), \mathrm{id}) \arrow[r, dashed] \arrow[d, two heads, "\delta_{n,\infty}"'] & (M(E)/E_n, \tau) \arrow[d, two heads, "\pi_{n,\infty}"] \\
C(X, \mathrm{id}) \arrow[r, "\phi"'] & (C(E), \tau)
\end{tikzcd}$$

Since all the vertical maps are quotient maps, we can fit a $C^{*,\tau}$-homomorphism on the dashed arrow in such a way that the diagram continues to commute. Call this homomorphism μ_n. We claim that $\mu_m \circ \lambda$ is a lift of ϕ. To see that, we compute

$$\pi_{m,\infty} \circ \mu_m \circ \lambda = \phi \circ \delta_{m,\infty} \circ \lambda = \phi.$$

$\square$

6 Variations on Lin's theorem

From here on out we more less just follow the proof in [9], modifying their techniques to keep track of reflections.

In this section we write M_n for $M_n(\mathbb{C})$.

6.1 Approximating normal elements

The following lemma gives a kind of self-τ stable rank for (M_n, τ).

Lemma 6.1. *Let τ be a reflection on M_n. For any $\varepsilon > 0$ and any self-τ matrix $A \in (M_n, \tau)$ we can find a self-τ invertible matrix B such that $\|A - B\| < \varepsilon$.*

Proof. If A is invertible there is nothing to prove. So suppose A is not invertible. Consider the path of self-τ matrices $B_t = (1-t)A + tI$. Define a function $p\colon [0,1] \to \mathbb{C}$ by

$$p(t) = \det(B_t).$$

By definition of det and B_t the function p is a polynomial. Since $p(0) = \det(B_0) = \det(A) = 0$ and $p(1) = \det(B_1) = \det(I) = 1$, p is not constant. Hence it has only finitely many zeros. Thus for any $\varepsilon > 0$ we can find a t_0 such that $0 < t_0 < \varepsilon/(\|A - I\|)$ and $p(t_0) \neq 0$. Then B_{t_0} is self-τ and invertible, and

$$\|A - B_{t_0}\| = \|At_0 - It_0\| \leq t_0\|A - I\| < \varepsilon.$$

$\square$

Lemma 6.2. *Let a be a self-τ invertible element in a unital $C^{*,\tau}$-algebra (A, τ). Then a can be written as $a = up$ where u is a self-τ-unitary and $p = (a^*a)^{1/2}$.*

Proof. Since a is invertible so is a^*a. Hence we can define $u = a(a^*a)^{-1/2}$ and $p = (a^*a)^{1/2}$. Then u is a unitary and $up = a$. By using Lemma 3.6 and standard functional calculus tricks, we get

$$u^\tau = ((a^*a)^{-1/2})^\tau a^\tau = ((a^*a)^\tau)^{-1/2} a = (aa^*)^{-1/2} a = a(a^*a)^{-1/2} = u$$

$\square$

Let (n_j) be a sequence of natural numbers and let τ_j be a reflections on M_{n_j}. Define

$$(M, \tau) = \prod_j (M_{n_j}, \tau_j), \quad (A, \tau) = \bigoplus_j (M_{n_j}, \tau_j).$$

Let $\pi\colon (M, \tau) \to (M/A, \tau)$ denote the quotient map.

Lemma 6.3. *For any self-τ element $a \in (M/A, \tau)$ there exists a self-τ unitary $u \in (M/A, \tau)$ such that $a = up$, where $p = (a^*a)^{1/2}$.*

Proof. Let $x = (x_j)$ be any self-τ lift of a. Using Lemma 6.1 we can for all $j \in \mathbb{N}$ find an invertible self-τ element $y_j \in (M_{n_j}, \tau_j)$ such that $\|x_j - y_j\| < 1/j$. Then the sequence (y_j) is in (M, τ) and $\pi(y) = \pi(x) = a$. By Lemma 6.2 we can, for each $j \in \mathbb{N}$, find a self-τ unitary $v_j \in (M_{n_j}, \tau_j)$ such that $y_j = v_j q_j$, where $q_j = (y_j^* y_j)^{1/2}$. If we let $v = (v_j)$ and $q = (q_j)$ then $y = vq$ and v is a self-τ unitary. Now put $u = \pi(v)$ and $p = \pi(q)$. Then $a = \pi(y) = \pi(v)\pi(q) = up$, u is a self-τ unitary, and

$$p = \pi(q) = \pi((y^* y)^{1/2}) = (\pi(y)^* \pi(y))^{1/2} = (a^* a)^{1/2}.$$

□

Lemma 6.4. *If $x \in (M/a, \tau)$ is normal and self-τ then for every $\varepsilon > 0$ there is a normal self-τ invertible element $y \in (M/A, \tau)$ such that $\|x - y\| < \varepsilon$.*

Proof. By Lemma 6.3 we can write $x = up$ where u is a self-τ unitary and $p = (x^* x)^{1/2}$. Since we assumed x to be normal u and p commute by standard functional calculus. Define $y = u(p + (\varepsilon/2)I)$, where I is the unit in M/A. Since y is the product of two commuting normal and invertible elements it is normal and invertible. By Lemma 3.6 we have

$$p^\tau = \big((x^* x)^\tau\big)^{1/2} = \big(x x^*\big)^{1/2} = \big(x^* x\big)^{1/2} = p.$$

From that it follows that y is self-τ. Finally we see that

$$\|x - y\| = \|up - (up + (\varepsilon/2)u)\| = \|(\varepsilon/2)u\| = \varepsilon/2 < \varepsilon.$$

□

Lemma 6.5. *Let $\lambda \in \mathbb{C}$ be given. If $x \in (M/A, \tau)$ is normal and self-τ then for every $\varepsilon > 0$ there is a normal self-τ element $y \in (M/A, \tau)$ with $\lambda \notin \sigma(y)$, and such that $\|x - y\| < \varepsilon$.*

Proof. Let $\tilde{x} = x - \lambda I$. Then $\tilde{x}$ is normal and self-τ so by Lemma 6.4 we can find a normal, self-τ and invertible $\tilde{y} \in M/A$ such that $\|\tilde{y} - \tilde{x}\| < \varepsilon$. Let $y = \tilde{y} + \lambda I$. Then y is normal and self-τ, and

$$\|y - x\| = \|\tilde{y} + \lambda I - x\| = \|\tilde{y} - (x - \lambda I)\| = \|\tilde{y} - \tilde{x}\| < \varepsilon.$$

We note that since 0 is not in the spectrum of $\tilde{y}$ we have $\lambda \notin \sigma(y)$. □

Lemma 6.6. *Let F be an at most countable subset of $\mathbb{C}$. If $x \in (M/A, \tau)$ is normal and self-τ then for every $\varepsilon > 0$ there is a normal self-τ element $y \in (M/A, \tau)$ with $F \cap \sigma(y) = \emptyset$, and such that $\|x - y\| < \varepsilon$.*

Proof. Let X be the set of normal and self-τ elements in $(M/A, \tau)$. This is a closed subset of M/A, so it is a complete metric space. Let $F = \{\lambda_1, \lambda_2, \ldots\}$. For each $n \in \mathbb{N}$ let U_n be the set of self-τ normal elements in $(M/A, \tau)$ that do not have λ_n in their spectrum. By Lemma 6.5 all the U_n are dense in X. Since the set of invertible elements in a C^*-algebra is open all the U_n are open in the relative topology of X. By Baire's theorem the set $\bigcap_n U_n$ is dense in X. That is, the set of normal self-τ elements whose spectrum does not contain F is dense in the set of normal self-τ elements. $\square$

For any complex number z we denote by $\Re(z)$ and $\Im(z)$ the real and imaginary parts of z. For all $\varepsilon > 0$ define

$$\begin{aligned}\Gamma_\varepsilon &= \{z \in \mathbb{C} \mid \Re(z) \in \varepsilon\mathbb{Z} \text{ or } \Im(z) \in \varepsilon\mathbb{Z}\},\\ \Sigma_\varepsilon &= \{z \in \mathbb{C} \mid \Re(z) \in \varepsilon(\mathbb{Z} + \frac{1}{2}) \text{ and } \Im(z) \in \varepsilon(\mathbb{Z} + \frac{1}{2})\}.\end{aligned}$$

Proposition 6.7. *If $x \in (M/A, \tau)$ is normal and self-τ then for every $\varepsilon > 0$ there is a normal self-τ element $y \in (M/A, \tau)$ with $\sigma(y) = \Gamma_\varepsilon$, and such that $\|x - y\| < \varepsilon$.*

Proof. By Lemma 6.6 we can find a normal and self-τ element $\tilde{y} \in (M/A, \tau)$ with

$$\sigma(\tilde{y}) \cap \Sigma_\varepsilon = \emptyset, \quad \text{and} \quad \|\tilde{y} - x\| < \left(1 - \frac{\sqrt{2}}{2}\right)\varepsilon.$$

There is a continuous retraction $f\colon \mathbb{C} \setminus \Sigma_\varepsilon \to \Gamma_\varepsilon$ with $|f(z) - z| < (1 - \frac{\sqrt{2}}{2})$ for all z. Let $y = f(\tilde{y})$. Then y is normal, has the right spectrum, and is ε close to x. By Lemma 3.6 we have

$$y^\tau = f(\tilde{y})^\tau = f(\tilde{y}^\tau) = f(\tilde{y}) = y.$$

$\square$

6.2 The proof of Theorem 1

Proposition 6.8. *Suppose (A_n, τ_n) is a sequence of $C^{*,\tau}$-algebras. If x is a normal self-τ element in*

$$(Q, \tau) = \prod_{n=1}^{\infty} (A_n, \tau_n) \Big/ \bigoplus_{n=1}^{\infty} (A_n, \tau_n)$$

with spectrum contained in some finite graph, then there is a lift of x to a normal self-τ element in $\prod_{n=1}^{\infty} (A_n, \tau_n)$.

Proof. Let Γ be a finite graph such that $\sigma(x) \subseteq \Gamma$. By Lemma 3.6 the map $f \mapsto f(x)$ is a $C^{*,\tau}$-homomorphism from $C(\Gamma, \mathrm{id})$ to (Q, τ). Since $C(\Gamma, \mathrm{id})$ is semiprojective, we can find an $m \in \mathbb{N}$ and a normal self-τ element

$$y \in \prod_{n=1}^{\infty} (A_n, \tau_n) \Big/ \bigoplus_{n=1}^{m} (A_n, \tau)\ ,$$

such that y is a lift of x. Identifying

$$\prod_{n=1}^{\infty}(A_n,\tau_n)\Big/\bigoplus_{n=1}^{m}(A_n,\tau)$$

with

$$\prod_{n=m}^{\infty}(A_n,\tau),$$

we see that if we pad y with leading zeros we get a self-τ and normal lift of x in $\prod_{n=1}^{\infty}(A_n,\tau_n)$. □

We are now ready to prove real versions of Lin's theorem. First we do the case of normal matrices.

Theorem 6.9. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $n \in \mathbb{N}$, any reflection τ on M_n and self-τ matrix $X \in (M_n,\tau)$ with $\|X\| \le 1$ and*

$$\|X^*X - XX^*\| < \delta,$$

there exists a normal self-τ matrix $X' \in (M_n,\tau)$ with

$$\|X - X'\| < \varepsilon.$$

Proof. Suppose there was an ε that had no accompanying δ. Then there must exist a sequence (n_j) of natural numbers, reflections τ_j on M_{n_j}, and self-τ contractive matrices $X_j \in (M_{n_j},\tau_j)$ such that

$$\|X_j^*X_j - X_jX_j^*\| \to 0,$$

but every X_j is at least ε away from all normal self-τ matrices in (M_{n_j},τ_j).

Let, as in Section 6.1,

$$(M,\tau) = \prod_j (M_{n_j},\tau_j), \quad (A,\tau) = \bigoplus_j (M_{n_j},\tau_j).$$

Let $x = (X_j)$ and let $y = \pi(x)$, where π is the quotient map from (M,τ) to $(M/A,\tau)$. Then y is a normal and self-τ element. By Lemma 6.6 we can find a normal self-τ element $z \in (M/A,\tau)$ with spectrum contained in a finite graph and $\|y - z\| < \varepsilon/4$. Using Proposition 6.8 we can find a normal self-τ element $x' \in (M,\tau)$ such that $\pi(x') = z$. The definition of the norm in $(M/A,\tau)$ tells us that there exists $(A_j) = a \in A$ such that

$$\|(x - x') - a\| = \|y - z\| + \varepsilon/4 < \varepsilon/2.$$

Now pick a j_0 such that $\|A_{j_0}\| < \varepsilon/2$. Then we have

$$\|X_{j_0} - X'_{j_0}\| \le \|(X_{j_0} - X'_{j_0}) - A_{j_0}\| + \|A_{j_0}\| < \|(x - x') - a\| + \varepsilon/2 < \varepsilon.$$

Which contradicts our assumption about all the X_j being at least ε away from any normal self-τ element. □

Theorem 6.10. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $n \in \mathbb{N}$, any reflection τ on M_n and any pair $A, B \in (M_n, \tau)$ of self-adjoint, self-τ matrices such that $\|A\|, \|B\| \leq 1$ and*

$$\|AB - BA\| < \delta,$$

there exists a commuting pair $A', B' \in (M_n, \tau)$ of self-adjoint and self-τ matrices with

$$\|A - A'\| + \|B - B'\| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Use Theorem 6.9 to find a δ matching $\varepsilon/2$. Let $A, B \in (M_n, \tau)$ be given as in the theorem. Define $X = (A + iB)/2$. Then

$$\|XX^* - X^*X\| = \|AB - BA\| < \delta.$$

Hence we can find a normal self-τ matrix $X' \in M_n$ such that $\|X - X'\| < \varepsilon/2$. Now let

$$A' = X' + X'^* \quad \text{and} \quad B' = -i(X' - X'^*).$$

Then A' and B' are self-adjoint and self-τ. Since X' is normal they commute. As

$$\|A - A'\| = \|(X + X^*) - (X' + X'^*)\| \leq \|X - X'\| + \|X^* - X'^*\| < \varepsilon,$$

and likewise for $\|B - B'\|$, A' and B' show that we can approximate A and B by commuting self-adjoint, self-τ matrices. $\square$

Setting τ equal the transpose in Theorem 6.10 we obtain the extension of Lin's theorem to real matrices. Using the dual operation, $\tau = \sharp$, we obtain the extension of Lin's theorem to self-dual matrices.

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