EXPANSIONS OF SUBFIELDS OF THE REAL FIELD BY A DISCRETE SET

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Abstract. Let K be a subfield of the real field, $D \subseteq K$ be a closed and discrete set and $f : D^n \to K$ be such that $f(D^n)$ is somewhere dense. Then (K, f) defines $\mathbb{Z}$. We present several applications of this result. We show that K expanded by predicates for different cyclic multiplicative subgroups defines $\mathbb{Z}$. Moreover, we prove that every definably complete expansion of a subfield of the real field satisfies an analogue of the Baire Category Theorem.

1. Introduction

Let K be a subfield of the field of real numbers.

Theorem A. Let $D \subseteq K$ be discrete and let $f : D^n \to K$ be such that $f(D)$ is somewhere dense. Then (K, f) defines $\mathbb{Z}$.

This result generalizes earlier work of the author in [6] where Theorem A is shown in the case that $K = \mathbb{R}$ and D is closed and discrete. The proof in [6] relies crucially on the topological completeness of $\mathbb{R}$ and hence does not work for subfields of the real field. One can even construct a subfield K and a function $f : D \to K$ that satisfy the assumptions of Theorem A, but the parameter-free formula that defines $\mathbb{Z}$ in $(\mathbb{R}, f)$ does not define $\mathbb{Z}$ in (K, f). The work in the current paper shows how results from [6] can still be used to establish Theorem A.

There are discrete subsets D of subfields K of $\mathbb{R}$ that are closed in the induced topology on K, but that are not closed in the order topology on $\mathbb{R}$. Such discrete sets may even fail to be well ordered by the ordering on $\mathbb{R}$. To make use of the results of [6] we etablish the following theorem.

Theorem B. Let $D \subseteq K$ be a discrete set. Then there is a discrete set $E \subseteq K$ such that E is closed as a subset of $\mathbb{R}$, (K, D) and (K, E) are interdefinable and there is a surjection $g : E \to D$ definable in (K, D).

The proof of Theorem A will be presented in the section 4. In section 2 we prove a generalization of Miller's Lemma on Asymptotic Extraction of Groups from [7] that plays a key role in the proof of Theorem A. Section 3 gives a proof of Theorem B. In the rest of this section, several applications of Theorem A and B will be discussed.

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Two discrete subgroups. For any $\alpha \in K^\times$, let

$$\alpha^{\mathbb{Z}} := \{\alpha^k : k \in \mathbb{Z}\}.$$

In [1] van den Dries established that the structure $(K, \alpha^{\mathbb{Z}})$ is model theoretically tame, when K is a real closed field subfield of the real field. In particular, he showed that $\mathbb{Z}$ is not definable in that structure. Theorem A allows us to show that this is not the case in the structure $(K, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$.

Theorem C. Let $\alpha, \beta \in K_{>0}$ with $\log_\alpha(\beta) \notin \mathbb{Q}$. Then $(K, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ defines $\mathbb{Z}$.

Proof. The set $\alpha^{\mathbb{N}} \cup \beta^{\mathbb{N}}$ is discrete and definable in $(K, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$. Let $g : K_{>0} \times K_{>0} \to K$ be the function mapping (a, b) to $\frac{a}{b}$. The image of $(\alpha^{\mathbb{N}} \cup \beta^{\mathbb{N}}) \times (\alpha^{\mathbb{N}} \cup \beta^{\mathbb{N}})$ under g is dense in $K_{>0}$. Hence $(K, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ defines $\mathbb{Z}$ by Theorem A. $\square$

An analogue of the Baire Category Theorem. An expansion $\mathcal{K}$ of K is *definably complete* if every bounded subset of K definable in $\mathcal{K}$ has a supremum in K. Given a subset Y of K^2 and $a \in K$, we denote $\{b : (b, a) \in Y\}$ by Y_a.

Theorem D. Let $\mathcal{K}$ be a definably complete expansion of K. Then $\mathcal{K}$ is definably Baire; that is there exists *no* set $Y \subseteq K^2_{>0}$ definable in $\mathcal{K}$ such that

(i) Y_t is nowhere dense for $t \in K_{>0}$,
(ii) $Y_s \subseteq Y_t$ for $s, t \in K_{>0}$ with $s < t$, and
(iii) $\bigcup_{t \in K_{>0}} Y_t = K$.

Proof. Suppose $\mathcal{K}$ is not definably Baire. By [3, Corollary 6.6], there is a closed and discrete set $D \subseteq K$ definable in $\mathcal{K}$ and $f : D \to K$ definable in $\mathcal{K}$ such that the image of f is dense in K. By Theorem A, $\mathbb{Z}$ is definable in $\mathcal{K}$. Thus $\mathcal{K}$ is Baire by [3, Lemma 6.2]. $\square$

Definable versions of standard facts from real analysis hold in definably complete expansions of ordered fields that satisfy the conclusion of Theorem D. For details, see the work of Fornasiero and Servi in [4].

Optimality of dichotomies over $\mathbb{R}$. By Theorem B, the dichotomy in [6, Theorem 1.2] extends to discrete subsets of $\mathbb{R}$ as follows.

Theorem E. Let $\mathcal{R}$ be an o-minimal expansion of $\mathbb{R}$ and let $D \subseteq \mathbb{R}$ be discrete. Then either

- $(\mathcal{R}, D)$ defines $\mathbb{Z}$ or
- every subset of $\mathbb{R}$ definable in $(\mathcal{R}, D)$ has interior or is nowhere dense.

However, by the following result neither in Theorem E nor in [6, Theorem 1.2] can the statement *'is nowhere dense'* be replaced by *'is a finite union of discrete sets'*.

Theorem F. There is a closed and discrete set $D \subseteq \mathbb{R}$ such that $(\mathbb{R}, D)$ does not define $\mathbb{Z}$, but defines a set that neither has interior nor is a union of finitely many discrete sets.

Proof. By [5, 2.3] there is a discrete set D such that $(\mathbb{R}, D)$ does not define $\mathbb{Z}$, but sets on every level of the projective hierarchy. By Theorem B, we can assume that D is closed. Since the union of finitely many discrete sets and an open sets is F_σ, there is a set definable in $(\mathbb{R}, D)$ that is not of that form. $\square$

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Notation. In the rest of the paper K will always be a fixed subfield of $\mathbb{R}$. As before, we do not distinguish between the field K and its underlying set. We will use a, b, c for elements of K. The letters l, n, m, N will always denote natural numbers. When we say definable, we mean definable with parameters. Given a subset A of $K^n \times K^m$ and $a \in K^m$, we denote the set $\{b : (b, a) \in A\}$ by A_a.

2. Asymptotic extraction

We will show the following generalization of Miller's Lemma on Asymptotic Extraction of Groups from [7].

Lemma 1. Let $\mathcal{K}$ be an expansion of K and let $S \subseteq K_{>0} \times K^l$ be definable in $\mathcal{K}$ such that for every $n \in \mathbb{N}$ and every $\varepsilon \in K_{>0}$, there is $b \in K^l$ such that

(1) $S_b \subseteq \bigcup_{m \in \mathbb{N}, m \leq n} (m - \varepsilon, m + \varepsilon)$, and
(2) $|S_b \cap (m - \varepsilon, m + \varepsilon)| = 1$ for $m \leq n$.

Then $\mathcal{K}$ defines $\mathbb{Z}$.

Proof. For $\varepsilon \in K_{>0}$ define B_ε as the set of all $b \in K^l$ that satisfy the following two properties:

(i) $|a_1 - a_2| > 1 - \varepsilon$, for all $a_1, a_2 \in S_b$ with $a_1 \neq a_2$ and
(ii) $|a_1 - a_2| < 1 + \varepsilon$ for all $a_1, a_2 \in S_b$ with $S_b \cap (a_1, a_2) = \emptyset$.

For $b \in B_\varepsilon$, let $\lambda(b)$ be the smallest element of S_b. Such an element exists, since $S_b \subseteq K_{>0}$. Set

$$S'_b := \{a - \lambda(b) : a \in S_b\}.$$

Finally, define

$$W := \{c \in K : \forall \varepsilon \in K_{>0} \exists b \in B_\varepsilon (c - \varepsilon, c + \varepsilon) \cap S'_b \neq \emptyset\}.$$

We will finish the proof by showing that $W = \mathbb{N}$.
Let $n \in \mathbb{N}$ and $\varepsilon \in K_{>0}$. By our assumption on S, there is $b \in K^l$ such that

$$S_b \subseteq \bigcup_{m \leq n} (m - \frac{\varepsilon}{2}, m + \frac{\varepsilon}{2}) \text{ and } |S_b \cap (m - \frac{\varepsilon}{2}, m + \frac{\varepsilon}{2})| = 1$$

for $m \leq n$. Hence $|a_1 - a_2| \in (1 - \varepsilon, 1 + \varepsilon)$ for two adjacent elements $a_1, a_2 \in S_b$. Thus $b \in B_\varepsilon$. Since $\lambda(b) \in (0, \frac{\varepsilon}{2})$, we have that $|S'_b \cap (n - \varepsilon, n + \varepsilon)| = 1$. Hence $n \in W$.
Let $c \in K$ be such that $c \in (n, n+1)$ for some $n \in \mathbb{N}$. Let $\varepsilon \in K_{>0}$ be such that $2(n+1)\varepsilon \leq \min\{c - n, n + 1 - c\}$ and let $b \in B_\varepsilon$. Since $b \in B_\varepsilon$,

$$S'_b \cap (n, n+1) \subseteq (n - n\varepsilon, n + n\varepsilon) \cap (n + 1 - (n+1)\varepsilon, n + 1 + (n+1)\varepsilon).$$

Because of our choice of ε, we have $c - \varepsilon > n + n\varepsilon$ and $c + \varepsilon < n + 1 - (n+1)\varepsilon$. Hence $(c - \varepsilon, c + \varepsilon) \cap S'_b = \emptyset$ and $c \notin W$. $\square$

3. Proof of Theorem B

We say a set $X \subseteq K$ is *closed as a subset of* $\mathbb{R}$ if it is closed in the order topology on $\mathbb{R}$.

Lemma 2. Let $D \subseteq K_{>0}$ be discrete and closed as a subset of $\mathbb{R}$. There are $E \subseteq K_{>0}$ and a bijection $g : D \to E$ such that g is definable in (K, D) and $|a - b| \geq 1$ for all distinct $a, b \in E$.

Proof. Let $\sigma : D \to D$ be the successor function on the well-ordered set $(D, <)$. Define $g : D \to K$ by

$$d \mapsto d \cdot \max\left(\{(\sigma(e) - e)^{-1} \ : \ e \in D, e < d\} \cup \{1\}\right).$$

The maximum in the definition of g always exists in K, because the set $\{e \in D : e < d\}$ is finite. The function g is strictly increasing and definable in (K, D). The image of D under g is discrete and closed as subset of $\mathbb{R}$. By construction, the distance between two elements of $g(D)$ is at least 1. □

Lemma 3. Let $D \subseteq K_{>0}$ be an infinite discrete set. Then (K, D) defines an unbounded discrete set E that is closed as a subset of $\mathbb{R}$.

Proof. For every $\varepsilon \in K_{>0}$, we define[1]

$$B_\varepsilon := \{d \in D : (d - \varepsilon, d + \varepsilon) \cap D = \{d\}\}.$$

Note that $B_\varepsilon \supseteq B_\delta$, for $\varepsilon, \delta \in K_{>0}$ with $\varepsilon \leq \delta$. If there is $\varepsilon \in K$ such that B_ε is infinite, then this B_ε is unbounded, discrete and closed as a subset of $\mathbb{R}$. So we can reduce to the case that B_ε is finite for every $\varepsilon \in K$.
Let $g : K_{>0} \to D$ be the function that maps

$$\varepsilon \mapsto \text{the smallest } d \in B_\varepsilon \setminus \left(\bigcup_{\delta > \varepsilon, B_\delta \neq B_\varepsilon} B_\delta \right).$$

Then $g(K)$ is infinite, since D is. Consider the function $h : K_{>0} \to K$ defined by

$$\varepsilon \mapsto \max\left(\{(d_1 - d_2)^{-1}, d_1 - d_2 : d_1, d_2 \in g(K_{\geq \varepsilon}), d_1 > d_2\} \cup \{1\}\right).$$

The maximum in the definition of h exists in K, since B_ε is finite. Since $g(K)$ is infinite, $h(K)$ must be unbounded. Since $g(K_{\geq\varepsilon})$ is finite, $h(K) \cap (0, a)$ is finite for every $a \in K$. Hence $h(K)$ is closed as a subset of $\mathbb{R}$. □

Proof of Theorem B. Let D be a discrete subset of K. We can assume that $D \subseteq K_{>0}$. By Lemma 2 and 3, there is an infinite set $A \subseteq K_{>0}$ definable in (K, D) such that $|a_1 - a_2| \geq 1$ for all $a_1, a_2 \in A$ with $a_1 \neq a_2$. Let $\sigma : A \to A$ be the successor function on the well ordered set $(A, <)$. We now construct a discrete set E that is closed as a subset of $\mathbb{R}$ and encodes all the information about D. For every $a \in A$, set

$$B_a := \{d \in D : 0 < d < a \text{ and } (d - a^{-1}, d + a^{-1}) \cap D = \{d\}\}.$$

The set B_a is finite and definable in (K, D) for every $a \in A$. Moreover, $B_{a_1} \subseteq B_{a_2}$ for $a_1, a_2 \in A$ with $a_1 \leq a_2$. Since D is discrete, $D = \bigcup_{a \in A} B_a$. Further for $a \in A$, define

$$C_a := \{a + \frac{d}{a} : d \in B_a\}.$$

[1]This definition was first used by Fornasiero in [2] for studying definably complete expansions of fields.

Then C_a is finite, definable in (K, D) and

$$C_a \subseteq (a, a+1) \subseteq (a, \sigma(a)).$$

Finally set $F := \bigcup_{a \in A} C_a$. Since $F \cap (a, \sigma(a)) = C_a$ is finite for every $a \in A$, the set F is discrete and closed as a subset of $\mathbb{R}$. Now define

$$E := F \cup \{-a \ : \ a \in A\}.$$

Then E is discrete and closed as a subset of $\mathbb{R}$, since A and F are. Moreover, A and F are definable in (K, E), because $A, F \subseteq K_{>0}$.
It is only left to show that D is definable in (K, E). Let $h : K \to A$ be a function mapping a real number x to the largest $a \in A$ with $a < x$ if such an a exists, and to 0 otherwise. Note that h is definable in (K, E), because A is. Define a function $g : K \to K$ by

$$g(a) := h(a)(a - h(a)).$$

The image of C_a under g is B_a for each $a \in A$, because $C_a \subseteq (a, \sigma(a))$. Hence the image of F under g is D, since $D = \bigcup_{a \in A} C_a$. Hence D is definable in (K, E), since g and F are. $\square$

Lemma 4. Let $D \subseteq K_{>0}$ be discrete and closed as a subset of $\mathbb{R}$. There are $E \subseteq K_{>0}$ and a bijection $g : D^n \to E$ such that g is definable in (K, D) and E discrete and closed as a subset of $\mathbb{R}$.

Proof. By Lemma 2, we can assume that the distance between two elements of D is at least 1. Let $h : K_{>0} \times K^n \to K$ be defined by

$$(x_0, x_1, ..., x_n) \mapsto x_0 + \sum_{i=1}^{n} \frac{x_i}{(nx_0)^i}.$$

Consider $g : D^n \to K$ defined by

$$(d_1, ..., d_n) := h(\max\{d_1, ..., d_n\}, d_1, ..., d_n).$$

It is easy to show that g is injective and $g(D^n)$ is discrete and closed as a subset of $\mathbb{R}$. $\square$

4. Proof of Theorem A

Let D be a discrete subset of K and let $f : D^n \to K$ be a function such that $f(D^n)$ is somewhere dense. By Theorem B we can assume that D is closed as a subset of $\mathbb{R}$. By Lemma 2 and 4 we can assume that $n = 1$ and that the distance between two distinct elements of D is at least 1. By composing f by a semialgebraic function we can even assume that $f(D) \subseteq (1, 2)$.

We recall several definitions from [6]. Let $\varphi(x, y)$ be the formula

$$\forall u \in D(f(u) < y < f(u)(1 + u^{-2})) \to (u < x^{\frac{1}{7}} \vee u > x).$$

Note that for all $a, b \in K$

$$(\mathbb{R}, f) \models \varphi(a, b) \text{ iff } (K, f) \models \varphi(a, b).$$

For $c \in \mathbb{R}$, define

$$A_c := \{d \in D : f(d) < c < f(d) \cdot (1 + d^{-2}) \wedge \varphi(d)\}.$$

Further for $c \in \mathbb{R}$ let $v_c : D \setminus \{c\} \to \mathbb{R}$ be given by

$$v_c(x) := \frac{x^{-2} f(x)}{c - f(x)}.$$

The following Fact is an immediate corollary of the proof of [6, Theorem 1.1] (see statements (1) and (2) on p. 2167 of [6]).

Fact 5. [6] There are $c \in \mathbb{R}$, $N \in \mathbb{N}$ and $d \in D$ such that

(i) $\nu_c(A_c \cap D_{>d}) \subseteq \bigcup_{m \in \mathbb{N}_{>N}} (m, m + \frac{1}{m})$,
(ii) $|\nu_c(A_c \cap D_{>d}) \cap (m, m + \frac{1}{m})| = 1$ for $m > N$, and
(iii) ν_c is an increasing function on $A_c \cap D_{>d}$.

Lemma 6. For every $n \in \mathbb{N}$, there are $c \in K$, $d_1, d_2 \in D$ and $N \in \mathbb{N}$ such that

(i) $\nu_c(A_c \cap [d_1, d_2]) \subseteq \bigcup_{m \in [N, N+n]} (m, m + \frac{1}{m})$ and
(ii) $|\nu_c(A_c \cap [d_1, d_2]) \cap (m, m + \frac{1}{m})| = 1$ for $m \in [N, N + n]$.
(iii) ν_c is an increasing function on $A_c \cap [d_1, d_2]$.

Proof. By Fact 5 there are $c \in \mathbb{R}$, $N \in \mathbb{N}$ and $d_1, d_2 \in D$ such that (i)-(iii) hold. Since D is closed and discrete, $D \cap [d_1, d_2]$ is finite. As A_c and ν_c depend continuously on c, we can take $c' \in K$ so close to c such that

$$A_c \cap [d_1, d_2] = A_{c'} \cap [d_1, d_2]$$

and

$$\nu_c(d) \in (m, m + \frac{1}{m}) \text{ iff } \nu_{c'}(d) \in (m, m + \frac{1}{m})$$

for every $d \in A_{c'} \cap [d_1, d_2]$. Hence (i)-(iii) holds for c', N, d_1, d_2. □

Proof of Theorem A. Let $S \subseteq K_{>0} \times K^3$ be

$$\{(a, b_1, b_2, b_3) \in K_{>0} \times K^3 : b_2, b_3 \in A_{b_1} \wedge a + \nu_{b_1}(b_2) \in \nu_{b_1}(A_{b_1} \cap [b_2, b_3])\}.$$

We will now show that S satisfies the assumption of Lemma 1. Let $n \in \mathbb{N}$ and $\varepsilon \in K_{>0}$. Choose $N \in \mathbb{N}$ so large such that $N^{-1} < \varepsilon$. By Lemma 6, there is $c \in K$ and $d_1, d_2 \in A_c$ such that

$$\nu_c(A_c \cap [d_1, d_2]) \subseteq \bigcup_{m \in [N, N+n]} (m, m + \frac{1}{m})$$

and

$$|\nu_c(A_c \cap [d_1, d_2]) \cap (m, m + \frac{1}{m})| = 1, \text{ for } m \in [N, N + n].$$

Since $N^{-1} < \varepsilon$, we get that

$$S_{(c,d_1,d_2)} \subseteq \bigcup_{m \leq n} (m - \varepsilon, m + \varepsilon)$$

and $|S_{(c,d_1,d_2)} \cap (m - \varepsilon, m + \varepsilon)| = 1$ for $m \leq n$. □

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