Vanishing viscosity limit for a coupled Navier-Stokes/Allen-Cahn system

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Abstract. In this paper, we study the vanishing viscosity limit for a coupled Navier-Stokes/Allen-Cahn system in a bounded domain. We first show the local existence of smooth solutions of the Euler/Allen-Cahn equations by modified Galerkin method. Then using the boundary layer function to deal with the mismatch of the boundary conditions between Navier-Stokes and Euler equations, and assuming that the energy dissipation for Navier-Stokes equation in the boundary layer goes to zero as the viscosity tends to zero, we prove that the solutions of the Navier-Stokes/Allen-Cahn system converge to that of the Euler/Allen-Cahn system in a proper small time interval. In addition, for strong solutions of the Navier-Stokes/Allen-Cahn system in 2D, the convergence rate is $c\nu^{1/2}$.

Keywords. Navier–Stokes, Euler, Allen–Cahn, vanishing viscosity limit
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1 Introduction

In this paper, we are concerned with the vanishing viscosity limit for the following Navier-Stokes/Allen-Cahn system in $\Omega \times (0, +\infty)$:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P_1 = \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi), \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = \gamma(\Delta \phi - f(\phi)), \tag{1.3}$$

with initial data

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \phi(x, 0) = \phi_0(x), \quad x \in \Omega, \tag{1.4}$$

and boundary conditions:

$$\mathbf{u}(x, t) = 0, \quad \partial_{\mathbf{n}}\phi(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty). \tag{1.5}$$

Here, $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$.

System (1.1)-(1.5) can be viewed as a phase field model, which describes the motion of a mixture of two incompressible viscous fluids with the same density and viscosity (see [13, 29]). The two fluids are macroscopically immiscible and separated by a thin interface. (1.1) is the linear momentum equation, where $\mathbf{u}$ is the velocity field of the mixture, ϕ and P_1 denote the phase function and pressure, respectively. $\nabla\phi \otimes \nabla\phi$ denotes the induced elastic stress, which is a

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$n \times n$ matrix whose (i,j)-th entry is $\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_j}$ for $1 \le i,j \le n$. (1.2) implies the incompressibility of both fluids in the mixture and (1.3) is the phase equation of Allen–Cahn type. $f(\phi) = \frac{4}{\varepsilon^2}(\phi^3 - \phi)$. ν, λ, γ, ε are positive constants, representing the kinematic viscosity, the surface tension, the mobility and the width of the interface, respectively. $\mathbf{u}|_{\partial\Omega} = 0$ is non-slip boundary condition. And $\partial_{\mathbf{n}}\phi|_{\partial\Omega} = 0$ means that the diffused interface perpendicularly contacts the boundary of the domain, where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$.

From another point of view, system (1.1)-(1.3) is closely related to liquid crystal model, Magnetohydrodynamics (MHD) equations, and viscoelastic system with infinite Weissenberg number, see [28].

System (1.1)-(1.5) has a basic energy law

$$\frac{1}{2}\frac{d}{dt}\left(|\mathbf{u}|^2 + \lambda|\nabla\phi|^2 + \frac{2\lambda}{\varepsilon^2}|(\phi^2-1)|^2\right) = -(\nu|\nabla\mathbf{u}|^2 + \lambda\gamma|\Delta\phi - f(\phi)|^2), \tag{1.6}$$

where $|\cdot|$ denotes the L^2-norm. (1.6) can be derived by multiplying (1.1) by $\mathbf{u}$, (1.3) by $\lambda(-\Delta\phi + f(\phi))$, then adding them up and integrating over Ω, also using (1.2) and (1.5). Thanks to this important energy law (1.6), it can be proved that global weak solutions of system (1.1)-(1.5) exist in both 2D and 3D case. With regular initial data, strong solutions exist globally in 2D, but locally in 3D in a short time interval $[0, T_\nu)$, where T_ν depends on ν. Hence we just consider weak solutions of (1.1)-(1.5) in the 3D case in the proof of vanishing viscosity limit.

At the limit case, namely $\nu = 0$, Navier-Stokes/Allen-Cahn system formally becomes the following Euler/Allen-Cahn system:

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla P_2 = -\lambda\nabla\cdot(\nabla\psi\otimes\nabla\psi), \tag{1.7}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1.8}$$

$$\psi_t + (\mathbf{v}\cdot\nabla)\psi = \gamma(\Delta\psi - f(\psi)), \tag{1.9}$$

in $\Omega \times (0, +\infty)$ with initial data

$$\mathbf{v}(x,0) = \mathbf{u}_0(x), \quad \psi(x,0) = \phi_0(x), \quad x \in \Omega, \tag{1.10}$$

and boundary conditions:

$$\mathbf{v}\cdot\mathbf{n}(x,t) = 0, \quad \partial_{\mathbf{n}}\psi(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \tag{1.11}$$

where $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0$ describes that the boundary is impermeable. The basic energy law of system (1.7)-(1.11) is

$$\frac{1}{2}\frac{d}{dt}\left(|\mathbf{v}|^2 + \lambda|\nabla\psi|^2 + \frac{2\lambda}{\varepsilon^2}|(\psi^2-1)|^2\right) = -\lambda\gamma|\Delta\psi - f(\psi)|^2. \tag{1.12}$$

The main purpose of this article is to show that the solution of the viscid system (1.1)-(1.5) converges to that of the inviscid system (1.7)-(1.11) in a short time period as the viscosity goes to zero. The inviscid limit helps us to understand turbulent phenomena governed by the viscous equations, see Navier-Stokes equations for an example (see e.g. [1, 7, 8, 19, 23, 24]).

Let us first recall the classical issue of vanishing viscosity limit of Navier-Stokes equations in a bounded domain. In order to deal with the mismatch of the boundary conditions between Navier-Stokes system and Euler system, Kato in [7] introduced a boundary layer function defined on a boundary trip $\Gamma_{c\nu}$ with width $c\nu$, and proved that the solution $\mathbf{u}^\nu$ of Navier-Stokes equation converges to that of Euler equation in L^2-norm, provided

$$\nu \int_0^T |\nabla \mathbf{u}^\nu(t)|^2_{L^2(\Gamma_{c\nu})} dt \to 0, \text{ as } \nu \to 0, \tag{1.13}$$

where $[0,T]$ is the interval on which the smooth solution of Euler equation exists. Later, Temam and Wang in [23] and [25] improved Kato's condition by replacing the total gradient in (1.13) with tangential derivatives only, but required a slightly thicker boundary layer.

With the same boundary layer function in [7], and under the same assumption (1.13), we can also prove that weak solution of (1.1)-(1.5) converges to smooth solution of (1.7)-(1.11) in the basic energy space in $[0,T]$ as the viscosity tends to zero. In particular, for 2D case, strong solution of (1.1)-(1.5) converges to smooth solution of (1.7)-(1.11) at the rate $c\nu^{1/2}$. The main difficulty in the proof is to deal with coupled terms of $\mathbf{u}$, $\mathbf{v}$, ϕ and ψ. A key point is that the highest order coupled terms can be canceled, similar to the derivation of the energy law (1.6). To estimate other coupled terms, high regularity of solutions to (1.7)-(1.11) is required. Therefore we first establish the local existence and uniqueness of smooth solutions to Euler/Allen-Cahn system (1.7)-(1.11). This proof requires the energy law (1.12), so we employ the modified Galerkin method introduced in [11], namely, we only project (1.7) into finite dimensional space, but solve (1.9) for any given $\mathbf{v}$ by the fixed point theorem, such that the approximate solutions still satisfies the energy law (1.12).

This article is organized as follows. In section 2, we introduce mathematical preliminaries and state our main results. In section 3, we establish the local existence and uniqueness of smooth solutions to the Euler/Allen-Cahn system. In section 4, we prove that the solution of the Navier-Stokes/Allen-Cahn equations converges to that of the Euler/Allen-Cahn equations under the condition (1.13). In the last section, we show some results for related systems.

2 Preliminaries and Main Results

Let $C^\infty_{0,div}(\Omega)$ be the space of all divergence free vectors in $(C^\infty_0(\Omega))^n$ $(n = 2,3)$. We denote by H the closure of $C^\infty_{0,div}(\Omega)$ in $(L^2(\Omega))^n$. Moreover, we set

$$V = (H^1_0(\Omega))^n \cap H, \qquad V_2 = (H^2(\Omega))^n \cap V,$$
$$\mathcal{W} = \{\phi \in C^\infty(\Omega),\ \partial_{\mathbf{n}}\phi|_{\partial\Omega} = 0\}, \qquad \Phi_s = \text{the closure of } \mathcal{W} \text{ in } H^s(\Omega),\ s \in \mathbb{N}^+,$$
$$X_s = \{\mathbf{v} \in (H^s(\Omega))^n |\ \nabla\cdot\mathbf{v} = 0, \text{ in } \Omega, \quad \mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \partial\Omega\},\ s \in \mathbb{N}.$$

In particular, $H^0(\Omega) = L^2(\Omega)$. Obviously, $H \subset X_0$ (continuous embedding), $X_s \subset\subset X_0$ (compact embedding) for $s \geq 1$.

Let Γ_δ be the boundary strip of width δ. We shall take $\delta = c\nu$ with $c > 0$ being a small constant. We denote $|\cdot|$ and $(\cdot,\cdot)$ as the norm and scalar product in $L^2(\Omega)$ or $(L^2(\Omega))^n$. For any positive integer s, we take $|\cdot|_s$ and $((\cdot,\cdot))_s$ as the norm and scalar product in $H^s(\Omega)$ or $(H^s(\Omega))^n$, $((f,g))_s = \sum_{|\alpha|\le s}(D^\alpha f, D^\alpha g)$ where D^α is a multi-index derivation. We also denote $|\cdot|_X$ as the norm in other Banach spaces X.

Following almost the same arguments as in [11], for any fixed $\nu > 0$, we have the following well-posedness result for the Navier-Stokes/Allen-Cahn system:

Theorem 2.1. *Assume that* $n = 2, 3$. $\mathbf{u}_0 \in H$ *and* $\phi_0 \in \Phi_1$. *Then the system* (1.1)-(1.5) *has a global weak solution* $(\mathbf{u}, \phi)$ *such that for all* $T \in (0, +\infty)$

$$\mathbf{u} \in L^\infty(0,T;H) \cap L^2(0,T;V),$$
$$\phi \in L^\infty(0,T;\Phi_1) \cap L^2(0,T;\Phi_2).$$

Moreover, if $\mathbf{u}_0 \in V$ *and* $\phi_0 \in \Phi_2$, *then in the* 2D *case, system* (1.1)-(1.5) *admits a unique global strong solution such that for all* $T \in (0, +\infty)$

$$\mathbf{u} \in L^\infty(0,T;V) \cap L^2(0,T;V_2), \tag{2.14}$$
$$\phi \in L^\infty(0,T;\Phi_2) \cap L^2(0,T;\Phi_3); \tag{2.15}$$

while in the 3D *case, there exists a* $T_\nu > 0$ *depending on* $|\mathbf{u}_0|_1$, $|\phi_0|_2$ *and* ν *such that* (1.1)-(1.5) *has a unique strong solution in* $[0, T_\nu)$ *and* (2.14)-(2.15) *hold for* $T < T_\nu$.

For weak solution $(\mathbf{u}, \phi)$ to system (1.1)-(1.5), we easily deduce from energy law (1.6) that for any $T > 0$,

$$\nu \int_0^T |\nabla \mathbf{u}|^2(t)dt \le C, \qquad \int_0^T |\phi|_2^2(t)dt \le C, \tag{2.16}$$

where C depends on $\mathbf{u}_0, \phi_0$ and $|\Omega|$.

We also prove the local existence of smooth solutions to Euler/Allen-Cahn system.

Theorem 2.2. *Assume that* $n = 2, 3$. $\mathbf{v}_0 \in X_s$, $\psi_0 \in \Phi_{s+1}$, *where* $s \ge 3$ *is an integer. Then there exists a small constant* $T_* > 0$ *depending on* $|\mathbf{v}_0|_s$ *and* $|\psi_0|_{s+1}$, *such that the system* (1.7)-(1.11) *admits a smooth solution* $(\mathbf{v}, P_2, \psi)$ *on* $[0, T_*]$ *satisfying*

$$\mathbf{v} \in L^\infty(0,T_*;X_s), \quad P_2 \in L^\infty(0,T_*;H^{s+1}(\Omega)), \quad \psi \in L^\infty(0,T_*;\Phi_{s+1}). \tag{2.17}$$

Moreover, $(\mathbf{v}, \psi)$ *is unique in the sense of* $\big(L^\infty(0,T_*;L^2(\Omega)), L^\infty(0,T_*;H^1(\Omega))\big)$*-norm.*

Finally, we show the convergence of solutions of Navier-Stokes/Allen-Cahn system to that of Euler/Allen-Cahn system.

Theorem 2.3. *Assume that* $n = 2, 3$. *Let* $(\mathbf{u}, \phi)$ *be a global weak solution to* (1.1)-(1.5), *and* $(\mathbf{v}, \psi)$ *a local smooth solution to* (1.7)-(1.11) *on* $\Omega \times [0, T_*]$. *Assume that*

$$\nu \int_0^{T_*} |\nabla \mathbf{u}|^2_{L^2(\Gamma_{c\nu})}(t)dt \to 0 \ \text{as} \ \nu \to 0. \tag{2.18}$$

Then

$$|\mathbf{u}(t)-\mathbf{v}(t)|^2+\lambda|\phi(t)-\psi(t)|_1^2 \to 0 \text{ as } \nu\to 0, \quad \text{for a.e. } t\in[0,T_*]. \tag{2.19}$$

Moreover, if $n=2$ *and* $(\mathbf{u},\phi)$ *is a global strong solution to* (1.1)-(1.5)*, then the assumption* (2.18) *is automatically satisfied, and the convergence rate in* (2.19) *is* $c\nu^{1/2}$*.*

Remark 2.1. *The above theorems remain true if we replace the Neumann boundary condition for* ϕ *and* ψ *with homogeneous Dirichlet boundary condition.*

3 Local smooth solutions to Euler/Allen-Cahn system

In this section, we give the proof of Theorem 2.2 in the 3D case. The 2D case can be proved similarly. We shall apply a modified Galerkin method, following the spirit of [11]. We choose an orthonormal complete basis $\{\mathbf{w}_k\}_{k=1}^{\infty}\subset X_s$ $(s\ge 3)$ used in [20] for Euler equation, which satisfies

$$\begin{cases} ((\mathbf{w}_k,\mathbf{v}))_s=\lambda_k(\mathbf{w}_k,\mathbf{v}), \quad \forall \mathbf{v}\in X_s. \\ (\mathbf{w}_k,\mathbf{w}_j)=\delta_{kj}, i.e., ((\mathbf{w}_k,\mathbf{w}_j))_s=\lambda_k\delta_{kj}. \end{cases} \tag{3.20}$$

Here, $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$ and $\lambda_n\to\infty$ as $n\to\infty$.

For fixed integer $m>0$, denote X_{sm} as the space spanned by $\{\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_m\}$. Let $P_m: X_0\to X_{sm}$ be the orthonormal projection. We consider the approximate problem:

$$\frac{\partial \mathbf{v}_m}{\partial t}=P_m\left(-(\mathbf{v}_m\cdot\nabla)\mathbf{v}_m-\lambda\nabla\cdot(\nabla\psi_m\otimes\nabla\psi_m)\right), \qquad \mathbf{v}_m\in X_{sm}, \tag{3.21}$$

$$\frac{\partial \psi_m}{\partial t}+(\mathbf{v}_m\cdot\nabla)\psi_m=\gamma\left(\Delta\psi_m-f(\psi_m)\right), \tag{3.22}$$

$$\mathbf{v}_m(0,x)=P_m\mathbf{v}_0(x), \quad \psi_m(0,x)=\psi_0(x), \tag{3.23}$$

$$\mathbf{v}_m\cdot\mathbf{n}|_{\partial\Omega}=0, \quad \partial_{\mathbf{n}}\psi_m|_{\partial\Omega}=0. \tag{3.24}$$

Here, we only project (1.7) into finite dimensional space X_{sm}, but do not project (1.9). We shall see later in (3.28) that, the approximate solutions $(\mathbf{v}_m,\psi_m)$ still satisfies energy law. This guarantees the global existence of $(\mathbf{v}_m,\psi_m)$.

We apply Schauder fixed point theorem to deduce the local existence of solutions to (3.21)-(3.24). The arguments are similar to that in [11, Therem 2.1]. For readers' convenience, we give the outline of the proof here. Consider a closed, convex subset in $(C[0,T_0])^m$:

$$D=\left\{(g_m^1,g_m^2,\cdots,g_m^m)\,\middle|\,\left(\sum_{i=1}^m|g_m^i(t)|^2\right)^{1/2}\le M,\ 0\le t\le T_0;\ g_m^i(0)=(\mathbf{v}_0,\mathbf{w}_i), i=1,2,\cdots,m.\right\}$$

where M and T_0 are positive constants to be determined later. Given a $(g_m^1,g_m^2,\cdots,g_m^m)\in D$, we get a $\mathbf{v}_m(x,t)=\sum_{i=1}^m g_m^i(t)\mathbf{w}_i(x)$ satisfying $|\mathbf{v}_m|_{L^\infty(\Omega)}\le c_mM$ for $t\in[0,T_0]$, where c_m is a constant depending on m. With this $\mathbf{v}_m$ in (3.22), there exists a regular solution ψ_m on $\Omega\times[0,T_0]$, and

$$|\nabla\psi_m|^2(t)\le e^{c_m^2M^2t}\left(|\nabla\psi_0|^2+2\int_\Omega F(\psi_0)dx\right)=: e^{c_m^2M^2t}C(|\psi_0|_1,|\Omega|), \quad t\in[0,T_0].$$

Then we substitute this ψ_m into (3.21) and look for a solution $\tilde{\mathbf{v}}_m = \sum_{i=1}^m \tilde{g}_m^i(t)\mathbf{w}_i(x)$. It is standard that (3.21) is equivalent to an ODE system of $\tilde{g}_m^i(t)$ $(i = 1, 2, \cdots, m)$, with initial data $\tilde{g}_m^i(0) = (\mathbf{v}_0, \mathbf{w}_i)$. There exists a local solution $(\tilde{g}_m^1, \tilde{g}_m^2, \cdots, \tilde{g}_m^m) \in (C^1[0, T_0])^m$. Moreover, it turns out that

$$\begin{aligned} \left(\sum_{i=1}^m |\tilde{g}_m^i(t)|^2\right)^{1/2} &\leq \left(\sum_{i=1}^m |\tilde{g}_m^i(0)|^2\right)^{1/2} + \frac{\lambda C(m, |\psi_0|_1, \Omega)}{M^2}\left(e^{c_m^2 M^2 t} - 1\right) \\ &= \left(\sum_{i=1}^m |(\mathbf{v}_0, \mathbf{w}_i)|^2\right)^{1/2} + \frac{\lambda C(m, |\psi_0|_1, \Omega)}{M^2}\left(e^{c_m^2 M^2 t} - 1\right) \end{aligned} \tag{3.25}$$

for $t \in [0, T_0]$. Let $M = 2\left(\sum_{i=1}^m |(\mathbf{v}_0, \mathbf{w}_i)|^2\right)^{1/2} + 2$. Then there exits a small T_0 such that the right hand side of (3.25) is not more than M. Therefore, $\left(\sum_{i=1}^m |\tilde{g}_m^i(t)|^2\right)^{1/2} \leq M$ for $t \in [0, T_0]$. Thus $(\tilde{g}_m^1, \tilde{g}_m^2, \cdots, \tilde{g}_m^m) \in D$. And the mapping $\mathcal{L} : (g_m^1, g_m^2, \cdots, g_m^m) \mapsto (\tilde{g}_m^1, \tilde{g}_m^2, \cdots, \tilde{g}_m^m)$ is a compact operator in $(C[0, T_0])^m$. Applying Schauder fixed point theorem, $\mathcal{L}$ has a fixed point. Thus we obtain the following lemma:

Lemma 3.1. *There exists a $T_0 > 0$, depending on $\mathbf{v}_0, \psi_0, m$ and Ω, such that* (3.21)-(3.24) *has a weak solution $(\mathbf{v}_m, \psi_m)$ in $\Omega \times [0, T_0]$.*

In order to show the global existence of weak solution to (3.21)-(3.24), we establish a priori estimates next. Assume $(\mathbf{v}_m, \psi_m)$ is a weak solution to (3.21)-(3.24) in $\Omega \times [0, T]$ for certain $T > 0$. Taking the inner product of (3.21) with $\mathbf{v}_m$ in $L^2(\Omega)$, using the fact that $((\mathbf{v}_m \cdot \nabla)\mathbf{v}_m, \mathbf{v}_m) = 0$ and $\nabla \cdot (\nabla\psi_m \otimes \nabla\psi_m) = \nabla\left(\frac{|\nabla\psi_m|^2}{2}\right) + \Delta\psi_m \nabla\psi_m$, we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{v}_m|^2 = -\lambda\left(\Delta\psi_m \nabla\psi_m, \mathbf{v}_m\right). \tag{3.26}$$

Taking the inner product of (3.22) with $\gamma\left(\Delta\psi_m - f(\psi_m)\right)$ in $L^2(\Omega)$, we have

$$\frac{d}{dt}\left(\frac{\lambda}{2}|\nabla\psi_m|^2 + \lambda\int_\Omega F(\psi_m)dx\right) + \lambda\gamma|\Delta\psi_m - f(\psi_m)|^2 = \lambda\left((\mathbf{v}_m \cdot \nabla)\psi_m, \Delta\psi_m\right), \tag{3.27}$$

where we used $((\mathbf{v}_m \cdot \nabla)\psi_m, f(\psi_m)) = \int_\Omega \mathbf{v}_m \cdot \nabla F(\psi_m) = 0$ and $F(\psi_m) = \frac{1}{\varepsilon^2}(\psi_m^2 - 1)^2$. Adding the above two resultant, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}|\mathbf{v}_m|^2 + \frac{\lambda}{2}|\nabla\psi_m|^2 + \lambda\int_\Omega F(\psi_m)dx\right) + \lambda\gamma|\Delta\psi_m - f(\psi_m)|^2 = 0. \tag{3.28}$$

Hence,

$$\sup_{0\leq t\leq T}\left(|\mathbf{v}_m|^2(t) + \lambda|\nabla\psi_m|^2(t) + \lambda\int_\Omega F(\psi_m)dx\right) \leq |\mathbf{v}_0|^2 + \lambda|\nabla\psi_0|^2 + \lambda\int_\Omega F(\psi_0)dx. \tag{3.29}$$

By Lemma 3.1 and the above inequality, we have

Theorem 3.1. *For any integer $m > 0$, $\mathbf{v}_0 \in X_0$, $\psi_0 \in \Phi_1$, system* (3.21)-(3.24) *has a weak solution $(\mathbf{v}_m, \psi_m)$ in $\Omega \times [0, \infty)$.*

In what follows, we do high order energy estimates, so that we can pass to limits as $m \to \infty$. Firstly, since $\mathbf{v}_m \in X_{sm}$, we can rewrite (3.21) as

$$\frac{\partial \mathbf{v}_m}{\partial t} = -(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m - \lambda \Delta\psi_m \nabla\psi_m - \nabla q_m, \tag{3.30}$$

for a certain q_m. By taking divergence operator on both sides of (3.30) on Ω, and taking scalar product of both sides of (3.30) with $\mathbf{n}$ on $\partial\Omega$, we know that q_m satisfies

$$\begin{aligned} \Delta q_m &= -\lambda \nabla \cdot (\Delta\psi_m \nabla\psi_m) - \sum_{i,j} \nabla_j \mathbf{v}_{mi} \nabla_i \mathbf{v}_{mj}, \text{ in } \Omega \\ \frac{\partial q_m}{\partial \mathbf{n}} &= -\lambda \Delta\psi_m \frac{\partial \psi_m}{\partial \mathbf{n}} + \sum_{i,j} \mathbf{v}_{mi}\mathbf{v}_{mj}\Gamma_{ij}, \text{ on } \partial\Omega \end{aligned}$$

where the function Γ_{ij} depends only on $\partial\Omega$ (see [20, Lemma 1.1]). By elliptic estimates (see also [20, Lemma 1.2]), for $s \geq \frac{5}{2}$,

$$|\nabla q_m|_s \leq C\left(|\Delta\psi_m|_s |\nabla\psi_m|_s + |\mathbf{v}_m|_s^2\right). \tag{3.31}$$

Taking inner product of (3.30) with $\lambda_k g_{km} \mathbf{w}_k$ in $L^2(\Omega)$, and adding in k, $k = 1, 2 \cdots, m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|\mathbf{v}_m|_s^2 = ((-(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m - \lambda\Delta\psi_m\nabla\psi_m - \nabla q_m, \mathbf{v}_m))_s, \tag{3.32}$$

For $s \geq 3$, we estimate the righthand side of (3.32) to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\mathbf{v}_m|_s^2 &\leq |((\mathbf{v}_m \cdot \nabla\mathbf{v}_m, \mathbf{v}_m))_s| + |\lambda((\Delta\psi_m\nabla\psi_m, \mathbf{v}_m))_s| + |((\nabla q_m, \mathbf{v}_m))_s| \\ &\leq C|\mathbf{v}_m|_s^3 + C\lambda|\Delta\psi_m|_s|\nabla\psi_m|_s|\mathbf{v}_m|_s + |\nabla q_m|_s|\mathbf{v}_m|_s \\ &\leq C|\mathbf{v}_m|_s^3 + \frac{\lambda\gamma}{8}|\Delta\psi_m|_s^2 + C|\nabla\psi_m|_s^4 + C|\mathbf{v}_m|_s^4 + C\left(|\Delta\psi_m|_s|\nabla\psi_m|_s + |\mathbf{v}_m|_s^2\right)|\mathbf{v}_m|_s \\ &\leq \frac{\lambda\gamma}{4}|\Delta\psi_m|_s^2 + C|\nabla\psi_m|_s^4 + C|\mathbf{v}_m|_s^4 + C, \end{aligned} \tag{3.33}$$

where we used $((\mathbf{v}_m \cdot \nabla)\nabla^s\mathbf{v}_m, \nabla^s\mathbf{v}_m) = 0$ and (3.31).

Remark 3.1. *In the 2D case, $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$, $\forall \varepsilon > 0$. $|((\mathbf{v}_m \cdot \nabla\mathbf{v}_m, \mathbf{v}_m))_s|$ in (3.33) can be controlled by $|\mathbf{v}_m|_s^\alpha$ optimally, where $2 < \alpha < 3$. This term makes it difficult to obtain global existence of smooth solutions.*

Applying ∇^i ($i = 1, 2, \cdots, s$; $s \geq 3$) to (3.22) (here, $\nabla^i = \nabla\Delta^{(i-1)/2}$ if i is odd, $\nabla^i = \Delta^{i/2}$ if i is even), then taking the L^2 inner product of the resulting equation with $\lambda\nabla^i\Delta\psi_m$, and using the boundary condition $\frac{\partial\psi_m}{\partial\mathbf{n}}|_{\partial\Omega} = 0$ and higher order natural boundary conditions such as $\frac{\partial\Delta\psi_m}{\partial\mathbf{n}}|_{\partial\Omega} = 0$, we obtain

$$\frac{\lambda}{2}\frac{d}{dt}|\nabla^{i+1}\psi_m|^2 + \lambda\gamma|\nabla^i\Delta\psi_m|^2 = \lambda\left(\nabla^i(\mathbf{v}_m \cdot \nabla)\psi_m, \nabla^i\Delta\psi_m\right) + \lambda\gamma\left(\nabla^i f(\psi_m), \nabla^i\Delta\psi_m\right).$$

Adding the above resultant in i ($i = 1, 2, \cdots, s$; $s \geq 3$), we get

$$\frac{\lambda}{2}\frac{d}{dt}|\nabla\psi_m|_s^2 + \lambda\gamma|\Delta\psi_m|_s^2 = \lambda\sum_{i=1}^{s}\left(\nabla^i(\mathbf{v}_m \cdot \nabla)\psi_m, \nabla^i\Delta\psi_m\right) + \lambda\gamma\sum_{i=1}^{s}\left(\nabla^i f(\psi_m), \nabla^i\Delta\psi_m\right). \tag{3.34}$$

Next we estimate the righthand side terms.

$$
\begin{aligned}
& \lambda \left| \sum_{i=1}^{s} \left(\nabla^i (\mathbf{v}_m \cdot \nabla)\psi_m, \nabla^i \Delta\psi_m \right) \right| \\
\leq\ & \sum_{i=1}^{s} \lambda \left(|\nabla^i \mathbf{v}_m| |\nabla\psi_m|_{L^\infty} + |\mathbf{v}_m|_{L^\infty} |\nabla^{i+1}\psi_m| \right) |\nabla^i \Delta\psi_m| \\
\leq\ & \sum_{i=1}^{s} \lambda \left(|\nabla^i \mathbf{v}_m| |\nabla\psi_m|_2 + |\mathbf{v}_m|_2 |\nabla^{i+1}\psi_m| \right) |\nabla^i \Delta\psi_m| \\
\leq\ & \sum_{i=1}^{s} \left[\frac{\lambda\gamma}{4} |\nabla^i \Delta\psi_m|^2 + C|\nabla^i \mathbf{v}_m|^4 + C|\nabla\psi_m|_2^4 + C|\mathbf{v}_m|_2^4 + C|\nabla^{i+1}\psi_m|^4 \right] \\
\leq\ & \frac{\lambda\gamma}{8} |\Delta\psi_m|_s^2 + C|\mathbf{v}_m|_s^4 + C|\nabla\psi_m|_s^4, \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
& \lambda\gamma \left(\nabla^i f(\psi_m), \nabla^i \Delta\psi_m \right) \\
=\ & \frac{4\lambda\gamma}{\varepsilon^2} \left(\nabla^i (\psi_m^3), \nabla^i \Delta\psi_m \right) - \frac{4\lambda\gamma}{\varepsilon^2} \left(\nabla^i \psi_m, \nabla^i \Delta\psi_m \right) \\
=\ & \frac{12\lambda\gamma}{\varepsilon^2} \left(\psi_m^2 \nabla^i \psi_m, \nabla^i \Delta\psi_m \right) + \sum \left(C\nabla^{i_1}\psi_m \nabla^{i_2}\psi_m \nabla^{i_3}\psi_m, \nabla^i \Delta\psi_m \right) \\
& - \frac{4\lambda\gamma}{\varepsilon^2} \left(\nabla^i \psi_m, \nabla^i \Delta\psi_m \right) \\
=:\ & I_1 + I_2 + I_3, \qquad (3.36)
\end{aligned}
$$

where Σ represents the sum with respect to integers i_1, i_2, i_3 satisfying $0 \leq i_1 \leq i_2 \leq i_3 \leq i-1$ and $i_1 + i_2 + i_3 = i$.

Using the fact that ψ_m is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$ and $H^1 \subset L^6$, we get

$$
\begin{aligned}
|I_1| \leq\ & C|\psi_m|_{L^6}^2 |\nabla^i \psi_m|_{L^6} |\nabla^i \Delta\psi_m| \\
\leq\ & C|\psi_m|_1^2 |\nabla^i \psi_m|_1 |\nabla^i \Delta\psi_m| \\
\leq\ & C|\nabla\psi_m|_i |\nabla^i \Delta\psi_m| \\
\leq\ & \frac{\lambda\gamma}{8} |\nabla^i \Delta\psi_m|^2 + C|\nabla\psi_m|_i^2, \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
|I_2| \leq\ & C \sum |\nabla^{i_1}\psi_m|_{L^6} |\nabla^{i_2}\psi_m|_{L^6} |\nabla^{i_3}\psi_m|_{L^6} |\nabla^i \Delta\psi_m| \\
\leq\ & C \sum |\nabla^i \Delta\psi_m| \prod_{k=1}^{3} \left(C|\nabla\psi_m|^{1-a_k} |\nabla^i \Delta\psi_m|^{a_k} + C|\nabla\psi_m| \right) \\
\leq\ & C \sum |\nabla^i \Delta\psi_m| \left(|\nabla^i \Delta\psi_m|^{\frac{i}{i+1}} + 1 \right), \\
\leq\ & \frac{\lambda\gamma}{8} |\nabla^i \Delta\psi_m|^2 + C, \qquad (3.38)
\end{aligned}
$$

where we used the Gagliardo–Nirenberg interpolation inequalities (cf. [30]):

$$
|\nabla^{i_k}\psi_m|_{L^6} \leq C|\nabla\psi_m|^{1-a_k} |\nabla^i \Delta\psi_m|^{a_k} + C|\nabla\psi_m|, \quad a_k = \frac{i_k}{i+1}, \quad k = 1,2,3.
$$

and $a_1 + a_2 + a_3 = \frac{i}{i+1} < 1$.

$$|I_3| \le C|\nabla^i \psi_m||\nabla^i \Delta\psi_m| \le \frac{\lambda\gamma}{8}|\nabla^i \Delta\psi_m|^2 + C|\nabla^i \psi_m|^2. \tag{3.39}$$

Using the above estimates, we conclude from (3.34) that

$$\frac{\lambda}{2}\frac{d}{dt}|\nabla\psi_m|_s^2 + \frac{\lambda\gamma}{2}|\Delta\psi_m|_s^2 \le C|\nabla\psi_m|_s^4 + C|\mathbf{v}_m|_s^4 + C. \tag{3.40}$$

Adding (3.33) and (3.40), we conclude

$$\frac{1}{2}\frac{d}{dt}\left(|\mathbf{v}_m|_s^2 + \lambda|\nabla\psi_m|_s^2\right) + \frac{\lambda\gamma}{4}|\Delta\psi_m|_s^2 \le C|\mathbf{v}_m|_s^4 + C|\nabla\psi_m|_s^4 + C. \tag{3.41}$$

Denote $Y_m(t) = |\mathbf{v}_m|_s^2 + \lambda|\nabla\psi_m|_s^2$. (3.41) can be rewritten as

$$\frac{d}{dt}Y_m(t) \le C_1 Y_m^2(t) + C_2,$$

$$Y_m(0) = |\mathbf{v}_m|_s^2(0) + \lambda|\nabla\psi_m|_s^2(0) \le |\mathbf{v}_0|_s^2 + \lambda|\nabla\psi_0|_s^2.$$

Hence there exists a T_* depending only on C_1, C_2, and $|\mathbf{v}_0|_s$, $|\psi_0|_{s+1}$, such that $Y_m(t) \le N$ on $[0, T_*]$, where $N > 0$ is a constant independent of m. Therefore, as $m \to +\infty$,

$$\mathbf{v}_m \text{ remains bounded in } L^\infty(0, T_*; X_s), \tag{3.42}$$

$$\psi_m \text{ remains bounded in } L^\infty(0, T_*; \Phi_{s+1})). \tag{3.43}$$

By the equation (3.21) and (3.22), it is easy to know that

$$\frac{\partial \mathbf{v}_m}{\partial t} \text{ remains bounded in } L^\infty(0, T_*; X_{s-1}), \tag{3.44}$$

$$\frac{\partial \psi_m}{\partial t} \text{ remains bounded in } L^\infty(0, T_*; \Phi_{s-1}). \tag{3.45}$$

Using (3.42)-(3.45) and the standard compact embedding theorem, the passage to the limit is standard. We obtain the existence of $\mathbf{v} \in L^\infty(0, T_*; X_s)$, $\psi \in L^\infty(0, T_*; \Phi_{s+1})$ such that

$$\left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\xi}\right) = \left(-(\mathbf{v}\cdot\nabla)\mathbf{v} - \lambda\nabla\cdot(\nabla\psi\otimes\nabla\psi), \boldsymbol{\xi}\right), \quad \forall \boldsymbol{\xi} \in X_0, \quad \text{a.e. } t \in [0, T_*],$$

$$(\frac{\partial \psi}{\partial t}, \varphi) + ((\mathbf{v}_m\cdot\nabla)\psi, \varphi) = -\gamma(\nabla\psi, \nabla\varphi) - (f(\psi), \varphi), \quad \forall \varphi \in H^1(\Omega), \quad \text{a.e. } t \in [0, T_*].$$

The proof of global existence of smooth solutions to (1.7)-(1.11) is finished.

It is standard to prove the uniqueness of smooth solutions to (1.7)-(1.11). We omit it here. The proof of Theorem 2.2 is complete.

4 Vanishing viscosity limit

In this section, we prove Theorem 2.3. In the proof, C is a constant independent of ν.

To deal with the mismatch between the boundary condition of $\mathbf{u}$ and that of $\mathbf{v}$, we use the boundary layer function $\boldsymbol{\theta}$ constructed in [7]: $\boldsymbol{\theta} = div\left(\chi\left(\frac{\rho}{\delta}\right)\mathbf{A}\right)$ where $\rho = dist(\mathbf{x}, \partial\Omega)$ is a distance function, δ is a small positive constant, and χ is a smooth cut-off function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\chi(0) = 1, \quad \chi(r) = 0 \text{ for } r \geq 1.$$

$\mathbf{A}$ is a skew-symmetric matrix defined on $\Omega \times [0, T]$, such that

$$div\mathbf{A} = \mathbf{v} \text{ on } \partial\Omega, \ \mathbf{A} = 0 \text{ on } \partial\Omega. \tag{4.46}$$

Obviously, $\boldsymbol{\theta}$ is supported on Γ_δ. It was proved in [7] that $\boldsymbol{\theta}$ satisfies

$$div\boldsymbol{\theta} = 0 \text{ in } \Omega, \quad \boldsymbol{\theta} = \mathbf{v} \text{ on } \partial\Omega,$$

and $\boldsymbol{\theta}$ has the following estimates:

$$|\boldsymbol{\theta}|_{L^\infty} \leq C, \quad |\boldsymbol{\theta}| \leq C\delta^{\frac{1}{2}}, \quad |\boldsymbol{\theta}|_{L^4} \leq C\delta^{\frac{1}{4}}, \quad |\boldsymbol{\theta}_t| \leq C\delta^{\frac{1}{2}}, \tag{4.47}$$

$$|\nabla\boldsymbol{\theta}|_{L^\infty} \leq C\delta^{-1}, \quad |\nabla\boldsymbol{\theta}| \leq C\delta^{-\frac{1}{2}}, \quad |\rho\nabla\boldsymbol{\theta}|_{L^\infty} \leq C, \tag{4.48}$$

$$|\rho^2\nabla\boldsymbol{\theta}|_{L^\infty} \leq C\delta, \quad |\rho\nabla\boldsymbol{\theta}| \leq C\delta^{\frac{1}{2}}, \tag{4.49}$$

provided that $\mathbf{v}$ has the regularity:

$$|\nabla^2\mathbf{v}|_{L^\infty} \leq C, \quad |\nabla\mathbf{v}_t|_{L^\infty} \leq C, \quad |\mathbf{v}_t|_{L^\infty} \leq C. \tag{4.50}$$

Suppose $(\mathbf{u}, \phi)$ is a global weak solution to (1.1)-(1.5), and $(\mathbf{v}, \psi)$ is a local smooth solution to (1.7)-(1.11) on $\Omega \times [0, T_*]$. Denote $\mathbf{w} = \mathbf{u} - \mathbf{v} + \boldsymbol{\theta}$, $\zeta = \phi - \psi$, then $\mathbf{w}, \zeta$ satisfy the following equations:

$$\begin{aligned}
\mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} + \nabla q &= \nu\Delta\mathbf{u} - (\mathbf{w}\cdot\nabla)\mathbf{v} + (\boldsymbol{\theta}\cdot\nabla)\mathbf{v} + (\mathbf{u}\cdot\nabla)\boldsymbol{\theta} + \boldsymbol{\theta}_t \\
&\quad -\lambda\Delta\zeta\nabla\phi - \lambda\Delta\psi\nabla\zeta, && (4.51)\\
\nabla\cdot\mathbf{w} &= 0, && (4.52)\\
\zeta_t + (\mathbf{v}\cdot\nabla)\zeta + (\mathbf{w}\cdot\nabla)\phi &= \gamma\Delta\zeta + (\boldsymbol{\theta}\cdot\nabla)\phi - \frac{4\gamma}{\varepsilon^2}\zeta(\phi^2 + \phi\psi + \psi^2) + \frac{4\gamma}{\varepsilon^2}\zeta, && (4.53)\\
\mathbf{w}|_{t=0} = (\mathbf{u}_0 - \mathbf{v}_0 + \boldsymbol{\theta})(x), &\qquad \zeta|_{t=0} = (\phi_0 - \psi_0)(x), \quad x \in \Omega, && (4.54)\\
\mathbf{w}|_{\partial\Omega}(x,t) = 0, &\qquad \partial_{\mathbf{n}}\zeta|_{\partial\Omega}(x,t) = 0, \quad t \geq 0, && (4.55)
\end{aligned}$$

where $q = P_1 - P_2 + \frac{\lambda}{2}|\nabla\phi|^2 - \frac{\lambda}{2}|\nabla\psi|^2$.

Our goal is to estimate $|\mathbf{w}|(t)$ and $|\zeta|_{H^1}(t)$, we hope they go to zero, equivalently, $|\mathbf{u} - \mathbf{v}|(t)$ and $|\phi - \psi|_{H^1}(t)$ converge to zero as ν tends to zero.

Multiplying (4.51) by $\mathbf{w}$ and integrating over Ω, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\mathbf{w}|^2(t) + \nu|\nabla\mathbf{u}|^2 &= \nu(\nabla\mathbf{u}, \nabla(\mathbf{v} - \boldsymbol{\theta})) - ((\mathbf{w}\cdot\nabla)\mathbf{v}, \mathbf{w}) + ((\boldsymbol{\theta}\cdot\nabla)\mathbf{v}, \mathbf{w}) + ((\mathbf{u}\cdot\nabla)\boldsymbol{\theta}, \mathbf{u}) \\
&\quad - ((\mathbf{u}\cdot\nabla)\boldsymbol{\theta}, \mathbf{v}) + (\boldsymbol{\theta}_t, \mathbf{w}) - \lambda(\Delta\zeta\nabla\phi, \mathbf{w}) - \lambda(\Delta\psi\nabla\zeta, \mathbf{w}). \qquad (4.56)
\end{aligned}$$

Multiplying (4.53) by $\lambda\zeta$ and integrating over Ω, we get

$$\frac{\lambda}{2}\frac{d}{dt}|\zeta|^2 + \lambda\gamma|\nabla\zeta|^2 + \frac{4\lambda\gamma}{\varepsilon^2}\int_\Omega \zeta^2\left(\phi^2 + \phi\psi + \psi^2\right)dx$$

$$= \frac{4\lambda\gamma}{\varepsilon^2}|\zeta|^2 - \lambda(\mathbf{w}\cdot\nabla\phi,\zeta) + \lambda(\boldsymbol{\theta}\cdot\nabla\phi,\zeta). \tag{4.57}$$

Notice that $\phi^2+\phi\psi+\psi^2 = \frac{1}{2}\left(\phi^2+\psi^2+(\phi+\psi)^2\right) \geq 0$, thus $\int_\Omega \zeta^2\left(\phi^2+\phi\psi+\psi^2\right)dx \geq 0$. Multiplying (4.53) by $\lambda\Delta\zeta$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{\lambda}{2}\frac{d}{dt}|\nabla\zeta|^2 + \lambda\gamma|\Delta\zeta|^2 &= \lambda\left((\mathbf{v}\cdot\nabla)\zeta,\Delta\zeta\right) + \lambda\left((\mathbf{w}\cdot\nabla)\phi,\Delta\zeta\right) - \lambda\left((\boldsymbol{\theta}\cdot\nabla)\phi,\Delta\zeta\right)\\ &\quad + \frac{4\lambda\gamma}{\varepsilon^2}\left(\zeta(\phi^2+\phi\psi+\psi^2),\Delta\zeta\right) + \frac{4\lambda\gamma}{\varepsilon^2}|\nabla\zeta|^2. \end{aligned}\tag{4.58}$$

We observe that the term $-\lambda\left(\Delta\zeta\nabla\phi,\mathbf{w}\right)$ in (4.56) cancels out $\lambda\left((\mathbf{w}\cdot\nabla)\phi,\Delta\zeta\right)$ in (4.58), which is originated from the energy law (1.6). This is very important, otherwise we cannot control the two terms.

Next we estimate the other terms on the right-hand side of (4.56)-(4.58), using (4.47)-(4.49).

$$\begin{aligned}|\nu\left(\nabla\mathbf{u},\nabla(\mathbf{v}-\boldsymbol{\theta})\right)| &\leq \nu|\nabla\mathbf{u}||\nabla\mathbf{v}| + \nu|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)}|\nabla\boldsymbol{\theta}|_{L^2(\Gamma_\delta)}\\ &\leq \frac{\nu}{4}|\nabla\mathbf{u}|^2 + \nu|\nabla\mathbf{v}|^2 + C\nu\delta^{-\frac{1}{2}}|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)}, \end{aligned}\tag{4.59}$$

$$|((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{w})| \leq |\nabla\mathbf{v}|_{L^\infty}|\mathbf{w}|^2, \tag{4.60}$$

$$\begin{aligned}|((\boldsymbol{\theta}\cdot\nabla)\mathbf{v},\mathbf{w})| &\leq |\boldsymbol{\theta}||\nabla\mathbf{v}|_{L^\infty}|\mathbf{w}| \leq C\delta^{\frac{1}{2}}|\nabla\mathbf{v}|_{L^\infty}|\mathbf{w}|\\ &\leq C|\mathbf{w}|^2 + \delta|\nabla\mathbf{v}|^2_{L^\infty}, \end{aligned}\tag{4.61}$$

$$|((\mathbf{u}\cdot\nabla)\boldsymbol{\theta},\mathbf{u})| = \left|\int_{\Gamma_\delta}\left(\frac{1}{\rho}\mathbf{u}\cdot\rho^2\nabla\right)\boldsymbol{\theta}\frac{1}{\rho}\mathbf{u}dx\right| \leq \left|\frac{1}{\rho}\mathbf{u}\right|^2_{L^2(\Gamma_\delta)}\left|\rho^2\nabla\boldsymbol{\theta}\right|_{L^\infty(\Gamma_\delta)} \leq C\delta|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)},$$

$$|((\mathbf{u}\cdot\nabla)\boldsymbol{\theta},\mathbf{v})| = \left|\int_{\Gamma_\delta}\left(\frac{1}{\rho}\mathbf{u}\cdot\rho\nabla\right)\boldsymbol{\theta}\mathbf{v}dx\right| \leq \left|\frac{1}{\rho}\mathbf{u}\right|_{L^2(\Gamma_\delta)}|\rho\nabla\boldsymbol{\theta}|\,|\mathbf{v}|_{L^\infty(\Gamma_\delta)} \leq C\delta^{\frac{1}{2}}|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)}|\mathbf{v}|_{L^\infty(\Gamma_\delta)},$$

where we used (4.49) and the well-known inequality of Hardy-Littlewood (since $\mathbf{u}\in H_0^1(\Omega)$).

$$|(\boldsymbol{\theta}_t,\mathbf{w})| \leq |\boldsymbol{\theta}_t||\mathbf{w}| \leq C\delta^{\frac{1}{2}}|\mathbf{w}| \leq C|\mathbf{w}|^2 + C\delta, \tag{4.62}$$

$$\begin{aligned}|\lambda\left(\Delta\psi\nabla\zeta,\mathbf{w}\right)| &\leq \lambda|\Delta\psi|_{L^4}|\nabla\zeta|_{L^4}|\mathbf{w}|\\ &\leq \begin{cases} C\lambda|\Delta\psi|_{L^4}|\nabla\zeta|^{\frac{1}{2}}|\Delta\zeta|^{\frac{1}{2}}|\mathbf{w}| & (\text{if } n=2)\\ C\lambda|\Delta\psi|_{L^4}|\nabla\zeta|^{\frac{1}{4}}|\Delta\zeta|^{\frac{3}{4}}|\mathbf{w}| & (\text{if } n=3)\end{cases}\\ &\leq \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C|\nabla\zeta|^2 + C|\Delta\psi|^2_{L^4}|\mathbf{w}|^2, \end{aligned}\tag{4.63}$$

$$\begin{aligned}|\lambda(\mathbf{w}\cdot\nabla\phi,\zeta)| &\leq \lambda|\mathbf{w}||\nabla\phi||\zeta|_{L^\infty}\\ &\leq \begin{cases} C|\mathbf{w}||\nabla\phi||\zeta|^{\frac{1}{2}}|\Delta\zeta|^{\frac{1}{2}} & (\text{if } n=2)\\ C|\mathbf{w}||\nabla\phi||\zeta|^{\frac{1}{4}}|\Delta\zeta|^{\frac{3}{4}} & (\text{if } n=3)\end{cases}\\ &\leq \begin{cases} \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C|\mathbf{w}|^2 + C|\nabla\phi|^4|\zeta|^2 & (\text{if } n=2)\\ \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C|\mathbf{w}|^2 + C|\nabla\phi|^8|\zeta|^2 & (\text{if } n=3)\end{cases} \end{aligned}\tag{4.64}$$

$$|\lambda(\boldsymbol{\theta}\cdot\nabla\phi,\zeta)| \leq \lambda|\boldsymbol{\theta}||\nabla\phi||\zeta|_{L^\infty}$$

$$
\begin{aligned}
&\leq \begin{cases} C\delta^{\frac{1}{2}}|\nabla\phi||\zeta|^{\frac{1}{2}}|\Delta\zeta|^{\frac{1}{2}} & \text{(if } n=2) \\ C\delta^{\frac{1}{2}}|\nabla\phi||\zeta|^{\frac{1}{4}}|\Delta\zeta|^{\frac{3}{4}} & \text{(if } n=3) \end{cases} \\
&\leq \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C\delta|\nabla\phi|^2 + C|\zeta|^2, \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\mathbf{v}\cdot\nabla)\zeta,\Delta\zeta)| &\leq \lambda|\mathbf{v}|_{L^4}|\nabla\zeta|_{L^4}|\Delta\zeta| \\
&\leq \begin{cases} C\lambda|\mathbf{v}|_{L^4}|\nabla\zeta|^{\frac{1}{2}}|\Delta\zeta|^{\frac{3}{2}} & \text{(if } n=2) \\ C\lambda|\mathbf{v}|_{L^4}|\nabla\zeta|^{\frac{1}{4}}|\Delta\zeta|^{\frac{7}{4}} & \text{(if } n=3) \end{cases} \\
&\leq \begin{cases} \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C|\mathbf{v}|_{L^4}^4|\nabla\zeta|^2, & \text{(if } n=2) \\ \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C|\mathbf{v}|_{L^4}^8|\nabla\zeta|^2, & \text{(if } n=3) \end{cases} \qquad (4.66)
\end{aligned}
$$

$$
\begin{aligned}
|\lambda((\boldsymbol{\theta}\cdot\nabla)\phi,\Delta\zeta)| &\leq \lambda|\boldsymbol{\theta}|_{L^4(\Gamma_\delta)}|\nabla\phi|_{L^4(\Gamma_\delta)}|\Delta\zeta| \leq \lambda\delta^{\frac{1}{4}}|\nabla\phi|_{L^4(\Gamma_\delta)}|\Delta\zeta| \\
&\leq \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C\delta^{\frac{1}{2}}|\nabla\phi|_{L^4(\Gamma_\delta)}^2, \qquad (4.67)
\end{aligned}
$$

$$
\begin{aligned}
\frac{4\lambda\gamma}{\varepsilon^2}\left|\left(\zeta(\phi^2+\phi\psi+\psi^2),\Delta\zeta\right)\right| &\leq \frac{4\lambda\gamma}{\varepsilon^2}|\Delta\zeta||\zeta|_{L^6}|\phi^2+\phi\psi+\psi^2|_{L^3} \\
&\leq C|\Delta\zeta||\zeta|_{H^1}\left(|\phi|_{L^6}^2+|\psi|_{L^6}^2\right) \\
&\leq C|\Delta\zeta||\zeta|_{H^1}\left(|\phi|_{H^1}^2+|\psi|_{H^1}^2\right) \\
&\leq \frac{\lambda\gamma}{8}|\Delta\zeta|^2 + C\left(|\phi|_{H^1}^4+|\psi|_{H^1}^4\right)|\zeta|_{H^1}^2. \qquad (4.68)
\end{aligned}
$$

Adding (4.56), (4.57) and (4.58), then using the estimates above and taking $\delta = c\nu$, we finally obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(|\mathbf{w}|^2(t)+\lambda|\zeta|_1^2(t)\right) &\leq \left(C+|\nabla\mathbf{v}|_{L^\infty}+C|\Delta\psi|_{L^4}^2\right)|\mathbf{w}|^2 + C\left(1+|\mathbf{v}|_{L^4}^8+|\phi|_1^8+|\psi|_1^4\right)|\zeta|_1^2 \\
&+ \nu\left(C+|\nabla\mathbf{v}|^2+|\nabla\mathbf{v}|_{L^\infty}^2+C|\nabla\phi|^2\right) + C\nu^{\frac{1}{2}}\left(1+|\mathbf{v}|_{L^\infty(\Gamma_\delta)}\right)|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)} \\
&+ C\nu|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)}^2 + C\nu^{\frac{1}{2}}|\nabla\phi|_{L^4(\Gamma_\delta)}^2. \qquad (4.69)
\end{aligned}
$$

Denote

$$
\begin{aligned}
Y(t) &= |\mathbf{w}|^2+\lambda|\zeta|_1^2, \\
a(t) &= \nu\left(C+|\nabla\mathbf{v}|^2+|\nabla\mathbf{v}|_{L^\infty}^2+C|\nabla\phi|^2\right), \\
b(t) &= C\nu^{\frac{1}{2}}\left(1+|\mathbf{v}|_{L^\infty(\Gamma_\delta)}\right)|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)} + C\nu|\nabla\mathbf{u}|_{L^2(\Gamma_\delta)}^2 + C\nu^{\frac{1}{2}}|\nabla\phi|_{L^4(\Gamma_\delta)}^2.
\end{aligned}
$$

Since $(\mathbf{v},\psi)$ is a smooth local solution on $[0,T_*]$ and ϕ is bounded in $L^\infty(0,T_*;H^1(\Omega))$, the inequality (4.69) becomes

$$
\frac{dY(t)}{dt} \leq CY(t)+a(t)+b(t), \qquad (4.70)
$$

where $a(t)\leq C\nu$ and $Y(0)=|\boldsymbol{\theta}|^2(0)$. Recall (4.47) that $|\boldsymbol{\theta}|(t)\leq C\nu^{1/2}$. Hence,

$$
\sup_{0\leq t\leq T_*} Y(t) \leq e^{CT_*}\left(|\boldsymbol{\theta}|^2(0)+C\nu T_*+\int_0^{T_*} b(t)dt\right)
$$

$$\leq \quad e^{CT_*}\left(C\nu + C\nu T_* + \int_0^{T_*} b(t)dt\right), \tag{4.71}$$

where

$$\begin{aligned}\int_0^{T_*} b(t)dt \quad &\leq \quad CT_*^{\frac{1}{2}}\left(\nu\int_0^{T_*}|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)}dt\right)^{\frac{1}{2}} + C\nu\int_0^{T_*}|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)}dt + C\nu^{\frac{1}{2}}\int_0^{T_*}|\nabla\phi|^2_{L^4(\Gamma_\delta)}dt \\ &\leq \quad CT_*^{\frac{1}{2}}\left(\nu\int_0^{T_*}|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)}dt\right)^{\frac{1}{2}} + C\nu\int_0^{T_*}|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)}dt + C\nu^{\frac{1}{2}}, \end{aligned} \tag{4.72}$$

where we used (2.16) and the fact $H^1(\Omega) \subset L^4(\Omega)$. Assume

$$\nu\int_0^{T_*}|\nabla\mathbf{u}|^2_{L^2(\Gamma_\delta)}dt \to 0 \text{ as } \nu \to 0. \tag{4.73}$$

Then combining (4.71) and (4.72), we have

$$\sup_{0\leq t\leq T_*} Y(t) \to 0 \text{ as } \nu \to 0, \tag{4.74}$$

which equals to

$$\sup_{0\leq t\leq T_*}\left(|\mathbf{u}-\mathbf{v}|^2(t) + \lambda|\phi-\psi|^2_{H^1}(t)\right) \leq \sup_{0\leq t\leq T_*} Y(t) + |\boldsymbol{\theta}|^2(t) \to 0 \text{ as } \nu \to 0. \tag{4.75}$$

In 2D case, if $(\mathbf{u}, \phi)$ is a global strong solution to (1.1)-(1.5), then $|\nabla\mathbf{u}|^2(t)$ is uniformly bounded in $[0, \infty)$, thus the assumption (4.73) is automatically satisfied. (4.72) is simplified to

$$\int_0^{T_*} b(t)dt \leq C\nu^{1/2} \text{ for } \nu \in (0,1). \tag{4.76}$$

Then, using (4.76), (4.71) becomes

$$\sup_{0\leq t\leq T_*} Y(t) \leq e^{CT_*}\left(C\nu + C\nu T_* + C\nu^{1/2}\right) \leq C\nu^{1/2} \text{ for } \nu \in (0,1).$$

Therefore,

$$\sup_{0\leq t\leq T_*}\left(|\mathbf{u}-\mathbf{v}|^2(t) + \lambda|\phi-\psi|^2_{H^1}(t)\right) \leq \sup_{0\leq t\leq T_*} Y(t) + |\boldsymbol{\theta}|^2(t) \leq C\nu^{1/2} \text{ for } \nu \in (0,1). \tag{4.77}$$

The proof is complete.

5 Related Models and Results

Our system (1.1)-(1.3) is closely related to liquid crystal model, Magnetohydrodynamics (MHD) equations, and viscoelastic system (see [28] and the reference therein). In this section, we apply our previous approach to these models.

5.1 Liquid Crystal

If taking ϕ as a vector, say $\mathbf{d}$, system (1.1)-(1.3) can model the motion of liquid crystal flows:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} - \lambda\nabla\cdot(\nabla\mathbf{d}\otimes\nabla\mathbf{d}), \tag{5.78}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{5.79}$$

$$\mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{d} = \gamma(\Delta\mathbf{d} - f(\mathbf{d})), \tag{5.80}$$

where $\mathbf{d}$ represents the director field of liquid crystal molecules.

We consider the above system in 2D in an infinite channel $\mathbb{R}\times(0,1)$. We assume the velocity field $\mathbf{u}$ and director field $\mathbf{d}$ of liquid crystals are periodic with period 2π in the horizontal (x) direction. Set $\Omega = (0,2\pi)\times(0,1)$. $\mathbf{u}$ vanishes and the liquid crystal molecules align in a fixed direction $\mathbf{d}^*$ at the boundary of the channel (i.e. at $y=0,1$),

$$\mathbf{u} = 0, \quad \mathbf{d} = \mathbf{d}^*, \quad \text{on } (0,2\pi)\times\{0,1\}. \tag{5.81}$$

(5.78)-(5.80) is subject to initial data

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x,y), \quad \mathbf{d}|_{t=0} = \mathbf{d}_0(x,y), \text{ in } \Omega. \tag{5.82}$$

The above system is a simplified version of the Ericksen-Leslie model for the hydrodynamics of liquid crystals (cf. [2, 3, 6, 10]). Following exactly the same arguments as in [11], one obtains the well-posedness results of system (5.78)-(5.82).

The corresponding invisid liquid crystal model is

$$\mathbf{u}_t^{(0)} + (\mathbf{u}^{(0)}\cdot\nabla)\mathbf{u}^{(0)} = -\nabla P - \lambda\nabla\cdot(\nabla\mathbf{d}^{(0)}\otimes\nabla\mathbf{d}^{(0)}), \tag{5.83}$$

$$\nabla\cdot\mathbf{u}^{(0)} = 0, \tag{5.84}$$

$$\mathbf{d}_t^{(0)} + (\mathbf{u}^{(0)}\cdot\nabla)\mathbf{d}^{(0)} = \gamma(\Delta\mathbf{d}^{(0)} - f(\mathbf{d}^{(0)})), \tag{5.85}$$

$$\mathbf{u}^{(0)}, \mathbf{d}^{(0)} \text{ are } 2\pi\text{-periodic in } x, \qquad \mathbf{u}^{(0)}\cdot\mathbf{n} = 0, \quad \mathbf{d}^{(0)} = \mathbf{d}^*, \quad \text{on } (0,2\pi)\times\{0,1\}, \tag{5.86}$$

$$\mathbf{u}^{(0)}|_{t=0} = \mathbf{u}_0(x,y), \qquad \mathbf{d}^{(0)}|_{t=0} = \mathbf{d}_0(x,y), \text{ in } \Omega. \tag{5.87}$$

Following almost the same proof as that in Theorem 2.2 and Theorem 2.3, we have the follows.

Theorem 5.1. *Let $s \geq 3$ be an integer. Assume $\mathbf{u}_0 \in X_s$, $\mathbf{d}_0 \in H^{s+1}$. Then there exists a constant $T_* > 0$ depending on $|\mathbf{u}_0|_s$ and $|\mathbf{d}_0|_{s+1}$, such that system* (5.83)-(5.87) *has a smooth solution $(\mathbf{u}^{(0)}, P, \mathbf{d}^{(0)})$ in $[0,T_*]$ satisfying*

$$\mathbf{u}^{(0)} \in L^\infty(0,T_*;X_s), \quad P \in L^\infty(0,T_*;H^{s+1}(\Omega)), \quad \mathbf{d}^{(0)} \in L^\infty(0,T_*;H^{s+1}).$$

Moveover, $\left(\mathbf{u}^{(0)}, \mathbf{d}^{(0)}\right)$ is unique.

Theorem 5.2. *Let $(\mathbf{u},\mathbf{d})$ be a global strong solution to* (5.78)-(5.82)*, and $(\mathbf{u}^{(0)}, \mathbf{d}^{(0)})$ a local smooth solution to* (5.83)-(5.87) *on $\Omega\times[0,T_*]$. Then*

$$|\mathbf{u}(t) - \mathbf{u}^{(0)}(t)|^2 + \lambda|\mathbf{d}(t) - \mathbf{d}^{(0)}(t)|_1^2 \to 0 \text{ as } \nu\to 0, \quad \forall t\in[0,T_*]. \tag{5.88}$$

The convergence rate in (5.88) *is $c\nu^{1/2}$ for small $\nu\in(0,1)$.*

5.2 The MHD equations

The nondimensional form of viscous incompressible MHD equations is (cf. [18])

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - S(\mathbf{B}\cdot\nabla)\mathbf{B} + \nabla p &= \frac{1}{Re}\Delta\mathbf{u}, &(5.89)\\ \mathbf{B}_t + (\mathbf{u}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{u} &= \frac{1}{Rm}\Delta\mathbf{B}, &(5.90)\\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{B} = 0, & &(5.91)\end{aligned}$$

where $\mathbf{u}$, p and $\mathbf{B}$ denote the fluid velocity, its pressure and the magnetic field, respectively. Let L_*, U_*, B_*, ρ_* be the characteristic values for lengths, velocities, magnetic fields and densities of the fluid. The Reynolds number $R_e = \frac{L_* U_*}{\nu}$ where ν is the kinematic viscosity; the magnetic Reynolds number $R_m = L_* U_* \sigma\mu$ where μ is the magnetic permeability and σ the resistivity of the fluid; $S = \frac{M^2}{R_e R_m}\left(= \frac{B_*^2}{\mu\rho_* U_*^2}\right)$, where M is the Harmann number.

Due to $\nabla\cdot\mathbf{B} = 0$, we know in $2D$ case there exists a scalar function ϕ such that

$$\mathbf{B} = \nabla^{\mathbf{T}}\phi = \begin{pmatrix} -\partial_2\phi \\ \partial_1\phi \end{pmatrix},$$

Hence the above system in $2D$ is equivalent to the following equations

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \frac{1}{R_e}\Delta\mathbf{u} - S\nabla\cdot(\nabla\phi\otimes\nabla\phi), &(5.92)\\ \nabla\cdot\mathbf{u} &= 0, &(5.93)\\ \phi_t + (\mathbf{u}\cdot\nabla)\phi &= \frac{1}{R_m}\Delta\phi. &(5.94)\end{aligned}$$

Compared with the system (1.1)-(1.3), the nonlinear term $f(\phi)$ vanishes in (5.94).

Let $\nu \to 0$ and other parameters remain constants, we have $R_e \to +\infty$ and S, R_m are constants. System (5.92)-(5.94) formally becomes

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla P &= -S\nabla\cdot(\nabla\psi\otimes\nabla\psi), &(5.95)\\ \nabla\cdot\mathbf{v} &= 0, &(5.96)\\ \psi_t + (\mathbf{v}\cdot\nabla)\psi &= \frac{1}{R_m}\Delta\psi. &(5.97)\end{aligned}$$

We consider the vanishing viscosity limit of system (5.92)-(5.94) in the entire plane $\mathbb{R}^2$ as $R_e \to +\infty$. Omitting the terms with boundary layer function $\boldsymbol{\theta}$ in the proof of Theorem 2.3, and using the same estimates, we easily get similar result as follows.

Theorem 5.3. *Assume $\Omega = \mathbb{R}^2$. Let $(\mathbf{u}, \phi)$ be a global strong solution to* (5.92)-(5.94)*, and $(\mathbf{v}, \psi)$ a local smooth solution to* (5.95)-(5.97) *on $\mathbb{R}^2 \times [0, T_*]$. $(\mathbf{u}, \phi)$ and $(\mathbf{v}, \psi)$ are equipped with the same initial data. Then*

$$|\mathbf{u}(t) - \mathbf{v}(t)|^2 + \lambda|\phi(t) - \psi(t)|_1^2 \to 0 \text{ as } R_e \to +\infty, \quad \forall t \in [0, T_*]. \qquad (5.98)$$

The convergence rate in (5.98) *is cR_e^{-1}.*

Remark 5.1. *In the literature, inviscid limit for MHD equations has been studied as both* R_e *and* R_m *go to infinity, see [26, 27]. But in some industrial cases (see for example [4]),* $R_e \approx 10^5, R_m \approx 10^{-1}, S \approx 1$. *Hence it is also meaningful to consider the inviscid limit for MHD equations as only* $R_e \to +\infty$.

5.3 Viscoelasticity

We consider the following system describing the motion of incompressible viscoelastic fluids (cf. [16, 17]):

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \lambda \nabla \cdot \left(\frac{\partial W(\mathcal{F})}{\partial \mathcal{F}} \mathcal{F}^T\right), \tag{5.99}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{5.100}$$

$$\mathcal{F}_t + \mathbf{u} \cdot \nabla F = \nabla \mathbf{u} \mathcal{F}. \tag{5.101}$$

Here, $\mathbf{u}$ represents the velocity field of materials, p the pressure, ν the kinetic viscosity constant, and λ a parameter representing the competition between kinetic energy and elastic energy. $\mathcal{F}$ is the deformation tensor, and $W(\mathcal{F})$ the elastic energy functional.

Remark 5.2. *The dynamic of viscoelastic fluids can be described by the flow map. Let* X *be the original labeling (Lagrangian) coordinate of the particle, and* x *the current (Eulerian) coordinate. For a given velocity* $\mathbf{u}(x,t)$, *the flow map (particle trajectory)* $x(X,t)$ *is defined by the ODE :*

$$x_t = \mathbf{u}(x(X,t),t), \quad x(X,0) = X.$$

The deformation tensor $\tilde{\mathcal{F}}(X,t)$ *is defined as* $\tilde{\mathcal{F}}(X,t) = \frac{\partial x}{\partial X}$, *where we use the notation* $\tilde{\mathcal{F}}_{ij} = \frac{\partial x_i}{\partial X_j}$. *In the Eulerian coordinate, we can define* $\mathcal{F}(x,t) = \mathcal{F}(x(X,t),t) = \tilde{\mathcal{F}}(X,t)$. *The equation* (5.101) *is deduced by the chain rule. The equation* (5.99) *can be derived by the energetic variational approach, see [16, 28] for intrinsic mechanics in detail.*

Taking divergence of both sides of (5.101) and using $\nabla \cdot \mathbf{u} = 0$, we get the transport equation for $\nabla \cdot \mathcal{F}$ as

$$(\nabla \cdot \mathcal{F})_t + \mathbf{u} \cdot \nabla (\nabla \cdot \mathcal{F}) = 0. \tag{5.102}$$

Since $\mathcal{F}$ is the identity matrix at $t = 0$, $\nabla \cdot \mathcal{F}|_{t=0} = 0$, then (5.102) tells that

$$\nabla \cdot \mathcal{F} = 0, \quad \forall t \geq 0. \tag{5.103}$$

In 2D, this implies there exists a vector function $\boldsymbol{\phi} = (\phi_1, \phi_2)$, such that

$$\mathcal{F} = \nabla^T \boldsymbol{\phi} = \begin{pmatrix} -\partial_2 \boldsymbol{\phi} \\ \partial_1 \boldsymbol{\phi} \end{pmatrix}. \tag{5.104}$$

If we take $W(\mathcal{F})$ as the Hookean elasticity energy, i.e., $W(\mathcal{F}) = \frac{1}{2} tr(\mathcal{F}\mathcal{F}^T)$, and use (5.104), then the system (5.99)-(5.101) is equivalent to

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla q + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \boldsymbol{\phi} \otimes \nabla \boldsymbol{\phi}), \tag{5.105}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{5.106}$$

$$\boldsymbol{\phi}_t+(\mathbf{u}\cdot\nabla)\boldsymbol{\phi} = 0, \tag{5.107}$$

where the basic energy law is

$$\frac{1}{2}\frac{d}{dt}\left(|\mathbf{u}|^2+\lambda|\nabla\boldsymbol{\phi}|^2\right)=-\nu|\nabla\mathbf{u}|^2. \tag{5.108}$$

Due to the system has partial dissipation, it is difficult to obtain global classical solution in general. Consider the system (5.105)-(5.107) in the entire plane $\mathbb{R}^2$. The existence of global smooth solutions near equilibrium has been proved in [16]. Moreover, the authors proved the local existence of a classical solution near equilibrium to the inviscid system in $\mathbb{R}^2$:

$$\begin{aligned}\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla q-\lambda\nabla\cdot(\nabla\boldsymbol{\phi}\otimes\nabla\boldsymbol{\phi}),\\ \nabla\cdot\mathbf{u} &= 0,\\ \boldsymbol{\phi}_t+(\mathbf{u}\cdot\nabla)\boldsymbol{\phi} &= 0,\end{aligned}$$

by rewritting the above system as a quasilinear first order system.

Due to the lack of dissipation term in the equation for ϕ, we cannot show the validity of vanishing viscosity limit for system (5.105)-(5.107), by similar proof to that of Theorem 2.3. But we believe that the inviscid limit holds for solutions near equilibrium. We are going to investigate this problem in forthcoming work.

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