Reconstruction in quantum field theory with a fundamental length

Dedicated to the memory of Professor V. Ya. Fainberg

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Abstract

In this paper, we establish an analog of Wightman's reconstruction theorem for nonlocal quantum field theory with a fundamental length. In our setting, the Wightman generalized functions are defined on test functions analytic in a complex ℓ-neighborhood of the real space and are localizable at scales large compared to ℓ. The causality condition is formulated as continuity of the field commutator in an appropriate topology associated with the light cone. We prove that the relevant function spaces are nuclear and derive the kernel theorems for the corresponding classes of multilinear functionals, which provides the basis for the reconstruction procedure. Special attention is given to the accurate determination of the domain of the reconstructed quantum fields in the Hilbert space of states. We show that the primitive common invariant domain must be suitably extended to implement the (quasi)localizability and causality conditions.

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I. Introduction

The reconstruction theorem occupies a central position in the Wightman axiomatic approach [1] to quantum field theory (QFT). This theorem establishes the conditions under which a collection of tempered distributions is the set of vacuum expectation values of some local field theory. Moreover, it gives an explicit procedure for constructing the corresponding field operators in a Hilbert space of states. This reconstruction procedure is also applicable to nontempered distributions (known as ultradistributions), provided that the space of test functions on which they act contains a dense set of functions of compact support. In that event, the reconstructed fields belong to the Jaffe class [2] of strictly localizable fields, for which the local commutativity axiom, called also microcausality, can be formulated in the usual terms. The aim of this paper is to derive a reconstruction theorem for a larger class of functionals that are defined on test functions analytic in a complex ℓ-neighborhood of the real space. Accordingly, the momentum-space growth of these functionals is bounded by the exponential $\exp(\ell|p|)$. As shown in [3], such functionals are localizable at length scales large compared to ℓ and hence ℓ can be regarded as a fundamental length. A generalization of Wightman's approach to the nonlocal field theories with an exponential behavior of expectation values in momentum space was first proposed by Iofa and Fainberg [4, 5]. They replaced the microcausality condition (for the case of a single scalar field) with the requirement of symmetry of the analytic Wightman functions under the permutations of their arguments. Subsequently it was shown [6] that a natural way of formulating causality in nonlocal field theories with analytic test functions is by using a suitably adapted notion of carrier of an analytic functional. This notion plays a large part in various questions of complex and functional analysis and is basic for the Sato-Martineau theory of hyperfunctions, see, e.g., Hörmander's treatise [7].

Although the key ideas of nonlocal QFT with the exponential high-energy behavior of expectation values are contained in [4, 5, 6], no consideration has been given there to the features of reconstruction of nonlocal fields. We find it useful to close this gap because there is an intriguing intimate connection between nonlocal field theories of this kind, string theory [8] holographic models [9], and field theories on noncommutative spacetime [10, 11]. In deriving the nonlocal version of reconstruction theorem, we will take as starting point the up-to-date formulation [12] of quantum field theory with a fundamental length in terms of vacuum expectation values. A different strategy for reconstructing nonlocal fields was proposed by Brüning and Nagamachi [13]. While some observations made in [13] are very valuable, the definition of extended local commutativity introduced in that paper is misleading. This definition modifies the definition of quasilocality given in [6] and was motivated by the need to adapt the latter to the nonlocal model $:e^{g\phi^2}:(x)$, where ϕ is a free neutral scalar field. However a closer examination [12] shows that the analyticity domain of the n-point functions of this model is considerably larger than that found in [13] and, as a consequence, the nonlocal field $:e^{g\phi^2}:(x)$ fits in the original framework [4, 5, 6]. It is possible even to include a wider class of normal ordered functions of the free field, see Sec. VII below. Furthermore, the modified definition uses a projective limit as $\ell \to \infty$, which makes doubtful that the theory [13] has the local limit when the fundamental length approaches zero.

In the conventional formalism of local QFT, the basic test function space is taken to be the Schwartz space S of infinitely differentiable functions of fast decrease. An important point is that the space S is nuclear. This property is used in deriving most, if not all, of the results of the axiomatic approach [1, 14, 15] and is crucial for constructing the completed tensor algebra over the space $S(\mathbb{R}^4)$, which is an essential ingredient of Wightman's reconstruction theorem. For this reason, we begin the derivation of the nonlocal analog of this theorem with proving that the function spaces [6, 12] which replace S in QFT with a fundamental length are nuclear. Notice that our method of proving nuclearity is quite general and is also applicable to the function spaces [10] used in field theory on noncommutative space-time. The central problem with reconstructing nonlocal fields is finding the exact formulation of causality in terms of the fields, which must be equivalent to the quasilocality condition [6, 12] stated in terms of the Wightman generalized functions. It turns out that solving this problem requires an appropriate extension of the primitive common invariant domain of the reconstructed field operators in the Hilbert space.

The paper is organized as follows. In Sec. II, the basic definitions are given and the main result is stated. In Sec. III, we prove that the function spaces introduced in [6] and [12] are nuclear. The kernel theorems for the corresponding classes of multilinear functionals are established in Sec. IV. Making use of these results, we prove in Sec. V the reconstruction theorem for QFT with a fundamental length. Particular attention is given to the precise formulation of the quasilocality condition for the reconstructed fields, which substitutes for the microcausality axiom of local QFT. In Sec. VI, we show the equivalence of this condition to that given in [6, 12] in terms of the vacuum expectation values. Section VII contains concluding remarks. Appendices A and B present the proofs of some auxiliary statements about topological tensor products of locally convex vector spaces.

II. Basic definitions and the main result

A tempered distribution u on $\mathbb{R}^d$ is said to be supported in a closed set $M \subset \mathbb{R}^d$ if $(u, f) = 0$ for all test functions $f \in S(\mathbb{R}^d)$ that vanish in a neighborhood of M. Clearly, this definition is inapplicable in the case of analytic test functions. To obtain a suitable generalization, it is natural to consider instead the property that (u, f) tends to zero as f approaches zero in a neighborhood of M. But to formalize this simple idea, we must assign a topology to every open subset of $\mathbb{R}^d$, or in other words, define a presheaf of topological vector spaces. We recall the corresponding definitions introduced in [3, 6] for nonlocal QFT with a fundamental length.

Let ℓ be a positive number or $+\infty$. For each set $O \subset \mathbb{R}^d$, we define a space $A_\ell(O)$ in the following way. Let $\tilde{O}^l$ be the complex l-neighborhood of O, consisting of those points $z \in \mathbb{C}^d$ for which there is $x \in O$ such that $|z - x| \equiv \max_{1\le j\le d} |z_j - x_j| < l$. The space $A_\ell(O)$ consists of all analytic functions f on $\tilde{O}^\ell$ with the property that

$$\|f\|_{O,l,N} \stackrel{\text{def}}{=} \max_{|\kappa|\le N} \sup_{z\in\tilde{O}^l} |z^\kappa f(z)| < \infty \qquad \text{for all} \quad l < \ell \quad \text{and for any integer} \quad N. \tag{1}$$

The system of norms (1) is equivalent to the countable system $\|f\|_{O,\ell-1/N,N}$ and hence $A_\ell(O)$ is a metrizable locally convex space. We note that the same space corresponds to the closure $\bar{O}$ of O. Clearly, every element of $A_\ell(O)$ is completely determined by its values at real points. The space $A_\ell(\mathbb{R}^d)$ is naturally embedded in each of $A_\ell(O)$, $O \subset \mathbb{R}^d$, via the restriction map. Let $A'_\ell(\mathbb{R}^d)$ denote the topological dual of $A_\ell(\mathbb{R}^d)$, i.e., the space of all continuous linear functionals defined on $A_\ell(\mathbb{R}^d)$. According to what has been said, a closed set $M \subset \mathbb{R}^d$ can be regarded as a *carrier* of $u \in A'_\ell(\mathbb{R}^d)$, if u has a continuous extension to $A_\ell(M)$. By the Hahn-Banach theorem, this property is equivalent to the continuity of u in the topology induced on $A_\ell(\mathbb{R}^d)$ by that of $A_\ell(M)$. We refer the reader to [3, 6] for a more detailed motivation of this definition and for an explanation why the elements of $A'_\ell(\mathbb{R}^d)$ can be thought of as localizable in $\mathbb{R}^d$ at scales large compared to ℓ. For $O = \mathbb{R}$ and $\ell = 1/(eB) < \infty$, the space $A_\ell(O)$ coincides with the space $S^{1,B}$ in the notation used by Gelfand and Shilov [16]. As shown in the Secs. III and IV, the topological properties of the spaces $A_\ell(O)$ are similar to those of the Schwartz space S, which makes them convenient for use. In particular, these spaces are nuclear. Of prime importance is the property expressed by

$$A_\ell(O_1) \mathbin{\hat{\otimes}} A_\ell(O_2) = A_\ell(O_1 \times O_2), \tag{2}$$

where the superimposed hat is used to denote the completion of the tensor product with respect to its natural topology. We notice that, in general, a distinction should be made between the projective topology and the inductive topology of a tensor product, but in this case they coincide because any $A_\ell(O)$ is a Fréchet space. It follows from (2) that every separately continuous bilinear functional on $A_\ell(O_1) \times A_\ell(O_2)$ can be identified with a linear functional (i.e., with a generalized function) on $A_\ell(O_1 \times O_2)$. This statement is an analog of the famous Schwartz kernel theorem (called sometimes nuclear theorem) for S'. We shall also consider spaces with different indices and use the more general formula

$$A_{\ell_1}(O_1) \mathbin{\hat{\otimes}} A_{\ell_2}(O_2) = A_{\ell_1,\ell_2}(O_1 \times O_2),$$

where the space on the right-hand side consists of all analytic functions on $\tilde{O}_1^{\ell_1} \times \tilde{O}_2^{\ell_2}$ such that the norms

$$\|f\|_{O_1\times O_2, l_1, l_2, N} = \max_{|\kappa|\le N} \sup_{z\in \tilde{O}_1^{l_1}\times \tilde{O}_2^{l_2}} |z^\kappa f(z)|, \quad l_1 < \ell_1, \quad l_2 < \ell_2, \quad N = 0, 1, 2, \dots,$$

are finite. It should be noted that the spaces $A_\ell(O)$ with $\ell < \infty$ are not invariant under the linear coordinate transformations and $A_\infty(\mathbb{R}^d)$ is their maximal invariant subspace.

We content ourselves with proving a reconstruction theorem for the case of a scalar hermitian nonlocal field and proceed from the formulation [12] of nonlocal QFT in terms of the Wightman generalized functions. Before we begin, a few words about notation. As usual, we denote by $\mathcal{P}_+^\uparrow$ the proper orthochronous Poincaré group and by $f_{(a,\Lambda)}$ the function obtained by applying a transformation $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ to $f \in A_\infty(\mathbb{R}^{4n})$,

$$f_{(a,\Lambda)}(x_1, \dots, x_n) = f(\Lambda^{-1}(x_1 - a), \dots, \Lambda^{-1}(x_n - a)).$$

The open cone $\{x \in \mathbb{R}^4 \colon x^2 = (x^0)^2 - (\mathbf{x})^2 > 0\}$ is denoted by $\mathbb{V}$ and its upper (or forward) component is denoted by $\mathbb{V}^+$. We also use a "hat" notation to denote the Fourier transforms of functions and functionals. Our starting point is a set of analytic functionals $\{\mathcal{W}_n\}$ satisfying the following conditions.

$(a.1)$ *Initial functional domain.*

$$\mathcal{W}_n \in A'_\infty(\mathbb{R}^{4n}) \qquad \text{for} \quad n \ge 1.$$

(a.2) *Hermiticity.*

$$\overline{(\mathcal{W}_n, f)} = (\mathcal{W}_n, f^\dagger) \quad \text{for each } f \in A_\infty(\mathbb{R}^{4n}), \quad \text{with } f^\dagger(z_1, \dots, z_n) = \overline{f(\bar{z}_n, \dots, \bar{z}_1)}.$$

(a.3) *Positive definiteness.*

$$\sum_{k,m=0}^{N} (\mathcal{W}_{k+m}, f_k^\dagger \otimes f_m) \geq 0,$$

where $\mathcal{W}_0 = 1$, $f_0 \in \mathbb{C}$, and $\{f_1, \dots, f_N\}$ is an arbitrary finite set of test functions such that $f_k \in A_\infty(\mathbb{R}^{4k})$, $k = 1, \dots, N$.

(a.4) *Poincaré covariance.*

$$(\mathcal{W}_n, f) = (\mathcal{W}_n, f_{(a,\Lambda)}) \qquad \text{for all} \quad f \in A_\infty(\mathbb{R}^{4n}) \quad \text{and} \quad \text{for each} \quad (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

This property is equivalent to the existence of Lorentz invariant functionals $W_n \in A'_\infty(\mathbb{R}^{4(n-1)})$ such that

$$\mathcal{W}_n(x_1, \dots x_n) = W_n(x_1 - x_2, \dots, x_n - x_{n-1}), \quad n \geq 1.$$

(a.5) *Spectral condition.*

$$\operatorname{supp} \hat{W}_n \subset \underbrace{\bar{\mathbb{V}}^+ \times \cdots \times \bar{\mathbb{V}}^+}_{(n-1)}.$$

(a.6) *Cluster decomposition property.* If a is a spacelike vector, then for each $f \in A_\infty(\mathbb{R}^{4k})$ and for each $g \in A_\infty(\mathbb{R}^{4m})$,

$$(\mathcal{W}_{k+m}, f \otimes g_{(\lambda a, I)}) \longrightarrow (\mathcal{W}_k, f)(\mathcal{W}_m, g) \qquad \text{as} \quad \lambda \to \infty.$$

(a.7.1) *Quasilocalizability.* There exists $\ell < \infty$ such that every functional W_n has a continuous extension to the space $A_\ell(\mathbb{R}^{4(n-1)})$.

(a.7.2) *Quasilocality.* For any $n \geq 2$ and $1 \leq k \leq n-1$, the difference

$$W_n(\xi_1, \dots, \xi_{k-1}, \xi_k, \xi_{k+1}, \dots, \xi_{n-1}) \\ - W_n(\xi_1, \dots, \xi_{k-1} + \xi_k, -\xi_k, \xi_k + \xi_{k+1}, \dots, \xi_{n-1}) \quad (3)$$

extends continuously to the space $A_\ell(V_{(k)})$, where

$$V_{(k)} = \{\xi \in \mathbb{R}^{4(n-1)} \colon \xi_k^2 > 0\}. \quad (4)$$

It is significant that conditions (a.7.1) and (a.7.2) are formulated with respect to the relative coordinates $\xi_k = x_k - x_{k+1}$, $k = 1, \dots n-1$, see [12] for a discussion of this point. The reconstruction theorem will be divided into three parts represented by Theorems 1–3, whose proofs are given in Sec. V.

Theorem 1: *Let $\{\mathcal{W}_n\}$, $n = 1, 2, \dots,$ be a sequence of analytic functionals satisfying conditions (a.1)–(a.6). Then there exist a separable Hilbert space $\mathcal{H}$, a continuous unitary* *representation $U(a,\Lambda)$ of the group $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$, a unique state Ψ_0 invariant under $U(a,\Lambda)$, and a hermitian scalar field φ with an invariant dense domain $D \subset \mathcal{H}$ such that*

$$\langle \Psi_0, \varphi(f_1)\dots\varphi(f_n)\Psi_0\rangle = (\mathcal{W}_n, f_1\otimes\dots\otimes f_n), \qquad f_j \in A_\infty(\mathbb{R}^4), \quad j=1,\dots,n.$$

The field φ obeys all Wightman's axioms except for locality and with $A_\infty(\mathbb{R}^4)$ instead of the Schwartz space. Any other relativistic quantum field theory with the same vacuum expectation values is unitary equivalent to this one.

Since we have an analog of Schwartz's kernel theorem, the proof of Theorem 1 is completely analogous to that of the corresponding part of the Wightman reconstruction theorem [1].

Theorem 2: *From condition (a.7.1) it follows that the operator-valued generalized function $\varphi(f)$ defined on D by Theorem 1 with $f \in A_\infty(\mathbb{R}^4)$ extends continuously to the space $A_{\ell/2}(\mathbb{R}^4)$. Moreover, this condition is equivalent to the fact that every monomial $\varphi(f_1)\dots\varphi(f_n)$, $n \geq 1$, can be uniquely extended to an operator-valued generalized function over the space $A_\ell^{(r)}(\mathbb{R}^{4n})$ consisting of all functions of the form*

$$g^{(r)}(x_1,\dots,x_n) = g(x_1, x_1-x_2,\dots,x_{n-1}-x_n), \quad \text{where } g \in A_{\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4(n-1)}). \tag{5}$$

(The index r indicates changing to the relative coordinates.)

Remark 1: The extension is unique because $A_\infty(\mathbb{R}^{4n})$ is dense in $A_{\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4(n-1)})$. A simple proof of the fact that $A_\infty(\mathbb{R}^d)$ is dense in $A_\ell(\mathbb{R}^d)$ for any ℓ and d is given in Appendix of [12], and after minor changes in the notation, this proof applies also to the spaces $A_{\ell_1,\ell_2}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$, $d_1+d_2=d$. Besides the space $A_\infty(\mathbb{R}^{4n})$ is invariant under linear transformations of $\mathbb{R}^{4n}$, therefore it is dense in $A_\ell^{(r)}(\mathbb{R}^{4n})$.

Corollary: The operator-valued generalized function $\varphi(f)$ can be uniquely extended, with preserving hermiticity and the continuity in $f \in A_{\ell/2}(\mathbb{R}^4)$, to the domain D_ℓ spanned by D and all vectors of the form

$$\Psi(g^{(r)}) = \int dx_1\dots dx_n\, g^{(r)}(x_1,\dots,x_n)\prod_{i=1}^n \varphi(x_i)\Psi,$$

where $g \in A_{3\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4(n-1)})$ and $\Psi \in D$.

Proof: If $f \in A_{\ell/2}(\mathbb{R}^4)$ and $g \in A_{3\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4(n-1)})$, then the function

$$h(x,x_1,\dots,x_n) = f(x)g(x-x_1,x_2,\dots,x_n)$$

belongs to $A_{\ell/2,\ell}(\mathbb{R}^4\times\mathbb{R}^{4n})$ and hence $h^{(r)} = f\otimes g^{(r)}$ belongs to $A_\ell^{(r)}(\mathbb{R}^{4(n+1)})$. Therefore, we can define an extension of $\varphi(f)$ to D_ℓ by $\Psi(g^{(r)}) \to \Psi(f\otimes g^{(r)})$ with subsequent extension by linearity. However, we must show that $\Psi(g^{(r)})=0$ implies $\Psi(f\otimes g^{(r)})=0$. To this end, we approximate f by a sequence $f_\nu \in A_\infty(\mathbb{R}^4)$ and g by a sequence $g_\nu \in A_\infty(\mathbb{R}^{4n})$. Let Φ be an arbitrary element of D. Using the hermiticity of $\varphi(f_\nu)$ and Theorem 2, we obtain

$$\langle\Phi, \Psi(f_\nu\otimes g_\nu^{(r)})\rangle = \langle\varphi(f_\nu^\dagger)\Phi, \Psi(g_\nu^{(r)})\rangle \to \langle\varphi(f^\dagger)\Phi, \Psi(g^{(r)})\rangle = 0.$$

So, the required implication holds and we conclude that the extension of $\varphi(f)$ is well defined and continuous in $f \in A_{\ell/2}(\mathbb{R}^4)$. A similar reasoning with an additional approximation $\Phi_\nu \to \Phi \in D_\ell$ shows that the extension (which we denote by the same symbol) satisfies $\langle\Phi, \varphi(f)\Psi\rangle = \langle\varphi(f^\dagger)\Phi, \Psi\rangle$ for all $\Phi, \Psi \in D_\ell$. Now let $\psi(f)$ be another extension of $\varphi(f)$ to

D_ℓ, which is continuous in the topology of $A_{\ell/2}(\mathbb{R}^4)$. Using again the approximating sequences f_ν and g_ν and the hermiticity of $\varphi(f_\nu)$, we infer that $\langle\Phi, (\psi(f) - \varphi(f))\Psi(g^{(r)})\rangle = 0$ for all $\Phi \in D$, for every $g \in A_{3\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$, and for each $n = 1, 2, \ldots$. This completes the proof of Corollary of Theorem 2.

It should be noted that D_ℓ is not invariant either under the action of operators $\varphi(f)$, $f \in A_{\ell/2}(\mathbb{R}^4)$, or under the Lorentz group, but it follows from $(a.4)$ and $(a.7.1)$ that the operator-valued generalized function $\varphi(f)$ extends continuously to each space obtainable from $A_{\ell/2}(\mathbb{R}^4)$ by a Lorentz transformation Λ and this extension is well defined on $U(0,\Lambda)D_\ell$.

Theorem 3: *Let φ be the quantum field constructed by Theorem 1 and let D_ℓ be the domain specified in Corollary of Theorem 2. From condition (a.7.2) it follows that for any $\Psi, \Phi \in D_\ell$, the bilinear functional*

$$\langle\Phi, [\varphi(f_1), \varphi(f_2)]_-\Psi\rangle \tag{6}$$

on $A_{\ell/2}(\mathbb{R}^4) \times A_{\ell/2}(\mathbb{R}^4)$, which by (2) is identified with an element of $A'_{\ell/2}(\mathbb{R}^{4\cdot 2})$, extends continuously to the space $A_{\ell/2}(\mathbb{W})$, where

$$\mathbb{W} = \{(x, x') \in \mathbb{R}^{4\cdot 2} \colon (x - x') \in \mathbb{V}\}. \tag{7}$$

III. Nuclearity of the spaces $A_\ell(O)$

To prove Theorems 1–3, we need the following result.

Theorem 4: *$A_\ell(O)$ is a nuclear Fréchet space for each ℓ and for any $O \subset \mathbb{R}^d$.*

Proof: Let $A_{l,N}(O)$ be the space of all analytic functions f on $\tilde{O}^l$ with the property that $\|f\|_{O,l,N} < \infty$. It is easily seen that this normed space is complete. Indeed, if f_ν is a Cauchy sequence in $A_{l,N}(O)$, then it converges uniformly on $\tilde{O}^l$ and hence the limit function $f(z) = \lim_{\nu\to\infty} f_\nu(z)$ is analytic in this domain. There is a constant C such that $\|f_\nu\|_{O,l,N} \le C$. Therefore, $|z^\kappa f(z)| \le C$ for $|\kappa| \le N$ and for all $z \in \tilde{O}^l$. So, $f \in A_{l,N}(O)$. The space $A_\ell(O)$, being the projective limit of the complete spaces $A_{l,N}(O)$, is also complete. Besides, as noted above, it is metrizable and is hence a Fréchet space. It remains to show that $A_\ell(O)$ is nuclear. To this end, we recall some facts from functional analysis.

A convenient criterion of nuclearity is formulated by Pietsch [17] in terms of the Radon measure defined on the polars of neighborhoods of the origin. Let F be a locally convex space, F' be its dual, and $V \subset F$. The set of functionals $u \in F'$ such that $\sup_{f\in V} |(u, f)| \le 1$ is called the (absolute) polar of V and denoted by V°. The polar of each neighborhood of 0 is compact under the weak topology $\sigma(F', F)$, see Schaefer's textbook [18]. A Radon measure on a compact set Q is, by definition, a continuous linear form on the space $C(Q)$ of continuous functions on Q. A Radon measure μ is called positive if $\mu(\psi) \ge 0$ for all non-negative functions $\psi \in C(Q)$. Let $f \in F$ and let p_f be the semi-norm on F' defined by $p_f(u) = |(u, f)|$. The function $p_f(u)$ is continuous in the topology $\sigma(F', F)$ by the definition of the latter. If U is an absolutely convex absorbing set in F, then its associated Minkowski functional p_U is defined by

$$p_U(f) = \inf\{t > 0 \colon f \in tU\}. \tag{8}$$

By the Pietsch theorem, a locally convex space F is nuclear if and only if for every absolutely convex neighborhood U of 0 in F, there is an absolutely convex neighborhood V of 0 and a positive Radon measure μ on V° such that, for all $f \in F$,

$$p_U(f) \le \mu(p_f|_{V^\circ}), \tag{9}$$

where $p_f|_{V^\circ}$ is the restriction of p_f to V°. [In [17], Sec. 4.1.5, the right-hand side of (9) is written as $\int_{V^\circ} |(u, f)|\, d\mu$.]

In order to make use of this theorem, we represent the topology of $A_\ell(O)$ in a different form. Namely, the system of norms (1) is equivalent to the system

$$\|f\|'_{O,l,N} = \int_{\tilde{O}^l} (1+|z|)^N |f(z)|\, \mathrm{d}x\mathrm{d}y, \qquad l < \ell, \quad N = 0, 1, 2, \dots \quad (z = x + iy). \tag{10}$$

Indeed, taking into account that

$$(1+|z|)^N \le 2^N \max(1, |z|^N) = 2^N \max_{\kappa \le N} |z^\kappa|, \tag{11}$$

we obtain

$$\|f\|'_{O,l,N} \le \left(\int_{|y|\le l} \frac{\mathrm{d}x\mathrm{d}y}{(1+|z|)^{d+1}} \right) \sup_{z\in\tilde{O}^l} (1+|z|)^{N+d+1} |f(z)| \le C\, \|f\|_{O,l,N+d+1}.$$

On the other hand, the function $z^\kappa f(z)$ is analytic on $\tilde{O}^\ell$ and we may use Theorem 2.2.3 in [19], which shows that for each l' satisfying $l < l' < \ell$, there is a constant C' such that

$$\|f\|_{O,l,N} \le C' \max_{\kappa\le N} \sup_{\zeta\in\tilde{O}^l} \int_{|z-\zeta|\le l'-l} |z^\kappa f(z)|\, \mathrm{d}x\mathrm{d}y$$
$$\le C' \int_{\tilde{O}^{l'}} (1+|z|)^N |f(z)|\, \mathrm{d}x\mathrm{d}y = C'\, \|f\|'_{O,l',N}.$$

Thus, every absolutely convex neighborhood U of 0 in $A_\ell(O)$ contains a neighborhood of the form $\|f\|'_{O,l,N} < \epsilon$ and hence its Minkowski functional p_U satisfies the inequality

$$p_U(f) \le \epsilon^{-1} \|f\|'_{O,l,N}. \tag{12}$$

Now we apply the Pietsch theorem, taking

$$V = \{f \in A_\ell(O)\colon \sup_{\tilde{O}^l} (1+|z|)^{N+d+1} |f(z)| < 1\}. \tag{13}$$

Using (11), we see that V contains all functions f such that $\|f\|_{O,l,N+d+1} < 2^{-(N+d+1)}$ and is hence a neighborhood of 0. Let $z \in \tilde{O}^l$ and let $\delta_{z,N}$ be the continuous linear form on $A_\ell(O)$ defined by

$$(\delta_{z,N}, f) = (1+|z|)^{N+d+1} f(z).$$

Clearly, $\delta_{z,N} \in V^\circ$ and the map $\tilde{O}^l \to A'_\ell(O)\colon z \to \delta_{z,N}$ is continuous in the topology $\sigma(A'_\ell(O), A_\ell(O))$. If ψ is a continuous function on V°, then the function $z \to \psi(\delta_{z,N})$ is continuous and bounded on $\tilde{O}^l$ for any N. This enables us to define a Radon measure μ on $C(V^\circ)$ by the formula

$$\mu(\psi) = \epsilon^{-1} \int_{\tilde{O}^l} \frac{\psi(\delta_{z,N})}{(1+|z|)^{d+1}}\, \mathrm{d}x\mathrm{d}y. \tag{14}$$

Functional (14) is obviously bounded and positive. Furthermore, we have

$$\mu(p_f) = \epsilon^{-1}\int_{\tilde{O}^l}(1+|z|)^N|f(z)\,|\mathrm{d}x\mathrm{d}y = \epsilon^{-1}\|f\|'_{O,l,N}. \tag{15}$$

It follows from (12) and (15) that condition (9) is satisfied with this choice of V and μ. This completes the proof of Theorem 4.

Corollary: For any $O \subset \mathbb{R}^d$, the space $A_\ell(O)$ is a Fréchet-Schwartz (FS) and hence Montel space. In particular, it is barrelled, reflexive and separable.

We recall [18] that any nuclear Fréchet space can be represented as the projective limit of a decreasing sequence of Hilbert spaces with nuclear connecting maps. Every nuclear map is compact and the projective limit of a decreasing sequence of locally convex spaces with compact connecting maps is an FS space. These two classes of spaces have been introduced by Grothendieck [20, 21] A description of the properties of FS spaces is given, e.g., in Morimoto's monograph [22].

Remark 2: Clearly, $A_{\ell_1,\ell_2}(O_1 \times O_2)$ is also a nuclear Fréchet space for any ℓ_1, ℓ_2, and O_1, O_2. The proof is the same as above, with obvious changes in notation.

Remark 3: Theorem 4 gives a new simple proof of the well-known fact [23] that the spaces $S^{1,B}(\mathbb{R}^d) = A_{1/eB}(\mathbb{R}^d)$ are nuclear. In particular, so is $S^{1,0}(\mathbb{R}^d) = A_\infty(\mathbb{R}^d)$, which is the test function space for the ultra-hyperfunctions used in [13]. We also note that the nuclearity of $S^{1,B}(\mathbb{R}^d)$ implies the nuclearity of $S^1(\mathbb{R}^d) = \operatorname{inj}\lim_{B\to\infty} S^{1,B}(\mathbb{R}^d)$ by the hereditary properties of the inductive limits of countable families of locally convex spaces. This simple proof of nuclearity applies also to all Gelfand-Shilov spaces S^β and S^β_α with $\beta < 1$ and is easily adaptable to the spaces with $\beta > 1$.

The fact that every $A_\ell(O)$ is an FS space can be used in deriving other important properties of this presheaf of spaces. As an example, we prove the following decomposition theorem.

Theorem 5: *Let O_1 and O_2 be any two sets in $\mathbb{R}^d$. If a functional $u \in A'_\ell(\mathbb{R}^d)$ has a continuous extension to $A_\ell(O_1 \cup O_2)$, then it can be decomposed as $u = u_1 + u_2$, where u_1 and u_2 extend, respectively, to $A_\ell(O_1)$ and $A_\ell(O_2)$.*

Proof: We consider $A_\ell(O_1 \cup O_2)$ as a linear subspace of $A_\ell(O_1) \times A_\ell(O_2)$, by assigning to each $f \in A_\ell(O_1 \cup O_2)$ the pair of restrictions $f|\tilde{O}_1^\ell$, $f|\tilde{O}_2^\ell$. This subspace is closed because coincides either with the whole product space or with the kernel of the continuous map that takes each pair $(f_1, f_2) \in A_\ell(O_1) \times A_\ell(O_2)$ to the difference $f_1 - f_2$ belonging to the space of all rapidly decreasing analytic functions on $\tilde{O}_1^\ell \cap \tilde{O}_2^\ell$. As known, the product of a finite number of Fréchet spaces is a Fréchet space and so is any closed subspace of a Fréchet space. It follows that the topology induced on $A_\ell(O_1 \cup O_2)$ by that of $A_\ell(O_1) \times A_\ell(O_2)$ coincides with its original topology by the open mapping theorem. By the Hahn-Banach theorem the functional u has a continuous extension $\tilde{u}$ to the product space. Therefore we can write $u(f) = \tilde{u}(f|\tilde{O}_1^\ell, 0) + \tilde{u}(0, f|\tilde{O}_2^\ell)$, which completes the proof because the injections $A_\ell(O_{1,2}) \to A_\ell(O_1) \times A_\ell(O_2)$ are continuous.

IV. Kernel theorem for the spaces $A_\ell(O)$

Now we recall some basic facts [18] about the tensor products of locally convex spaces. Let F and G be such spaces. By the definition of the tensor product $F \otimes G$, there is a canonical bilinear map $(f, g) \to f \otimes g$ from $F \times G$ to $F \otimes G$, which is continuous if $F \otimes G$ is equipped with the projective topology τ_π. Furthermore, $F \otimes_\pi G$ has the following universality property: For any locally convex space E and for each continuous bilinear map $\beta\colon F \times G \to E$, there is a unique continuous linear map $\beta_*\colon F \otimes_\pi G \to E$ such that $\beta_*(f \otimes g) = \beta(f, g)$ for all $f \in F$ and $g \in G$. The linear map β_* is called the map associated with β. The completion of $F \otimes_\pi G$ is denoted by $F \hat{\otimes} G$.

The next theorem develops Grothendieck's construction given in [20], Chap. 2, Théoreme 13.

Theorem 6: *Let F, G, and H be complete locally convex spaces of scalar functions defined, respectively, on X, Y, and $X \times Y$. Let the topology of each of these spaces be not weaker than the topology of pointwise convergence. Suppose G is nuclear, H is barrelled, and the following conditions are satisfied:*

- (i) *For any $f \in F$ and $g \in G$, the function $(x, y) \to f(x)\,g(y)$ belongs to H and the corresponding bilinear map $\omega\colon F \times G \to H$ is continuous;*
- (ii) *For any $h \in H$ and for each $x \in X$, the function $y \to h(x, y)$ belongs to G and, if $v \in G'$, the function $h_v\colon x \to (v, h(x, \cdot))$ belongs to F;*
- (iii) *The bilinear map $G' \times H \to F\colon (v, h) \to h_v$ is separately continuous if G' is equipped with the strong topology.*

Then $F \hat{\otimes} G$ can be identified with H.

The proof of this theorem is presented in Appendix A. We shall soon see that the above conditions are easily verified and this gives a simple derivation of the desired kernel theorem for the class of spaces we are working with. It is worth noting that if F, G, and H are Fréchet spaces, then (iii) is a consequence of (i) and (ii), see [24] or [25].

Theorem 7: *For any $O_1 \subset \mathbb{R}^{d_1}$ and $O_2 \subset \mathbb{R}^{d_2}$, the space $A_{\ell_1}(O_1) \hat{\otimes} A_{\ell_2}(O_2)$ is identified with $A_{\ell_1,\ell_2}(O_1 \times O_2)$.*

Proof. We assume for simplicity that $\ell_1 = \ell_2 = \ell$ because the proof for $\ell_1 \neq \ell_2$ is in essence the same but cumbersome in notation. We use Theorem 6 with $F = A_\ell(O_1)$, $G = A_\ell(O_2)$, and $H = A_\ell(O_1 \times O_2)$. All these spaces are complete and nuclear by Theorem 4. We also recall that every Fréchet space is barrelled. Condition (i) is obviously fulfilled because $(\widetilde{O_1 \times O_2})^l = \tilde{O}^l \times \tilde{O}^l$ and we have the inequality

$$\|f(z_1)g(z_2)\|_{O_1\times O_2,l,N} \leq \|f\|_{O_1,l,N}\|g\|_{O_2,l,N}, \tag{16}$$

which demonstrates that the bilinear map $\omega\colon (f, g) \to f \otimes g$ is continuous at $(0, 0)$ and hence everywhere. Now let $h \in A_\ell(O_1 \times O_2)$ and $v \in A'_\ell(O_2)$. Then there are l_2 and N_2 such that

$$\|v\|_{O_2,l_2,N_2} \stackrel{\text{def}}{=} \sup_{f\in A_\ell(O_2)} \frac{|(v, f)|}{\|f\|_{O_2,l_2,N_2}} < \infty. \tag{17}$$

The function $h_v(z_1) = (v, h(z_1, \cdot))$ satisfies

$$\|h_v\|_{O_1,l_1,N_1} = \max_{|\kappa|\leq N_1} \sup_{z_1\in\tilde{O}_1^{l_1}} |z_1^\kappa h_v(z_1)| \leq \|v\|_{O_2,l_2,N_2}\|h\|_{O_1\times O_2,\max(l_1,l_2),N_1+N_2} \tag{18}$$

for each $l_1 < \ell$ and for all $N_1 = 0, 1, \dots$. We must show that $h_v(z_1)$ is analytic on $\tilde{O}_1^\ell$, i.e., has all partial derivatives in the complex variables z_{1j}, $j = 1, \dots, d_1$, at each point of this domain.

Let $z_1 \in \tilde{O}_1^l$, $l < l' < \ell$, and let L be the segment of the straight line joining the points z_{1j} and $z_{1j} + \Delta z_{1j}$, where $|\Delta z_{1j}| < (l' - l)/2$. The corresponding increment of the function h can be written as $\Delta_{z_{1j}} h(z_1, z_2) = \int_L h'_{z_{1j}}(z_{11}, \dots, \zeta_{1j}, \dots z_{1d_1}, z_2)\, d\zeta_{1j}$. Let $\tilde{L}^r$ be the complex r-neighborhood of L, with $r \le (l' - l)/2$. Using the Cauchy inequality, we obtain

$$|\Delta_{z_{1j}} h(z_1, z_2)| \le r^{-1} |\Delta z_{1j}| \sup_{\zeta_{1j} \in \tilde{L}^r} |h(z_{11}, \dots, \zeta_{1j}, \dots z_{1d_1}, z_2)|.$$

Therefore, the difference quotient $\Delta_{z_{1j}} h / \Delta z_{1j}$ considered at fixed $z_1 \in \tilde{O}_1^l$ as an element of $A_\ell(O_2)$ (parametrically depending on Δz_{1j}) satisfies the inequality

$$\left\| \frac{\Delta_{z_{1j}} h}{\Delta z_{1j}} \right\|_{O_2, l, N} \le \frac{2}{l' - l} \, \|h\|_{O_1 \times O_2, l', N}$$

for any N. Thus, the set of difference quotients is bounded in $A_\ell(O_2)$. It also follows from the Cauchy inequality that $h'_{z_{1j}} \in A_\ell(O_1 \times O_2)$. Using that $A_\ell(O_2)$ is a Montel space, we can choose a convergent sequence from the set of difference quotients. Clearly, its limit is $h'_{z_{1j}}$ because the topology of $A_\ell(O_2)$ is stronger than that of pointwise convergence. The uniqueness of this limit implies that the difference quotient converges to $h'_{z_{1j}}$ in $A_\ell(O_2)$. Therefore, $(v, \Delta_{z_{1j}} h / \Delta z_{1j}) \to (v, h'_{z_{1j}})$ as $\Delta z_{1j} \to 0$ and we conclude that condition (ii) is fulfilled.

Estimate (18) shows that the map $h \to h_v$ is continuous for every fixed v. Now we hold h fixed and let v belong to the space $A'_{l_2, N_2}(O_2)$ of functionals with finite norm (17). It follows also from (18) that the map $A'_{l_2, N_2}(O_2) \to A_\ell(O_1)\colon v \to h_v$ is continuous for each $l_2 < \ell$ and for every integer N_2. This amounts to saying that the corresponding map $A'_\ell(O_2) \to A_\ell(O_1)$ is continuous in the inductive limit topology determined on $A'_\ell(O_2)$ by the canonical injections $A'_{l_2, N_2}(O_2) \to A'_\ell(O_2)$. But this topology coincides with the strong topology of $A'_\ell(O_2)$ by the general open mapping theorem [26]. In fact, even Grothendieck's version (given in Introduction of [20]) of this theorem applies here because the strong dual of any FS space is a dual Fréchet-Schwartz (DFS) space (see [22]) and is hence ultrabornological or of type (β) in the terminology used by Grothendieck. This completes the proof of Theorem 7.

Because the projective tensor product is associative [26] it follows from Theorem 7 that

$$A_{\ell_1}(O_1) \hat{\otimes} \dots \hat{\otimes} A_{\ell_n}(O_n) = A_{\ell_1, \dots, \ell_n}(O_1 \times \dots \times O_n) \tag{19}$$

for any $\ell_j > 0$ and $O_j \subset \mathbb{R}^{d_j}$, $j = 1, \dots, n$.

Corollary [Kernel theorem for the spaces $A_\ell(O)$]: Let μ be a separately continuous multilinear map of $A_{\ell_1}(O_1) \times \dots \times A_{\ell_n}(O_n)$ into a locally convex space E. Then there is a unique continuous linear map $u_\mu\colon A_{\ell_1, \dots, \ell_n}(O_1 \times \dots \times O_n) \to E$ such that

$$\mu(f_1, \dots, f_n) = (u_\mu, f_1 \otimes \dots \otimes f_n)$$

for any $f_j \in A_{\ell_j}(O_j)$, $j = 1, \dots, n$.

For $n = 2$, this follows immediately from Theorem 7 because every separately continuous bilinear map of the product of two Fréchet spaces into a topological vector space is continuous [18]. For $n > 2$, we argue by induction, using the following simple lemma, which is a generalization of Lemma 3 in [27].

Lemma 1: *Let F, G, and E be locally convex spaces and let L be a sequentially dense subspace of F. Suppose in addition that G is barrelled and E is Hausdorff and complete. Then every separately continuous bilinear map $\beta\colon L\times G\to E$ has a unique extension to a separately continuous bilinear map $F\times G\to E$.*

Proof: For each fixed $g\in G$, the linear map $L\to E\colon f\to\beta(f,g)$ can be uniquely extended to F by continuity and thereby we obtain a map $\hat\beta\colon F\times G\to E$. We have only to show that $\hat\beta$ is linear and continuous in g for every fixed $f\in F$. We choose a sequence $f_\nu\in L$ convergent to f and consider the corresponding sequence of continuous linear maps $G\to E\colon g\to\beta(f_\nu,g)$. This sequence of maps converges pointwise to the map $g\to\hat\beta(f,g)$ and hence the latter is linear and continuous by the Banach-Steinhaus theorem because G is barrelled. The lemma is thus proved.

To complete the proof of the kernel theorem for $A_\ell(O)$, we use Lemma 1 with $F=A_{\ell_1}(O_1)\,\hat\otimes\ldots\hat\otimes\,A_{\ell_{n-1}}(O_{n-1})$, $G=A_{\ell_n}(O_n)$, and $L=A_{\ell_1}(O_1)\otimes\ldots\otimes A_{\ell_{n-1}}(O_{n-1})$.

Remark 4: This result contains, as a simple special case (with $\ell_1=\cdots=\ell_n=\infty$, $O_1=\cdots=O_n=\mathbb{R}^4$, and E a Banach space) the kernel theorem for ultra-hyperfunctions discussed by Brüning and Nagamachi [13]. Because their main effort was to cover multilinear maps with values in a Banach space, it is worth noting that the general kernel theorem for multilinear maps taking values in an arbitrary locally convex space is an immediate consequence of its simplest version for the complex-valued bilinear forms, see Appendix B.

It cannot be asserted that $A_\infty(\mathbb{R}^d)$ is dense in $A_\ell(O)$ for an arbitrary domain $O\subset\mathbb{R}^d$, although this is the case for $O=\mathbb{R}^d$. For this reason it is useful to introduce another class of spaces. Namely, we define $\mathcal{A}_\ell(O)$ to be the closure of $A_\infty(\mathbb{R}^d)$ regarded as a subspace of $A_\ell(O)$ and provide it with the topology induced by that of $A_\ell(O)$. Since $A_\infty(\mathbb{R}^d)$ is dense in $A_\ell(\mathbb{R}^d)$, the space $\mathcal{A}_\ell(O)$ can also be defined as the completion of $A_\ell(\mathbb{R}^d)$ with respect to the topology induced on it by that of $A_\ell(O)$. Analogously, $\mathcal{A}_{\ell_1,\ell_2}(O_1\times O_2)$ is the closure of $A_\infty(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ in $A_{\ell_1,\ell_2}(O_1\times O_2)$.

Theorem 8: *Every space $\mathcal{A}_\ell(O)$ is a nuclear Fréchet space and, for any $\ell_1,\ell_2>0$ and for any $O_1\subset\mathbb{R}^{d_1}$, $O_2\subset\mathbb{R}^{d_2}$, there is the canonical isomorphism*

$$\mathcal{A}_{\ell_1}(O_1)\,\hat\otimes\,\mathcal{A}_{\ell_2}(O_2)\cong\mathcal{A}_{\ell_1,\ell_2}(O_1\times O_2). \tag{20}$$

Proof: The first statement of the theorem follows from the well-known hereditary properties [18] of Fréchet and nuclear spaces. Now let $f\in\mathcal{A}_{\ell_1}(O_1)$, $g\in\mathcal{A}_{\ell_2}(O_2)$, and let sequences $f_\nu\in A_\infty(\mathbb{R}^{d_1})$, $g_\nu\in A_\infty(\mathbb{R}^{d_2})$ be such that $f_\nu\to f$ in $A_{\ell_1}(O_1)$ and $g_\nu\to g$ in $A_{\ell_2}(O_2)$. Using an analog of inequality (16) for the case $\ell_1\neq\ell_2$, it is easy to see that $f_\nu\otimes g_\nu\to f\otimes g$ in the topology of $A_{\ell_1,\ell_2}(O_1\times O_2)$. Therefore, $f\otimes g$ belongs to $\mathcal{A}_{\ell_1,\ell_2}(O_1\times O_2)$ and condition (i) of Theorem 6 is satisfied. Now let $h\in\mathcal{A}_{\ell_1,\ell_2}(O_1\times O_2)$, $h_\nu\in A_\infty(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$, and $h_\nu\to h$ in $A_{\ell_1,\ell_2}(O_1\times O_2)$. Then for every fixed z_1, the sequence $h_\nu(z_1,z_2)$ tends to the function $z_2\to h(z_1,z_2)$ in the topology of $A_{\ell_2}(O_2)$ and hence this function belongs to $\mathcal{A}_{\ell_2}(O_2)$. Let $v\in\mathcal{A}'_{\ell_2}(O_2)$. Arguing as in the proof of Theorem 7, we see that the functions $z_1\to(v,h_\nu(z_1,\cdot))$ belong to $A_\infty(\mathbb{R}^{d_1})$ because $\|v\|_{\mathbb{R}^{d_2},l_2,N_2}\leq\|v\|_{O_2,l_2,N_2}$. It follows from (an analog of) (18) that the sequence of these functions converges to the function $z_1\to(v,h(z_1,\cdot))$ in the topology of $A_{\ell_1}(O_1)$ as $\nu\to\infty$. So condition (ii) of Theorem 6 is fulfilled. Since $\mathcal{A}_{\ell_2}(O_2)$ is an FS space, we can now apply the arguments used at the end of the proof of Theorem 7 and conclude that condition (iii) of Theorem 6 is also fulfilled. This completes the proof.

V. Proof of the reconstruction theorem

The kernel theorem enables us to construct a tensor algebra $T(A_\infty(\mathbb{R}^4))$ in complete analogy with constructing the Borchers algebra associated with the Schwartz space $S(\mathbb{R}^4)$. Namely,

$$T(A_\infty(\mathbb{R}^4)) = \bigoplus_{n=0}^{\infty} T_n, \quad T_0 = \mathbb{C}, \quad T_n = A_\infty(\mathbb{R}^4)^{\hat{\otimes} n} = A_\infty(\mathbb{R}^{4n}) \quad \text{for} \quad n \geq 1. \tag{21}$$

The space $T(A_\infty(\mathbb{R}^4))$ consists of all terminating sequences of the form $\mathbf{f} = (f_0, f_1, \dots)$, where $f_n \in A_\infty(\mathbb{R}^{4n})$ and only a finite number of f_n's are different from zero. It is an involutory algebra under the multiplication

$$(\mathbf{f} \otimes \mathbf{g})_n = \sum_{k=0}^{n} f_k \otimes g_{n-k} \tag{22}$$

and with the involution

$$f_n^\dagger(z_1, \dots, z_n) = \overline{f_n(\bar{z}_n, \dots, \bar{z}_1)}. \tag{23}$$

Operations (22) and (23) are continuous in the natural topology of the direct sum. Furthermore, $T(A_\infty(\mathbb{R}^4))$ is a nuclear LF-space.

By conditions $(a.2)$ and $(a.3)$, the algebra $T(A_\infty(\mathbb{R}^4))$ can be equipped with the positive semidefinite hermitian form

$$s(\mathbf{f}, \mathbf{g}) = \sum_{k,m \geq 0} \left(\mathcal{W}_{k+m}, f_k^\dagger \otimes g_m \right), \tag{24}$$

which defines a nondegenerate inner product $\langle \cdot, \cdot \rangle$ on the quotient space $T(A_\infty(\mathbb{R}^4))/\ker s$. Completing the latter with respect to the corresponding norm $\| \cdot \|$, we obtain a Hilbert space $\mathcal{H}$. There is a natural continuous linear map $T(A_\infty(\mathbb{R}^4)) \to \mathcal{H}$. We denote by D its image, which is a dense subspace of $\mathcal{H}$, and by $\Psi_{\mathbf{f}}$ the image of $\mathbf{f}$ in $\mathcal{H}$. Since every space $A_\infty(\mathbb{R}^{4n})$ is separable, so is $\mathcal{H}$.

From $(a.4)$ it follows that $\mathbf{f} \in \ker s \Rightarrow \mathbf{f}_{(a,\Lambda)} \in \ker s$. Therefore, the action of $\mathcal{P}_+^\uparrow$ in $T(A_\infty(\mathbb{R}^4))$ induces a linear representation $(a, \Lambda) \to U(a, \Lambda)$ of this group in D, defined by

$$U(a, \Lambda)\Psi_{\mathbf{f}} = \Psi_{\mathbf{f}_{(a,\Lambda)}}. \tag{25}$$

The condition $(a.4)$ implies also that every operator $U(a, \Lambda)$ is isometric. Besides, it is bijective and hence can be extended by continuity to a unitary operator on the whole of $\mathcal{H}$. The map $(a, \Lambda) \to \langle \Psi_{\mathbf{f}}, U(a, \Lambda)\Psi_{\mathbf{f}} \rangle = s(\mathbf{f}, \mathbf{f}_{(a,\Lambda)})$ is continuous for each $\mathbf{f}$, because $\mathcal{P}_+^\uparrow$ acts continuously in every space $A_\infty(\mathbb{R}^{4n})$. Therefore, $\|\Psi_{\mathbf{f}} - U(a, \Lambda)\Psi_{\mathbf{f}}\| \to 0$ as $(a, \Lambda) \to (0, I)$. Using the unitarity of $U(a, \Lambda)$ in the same manner as in [1], we deduce that this operator is continuous in (a, Λ) on a general $\Psi \in \mathcal{H}$.

We denote by Ψ_0 the image of $(1, 0, 0, \dots)$ in $\mathcal{H}$. Clearly, $U(a, \Lambda)\Psi_0 = \Psi_0$ for all (a, Λ). There is no other translation invariant state in $\mathcal{H}$. Indeed, assume the converse, that such a state Φ_0 exists. Without loss of generality we can also assume that $\langle \Phi_0, \Psi_0 \rangle = 0$ and $\|\Phi_0\| = 1$. Since D is dense in $\mathcal{H}$, for any $\epsilon > 0$ there is a vector $\Psi_{\mathbf{f}}$ such that $\|\Psi_{\mathbf{f}} - \Phi_0\| < \epsilon$. Denoting $\Psi_{\mathbf{f}} - \Phi_0$ by Φ, we have

$$\langle \Psi_{\mathbf{f}}, U(\lambda a, I)\Psi_{\mathbf{f}} \rangle = \langle \Phi_0, U(\lambda a, I)\Phi_0 \rangle + \langle \Phi, U(\lambda a, I)\Phi_0 \rangle + \langle \Phi_0, U(\lambda a, I)\Phi \rangle + \langle \Phi, U(\lambda a, I)\Phi \rangle, \tag{26}$$

where $\langle\Phi_0, U(\lambda a, I)\Phi_0\rangle \equiv 1$. Applying the Schwarz inequality and using the unitarity of $U(\lambda a, I)$, we see that the sum of last three terms on the right-hand side of (26) is bounded in absolute value by $2\epsilon+\epsilon^2$. On the other hand, if a is spacelike and $\lambda\to\infty$, then from the cluster decomposition property $(a.6)$ it follows that

$$\langle\Psi_{\mathbf f}, U(\lambda a, I)\Psi_{\mathbf f}\rangle \equiv \sum_{k,m\geq 0}\Bigl(\mathcal{W}_{k+m}, f_k^\dagger\otimes (f_m)_{(\lambda a, I)}\Bigr)$$
$$\to \sum_{k,m\geq 0}(\mathcal{W}_k, f_k^\dagger)(\mathcal{W}_m, f_m) = \langle\Psi_{\mathbf f},\Psi_0\rangle\langle\Psi_0,\Psi_{\mathbf f}\rangle, \quad (27)$$

where $|\langle\Psi_{\mathbf f},\Psi_0\rangle| = |\langle\Phi,\Psi_0\rangle|<\epsilon$. Thus, we obtain a contradiction for sufficiently small ϵ.

Since $U(a,I)$ is unitary and continuous, it can be written as $U(a,I)=e^{ia^\nu P_\nu}$, where P_ν are commuting self-adjoint operators. From $(a.5)$ it follows that the spectrum of the energy-momentum operator P is contained in the closed forward cone $\overline{\mathbb{V}}^+$. This can be proved by the standard arguments [1, 14] because the Fourier transform of A_∞ contains the space $\mathcal{D}$ of smooth functions of compact support. Namely, the spectral representation of $U(a,I)$ shows that the statement about the spectrum of P amounts to saying that $\int \mathrm{d}a\rho(a)U(a,I)=0$ for all $\hat\rho$ belonging to $\mathcal{D}(\mathbb{R}^4)$ and supported in the complement of $\overline{\mathbb{V}}^+$. Let $\mathbf f$ be such that only one of its components f_k is nonzero and let $\mathbf g$ be such that only $g_m\neq 0$. Then we have

$$\langle\Psi_{\mathbf f}, \int \mathrm{d}a\,\rho(a)U(a,I)\Psi_{\mathbf g}\rangle = \left(\mathcal{W}_{k+m}, f_k^\dagger\otimes\int \mathrm{d}a\,\rho(a)(g_m)_{(a,I)}\right). \quad (28)$$

The Fourier transform of $\int \mathrm{d}a\,\rho(a)(g_m)_{(a,I)}$ is equal to $\hat\rho(p_1+\dots+p_m)\hat g_m(p_1,\dots,p_m)$. In terms of the functionals $\mathcal{W}_n$, condition $(a.5)$ means that $\operatorname{supp}\hat{\mathcal{W}}_n(p_1,\dots,p_n)$ is contained in the set defined by $p_j+\dots+p_n\in -\overline{\mathbb{V}}^+$, $j=1,2,\dots,n$. Because $(\mathcal{W}_n,h)=(2\pi)^{4n}(\hat{\mathcal{W}}_n,\hat h(-\cdot))$, we conclude that matrix element (28) vanishes if $\operatorname{supp}\hat\rho$ does not intersect $\overline{\mathbb{V}}^+$, which proves the statement.

For each $h\in A_\infty(\mathbb{R}^4)$, we define a linear operator $\varphi(h)$ on D by

$$\varphi(h)\Psi_{\mathbf f}=\Psi_{h\otimes\mathbf f}, \quad \text{where} \quad h\otimes\mathbf f=(0, hf_0, h\otimes f_1, h\otimes f_2,\dots). \quad (29)$$

It is well defined because if $\mathbf f\in\ker s$, then $h\otimes\mathbf f\in\ker s$. Indeed, we have $(h\otimes\mathbf f)^\dagger=\mathbf f^\dagger\otimes h^\dagger$ and hence $s(h\otimes\mathbf f,\mathbf g)=s(\mathbf f, h^\dagger\otimes\mathbf g)$ by definition (24). By the same argument,

$$\langle\Phi,\varphi(h)\Psi\rangle=\langle\varphi(h^\dagger)\Phi,\Psi\rangle \qquad \text{for all} \quad \Phi,\Psi\in D \quad \text{and} \quad h\in A_\infty(\mathbb{R}^4),$$

i.e., the field φ is hermitian. Clearly, $\varphi(h)D\subset D$. Since $A_\infty(\mathbb{R}^4)^{\otimes n}$ is dense in $A_\infty(\mathbb{R}^{4n})$, the vector Ψ_0 is cyclic for φ. All the matrix elements $\langle\Phi,\varphi(h)\Psi\rangle$, where $\Phi,\Psi\in D$, can be expressed in terms of the functionals $\mathcal{W}_n$ with fixed $(n-1)$ arguments and hence they belong to $A'_\infty(\mathbb{R}^4)$. From the relation $(h\otimes\mathbf f)_{(a,\Lambda)}=h_{(a,\Lambda)}\otimes\mathbf f_{(a,\Lambda)}$ and definition (25), it follows that

$$U(a,\Lambda)\varphi(h)U(a,\Lambda)^{-1}=\varphi(h_{(a,\Lambda)}).$$

Suppose now that $\tilde{\mathcal{H}}$, $\tilde U(a,\Lambda)$, and $\tilde\varphi$ define a field theory with a cyclic vacuum state $\tilde\Psi_0$ and with the same expectation values. Let $\tilde D_0$ be the vector subspace spanned by $\tilde\Psi_0$ and all vectors of the form

$$\underbrace{\tilde\varphi(f)\tilde\varphi(g)\dots\tilde\varphi(h)}_{n}\tilde\Psi_0, \quad (30)$$

where $n = 1, 2, \dots$ and all test functions are in $A_\infty(\mathbb{R}^4)$. We assert that the multilinear map taking each n-tuple

$$(f, g, \dots, h) \in \underbrace{A_\infty(\mathbb{R}^4) \times \dots A_\infty(\mathbb{R}^4)}_{n}$$

to vector (30) is separately continuous. We consider it as a function of one of variables, say, of g with all other variables held fixed. Let $\tilde{\Phi}$ be an arbitrary element of $\tilde{\mathcal{H}}$. Since $\tilde{D}_0$ is dense in $\tilde{\mathcal{H}}$, there exists a sequence $\tilde{\Phi}_\nu \in \tilde{D}_0$ such that $\tilde{\Phi}_\nu \to \tilde{\Phi}$. The sequence of continuous linear forms $g \to \langle \tilde{\Phi}_\nu, \tilde{\varphi}(f)\tilde{\varphi}(g)\dots\tilde{\varphi}(h)\tilde{\Psi}_0\rangle$ converges pointwise to the form $g \to \langle \tilde{\Phi}, \tilde{\varphi}(f)\tilde{\varphi}(g)\dots\tilde{\varphi}(h)\tilde{\Psi}_0\rangle$ and the latter is also continuous by the uniform boundedness principle which is applicable because $A_\infty(\mathbb{R}^4)$ is a Fréchet space. It follows that the map $A_\infty(\mathbb{R}^4) \to \tilde{\mathcal{H}}$: $g \to \tilde{\varphi}(f)\tilde{\varphi}(g)\dots\tilde{\varphi}_1(h)\tilde{\Psi}_0$ is weakly continuous. But for the class of Fréchet spaces, the weak continuity of a linear map is equivalent to the continuity in their original topology (see [18], Sec. IV.7.4), which proves our assertion. Using the kernel theorem for A_∞, we conclude that the multilinear map under consideration can be uniquely extended to a continuous linear map $A_\infty(\mathbb{R}^{4n}) \to \tilde{\mathcal{H}}$, which gives an exact meaning to the formal expression

$$\int dx_1 \dots dx_n f(x_1, \dots, x_n) \prod_{i=1}^{n} \tilde{\varphi}(x_i)\tilde{\Psi}_0, \qquad \text{where} \quad f \in A_\infty(\mathbb{R}^{4n}),$$

and also to the vectors

$$\tilde{\Psi}_{\mathbf{f}} = f_0\tilde{\Psi}_0 + \int dx f_1(x)\tilde{\varphi}(x)\tilde{\Psi}_0 + \int dx_1 dx_2 f_2(x_1, x_2)\tilde{\varphi}(x_1)\tilde{\varphi}(x_2)\tilde{\Psi}_0 + \dots .$$

Let V be the map taking $\Psi_{\mathbf{f}} \in \mathcal{H}$ to $\tilde{\Psi}_{\mathbf{f}} \in \tilde{\mathcal{H}}$. This map is well defined because $\mathbf{f} \in \ker s$ implies that $\langle \tilde{\Phi}, \tilde{\Psi}_{\mathbf{f}}\rangle = 0$ for all $\tilde{\Phi} \in \tilde{D}_0$ and hence $\tilde{\Psi}_{\mathbf{f}} = 0$. From the equality of the expectation values in the two theories, we also deduce that the operator V is isometric. Since D is dense in $\mathcal{H}$ and $\tilde{D}_0$ is dense in $\tilde{\mathcal{H}}$, this operator can be extended by continuity to a unitary operator from $\mathcal{H}$ onto $\tilde{\mathcal{H}}$. We have the chain of equalities

$$V\varphi(h)\Psi_{\mathbf{f}} = V\Psi_{h\otimes\mathbf{f}} = \tilde{\Psi}_{h\otimes\mathbf{f}} = \tilde{\varphi}(h)\tilde{\Psi}_{\mathbf{f}} = \tilde{\varphi}(h)V\Psi_{\mathbf{f}},$$

and hence $V\varphi(h)V^{-1} = \tilde{\varphi}(h)$. The transformation law of $\tilde{\varphi}$ under the Poincaré group implies that $\tilde{U}(a, \Lambda)\tilde{\Psi}_{\mathbf{f}} = \tilde{\Psi}_{\mathbf{f}_{(a,\Lambda)}}$. Therefore, an analogous chain of equalities gives $U(a, \Lambda) = V\tilde{U}(a, \Lambda)V^{-1}$, which completes the proof of Theorem 1.

We now turn to the proof of Theorem 2. Let φ be the field constructed above. First we show that under condition (a.7.1), the vector-valued generalized function $\varphi(g)\Psi_0$, $g \in A_\infty(\mathbb{R}^4)$, has a continuous extension to the space $A_{\ell/2}(\mathbb{R}^4)$. It follows from (a.7.1) that there are $l < \ell$ and N such that $\|W_2\|_{l,N} \equiv \sup_{\|f\|_{l,N}\le 1} |(W_2, f)| < \infty$. Using (11), the triangle inequality for the norm $|\zeta| = \max_j |\zeta_j|$, and the analyticity of g, we obtain the estimate

$$\|\varphi(g)\Psi_0\|^2 = (\mathcal{W}_2, g^\dagger \otimes g) = \left(W_2, \int \overline{g(x+\xi)}g(x)dx\right) \le \|W_2\|_{l,N} \left\| \int \overline{g(x+\xi)}g(x)dx \right\|_{l,N}$$

$$\le \|W_2\|_{l,N} \sup_{|\eta|<l} \sup_{\xi} (1 + |\xi + i\eta|)^N \left| \int \overline{g(x+\xi-i\eta)}g(x)dx \right|$$

$$\le (1+l)^N \|W_2\|_{l,N} \sup_{|\eta|<l} \sup_{\xi} \int (1 + |x+\xi|)^N |g(x+\xi-i\eta/2)| \, (1+|x|)^N |g(x-i\eta/2)| dx$$

$$\le 2^{2N+5}(1+l)^N \|W_2\|_{l,N} \|g\|_{l/2,N} \|g\|_{l/2,N+5} \int \frac{dx}{(1+|x|)^5} \le C\|g\|^2_{l/2,N+5},$$

which demonstrates the existence of the desired extension. Next we consider the vector-function

$$\Psi(g^{(r)}) = \int dx_1 \dots dx_n g(x_1, x_1 - x_2 \dots, x_{n-1} - x_n) \prod_{i=1}^{n} \varphi(x_i) \Psi_0, \tag{31}$$

where $n \geq 2$. The reasoning after formula (30) shows that it is well defined on the space $A_\infty(\mathbb{R}^{4n})$, for which the map $g \to g^{(r)}$ is an automorphism. We express the squared norm of vector (31) in terms of W_{2n}'s and denote by $\|g\|_{l/2,l,N}$ the norm of g in the space $A_{\ell/2,\ell,N}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$. Proceeding along the same lines as above and using in addition the elementary inequality $1 + |\xi| \leq (1 + |\xi_n|)(1 + \max_{j \neq n} |\xi_j|)$, we get

$$\|\Psi(g)\|^2 = \left(W_{2n}, \int \overline{g(x + \xi_n, -\xi_{n-1}, \dots - \xi_1)} g(x, \xi_{n+1}, \dots, \xi_{2n-1}) \mathrm{d}x \right)$$
$$\leq 2^{4N+5} (1 + l)^N \|W_{2n}\|_{l,N} \|g\|_{l/2,l,2N} \|g\|_{l/2,l,2N+5} \int \frac{\mathrm{d}x}{(1 + |x|)^5} \leq C' \|g\|^2_{l/2,l,2N+5}$$

and conclude that the linear map $g \to \Psi(g^{(r)})$ has a continuous extension to the space $A_{\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$. It is important that the extension is unique because $A_\infty(\mathbb{R}^{4n})$ is dense in this space, as already noted in Remark 1. If we replace Ψ_0 in (31) with an arbitrary $\Psi_{\mathbf{f}} \in D$, then the resulting vector-function also has a unique continuous extension to $A_{\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$. Indeed, it is a finite sum of vector functions of the previous form but with the difference that g is now replaced by $g \otimes f_m$, where $f_m \in A_\infty(\mathbb{R}^{4m})$. If $g \in A_{\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$, then $g \otimes f_m \in A_{\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n+m-1)})$ and the map $g \to g \otimes f_m$ is continuous. Thus, from $(a.7.1)$ it follows that every monomial $\prod_{i=1}^{n} \varphi(x_i)$, $n \geq 2$, has a unique continuous extension to an operator-valued generalized function defined on $A_\ell^{(r)}(\mathbb{R}^{4n})$. Conversely, in any field theory with such a property, the n-point vacuum expectation value W_n is well defined on $A_\ell(\mathbb{R}^{4(n-1)})$ by the formula

$$(W_n, f) = \left\langle \Psi_0, \int dx_1 \dots dx_n g_1(x_1) f(x_1 - x_2 \dots, x_{n-1} - x_n) \prod_{i=1}^{n} \varphi(x_i) \Psi_0 \right\rangle,$$

where g_1 is any element of $A_{\ell/2}(\mathbb{R}^4)$ such that $\int g_1(x)\, dx = 1$. Theorem 2 is proved.

Proof of Theorem 3: Combining Theorem 7 and Corollary of Theorem 2 we infer that the matrix element $\langle \Phi, [\varphi(x), \varphi(x')]_- \Psi \rangle$ is well defined as a continuous linear functional on $A_{\ell/2}(\mathbb{R}^{4 \cdot 2})$ for any fixed $\Phi, \Psi \in D_\ell$. For brevity, we denote this functional by $u_{\Phi,\Psi}$. Let

$$\Phi = \int dx_1 \dots dx_{k-1} \overline{h(x_{k-1}, x_{k-1} - x_{k-2}, \dots, x_2 - x_1)}\, \varphi(x_{k-1}) \cdots \varphi(x_1) \Psi_0 \tag{32}$$

and

$$\Psi = \int dx_{k+2} \dots dx_n g(x_{k+2}, x_{k+2} - x_{k+3}, \dots, x_{n-1} - x_n)\, \varphi(x_{k+2}) \cdots \varphi(x_n) \Psi_0, \tag{33}$$

where $k \geq 2$, $n \geq k + 2$, $h \in A_{3\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(k-2)})$, and $g \in A_{3\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-k-2)})$. Let $f(x, x')$ belong to $A_{\ell/2}(\mathbb{R}^{4 \cdot 2})$. We introduce the notation $\xi_{k-1} = x_{k-1} - x$, $\xi_k = x - x'$, $\xi_{k+1} = x' - x_{k+2}$. The number $(u_{\Phi,\Psi}, f)$ is equal to the value of functional (3) at the function

$$F(\xi_1, \dots, \xi_{n-1}) =$$
$$\int \mathrm{d}x\, h(x + \xi_{k-1},\ -\xi_{k-2}, \dots, -\xi_1)\ f(x, x - \xi_k) g(x - \xi_k - \xi_{k+1},\ \xi_{k+2}, \dots, \xi_{n-1}). \tag{34}$$

From $(a.7.2)$ it follows that functional (3) is bounded in one of the norms of the space $A_\ell(V_{(k)})$ with some indices $l < \ell$, N. Proceeding in a manner similar to that used in proving Theorem 2 and taking into account that the plane of integration in (34) may be shifted within the analyticity domain, we obtain

$$|(u_{\Phi,\Psi}, f)| \le C \sup_{|\eta|\le l} \sup_{\xi\in V_{(k)}} (1+|\xi|)^N |F(\xi+i\eta)|$$

$$\le C \sup_{|\eta|\le l} \sup_{\xi\in V_{(k)}} (1+\max_{j<k-1}|\xi_j|)^N(1+|\xi_{k-1}|)^N(1+|\xi_k|)^N(1+|\xi_{k+1}|)^N(1+\max_{j>k+1}|\xi_j|)^N|F(\xi+i\eta)|$$

$$\le C'\|h\|_{3l/2,l,2N}\|g\|_{3l/2,l,2N} \sup_{|\eta_k|\le l} \sup_{x} \sup_{\xi_k^2\ge 0}(1+|x|)^{2N+5}(1+|x-\xi_k|)^{2N}|f(x+i\eta_k/2, x-\xi_k-i\eta_k/2)|$$

$$\le C''\|h\|_{3l/2,l,2N}\|g\|_{3l/2,l,2N}\|f\|_{\mathbb{W},l/2,4N+5}.$$

Clearly, a similar estimate holds if Ψ_0 in (32), (33) is changed for an arbitrary vector in D and also in the event that $\Phi = \Psi_0$, or $\Psi = \Psi_0$, or $\Phi = \Psi = \Psi_0$. Therefore, every functional $u_{\Phi,\Psi}$, where $\Phi, \Psi \in D_\ell$, has a continuous extension to the space $A_{\ell/2}(\mathbb{W})$. This completes the proof of the reconstruction theorem.

VI. Two formulations of quasilocality

In this section, we show that the property stated in Theorem 3 faithfully enough reproduces the initial property $(a.7.2)$ of the Wightman functionals. Namely, if the commutator $[\varphi(x), \varphi(x')]_-$ in a field theory has this property, then functional (3) composed of the n-point vacuum expectation values of φ extends continuously to the space $A_{2\ell}(V_{(k)})$. We shall use a slightly different version of Lemma 1.

Lemma 2: *Let L be a sequentially dense subspace of a locally convex space F and let $G_1, \dots, G_n$ be barrelled spaces. Then every separately continuous multilinear form μ defined on $L \times G_1 \times \dots G_n$ has a unique extension to a separately continuous multilinear form on $F \times G_1 \times \dots G_n$.*

Proof: For each fixed $g_j \in G_j$, $j = 1, \dots, n$, the linear form $f \to \mu(f, g_1, \dots, g_n)$ can be uniquely extended to F by continuity. Letting $\hat{\mu}$ denote this extension, we have to show that it is linear and continuous in every g_j for any fixed $f \in F$ and $g_i \in G_i$, $i \ne j$. We set $j = 1$ without loss of generality. Choose a sequence $f_\nu \in L$ such that $f_\nu \to f$ in F and denote by $f_\nu^{(g_2,\dots,g_n)}$ the corresponding elements of G_1' defined by $f_\nu^{(g_2,\dots,g_n)}(g_1) = \mu(f_\nu, g_1, g_2, \dots, g_n)$. The sequence $f_\nu^{(g_2,\dots,g_n)}$ converges pointwise on G_1 to the functional $f^{(g_2,\dots,g_n)}(g_2) = \hat{\mu}(f, g_1, g_2, \dots, g_n)$ and hence this functional is linear and continuous because G_1 is barrelled. The lemma is thus proved.

Let φ be a scalar field with test functions in $A_\infty(\mathbb{R}^4)$. Suppose that every monomial $\prod_{i=1}^n \varphi(x_i)$ has a continuous extension to the space of all functions of form (5). Then, as shown in Sec. II, the operators $\varphi(f)$, $f \in A_{\ell/2}(\mathbb{R}^4)$, are well defined and act continuously on the linear span of the vacuum state Ψ_0 and all vectors of the form

$$\int dx_1 \dots dx_n\, g(x_1, x_1 - x_2 \dots, x_{n-1} - x_n) \prod_{i=1}^n \varphi(x_i)\Psi_0, \quad n \ge 1, \tag{35}$$

where g ranges over the space $A_{3\ell/2,\ell}(\mathbb{R}^4 \times \mathbb{R}^{4(n-1)})$. Suppose that the matrix element $\langle \Phi, [\varphi(x), \varphi(x')]_- \Psi\rangle$ has a continuous extension to $A_{\ell/2}(\mathbb{W})$ for any states Φ and Ψ of form (35).

Then *a fortiori* it has a unique continuous extension to the space $\mathcal{A}_{\ell/2}(\mathbb{W})$ introduced in Sec. IV. We take Φ and Ψ to be the vectors that are defined by (32) and (33) with the following choice of test functions:

$$h(x_{k-1}, -\xi_{k-2}, \dots, -\xi_1) = h_1(x_{k-1})\mathbf{h}(\xi_1, \dots, \xi_{k-2}), \quad h_1 \in A_{3\ell/2}(\mathbb{R}^4),\ \mathbf{h} \in A_\ell(\mathbb{R}^{4(k-2)}),$$

and

$$g(x_{k+2}, \xi_{k+2}, \dots, \xi_{n-1}) = g_1(x_{k+2})\mathbf{g}(\xi_{k+2}, \dots, \xi_{n-1}), \quad g_1 \in A_{3\ell/2}(\mathbb{R}^4),\ \mathbf{g} \in A_\ell(\mathbb{R}^{4(n-k-2)}).$$

By Theorem 8 and Lemma 2, which is applicable because any Fréchet space is barrelled, the matrix element under consideration extends uniquely to a trilinear separately continuous form $\mu(\mathbf{h}, f, \mathbf{g})$ on the space $A_\ell(\mathbb{R}^{4(k-2)}) \times F \times A_\ell(\mathbb{R}^{4(n-k-2)})$, where

$$F = A_{3\ell/2}(\mathbb{R}^4) \hat{\otimes} \mathcal{A}_{\ell/2}(\mathbb{W}) \hat{\otimes} A_{3\ell/2}(\mathbb{R}^4) = \mathcal{A}_{3\ell/2,\ell/2,3\ell/2}(\mathbb{R}^4 \times \mathbb{W} \times \mathbb{R}^4).$$

Let f_0 be a fixed element of $A_\infty(\mathbb{R}^4)$ and let $\mathbf{f} \in \mathcal{A}_{2\ell,\ell,2\ell}(\mathbb{R}^4 \times \mathbb{V} \times \mathbb{R}^4)$. Then the function

$$f(x_{k-1}, x, x', x_{k+2}) = f_0\left(\frac{x + x'}{2}\right)\mathbf{f}(x_{k-1} - x, x - x', x' - x_{k+2}) \tag{36}$$

belongs to F and the map $\iota\colon \mathcal{A}_{2\ell,\ell,2\ell}(\mathbb{R}^4 \times \mathbb{V} \times \mathbb{R}^4) \to F\colon \mathbf{f} \to f$ is continuous. Using Theorem 8 again, we infer that the trilinear form $\mu(\mathbf{h}, \iota(\mathbf{f}), \mathbf{g})$ extends uniquely to a continuous linear functional on the space

$$\begin{aligned} A_\ell(\mathbb{R}^{4(k-2)}) \hat{\otimes} \mathcal{A}_{2\ell,\ell,2\ell}(\mathbb{R}^4 \times \mathbb{V} \times \mathbb{R}^4) \hat{\otimes} A_\ell(\mathbb{R}^{4(n-k-2)}) \\ = \mathcal{A}_{\ell,2\ell,\ell,2\ell,\ell}(\mathbb{R}^{4(k-2)} \times \mathbb{R}^4 \times \mathbb{V} \times \mathbb{R}^4 \times \mathbb{R}^{4(n-k-2)}). \end{aligned} \tag{37}$$

Now we consider functional (3) with W_n taken to be the n-point Wightman function of φ. If $\mathbf{h} \in A_\infty(\mathbb{R}^{4(k-2)})$, $\mathbf{f} \in A_\infty(\mathbb{R}^{4\cdot 3})$, $\mathbf{g} \in A_\infty(\mathbb{R}^{4(k-n-2)})$, and $\int f_0(X)\mathrm{d}X = 1$, then $\mu(\mathbf{h}, \iota(\mathbf{f}), \mathbf{g})$ coincides with the value of functional (3) at the test function $\mathbf{h} \otimes \mathbf{f} \otimes \mathbf{g}$. The linear span of all functions of this form is dense in space (37), and we conclude that functional (3) has a unique continuous extension to this space. *A fortiori* it can be continuously extended to $\mathcal{A}_{2\ell}(V_{(k)})$ and, by the Hahn-Banach theorem, to $A_{2\ell}(V_{(k)})$.

VII. Concluding remarks

An interesting feature of the reconstruction theorem established in this paper is the necessity of using the extended domain $D_\ell \subset \mathcal{H}$ for the operator realization of the quasilocality condition. It cannot be replaced by the invariant domain D spanned by all vectors of the form

$$\int dx_1 \dots dx_n f(x_1, \dots, x_n) \prod_{i=1}^{n} \varphi(x_i)\Psi_0, \tag{38}$$

where $f \in A_\infty(\mathbb{R}^{4n})$, because such a simplified formulation is not equivalent to the initial assumption (a.7.2) for the Wightman functionals. At this point, there is a significant difference to the situation in local QFT [1, 14, 15], where the invariant domain D_0 spanned by all vectors $\varphi(f)\varphi(g)\dots\varphi(h)\Psi_0$, with $f, g, \dots, h$ ranging over $\mathcal{D}(\mathbb{R}^4)$, is large enough for formulating the microcausality condition. Then this condition is also satisfied for the field commutator acting on any vector of form (38) with $f \in S(\mathbb{R}^{4n})$, because $\mathcal{D}(\mathbb{R}^4)^{\otimes n}$ is dense in $S(\mathbb{R}^{4n})$ and the distributions supported in a given closed set form a closed set in the space of tempered distributions. On the contrary, the subspace of $A'_\ell(\mathbb{R}^d)$ consisting of those functionals that are carried by a closed set $M \subset \mathbb{R}^d$ is everywhere dense in $A'_\ell(\mathbb{R}^d)$. Indeed, if this were not the case, then by the Hahn-Banach theorem there would exist a nontrivial function $f \in A_\ell(\mathbb{R}^d)$ such that $(u, f) = 0$ for all u in this subspace because $A_\ell(\mathbb{R}^d)$ is reflexive. In particular, $(\delta_z, f) = f(z) = 0$ for all $z \in \tilde{M}^\ell$, but this contradicts the analyticity of f.

Some examples of nonlocal but quasilocalizable fields were discussed in [12, 13]. The simplest model of this kind is the normal ordered Gaussian function $:e^{g\phi^2}:(x)$ of a free neutral scalar field ϕ. As shown in [12], the vacuum expectation values

$$\mathcal{W}_n(x_1, \dots x_n) = \langle \Psi_0, :e^{g\phi^2}:(x_1) \dots :e^{g\phi^2}:(x_n)\Psi_0\rangle, \tag{39}$$

calculated by the Wick theorem, satisfy conditions $(a.7.1)$ and $(a.7.2)$ with $\ell = \sqrt{2g/3}$. Therefore, the field $:e^{g\phi^2}:(x)$ has the properties established by Theorems 2 and 3. Moreover, these properties are characteristic of any field φ expressed as

$$\varphi(x) = \sum_{r=0}^{\infty} \frac{d_r}{r!} :\phi^r:(x), \tag{40}$$

where the coefficients d_r satisfy the inequality $d_r^2 \leq C(2g)^r r!$. It is worth noting that these fields can be implemented directly in the Fock space $\mathcal{H}_0$ of the initial field ϕ without appealing to the reconstruction theorem. A simple method of analyzing the conditions for convergence of infinite series in Wick powers of a free field has been proposed in [28]. This method systematically uses the analyticity properties of Wightman functions and is well applicable to the nonlocalizable power series.

Conditions $(a.7.1)$ and $(a.7.2)$ can be weakened by assuming that the nonlocality parameter increases with n. Such a modification would be appropriate if a future investigation of physically relevant models (related, e.g., to string theory) would give sufficient grounds. For instance, it may be suggested that for any integer $k \geq 1$, there is a positive number $\gamma(k)$ such that the following requirements are fulfilled.

$(a.7.1)'$ Every functional $\mathcal{W}_n$, $n > 1$, has a continuous extension to each of the spaces

$$A_{\ell\gamma(k-1)}(\mathbb{R}^{4(k-1)}) \hat{\otimes} A_{\ell/2}(\mathbb{R}^{4\cdot 2}) \hat{\otimes} A_{\ell\gamma(n-k-1)}(\mathbb{R}^{4(n-k-1)}), \quad 1 \leq k \leq n-1.$$

$(a.7.2)'$ For any $n \geq 2$ and $k \leq n-1$, the difference

$$\mathcal{W}_n(x_1, \dots, x_k, x_{k+1}, \dots, x_n) - \mathcal{W}_n(x_1, \dots, x_{k+1}, x_k, \dots, x_n)$$

has a continuous extension to the space

$$A_{\ell\gamma(k-1)}(\mathbb{R}^{4(k-1)}) \hat{\otimes} A_{\ell/2}(\mathbb{W}) \hat{\otimes} A_{\ell\gamma(n-k-1)}(\mathbb{R}^{4(n-k-1)}).$$

It follows from $(a.7.1)'$ that the operator-valued generalized function $f \to \varphi(f)$ constructed by Theorem 1 extends continuously to the space $A_{\ell/2}(\mathbb{R}^4)$. Moreover, this condition is equivalent to the fact that every monomial $\prod_{j=1}^{n+1} \varphi(f_j)$ extends continuously to $A_{\ell/2}(\mathbb{R}^4)\hat{\otimes} A_{\ell\gamma(n)}(\mathbb{R}^{4n})$.

The operator-valued generalized function $\varphi(f)$, $f \in A_{\ell/2}(\mathbb{R}^4)$, is thereby defined on the linear span D'_ℓ of all vectors of the form

$$\int dx_1 \dots dx_n g(x_1, \dots, x_n) \prod_{i=1}^{n} \varphi(x_i)\Psi, \quad \text{where } g \in A_{\ell\gamma(n)}(\mathbb{R}^{4n}) \text{ and } \Psi \in D.$$

Condition $(a.7.2)'$ amounts to saying that, for any $\Psi, \Phi \in D'_\ell$, the functional (6) extends continuously to the space $A_{\ell/2}(\mathbb{W})$.

It is easy to see that conditions $(a.7.1)$ and $(a.7.2)$ imply $(a.7.1)'$ and $(a.7.2)'$ with $\gamma(k) = k + 1/2$. One or the other of these formulations is preferable according to which variables, ξ_j or x_j, are more convenient to work with.

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Appendix A: Proof of Theorem 6

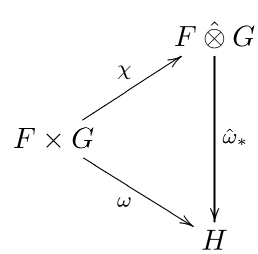
First, we recall Grothendieck's characterization [20] of the projective tensor product of complete nuclear spaces. As above, we use the standard notation $\sigma(F', F)$ for the weak topology on the dual of F. When F' is provided with this topology, one writes F'_σ. Let $\mathcal{B}(F'_\sigma, G'_\sigma)$ be the space of separately continuous bilinear forms on $F'_\sigma \times G'_\sigma$. As shown by Grothendieck (see also [18], Sec. IV.9.4), if F and G are complete locally convex spaces and at least one of them is nuclear, then $F \hat{\otimes} G$ can be identified with the space $\mathcal{B}_e(F'_\sigma, G'_\sigma)$ equipped with the topology τ_e of biequicontinuous convergence, which is determined by the set of seminorms

$$p_{U,V}(b) = \sup_{u \in U^\circ, v \in V^\circ} |b(u, v)|,$$

where U and V range, respectively, over bases of neighborhoods of 0 in F and G. The natural map $\chi\colon F \times G \to \mathcal{B}(F'_\sigma, G'_\sigma)$ takes each pair (f, g) to the bilinear form

$$(f \otimes g)(u, v) = (u, f)(v, g), \qquad u \in F', v \in G'. \tag{A1}$$

We now apply this construction to our situation. Let $\omega_*\colon F \otimes G \to H$ be the linear map associated with ω, and let $\hat{\omega}_*$ be its extension by continuity to $F \hat{\otimes} G = \mathcal{B}_e(F'_\sigma, G'_\sigma)$. By definition, $\hat{\omega}_*$ is a unique continuous map for which the diagram



is commutative. All we need to do is to show that $\hat{\omega}_*$ is an algebraic and topological isomorphism. We first prove that $\hat{\omega}_*$ is injective. By definition,

$$\hat{\omega}_*(f \otimes g)(x, y) = \omega(f, g)(x, y) = f(x)g(y), \qquad f \in F,\, g \in G. \tag{A2}$$

Let $b \in \ker \hat{\omega}_*$ and let $\{b_\gamma\}_{\gamma \in \Gamma}$ be a net in $F \otimes G$ such that $b_\gamma \to b$ in the topology of $F \hat{\otimes} G$. Then $\hat{\omega}_*(b_\gamma) \to 0$ in H. We define δ_x and δ_y to be the linear functionals on F and G such that $(\delta_x, f) = f(x)$ and $(\delta_y, g) = g(y)$. Since the topologies of F and G are not weaker than that of pointwise convergence, these functionals are continuous and belong, respectively, to F' and G'. For the bilinear forms b_γ, we have the relation

$$b_\gamma(\delta_x, \delta_y) = \hat{\omega}_*(b_\gamma)(x, y). \tag{A3}$$

Indeed, if $b_\gamma = f \otimes g$, then (A3) follows immediately from (A1), (A2), and each element of $F \otimes G$ is a finite sum of elements of this form. Passing to the limit, we obtain $b(\delta_x, \delta_y) = 0$, because the topology of $\mathcal{B}_e(F'_\sigma, G'_\sigma)$ and H is not weaker than the topology of pointwise convergence. By the Hahn-Banach theorem the sets $\{\delta_x \in F' \colon x \in X\}$ and $\{\delta_y \in G' \colon y \in Y\}$ are total[2] in F'_σ and G'_σ because the duals of the latter coincide with F and G. Therefore, b is identically zero and we conclude that $\ker \hat{\omega}_* = 0$.

To prove that $\hat{\omega}_*$ is surjective, we use that every complete nuclear space is semi-reflexive and hence the strong topology of G' is compatible with the duality between G and G', see [18], Sec. IV.5. Because of this, condition (iii) implies that, for each fixed $h \in H$, the map $v \to h_v$ is weakly continuous, i.e., is continuous under the topologies $\sigma(G', G)$ and $\sigma(F, F')$ (ibid, Sec. IV.7.4). Therefore, the bilinear form $b_h \colon (u, v) \to (u, h_v)$ belongs to $\mathcal{B}(F'_\sigma, G'_\sigma)$. Clearly, we have the identity $b_h(\delta_x, \delta_y) = h(x, y)$. Considering a net in $F \otimes G$ which converges to b_h in $F \hat{\otimes} G = \mathcal{B}_e(F'_\sigma, G'_\sigma)$ and using again that the topology of $\mathcal{B}_e(F'_\sigma, G'_\sigma)$ and H is not weaker than that of pointwise convergence, we obtain the equality $\hat{\omega}_*(b_h)(x, y) = h(x, y)$. Since it holds for all x and y, we conclude that $\hat{\omega}_*(b_h) = h$ and ω_* is hence surjective. It remains to prove that the inverse map ω_*^{-1} is continuous. We must show that for each neighborhood U of 0 in F and for each neighborhood V of 0 in G, there is a neighborhood W of 0 in H such that $p_{U,V}(b_h) \leq 1$ for all $h \in W$, or equivalently,

$$\sup_{h \in W, u \in U^\circ, v \in V^\circ} |(u, h_v)| \leq 1. \tag{A4}$$

The set $\{h_v \in F \colon v \in V^\circ\}$ is bounded because the polar V° is bounded in G' and the map $v \to h_v$ is continuous for every fixed $h \in H$. Therefore, the family of continuous linear maps $H \to F \colon h \to h_v$, where v ranges V°, is pointwise bounded. Since H is assumed to be barrelled, it follows that this family is equicontinuous (ibid, Sec. III.4.2). Thus, there exists a neighborhood W of 0 in H such that $h_v \in U$ for all $h \in W$ and for all $v \in V^\circ$. Then (A4) holds by the definition of the polar. This completes the proof of Theorem 6.

Appendix B: Two forms of the kernel theorem

Theorem 9. *Let F, G, and H be Fréchet spaces, and let ω be a continuous bilinear map from $F \times G$ to H. Suppose that for each continuous bilinear form $b \colon F \times G \to \mathbb{C}$, there is a*

[2] A subset of a locally convex space E is called total in E if its linear span is dense in E.

unique linear functional $u_b \in H'$ *such that* $b = u_b \circ \omega$. *Then*

$$F \hat{\otimes} G = H. \tag{B1}$$

More precisely, the continuous extension of the map $\omega_* \colon F \otimes G \to H$ *to the completion of the projective tensor product* $F \otimes_\pi G$ *is an algebraic and topological isomorphism.*

Proof. The dual of $F \otimes_\pi G$ is identified with the space $\mathcal{B}(F, G)$ of all continuous bilinear forms on $F \times G$ (see [18], Sec. III.6.2). We denote by $\hat{\omega}_*$ the continuous extension of ω_* to $F \hat{\otimes} G$ and consider the dual map $\hat{\omega}'_* \colon H' \to (F \hat{\otimes} G)' = \mathcal{B}(F, G)$. By definition,

$$\hat{\omega}'_*(u)(x \otimes y) = u(\omega(x \otimes y)) = (u \circ \omega)(x, y), \quad u \in H'.$$

It follows from our assumptions that the map $\hat{\omega}'_*$ is bijective. Therefore, $\hat{\omega}_*$ is injective and has a dense image. Moreover, $F \hat{\otimes} G$ is a Fréchet space (ibid, Sec. III.6.3) and hence the image of $\hat{\omega}_*$ is closed because that of $\hat{\omega}'_*$ is weakly closed (ibid, Sec. IV.7.7). Thus, $\hat{\omega}_*$ is an algebraic isomorphism. Using the open mapping theorem, we conclude that this map is also a topological isomorphism, which completes the proof.

Corollary: Let the assumptions of Theorem 9 be satisfied, and let E *be a locally convex space. Then for every separately continuous bilinear map* $\beta \colon F \times G \to E$*, there is a unique continuous linear map* $u_\beta \colon H \to E$ *such that* $\beta = u_\beta \circ \omega$.

Indeed, this statement expresses the category theoretical meaning of formula (B1).

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