Topological algebras of rapidly decreasing matrices and generalizations

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Abstract

It is a folklore fact that the rapidly decreasing matrices of countable size form an associative topological algebra whose set of quasi-invertible elements is open, and such that the quasi-inversion map is continuous. We provide a direct proof, which applies more generally to a large class of algebras of weighted matrices with entries in a Banach algebra.

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If $(\mathcal{A}, \|.\|)$ is a Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{W}$ a non-empty set of monotonically increasing functions $f\colon \mathbb{N} \to]0,\infty[$, we define $M(\mathcal{A},\mathcal{W})$ as the set of all $T = (t_{ij})_{i,j\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}\times\mathbb{N}}$ such that

$$\|T\|_f := \sup_{i,j\in\mathbb{N}} f(i\vee j)\|t_{ij}\| \,<\, \infty$$

for all $f \in \mathcal{W}$, where $i \vee j$ denotes the maximum of i and j. It is clear that $M(\mathcal{A},\mathcal{W})$ is a vector space; we give it the locally convex Hausdorff vector topology defined by the set of norms $\{\|.\|_f\colon f \in \mathcal{W}\}$. We show:

Theorem. *Assume there exists $g \in \mathcal{W}$ such that $C_g := \sum_{n=1}^{\infty} \frac{1}{g(n)} < \infty$. If $R = (r_{ij})_{i,j\in\mathbb{N}}, S = (s_{ij})_{i,j\in\mathbb{N}} \in M(\mathcal{A},\mathcal{W})$ and $i, j \in \mathbb{N}$, then the series*

$$t_{ij} := \sum_{k=1}^{\infty} r_{ik}s_{kj}$$

converges absolutely in $\mathcal{A}$. Moreover, $RS := (t_{ij})_{i,j\in\mathbb{N}} \in M(\mathcal{A},\mathcal{W})$, and the multiplication defined in this way makes $M(\mathcal{A},\mathcal{W})$ a locally m-convex, associative topological algebra which is complete as a topological vector space, has an open set of quasi-invertible elements, and whose quasi-inversion map is continuous.

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Recall that an element x in an associative (not necessarily unital) algebra A is called *quasi-invertible* if there exists $y \in A$ such that $xy = yx$ and $x + y - xy = 0$. The element $q(x) := y$ is then unique and is called the *quasi-inverse* of x. Locally convex topological algebras A with an open set $Q(A)$ of quasi-invertible elements and continuous quasi-inversion map $q\colon Q(A) \to A$ are called *continuous quasi-inverse algebras* (and *continuous inverse algebras* if they have, moreover, a unit element). See [6] for information on such algebras as well as [2] and [4], where such algebras are inspected due to their usefulness in infinite-dimensional Lie theory. Also recall that a topological algebra A is called *locally m-convex* if its vector topology can be defined using a set of seminorms $p\colon A \to [0, \infty[$ which are sub-multiplicative, i.e., $p(xy) \leq p(x)p(y)$ for all $x, y \in A$. If A is, moreover, complete as a topological vector space, this means that A is a projective limit of Banach algebras [5].

If we take $\mathcal{W} := \{f_m \colon m \in \mathbb{N}_0\}$ with $f_m(n) := n^m$, and $\mathcal{A} := \mathbb{C}$, then $M(\mathbb{C}, \mathcal{W})$ is the so-called algebra of rapidly decreasing matrices. It is a folklore fact that this algebra is a continuous quasi-inverse algebra, but the authors do not know a reference for a direct proof. Our original motivation was to close this gap in the literature and give a self-contained discussion of this algebra. But then it became clear that a whole new class of algebras can be treated by the same argument, and that only a simple condition (the existence of g with $C_g < \infty$) has to be imposed on the set of weights.

Proof of the theorem. Step 1. Let $R = (r_{ij})_{i,j\in\mathbb{N}}$ and $S = (s_{ij})_{i,j\in\mathbb{N}}$ be in $M(\mathcal{A}, \mathcal{W})$. We show that the series $t_{ij} := \sum_{k=1}^{\infty} r_{ik}s_{kj}$ converge absolutely in $\mathcal{A}$, and that $T := (t_{ij})_{i,j\in\mathbb{N}} \in M(\mathcal{A}, \mathcal{W})$. To this end, let $f, g \in \mathcal{W}$ with $C_g < \infty$. If $i \geq j$, we have $i \vee j = i \leq i \vee k$ for all $k \in \mathbb{N}$, hence $f(i \vee j) \leq f(i \vee k)$ by monotonicity and thus

$$\begin{aligned} f(i \vee j) \sum_{k=1}^{\infty} \|r_{ik}\|\,\|s_{kj}\| &= \sum_{k=1}^{\infty} f(i \vee j)\|r_{ik}\|\,\|s_{kj}\| \leq \sum_{k=1}^{\infty} f(i \vee k)\|r_{ik}\|\,\|s_{kj}\| \\ &\leq \|R\|_f \sum_{k=1}^{\infty} \|s_{kj}\| \leq \|R\|_f \sum_{k=1}^{\infty} \underbrace{g(k \vee j)\|s_{kj}\|}_{\leq \|S\|_g} \frac{1}{g(k \vee j)} \\ &\leq \|R\|_f \|S\|_g \sum_{k=1}^{\infty} \frac{1}{g(k \vee j)} \leq C_g\, \|R\|_f \|S\|_g < \infty, \quad (1) \end{aligned}$$

using that g is monotonically increasing for the penultimate inequality. If

$i \leq j$, the same argument shows that

$$f(i \vee j) \sum_{k=1}^{\infty} \|r_{ik}\| \, \|s_{kj}\| \leq C_g \, \|R\|_g \|S\|_f < \infty. \tag{2}$$

In particular, in either case $\sum_{k=1}^{\infty} \|r_{ik}\| \, \|s_{kj}\| < \infty$, whence indeed $\sum_{k=1}^{\infty} r_{ik} s_{kj}$ converges absolutely. Now (1) and (2) show that $SR := T := (t_{ij})_{i,j \in \mathbb{N}} \in M(\mathcal{A}, \mathcal{W})$, with

$$\|SR\|_f \leq C_g \left(\|R\|_f \|S\|_g \vee \|R\|_g \|S\|_f \right). \tag{3}$$

Step 2: We show that the multiplication just defined is associative. To this end, let $R = (r_{ij})_{i,j \in \mathbb{N}}$, $S = (s_{ij})_{i,j \in \mathbb{N}}$ and $T = (t_{ij})_{i,j \in \mathbb{N}}$ be in $M(\mathcal{A}, \mathcal{W})$. Let R', S' and T' be the matrices with entries $\|r_{ij}\|$, $\|s_{ij}\|$ and $\|t_{ij}\|$, respectively. Then $R', S', T' \in M(\mathbb{R}, \mathcal{W})$, as is clear from the definitions. Hence

$$\begin{aligned} \sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} \|r_{i\ell}\| \, \|s_{\ell k}\| \, \|t_{kj}\| &= \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} \|r_{i\ell}\| \, \|s_{\ell k}\| \, \|t_{kj}\| = \sum_{\ell=1}^{\infty} \|r_{i\ell}\| (S'T')_{\ell j} \\ &= (R'(S'T'))_{ij} \in \mathbb{R} \end{aligned}$$

(where the first equality is a well-known elementary fact, which can also be infered by applying Fubini's Theorem to the counting measures on $\mathbb{N}^2$ and $\mathbb{N}$). Thus $\sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} \|r_{i\ell} s_{\ell k} t_{kj}\| < \infty$, showing that the family $(r_{ik} s_{k\ell} t_{\ell j})_{(k,\ell) \in \mathbb{N} \times \mathbb{N}}$ of elements of $\mathcal{A}$ is absolutely summable. As a consequence,

$$\begin{aligned} ((RS)T)_{ij} &= \sum_{k=1}^{\infty} (RS)_{ik} t_{kj} = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} r_{i\ell} s_{\ell k} t_{kj} = \sum_{(k,\ell) \in \mathbb{N} \times \mathbb{N}} r_{i\ell} s_{\ell k} t_{kj} \\ &= \sum_{\ell=1}^{\infty} \sum_{k=1}^{\infty} r_{i\ell} s_{\ell k} t_{kj} = \sum_{\ell=1}^{\infty} r_{i\ell} (ST)_{\ell j} = (R(ST))_{ij} \end{aligned}$$

using [1, 5.3.6] for the third and fourth equalities. Thus $(RS)T = R(ST)$.

Step 3. The locally convex space $M(\mathcal{A}, \mathcal{W})$ is complete. To see this, note first that $M(\mathcal{A}, \{f\})$ (with the norm $\|.\|_f$) is a Banach space isomorphic to the space $\ell^\infty(\mathcal{A})$ of bounded $\mathcal{A}$-valued sequences, for each $f \in \mathcal{W}$. Next, after replacing $\mathcal{W}$ with the set of finite sums of elements of $\mathcal{W}$ (which changes neither $M(\mathcal{A}, \mathcal{W})$ as a set, nor its topology), we may assume henceforth that $\mathcal{W} + \mathcal{W} \subseteq \mathcal{W}$ and hence that $\mathcal{W}$ is upward directed. Then $M(\mathcal{A}, \mathcal{W})$ is the projective limit of the complete spaces $M(\mathcal{A},\{f\})$ ($f \in \mathcal{W}$) and hence complete.

Step 4. We show that the set Q of quasi-invertible elements in $M(\mathcal{A},\mathcal{W})$ is open. By [2, Lemma 2.6], we need only check that Q is a 0-neighbourhood. To this end, choose $g \in \mathcal{W}$ such that $C_g < \infty$. Then

$$\Big\{T \in M(\mathcal{A},\mathcal{W})\colon \|T\|_g < \frac{1}{C_g}\Big\} \subseteq Q\,.$$

Indeed, pick T in the left hand side. We claim that

$$(\forall n \in \mathbb{N}) \qquad \|T^n\|_f \leq (C_g\|T\|_g)^{n-1}\|T\|_f \tag{4}$$

for each $f \in \mathcal{W}$. If this is true, then $\sum_{n=1}^{\infty} T^n$ converges in each of the Banach spaces $(M(\mathcal{A},\{f\}),\|.\|_f)$ and hence also in the projective limit $M(\mathcal{A},\mathcal{W})$. Now the usual argument shows that $\sum_{n=1}^{\infty} T^n$ is the quasi-inverse of T.

To prove the claim, we proceed by induction. If $n = 1$, then $\|T\|_f = (C_g\|T\|_g)^0\|T\|_f$. If the claim holds for $n-1$ in place of n, writing $A^n = A^{n-1}A$ we deduce from (3) that

$$\|T^n\|_f \leq C_g\left(\|T^{n-1}\|_f\|T\|_g \vee \|T^{n-1}\|_g\|T\|_f\right). \tag{5}$$

Now

$$C_g\|T^{n-1}\|_f\|T\|_g \leq C_g(C_g\|T\|_g)^{n-2}\|T\|_f\|T\|_g = (C_g\|T\|_g)^{n-1}\|T\|_f \tag{6}$$

by induction. Likewise,

$$C_g\|T^{n-1}\|_g\|T\|_f \leq C_g(C_g\|T\|_g)^{n-2}\|T\|_g\|T\|_f = (C_g\|T\|_g)^{n-1}\|T\|_f, \tag{7}$$

applying the inductive hypothesis to g and g in place of f and g. Combining (5), (6) and (7), we see that $\|T^n\|_f \leq (C_g\|T\|_g)^{n-1}\|T\|_f$, which completes the inductive proof.

Step 5. $M(\mathcal{A},\mathcal{W})$ is locally m-convex. To see this, pick $g \in \mathcal{W}$ with $C_g < \infty$. After replacing $\mathcal{W}$ with $\{f+g\colon f \in \mathcal{W}\}$ (which changes neither $M(\mathcal{A},\mathcal{W})$ as a set nor its topology), we may assume henceforth that $C_f < \infty$ for each $f \in \mathcal{W}$. We may therefore choose $g := f$ in (3) and obtain

$$\|RS\|_f \leq C_f\|R\|_f\|S\|_f\,.$$

Let $h := C_f \cdot f$. Then $C_h = \frac{1}{C_f}\sum_{n=1}^\infty \frac{1}{f(n)} = 1$ and $\|.\|_f$ is equivalent to the norm $\|.\|_h$, which is submultiplicative as $\|RS\|_h \leq C_h\|R\|_h\|S\|_h = \|R\|_h\|S\|_h$.

Step 6. Continuity of quasi-inversion. Since we assume that $C_f < \infty$ for each $f \in \mathcal{W}$, we know from Step 5 that $M(\mathcal{A}, \{f\})$ is a Banach algebra, with respect to a submultiplicative norm $\|.\|_h$ which is equivalent to $\|.\|_f$. Now, as we assume that $\mathcal{W} + \mathcal{W} \subseteq \mathcal{W}$ (see Step 3), $M(\mathcal{A}, \mathcal{W})$ is the projective limit of the Banach algebras $M(\mathcal{A}, \{f\})$ ($f \in \mathcal{W}$). Because quasi-inversion is continuous in each of the Banach algebras, and continuity of maps into projective limits can be checked componentwise, it follows that quasi-inversion is continuous on $Q \subseteq M(\mathcal{A}, \mathcal{W})$. □

Remark. Our results were first recorded in the unpublished thesis [3].

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