A Natural Occurrence of Shift Equivalence

Fatma Muazzez Şimşir
Çınar Sokak No:115/2
06170 Ankara, Turkey
E-mail: fsimsir@gmail.com

Cem Tezer
Department of Mathematics,
Middle East Technical University
06535 Ankara, Turkey
E-mail: rauf@metu.edu.tr

Abstract

A natural occurrence of shift equivalence in a purely algebraic setting constitutes the subject matter of the following short exposition.

1 Introduction

Group endomorphisms $\alpha : G \longrightarrow G$, $\beta : H \longrightarrow H$, are said to be *conjugate* if there exists an isomorphism $\theta : G \longrightarrow H$ such that $\theta \circ \alpha = \beta \circ \theta$. $\alpha : G \longrightarrow G$ and $\beta : H \longrightarrow H$ are said to be *shift equivalent* if there exist group endomorphisms $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow G$ and $n \in \mathbb{Z}^+$ such that the relations

$$\begin{aligned} \varphi \circ \alpha &= \beta \circ \varphi \\ \psi \circ \beta &= \alpha \circ \psi \\ \psi \circ \varphi &= \alpha^n \\ \varphi \circ \psi &= \beta^n \end{aligned}$$

hold, equivalently, the diagrams

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"] \\ H \arrow[r, "\beta"] & H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\alpha"] & G \\ H \arrow[r, "\beta"] \arrow[u, "\psi"] & H \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\alpha^n"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"] \\ H \arrow[r, "\beta^n"] \arrow[ur, "\psi"] & H \end{tikzcd}$$

commute.This state of affairs is described by saying that φ, ψ effect a shift equivalence of α to β of lag $n \in \mathbb{Z}^+$.

The concept of shift equivalence was introduced by R. F. Williams [W1], [W2] in

2010 *Mathematics Subject Classification*: Primary 37C15; Secondary 37B10, 37B45.
Key words and phrases: Shift equivalence, simple direct limit.

the context of topological dynamics. The fact that shift equivalence is an equivalence relation among group endomorphisms can be demostrated by a straightforward argument. [T1]
Clearly both conjugacy and shift equivalence can be defined in any category and the former constitutes a special case of the latter in two ways:

- A shift equivalence with lag 0 is a conjugacy.

- A shift equivalence between two automorphisms is a conjugacy.

The simple result presented here was independently observed by Yu. I. Ustinov in a short report [U]. In our opinion this is the most straightforward and natural occurrence of shift equivalence as complete invariant. Although by no means entirely novel, we feel that this elegant result deserves to be available to a wider public in the form of an independent exposition.
Another very natural occurrence of shift equivalence arises in shape and homotopy theory, [T2].

2 Statement and proof of the main result:

Given a group endomorphism $\alpha : G \longrightarrow G$ the *simple direct limit* of α, denoted by $\mathfrak{G} = \lim_{\rightarrow}(G, \alpha)$, is the set of equivalence classes in $G \times \mathbb{Z}^+$ under the equivalence relation $\sim$ where

$$(g, n) \sim (g', n')$$

iff

$$\alpha^{N-n}(g) = \alpha^{N-n'}(g')$$

for some $N \geq n, n'$.

$\sim$ can be easily checked to be an equivalence relation. $\mathfrak{G}$ has a natural group structure with respect to the binary operation

$$(g, n)(g', n') = (\alpha^{n'}(g)\alpha^{n}(g'), n + n')$$

where, by abuse of notation, we let (g, n) stand for the equivalence class it represents. Again it can be routinely checked that this is a well-defined operator satisfying all group axioms. There are two natural isomorphisms on $\mathfrak{G}$: Firstly,

$$\check{\alpha} : \mathfrak{G} \longrightarrow \mathfrak{G}$$

defined by

$$\check{\alpha}((g, n)) = (\alpha(g), n)$$

secondly,

$$s_\alpha : \mathfrak{G} \longrightarrow \mathfrak{G}$$

(which we like to call the "coshift") defined by

$$s_\alpha((g,n)) = (g, n+1).$$

Again, well-definedness and morphology need checking. To see that $\breve{\alpha}$ and s_α are isomorphisms it is enough to observe that

$$\breve{\alpha} \circ s_\alpha = s_\alpha \circ \breve{\alpha} = Id(\mathfrak{G}).$$

Theorem 2.1. *Let G and H be finitely generated groups , $\alpha : G \longrightarrow G$, $\beta : H \longrightarrow H$ group endomorphisms, $\mathfrak{G} = \lim_{\rightarrow}(G,\alpha)$, $\mathfrak{H} = \lim_{\rightarrow}(H,\beta)$. The isomorphisms $s_\alpha : \mathfrak{G} \longrightarrow \mathfrak{G}$, $s_\beta : \mathfrak{H} \longrightarrow \mathfrak{H}$ are conjugate iff α, β are shift equivalent.*

Proof. Given a subset K of a group, let $\langle K \rangle$ denote the subgroup generated by K. There exist finite sets $A \subseteq G$, $B \subseteq H$ such that $G = \langle A \rangle$, $H = \langle B \rangle$. Assume first that s_α and s_β , or equivalently, $\breve{\alpha}$ and $\breve{\beta}$ are conjugate: There exists an isomorphism

$$T : \mathfrak{G} \longrightarrow \mathfrak{H}$$

such that

$$T \circ \breve{\alpha} = \breve{\beta} \circ T.$$

Let

$$i_\alpha : G \longrightarrow \mathfrak{G}$$

$$i_\beta : H \longrightarrow \mathfrak{H}$$

be the natural injections defined by

$$i_\alpha(g) = (g, 0) \in \mathfrak{G}$$

$$i_\beta(h) = (h, 0) \in \mathfrak{H}.$$

We have

$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle.$$

Clearly $T \circ i_\alpha(A)$ is a finite subset of $\mathfrak{H}$. Hence there exists $k \in \mathbf{Z}^+$ such that

$$T \circ i_\alpha(G) \subseteq \langle T \circ i_\alpha(A) \rangle \subseteq H \times \{k\}.$$

Therefore,

$$\breve{\beta}^k \circ T \circ i_\alpha(G) \subseteq H \times \{0\}.$$

We define

$$\varphi = i_\beta^{-1} \circ \breve{\beta}^k \circ T \circ i_\alpha : G \longrightarrow H.$$

Similarly, there exists a sufficiently large $l \in \mathbf{Z}_+$ such that

$$\psi = i_\alpha^{-1} \circ \breve{\alpha}^l \circ T \circ i_\alpha : H \longrightarrow G$$

is a well defined homomorphism. We claim that φ and ψ effect a shift equivalence of α to β with lag $k+l \in \mathbf{Z}^+$: Clearly

$$\varphi \circ \alpha = \beta \circ \varphi$$

$$\psi \circ \beta = \alpha \circ \psi.$$

Moreover,

$$\psi \circ \varphi = i_\alpha^{-1} \circ \check{\alpha}^l \circ T^{-1} \circ i_\beta \circ i_\beta^{-1} \circ \check{\beta}^k \circ T \circ i_\alpha = \alpha^{k+l}.$$

Similarly,

$$\varphi \circ \psi = \beta^{k+l}.$$

Conversely assume, that there exist $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow G$ and $n \in \mathbf{Z}^+$ such that $\varphi \circ \alpha = \beta \circ \phi$, $\psi \circ \beta = \alpha \circ \psi$, $\psi \circ \varphi = \alpha^n$, $\phi \circ \psi = \beta^n$. Consider the map

$$E : \mathfrak{G} \longrightarrow \mathfrak{H}$$

defined by

$$E((g,m)) = (\varphi(g), m)$$

and note that E is well-defined: If $\alpha^{l-m}(g) = \alpha^{l-m'}(g')$, then

$$\varphi \circ \alpha^{l-m}(g) = \varphi \circ \alpha^{l-m'}(g')$$

hence

$$\beta^{l-m} \circ \varphi(g) = \beta^{l-m'} \circ \varphi(g').$$

We have also $E \circ \check{\alpha} = \check{\beta} \circ E$ owing to $\varphi \circ \alpha = \beta \circ \varphi$, once again. Similarly define

$$F : \mathfrak{H} \longrightarrow \mathfrak{G}$$

by

$$F((h,m)) = (\psi(h), m).$$

We observe

$$\begin{aligned} F \circ E((g,m)) &= F((\varphi(g), m)) = (\psi \circ \varphi(g), m) \\ &= (\alpha^n(g), m) \\ &= \check{\alpha}^n(g,m). \end{aligned}$$

Thus $F \circ E = \check{\alpha}^n$. The right hand side is an isomorphism, E is an isomorphism, too, which commutes with $\check{\alpha}$ and $\check{\beta}$. □

References

[T1] C. Tezer, *The shift on the inverse limit of a covering projection*, Israel Journal of Mathematics 59 (1987), 129–149.

[T2] C. Tezer, *Shift equivalence in homotopy*, Mathematische Zeitschrift 210 (1992), 197–201.

[U] Yu. I. Ustinov, *Algebraic invariants of topological conjugacy of solenoids*, Mat. Zametki 42 (1987), 132–144.

[W1] R. F. Williams, *Non-wandering 1-dimensional sets*, Topology 6 (1967), 473–487.

[W2] R. F. Williams, *Classification of 1-dimensional attractors*, Proc. Symp. Pure Math 14 (1970), 341–361.