Determination of the Integrated Sidelobe Level of Sets of Rotated Legendre Sequences

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***Abstract*—Sequences sets with low aperiodic auto- and cross-correlations play an important role in many applications like communications, radar and other active sensing applications. The use of antipodal sequences reduces hardware requirements while increases the difficult of the task of signal design. In this paper we present a method for the computation of the Integrated Sidelobe Level (ISL), and we use it to calculate the asymptotic expression for the ISL of a set of sequences formed by different rotations of a Legendre sequence.**

***Index Terms*—Integrated Sidelobe Level, antipodal sequences, Legendre Sequences, auto-correlation, cross-correlation.**

I. Introduction

THE design of sequences set with good correlation properties is present in many fields of engineering such as radar, sonar, communications, medical imaging and so on. Good auto-correlation properties means that any sequence in the set is nearly uncorrelated with its own shifted version while good cross-correlation means that any member of the sequences set is nearly uncorrelated with any other members at any shift.

A commonly used metric of the goodness of the correlation is the *Integrated Sidelobe Level* (ISL). The ISL of a set of M sequences each of N (possibly complex) symbols that we will indicate with $x_j^{(p)}$ with $j = 0, \ldots, N-1$ and $p = 0, \ldots, M-1$ is defined as

$$\mathrm{ISL} = \sum_{p=0}^{M-1} \sum_{\substack{k=-N+1 \\ k\neq 0}}^{N-1} \left|X_{x^{(p)}x^{(p)}}(k)\right|^2 + \sum_{p=0}^{M-1} \sum_{\substack{q=0 \\ p\neq q}}^{M-1} \sum_{k=-N+1}^{N-1} |X_{x^{(p)}x^{(q)}}(k)|^2$$

where

$$X_{x^{(p)}x^{(p)}}(k) = \sum_{j=\max\{0,-k\}}^{\min\{N-k,N\}-1} x_j^{(p)} x_{j+k}^{*(p)} \quad k = -N+1 \ldots N-1$$

is is the auto-correlation of the sequence $\boldsymbol{x}^{(p)}$, and

$$X_{x^{(p)}x^{(q)}}(k) = \sum_{j=\max\{0,-k\}}^{\min\{N-k,N\}-1} x_j^{(p)} x_{j+k}^{*(q)} \quad k = -N+1 \ldots N-1$$

is the cross-correlation between the sequences $\boldsymbol{x}^{(p)}$ and $\boldsymbol{x}^{(q)}$.

Good set of sequences are those having a low ISL value. Due to the strong interest in the design of sequences with low ISL value, many algorithms have been suggested for its minimization. Our purpose is to develop an analytical expression that may drive optimization in some particular difficult cases, most notably when the antipodal constrain ($x_j^p = \pm 1$) is imposed.

To facilitate the discussion, denote the sum of squares corresponding to the auto-correlation terms as

$$\mathbb{X}_{x^{(p)}x^{(p)}} = \sum_{\substack{k=-N+1 \\ k\neq 0}}^{N-1} \left|X_{x^{(p)}x^{(p)}}(k)\right|^2 \tag{1}$$

and the sum of squares corresponding to the cross-correlation terms as

$$\mathbb{X}_{x^{(p)}x^{(q)}} = \sum_{k=-N+1}^{N-1} |X_{x^{(p)}x^{(q)}}(k)|^2 \qquad p \neq q \tag{2}$$

so that

$$\mathrm{ISL} = \sum_{p=0}^{M-1} \mathbb{X}_{x^{(p)}x^{(p)}} + \sum_{p=0}^{M-1} \sum_{\substack{q=0 \\ p\neq q}}^{M-1} \mathbb{X}_{x^{(p)}x^{(q)}} \tag{3}$$

A general method for the calculation of $\mathbb{X}_{x^{(p)}x^{(p)}}$ of any sequences of odd length is presented in [1], [2]. This method hinges on generating functions and writes correlations as proper sums of their values on the unit circle in the complex plane. The method works well when we have analytical insights on the generating functions.

Extending the ideas of [1], in section II we devise a general method for the calculation of $\mathbb{X}_{x^{(p)}x^{(q)}}$ in (2) of any pair of real sequences of odd length and thus, together with the result in [1], [2], the ISL for a set of sequences. In section III we use this method to obtain an asymptotic expression for the ISL value of a set formed by different rotations of Legendre sequences. Some minor results about an optimization procedure based on the latter expression are reported in [3], where we find the optimal rotations that minimize the ISL for any sequences length N.

Throughout the paper we use the following asymptotic notation.

We say that

- two sequences a_N and b_N are asymptotically equivalent, $a_N \sim b_N$ iff

$$\lim_{N\to\infty} \frac{a_N}{b_N} = 1$$

- a_N is asymptotically bounded by b_N, $a_N = O(b_N)$ iff

$$\exists M > 0 \text{ and } \exists N_o \quad \Big| \quad |a_N| \leq M\,|b_N| \quad \forall N > N_o$$

II. Calculation of the cross-correlation terms in the ISL

Let $a_0, a_1, \ldots, a_{N-1}$ and $b_0, b_1, \ldots, b_{N-1}$ be two real sequences of length N, we want to obtain an expression for $\mathbb{X}_{ab}$.

If we define the generating functions of the two sequences as

$$Q_a(z) = \sum_{j=0}^{N-1} a_j z^j \qquad Q_b(z) = \sum_{j=0}^{N-1} b_j z^j$$

we have that

$$Q_a(z)Q_b^*(z) = \sum_{k=-N+1}^{N-1} X_{ab}(k) z^{-k}$$

and thus

$$|Q_a(z)Q_b^*(z)|^2 = \sum_{k=-N+1}^{N-1} \sum_{l=-N+1}^{N-1} X_{ab}(k) X_{ab}(l) z^{-k+l}$$

Now, set $\varepsilon_j = e^{\frac{2\pi \mathbf{i}}{N} j}$ and note that for $k, l = -N+1, \ldots, N-1$,

$$\sum_{j=0}^{N-1} \varepsilon_j^{-k+l} = \begin{cases} N & \text{if } -l+k = -N, 0, N \\ 0 & \text{otherwise} \end{cases}$$

Hence, if we define

$$S' = \sum_{j=0}^{N-1} \left|Q_a(\varepsilon_j)Q_b^*(\varepsilon_j)\right|^2 = N \sum_{k=-N+1}^{N-1} X_{ab}^2(k) + N \sum_{k=1}^{2N-1} X_{ab}(k) X_{ab}(k-N) + N \sum_{k=-N+1}^{-1} X_{ab}(k) X_{ab}(k+N)$$

and (for N odd)

$$S'' = \sum_{j=0}^{N-1} \left|Q_a(-\varepsilon_j)Q_b^*(-\varepsilon_j)\right|^2 = N \sum_{k=-N+1}^{N-1} X_{ab}^2(k) + -N \sum_{k=1}^{2N-1} X_{ab}(k) X_{ab}(k-N) - N \sum_{k=-N+1}^{-1} X_{ab}(k) X_{ab}(k+N)$$

we can express $\mathbb{X}_{ab}$ (i.e. the sum of squares of cross-correlations as in (2)) as

$$\mathbb{X}_{ab} = \sum_{k=-N+1}^{N-1} X_{ab}^2(k) = \frac{S' + S''}{2N}$$

To compute S'' we use the Lagrange interpolation polynomials to calculate the values of $Q_a(-\varepsilon_j)$ from $Q_a(\varepsilon_k)$ for $j, k = 0, \ldots, N-1$. In this special case the data points (ε_k) coincide with the complex roots of unity and, for N odd, the Lagrange base polynomials simply reduce to $\frac{2}{N}\frac{\varepsilon_k}{\varepsilon_j + \varepsilon_k}$ [4, p. 89]. Then

$$Q_a(-\varepsilon_j) = \frac{2}{N} \sum_{k=0}^{N-1} \frac{\varepsilon_k}{\varepsilon_j + \varepsilon_k} Q_a(\varepsilon_k) \tag{4}$$

By substituting (4) into S'' and developing the product $\left|Q_a(-\varepsilon_j)Q_b^*(-\varepsilon_j)\right|^2$ we get

$$\begin{aligned} S'' = & \frac{16}{N^4} \sum_{j=0}^{N-1} \left[\sum_{k_1=0}^{N-1} \frac{\varepsilon_{k_1}}{\varepsilon_j + \varepsilon_{k_1}} Q_a(\varepsilon_{k_1}) \sum_{l_1=0}^{N-1} \frac{\varepsilon_{l_1}^*}{\varepsilon_j^* + \varepsilon_{l_1}^*} Q_a^*(\varepsilon_{l_1}) \right. \\ & \left. \sum_{k_2=0}^{N-1} \frac{\varepsilon_{k_2}}{\varepsilon_j + \varepsilon_{k_2}} Q_b(\varepsilon_{k_2}) \sum_{l_2=0}^{N-1} \frac{\varepsilon_{l_2}^*}{\varepsilon_j^* + \varepsilon_{l_2}^*} Q_b^*(\varepsilon_{l_2}) \right] \\ = & \frac{16}{N^4} \sum_{k_1=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{k_2=0}^{N-1} \sum_{l_2=0}^{N-1} Q_a(\varepsilon_{k_1}) Q_a^*(\varepsilon_{l_1}) Q_b(\varepsilon_{k_2}) Q_b^*(\varepsilon_{l_2}) \\ & \sum_{j=0}^{N-1} \frac{\varepsilon_{k_1}}{\varepsilon_j + \varepsilon_{k_1}} \frac{\varepsilon_{l_1}^*}{\varepsilon_j^* + \varepsilon_{l_1}^*} \frac{\varepsilon_{k_2}}{\varepsilon_j + \varepsilon_{k_2}} \frac{\varepsilon_{l_2}^*}{\varepsilon_j^* + \varepsilon_{l_2}^*} \end{aligned}$$

in which we may exploit the fact that $\varepsilon_j^* = 1/\varepsilon_j$ to write

$$\begin{aligned} S'' = \frac{16}{N^4} \sum_{k_1=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{k_2=0}^{N-1} \sum_{l_2=0}^{N-1} \varepsilon_{k_1} \varepsilon_{k_2} Q_a(\varepsilon_{k_1}) Q_a^*(\varepsilon_{l_1}) Q_b(\varepsilon_{k_2}) Q_b^*(\varepsilon_{l_2}) \\ \times \left\{ \sum_{j=0}^{N-1} \frac{1}{\varepsilon_j + \varepsilon_{k_1}} \frac{\varepsilon_j}{\varepsilon_j + \varepsilon_{l_1}} \frac{1}{\varepsilon_j + \varepsilon_{k_2}} \frac{\varepsilon_j}{\varepsilon_j + \varepsilon_{l_2}} \right\} \end{aligned} \tag{5}$$

Let us define now the innermost sum of (5) as

$$\begin{aligned} W(k_1, l_1, k_2, l_2) &= \sum_{j=0}^{N-1} \frac{1}{\varepsilon_j + \varepsilon_{k_1}} \frac{\varepsilon_j}{\varepsilon_j + \varepsilon_{l_1}} \frac{1}{\varepsilon_j + \varepsilon_{k_2}} \frac{\varepsilon_j}{\varepsilon_j + \varepsilon_{l_2}} \\ &= \sum_{j=0}^{N-1} f_{k_1, l_1, k_2, l_2}(\varepsilon_j) \end{aligned}$$

with

$$f_{p,q,r,s}(z) = \frac{z^2}{(z+\varepsilon_p)(z+\varepsilon_q)(z+\varepsilon_r)(z+\varepsilon_s)}$$

Depending on p, q, r, s, the rational function $f_{p,q,r,s}(z)$ can be transformed into a specific sum of simple rational parts. Each of these rational parts can be summed separately. This path is fully developed in [1] and we here exploit the results therein.

In particular we have that

A) for $0 \le p < N$

$$W(p,p,p,p) = \frac{1}{16} \left(\frac{1}{3} N^4 + \frac{2}{3} N^2 \right) \frac{1}{\varepsilon_p^2}$$

B) for $0 \le p \ne q < N$

$$\begin{aligned} & W(p,p,p,q) = W(p,p,q,p) = W(p,q,p,p) = \\ & W(q,p,p,p) = \frac{1}{8} N^2 \left(\frac{\varepsilon_q + \varepsilon_p}{\varepsilon_p (\varepsilon_q - \varepsilon_p)^2} \right) \end{aligned}$$

C) for $0 \le p \ne q \ne r < N$

$$\begin{aligned} & W(p,p,q,r) = W(p,p,r,q) = W(p,q,p,r) = \\ & W(p,r,p,q) = W(p,q,r,p) = W(p,r,q,p) = \\ & W(q,p,r,p) = W(r,p,q,p) = W(q,r,p,p) = \\ & W(r,q,p,p) = -\frac{1}{4} N^2 \frac{1}{\varepsilon_q - \varepsilon_p} \frac{1}{\varepsilon_r - \varepsilon_p} \end{aligned}$$

$D)$ for $0 \leq p \neq q < N$

$$W(p,p,q,q) = W(p,q,p,q) = W(p,q,q,p) = -\frac{1}{2}N^2 \frac{1}{(\varepsilon_p - \varepsilon_q)^2}$$

$E)$ for $0 \leq p \neq q \neq r \neq s < N$

$$W(p,q,r,s) = 0$$

Taking into account all the above cases we may write $S'' = \frac{16}{N^4}(\alpha+\beta+\gamma+\delta)$, where the terms α, β, γ, and δ correspond to the contributions of the cases A, B, C and D respectively.

For the cases included in $A)$ we have that

$$\alpha = \frac{1}{16}\left(\frac{1}{3}N^4 + \frac{2}{3}N^2\right)\sum_{p=0}^{N-1}\left|Q_a(\varepsilon_p)Q_b(\varepsilon_p)\right|^2 \tag{6}$$

for the cases in $B)$ we have

$$\beta = \frac{1}{8}N^2 \sum_{\substack{p,q=0\\p\neq q}}^{N-1}\left\{\left(\frac{\varepsilon_q+\varepsilon_p}{\varepsilon_p(\varepsilon_q-\varepsilon_p)^2}\right)\times \left[\varepsilon_p^2\left|Q_a(\varepsilon_p)\right|^2 Q_b(\varepsilon_p)Q_b^*(\varepsilon_q) + \varepsilon_p\varepsilon_q\left|Q_a(\varepsilon_p)\right|^2 Q_b(\varepsilon_q)Q_b^*(\varepsilon_p) + \varepsilon_p^2 Q_a(\varepsilon_p)Q_a^*(\varepsilon_q)\left|Q_b(\varepsilon_p)\right|^2 + \varepsilon_q\varepsilon_p Q_a(\varepsilon_q)Q_a^*(\varepsilon_p)\left|Q_b(\varepsilon_p)\right|^2\right]\right\} \tag{7}$$

for $C)$ we have

$$\gamma = \frac{1}{4}N^2 \sum_{\substack{p,q,r=0\\p\neq q\neq r}}^{N-1}\left\{\frac{-1}{(\varepsilon_q-\varepsilon_p)(\varepsilon_r-\varepsilon_p)}\times \left[\varepsilon_p\varepsilon_q\left|Q_a(\varepsilon_p)\right|^2 Q_b(\varepsilon_q)Q_b^*(\varepsilon_r) + \varepsilon_p\varepsilon_r\left|Q_a(\varepsilon_p)\right|^2 Q_b(\varepsilon_r)Q_b^*(\varepsilon_q) + \varepsilon_p^2 Q_a(\varepsilon_p)Q_a^*(\varepsilon_q)Q_b(\varepsilon_p)Q_b^*(\varepsilon_r) + \varepsilon_p^2 Q_a(\varepsilon_p)Q_a^*(\varepsilon_r)Q_b(\varepsilon_p)Q_b^*(\varepsilon_q) + \varepsilon_p\varepsilon_r Q_a(\varepsilon_p)Q_a^*(\varepsilon_q)Q_b(\varepsilon_r)Q_b^*(\varepsilon_p) + \varepsilon_p\varepsilon_q Q_a(\varepsilon_p)Q_a^*(\varepsilon_r)Q_b(\varepsilon_q)Q_b^*(\varepsilon_p) + \varepsilon_q\varepsilon_r Q_a(\varepsilon_q)Q_a^*(\varepsilon_p)Q_b(\varepsilon_r)Q_b^*(\varepsilon_p) + \varepsilon_r\varepsilon_q Q_a(\varepsilon_r)Q_a^*(\varepsilon_p)Q_b(\varepsilon_q)Q_b^*(\varepsilon_p) + \varepsilon_q\varepsilon_p Q_a(\varepsilon_q)Q_a^*(\varepsilon_r)\left|Q_b(\varepsilon_p)\right|^2 + \varepsilon_r\varepsilon_p Q_a(\varepsilon_r)Q_a^*(\varepsilon_q)\left|Q_b(\varepsilon_p)\right|^2\right]\right\} \tag{8}$$

and for $D)$

$$\delta = \frac{1}{2}N^2 \sum_{\substack{p,q=0\\p\neq q}}^{N-1}\left\{\frac{-1}{(\varepsilon_p-\varepsilon_q)^2}\times \left[\varepsilon_p\varepsilon_q\left|Q_a(\varepsilon_p)Q_b(\varepsilon_q)\right|^2 + \varepsilon_p^2 Q_a(\varepsilon_p)Q_a^*(\varepsilon_q)Q_b(\varepsilon_p)Q_b^*(\varepsilon_q) + \varepsilon_p\varepsilon_q Q_a(\varepsilon_p)Q_a^*(\varepsilon_q)Q_b(\varepsilon_q)Q_b^*(\varepsilon_p)\right]\right\} \tag{9}$$

Summarizing, we can write the sum of squares corresponding to cross-correlations terms of the ISL as

$$\mathbb{X}_{ab} = \frac{1}{2N}\sum_{j=0}^{N-1}\left|Q_a(\varepsilon_j)Q_b^*(\varepsilon_j)\right|^2 + \frac{16}{N^4}(\alpha+\beta+\gamma+\delta)$$

where the quantities α, β, γ, δ are defined in (6), (7), (8), (9).

With the method presented above in conjunction with the method presented in [1], we can have an analytical expression for the ISL for any set of real sequences of odd length. The computation of the above equations seems to be hard at a first look, but in a number of cases, in particular for sequences from difference sets [2] may lead to significant results.

In the following, we use this method to evaluate the asymptotic trend of the ISL of a set of sequences made up by different Rotations of a Legendre Sequence (RLS set) when N grows to infinity.

III. Legendre Sequences

The *Legendre Sequence* (LS) $\ell_0,\ldots,\ell_{N-1}$ exists for any prime N and is defined as

$$\ell_0 = 1$$
$$\ell_j = \begin{cases} 1 & \text{if } j \text{ is a square} \pmod N \\ -1 & \text{if } j \text{ is a nonsquare} \pmod N \end{cases}$$

A LS may be cyclically rotated t_a positions to the left to obtain a *Rotated Legendre Sequence* (RLS) a_j defined as

$$a_j = \ell_{j+t_a \pmod N} = \ell_{j+f_aN \pmod N}$$

with $f_a = t_a/N \in [0,1]$.

The asymptotic value of $\mathbb{X}_{aa}$ for the family of RLS was calculated in [5] and [1] [1] noting that the asymptotic value of the modulus of the generating function of the LS ($|Q_\ell(\varepsilon_j)|$) is independent of j, yielding

$$\frac{\mathbb{X}_{aa}}{N^2} \sim \frac{2}{3} - 4\left|f_a - \frac{1}{2}\right| + 8\left(f_a - \frac{1}{2}\right)^2 \tag{10}$$

We follow the same path as in [1] but for the calculation of the cross-correlations terms of the ISL $\mathbb{X}_{ab}$.

To proceed, remember that the generating function of the LS is

[1] The first contribution relies on a "Postulate of Mathematical Ergodicity" to arrive at a result which is formally proved by the second.

$$Q_\ell(\varepsilon_j)=\begin{cases}1+\ell_j\sqrt{N} & \text{if } j\neq 0 \text{ and } N=1 \pmod 4\\ 1+\mathbf{i}\ell_j\sqrt{N} & \text{if } j\neq 0 \text{ and } N=3 \pmod 3\\ 1 & \text{if } j=0\end{cases} \tag{11}$$

Moreover, if we denote by $Q_a(\varepsilon_j)$ the generating function of the RLS $a_j=\ell_{j+t_a \pmod N}$, then

$$Q_a(\varepsilon_j)=\varepsilon_j^{-t_a}Q_\ell(\varepsilon_j)$$

Assume now that the two sequences a_j and b_j are obtained by rotating ℓ_j by, respectively, t_a and t_b positions to the left. We may compute S' as

$$S'=\sum_{j=0}^{N-1}\left|\varepsilon_j^{-t_a}Q_\ell(\varepsilon_j)\varepsilon_j^{t_b}Q_\ell^*(\varepsilon_j)\right|^2=\sum_{j=0}^{N-1}\left|Q_\ell(\varepsilon_j)\right|^4$$

from (11) we know immediately that $\left|Q_\ell(\varepsilon_j)\right|^4\sim N^2$, then $S'\sim N^3$. Let us now compute the asymptotic values of α, β, γ and δ in (6), (7), (8), (9) for any pair of RLS.

- For α in (6) we have

$$\begin{aligned}\alpha&=\frac{1}{16}\left(\frac{1}{3}N^4+\frac{2}{3}N^2\right)S'\\&\sim\frac{1}{48}N^7\end{aligned}$$

- For β in (7)

$$\begin{aligned}\beta=&\frac{1}{8}N^2\sum_{\substack{p,q=0\\p\neq q}}^{N-1}\Bigg\{\left(\frac{\varepsilon_q+\varepsilon_p}{\varepsilon_p(\varepsilon_q-\varepsilon_p)^2}\right)\times\\&\Big[\varepsilon_p^2\left|Q_\ell(\varepsilon_p)\right|^2\varepsilon_{p-q}^{t_b}Q_\ell(\varepsilon_p)Q_\ell^*(\varepsilon_q)+\\&\varepsilon_p\varepsilon_q\left|Q_\ell(\varepsilon_p)\right|^2\varepsilon_{q-p}^{t_b}Q_\ell(\varepsilon_q)Q_\ell^*(\varepsilon_p)+\\&\varepsilon_p^2\varepsilon_{p-q}^{t_a}Q_\ell(\varepsilon_p)Q_\ell^*(\varepsilon_q)\left|Q_\ell(\varepsilon_p)\right|^2+\\&\varepsilon_q\varepsilon_p\varepsilon_{q-p}^{t_a}Q_\ell(\varepsilon_q)Q_\ell^*(\varepsilon_p)\left|Q_\ell(\varepsilon_p)\right|^2\Big]\Bigg\}\\\sim&\frac{1}{8}N^2\sum_{\substack{p,q=0\\p\neq q}}^{N-1}\Bigg\{\left(\frac{\varepsilon_q+\varepsilon_p}{\varepsilon_p(\varepsilon_q-\varepsilon_p)^2}\right)\times\\&\Big(N^2\ell_p\ell_q\varepsilon_p^2\varepsilon_{p-q}^{t_b}+N^2\ell_p\ell_q\varepsilon_p\varepsilon_q\varepsilon_{q-p}^{t_b}+\\&N^2\ell_p\ell_q\varepsilon_p^2\varepsilon_{p-q}^{t_a}+N^2\ell_p\ell_q\varepsilon_p\varepsilon_q\varepsilon_{q-p}^{t_a}\Big)\Bigg\}\\=&\frac{1}{8}N^4\sum_{\substack{p,q=0\\p\neq q}}^{N-1}\Bigg\{\left(\frac{\ell_p\ell_q}{(1-\varepsilon_{p-q})^2}\right)\times\\&\Big(\varepsilon_{p-q}^{t_b+1}+\varepsilon_{p-q}^{t_b+2}+\varepsilon_{p-q}^{1-t_b}+\varepsilon_{p-q}^{-t_b}+\\&\varepsilon_{p-q}^{t_a+2}+\varepsilon_{p-q}^{t_a+1}+\varepsilon_{p-q}^{1-t_a}+\varepsilon_{p-q}^{-t_a}\Big)\Bigg\}\\=&\frac{1}{8}N^4\sum_{\substack{k=-N+1\\k\neq 0}}^{N-1}\left(X_{\ell\ell}(k)+X_{\ell\ell}(N-k)\right)\\&\frac{\varepsilon_k^{t_b+1}+\varepsilon_k^{t_b+2}+\varepsilon_k^{1-t_b}+\varepsilon_k^{-t_b}+\varepsilon_k^{t_a+2}+\varepsilon_k^{t_a+1}+\varepsilon_k^{1-t_a}+\varepsilon_k^{-t_a}}{(1-\varepsilon_k)^2}\end{aligned}$$

Note that $X_{\ell\ell}(k)+X_{\ell\ell}(N-k)$ is the periodic correlation [2] of the LS. Then, from [5] and [6] we know that $|X_{\ell\ell}(k)+X_{\ell\ell}(N-k)|\leq 3$ for Legendre sequences. Then, using the fact that $\sum_{k=1}^{N-1}\frac{1}{|1-\varepsilon_k|^2}=O(N^2)$ (see (12) and (15) below and set $t=0$), and using the triangle inequality we get that $\beta=O(N^6)$.

- For the calculation of γ in (8), following the same steps we did for β we have

$$\begin{aligned}\gamma\sim&\frac{1}{4}N^4\sum_{\substack{p,q,r=0\\p\neq q\neq r}}^{N-1}\Bigg\{-\frac{\ell_q\ell_r}{(1-\varepsilon_{p-q})(1-\varepsilon_{p-r})}\Big(\varepsilon_{p-r}\varepsilon_{q-r}^{t_b}+\\&\varepsilon_{p-q}\varepsilon_{q-r}^{-t_b}+\varepsilon_{p-r}^{t_b+1}\varepsilon_{p-q}^{t_a+1}+\varepsilon_{p-r}^{t_a+1}\varepsilon_{p-q}^{t_b+1}+\\&\varepsilon_{p-r}^{-t_b}\varepsilon_{p-q}^{t_a+1}+\varepsilon_{p-r}^{t_a+1}\varepsilon_{p-q}^{-t_b}+\varepsilon_{p-q}^{-t_a}\varepsilon_{p-r}^{-t_b}+\\&\varepsilon_{p-r}^{-t_a}\varepsilon_{p-q}^{-t_b}+\varepsilon_{p-r}\varepsilon_{q-r}^{t_a}+\varepsilon_{p-q}\varepsilon_{q-r}^{-t_a}\Big)\Bigg\}\\=&\frac{1}{4}N^4\sum_{\substack{u,v=-N+1\\u\neq v\neq 0}}^{N-1}\Bigg\{-\frac{X_{\ell\ell}(v-u)+X_{\ell\ell}(N-(v-u))}{(1-\varepsilon_v)(1-\varepsilon_u)}\Big(\varepsilon_u\varepsilon_{u-v}^{t_b}+\\&\varepsilon_v\varepsilon_{u-v}^{-t_b}+\varepsilon_u^{t_b+1}\varepsilon_v^{t_a+1}+\varepsilon_u^{t_a+1}\varepsilon_v^{t_b+1}+\\&\varepsilon_u^{-t_b}\varepsilon_v^{t_a+1}+\varepsilon_u^{t_a+1}\varepsilon_v^{-t_b}+\varepsilon_v^{-t_a}\varepsilon_u^{-t_b}+\\&\varepsilon_u^{-t_a}\varepsilon_v^{-t_b}+\varepsilon_u\varepsilon_{u-v}^{t_a}+\varepsilon_v\varepsilon_{u-v}^{-t_a}\Big)\Bigg\}\end{aligned}$$

and again we have that $\gamma=O(N^6)$

- For δ in (9) we have

$$\begin{aligned}\delta=&\frac{1}{2}N^2\sum_{\substack{p,q=0\\p\neq q}}^{N-1}\Bigg\{\frac{-1}{(\varepsilon_p-\varepsilon_q)^2}\times\Big[\varepsilon_p\varepsilon_q\left|Q_\ell(\varepsilon_p)Q_\ell(\varepsilon_q)\right|^2+\\&\varepsilon_p^2\varepsilon_{p-q}^{t_a}Q_\ell(\varepsilon_p)Q_\ell^*(\varepsilon_q)\varepsilon_{p-q}^{t_b}Q_\ell(\varepsilon_p)Q_\ell^*(\varepsilon_q)+\\&\varepsilon_p\varepsilon_q\varepsilon_{p-q}^{t_a}Q_\ell(\varepsilon_p)Q_\ell^*(\varepsilon_q)\varepsilon_{q-p}^{t_b}Q_\ell(\varepsilon_q)Q_\ell^*(\varepsilon_p)\Big]\Bigg\}\\\sim&-\frac{1}{2}N^4\sum_{\substack{p,q=0\\p\neq q}}^{N-1}\left\{\frac{\varepsilon_{q-p}+\varepsilon_{q-p}^{-t_a-t_b}+\varepsilon_{q-p}^{1-t_a+t_b}}{(1-\varepsilon_{q-p})^2}\right\}\\=&-\frac{1}{2}N^4\sum_{\substack{k=-N+1\\k\neq 0}}^{N-1}\frac{\left(\varepsilon_k+\varepsilon_k^{-t_a-t_b}+\varepsilon_k^{1-t_a+t_b}\right)}{(1-\varepsilon_k)^2}(N-|k|)\\=&-N^4\sum_{k=1}^{N-1}\frac{\left(\varepsilon_k+\varepsilon_k^{-t_a-t_b}+\varepsilon_k^{1-t_a+t_b}\right)}{(1-\varepsilon_k)^2}(N-|k|)\end{aligned}$$

Larger values of the summand are those for k close to 1, which make the denominator close to zero and numerator $\sim cN$ for some constant c (for k close to $N-1$, the denominator becomes also close to zero but the numerator is $O(1)$).

Exploiting this and using the small angle approximation for the complex exponential, we may write

$$\delta \sim -N^5 \sum_{k=1}^{N-1} \frac{\varepsilon_k + \varepsilon_k^{-t_a - t_b} + \varepsilon_k^{1-t_a+t_b}}{-\frac{4\pi^2}{N^2} k^2} \tag{12}$$

To continue, we recall the definition of the Dilogarithm function and its series expansion valid for $|z| \leq 1$

$$\mathrm{Li}_2\left(z\right) = -\int_0^1 \frac{ln(1-zt)}{t} dt = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{13}$$

Taking the real part of (13) and evaluating on the unit circle gives [7, eq. (8.7)]

$$\mathrm{Re}\left\{\mathrm{Li}_2\left(e^{\mathbf{i}\theta}\right)\right\} = \mathrm{Re}\left\{\sum_{k=1}^{\infty} \frac{e^{\mathbf{i}k\theta}}{k^2}\right\} = \frac{1}{6}\pi^2 - \frac{1}{4}|\theta|\left(2\pi - |\theta|\right) \tag{14}$$

Exploiting (14) and concentrating on the first period $0 \leq \frac{t}{N} \leq 1$ we obtain

$$\mathrm{Re}\left\{\sum_{k=1}^{\infty} \frac{\varepsilon_k^t}{k^2}\right\} = \pi^2 \left[\frac{1}{6} - \left[\frac{t}{N}\right]_1 \left(1 - \left[\frac{t}{N}\right]_1\right)\right] \tag{15}$$

where $[\cdot]_1 = \cdot \pmod 1$.

Hence, since we know that δ is real

$$\begin{aligned}
\delta \sim & \frac{1}{4}N^7 \left\{\frac{1}{6} + \frac{1}{6} - \left[-\frac{t_a + t_b}{N}\right]_1 \left(1 - \left[-\frac{t_a + t_b}{N}\right]_1\right) + \right.\\
& \left. \frac{1}{6} - \left[\frac{t_b - t_a}{N}\right]_1 \left(1 - \left[\frac{t_b - t_a}{N}\right]_1\right)\right\} \\
= & \frac{1}{4}N^2 \left\{\frac{1}{2} - [-f_a - f_b]_1 \left(1 - [-f_a - f_b]_1\right) - \right.\\
& \left. [f_b - f_a]_1 \left(1 - [f_b - f_a]_1\right)\right\} \\
= & \frac{1}{4}N^2 \left\{\frac{1}{2} - [f_a + f_b]_1 \left(1 - [f_a + f_b]_1\right) - \right.\\
& \left. [f_a - f_b]_1 \left(1 - [f_a - f_b]_1\right)\right\}
\end{aligned}$$

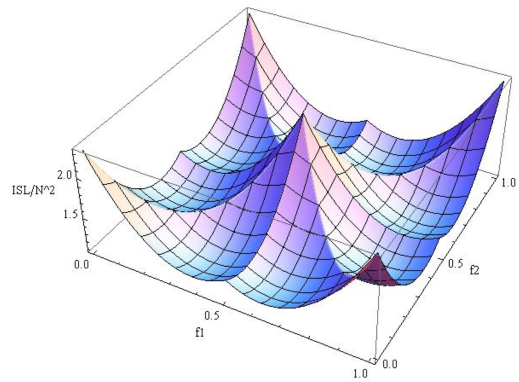
where we have defined $f_a = \frac{t_a}{N}$ and $f_b = \frac{t_b}{N}$. Then, exploiting the symmetries of a quadratic form of a modulus function we have for $0 \leq f_a, f_b \leq 1$

$$[f_a + f_b]_1 \left(1 - [f_a + f_b]_1\right) = \frac{1}{4} - \left(|f_a + f_b - 1| - \frac{1}{2}\right)^2$$

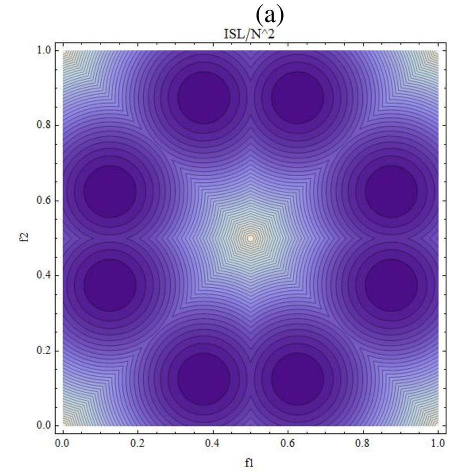
$$[f_a - f_b]_1 \left(1 - [f_a - f_b]_1\right) = \frac{1}{4} - \left(|f_a - f_b| - \frac{1}{2}\right)^2$$

so that

$$\delta \sim \frac{1}{4}N^7 \left[\left(|f_a + f_b - 1| - \frac{1}{2}\right)^2 + \left(|f_a - f_b| - \frac{1}{2}\right)^2\right]$$



(a)



(b)

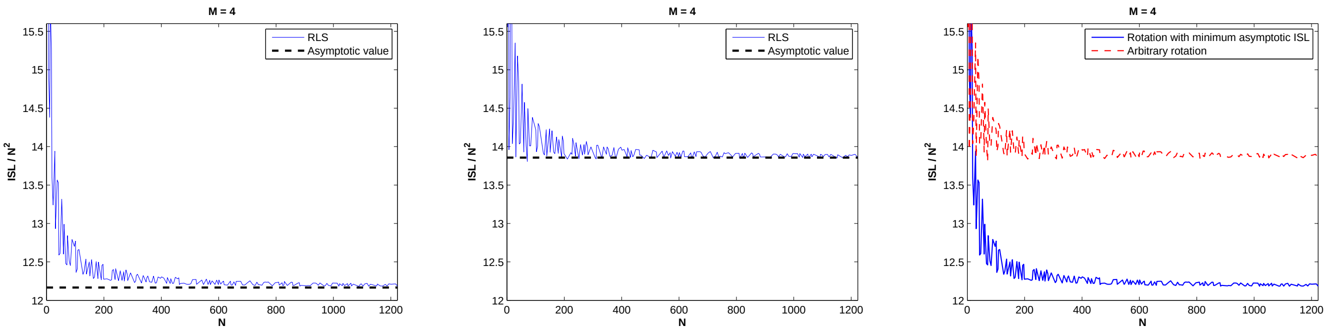
Fig. 1. Plots of ISL for $M = 2$ as a function of f_1 and f_2: (a) 3D-view, (b) iso-ISL lines

Based on the above we are now interested in computing the asymptotic value of

$$\begin{aligned}
\frac{1}{N^2}\mathbb{X}_{ab} = & \frac{1}{2N^3}\left(S' + S''\right) \sim \frac{1}{2N^3}\left[N^3 + \frac{16}{N^4}\left(\alpha + \beta + \gamma + \delta\right)\right] \\
& \sim \frac{2}{3} + 2\left(|f_a + f_b - 1| - \frac{1}{2}\right)^2 + 2\left(|f_a - f_b| - \frac{1}{2}\right)^2
\end{aligned} \tag{16}$$

Going back to our original problem for calculation of the ISL value of a set of M sequences $x_j^{(p)}$ with $j = 0, \ldots, N-1$ and $p = 0, \ldots, M-1$, where each $x^{(p)}$ is made by a different rotation f_p of a LS (RLS set), replacing (10) and (16) into (3) we finally have that

$$\begin{aligned}
\frac{\mathrm{ISL}}{N^2} \sim & \sum_{p=0}^{M-1} \frac{2}{3} - 4\left|f_p - \frac{1}{2}\right| + 8\left(f_p - \frac{1}{2}\right)^2 + \\
& \sum_{p=0}^{M-1} \sum_{\substack{q=0 \\ p \neq q}}^{M-1} \frac{2}{3} + 2\left(\left|f_p + f_q - 1\right| - \frac{1}{2}\right)^2 + \\
& 2\left(\left|f_p - f_q\right| - \frac{1}{2}\right)^2
\end{aligned} \tag{17}$$



(a) (b) (c)

Fig. 2. Plots of ISL for $M = 4$ as a function of N: (a) Rotations minimizing asymptotic ISL, (b) Arbitrary rotations. In dotted line the asymptotic value. (c) Comparison of the cases in a) and b).

As an example, Figure 1 reports the 3D and contour plot of the right-hand side of (17) for $M = 2$. Direct visual inspection of that Figure confirms that minima exists and can be easily identified. We offered a preliminary exploit of the result in [3] where an optimization procedure was developed to find the optimal rotations that minimize the ISL for any sequences length N.

As another example, in Figure 2 we plot the ISL for $M = 4$ as a function of the sequence length N. In case a) the values of rotation are those that minimize the asymptotic ISL, while in case b) we use an arbitrary rotation. In both cases we can see that the trend of the plots is in agreement with the asymptotic value calculated. In the same figure in case c), we plot together both curves in a) and b) to show that the one that achieves the minimum asymptotic value of ISL, also achieves the minimum ISL value for sequences length greater than approximately 20. For different choices of rotations and different number of sequences (M), the behavior is the same than presented.

IV. Conclusion

We apply a method based on generating functions, which has already been proposed for the calculation of the ISL of a sequence, to the calculation of the cross-correlation components of the ISL of a set of sequences.

The apparent complexity of the resulting expressions can be tackled in the asymptotic conditions for sequences whose generating function has a relatively simple trend.

Since this is the case of Legendre sequences, we are able to derive an analytical expression for the asymptotic ISL of sets of rotated Legendre sequences.

Such an expression can be exploited to drive the optimization procedure needed to construct small-ISL sets of antipodal sequences with potential applications to communication and active sensing systems.

References

[1] T. Høholdt, H.J. Jensen, "Determination of the merit factor of legendre sequences," *IEEE Transactions on Information Theory*, vol. 34, no. 1, pp. 161–164, Jan. 1988.

[2] T. Høholdt, "The merit factor problem for binary sequences," in *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, vol. 3857 of *Lecture Notes in Computer Science*, pp. 51–59. Springer Berlin / Heidelberg, 2006.

[3] J. Haboba, R. Rovatti, G. Setti, "Integrated sidelobe level of sets of rotated legendre sequences," *Presented in ICASSP 2011*.

[4] G. Polya, G. Szegö, *Aufgeben und Lehrs¨atze aus der Analyse II*, Berlin: Springer, 1925.

[5] M.J.E. Golay, "The merit factor of legendre sequences," *IEEE Transactions on Information Theory*, vol. 29, no. 6, pp. 934–936, Oct. 1983.

[6] S. W. Golomb, G. Gong, *Signal Design for Good Correlation: For Wireless Communication, Cryptography, and Radar*, Cambridge University Press, 2005.

[7] L.C. Maximon, "The dilogarithm function for complex argument," *Proceedings of the Royal Society, part A: Mathematical, Physical & Engineering Sciences*, vol. 459, pp. 2807–2819, 2003.