ON INTEGRABILITY OF NONAUTONOMOUS NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We show, in general, how to transform nonautonomous nonlinear Schrödinger equation with quadratic Hamiltonians into the standard autonomous form that is completely integrable by the familiar inverse scattering method in nonlinear science. Derivation of the corresponding equivalent nonisospectral Lax pair is outlined.

1. Introduction

Recently several nonautonomous (with time-dependent coefficients) and inhomogeneous (with space-dependent coefficients) nonlinear Schrödinger equations have been discussed as (possible) new integrable systems [6], [7], [13], [17], [35], [41], [45], [46], [51], [58], [59], [67], [71], [72], [73], [74], [75], [77], [82], [85], [86], [87], [93] (see also [2], [3], [8], [15], [16], [18], [38], [48], [81] and references therein for earlier works). They arise in the theory of Bose–Einstein condensation [30], [66], fiber optics [5], [40], superconductivity and plasma physics [15], [16], [62], [63].

As pointed out in recent papers [42] and [48] (see also [2], [3], [18], [29], [38], [60]), all these systems can be reduced by a set of transformations to the standard autonomous nonlinear Schrödinger equation, which explains their integrability properties because this equation is a well-known complete integrable system with Lax pair [50], [90], [91], [92], conservation laws and N-soliton solutions, solvable through the inverse scattering method [2], [3], [4], [47], [63], [69]. Integration techniques of the nonlinear Schrödinger equation include also Painlevé analysis [11], [19], [20], [21], [22], [28], [29], [42], [47], [61], [79], [83], Hirota method [43], [44], [47], Bäcklund transform [10], [14], [47] and Hamiltonian approach [1], [4], [36], [37], [56], [57], [63] among others [31], [54], [64], [68].

We show here how to construct these transformations explicitly (in quadratures) for the most general variable quadratic Hamiltonian. A simple relation with the Green function of a linear problem, which seems has been missing in the available literature, is emphasized. The corresponding equivalent Lax pair is also briefly discussed.

2. Transformation into Autonomous Form

The nonautonomous nonlinear Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = H\psi + h\left|\psi\right|^2 \psi, \tag{2.1}$$

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where the variable Hamiltonian H is an arbitrary quadratic form of operators $p = -i\partial/\partial x$ and x, namely,

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi, \tag{2.2}$$

(a, b, c, d, f and g are suitable real-valued functions of time only) under the following integrability condition[1] [77]:

$$h = h_0 a(t)\beta^2(t)\mu(t) = h_0\beta^2(0)\mu^2(0)\frac{a(t)\lambda^2(t)}{\mu(t)} \tag{2.3}$$

(h_0 is a real constant) can be transformed with the help of the substitution:

$$\psi(x,t) = \frac{1}{\sqrt{\mu(t)}}e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)\right)}\chi(\beta(t)x+\varepsilon(t),\gamma(t)) \tag{2.4}$$

into the autonomous form with respect to the new variables $\xi = \beta(t)x + \varepsilon(t)$ and $\tau = \gamma(t)$:

$$i\chi_\tau + h_0|\chi|^2\chi = \chi_{\xi\xi}, \tag{2.5}$$

which is completely integrable by advanced methods of the soliton theory [4], [47], [63], [70], [90], [91], [92] (see also [31] and references cited in the introduction). Equations (2.2)–(2.3) seem to represent the maximum nonautonomous and inhomogeneous one-dimensional integrable system of this kind. (Important special cases of the transformation (2.4) are discussed in Refs. [2], [15], [16], [18], [38], [42], [48], [67] and [77].)

Here, the real-valued functions α, β, γ, δ, ε and κ of time t only are given in terms of solutions of the following system of ordinary differential equations [23]:

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = 0, \tag{2.6}$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \tag{2.7}$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \tag{2.8}$$

and

$$\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2\alpha g, \tag{2.9}$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta, \tag{2.10}$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2. \tag{2.11}$$

The substitution [23]:

$$\alpha = \frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)} \tag{2.12}$$

reduces the Riccati equation (2.6) to the second order linear equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \tag{2.13}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{2.14}$$

[1] If the nonlinear term has the form $h|\psi|^p\psi$, popular in the mathematical literature, the integrability condition becomes $h = h_0 a\beta^2\mu^{p/2}$.

(Relations with the corresponding Ehrenfest theorem for the linear Hamiltonian are discussed in Ref. [25].)

The initial value problem for the system (2.6)–(2.11), which corresponds to the linear Schrödinger equation with a variable quadratic Hamiltonian (generalized harmonic oscillators [9], [33], [39], [84], [89]), can be explicitly solved in terms of solutions of characteristic equation (2.13) as follows [23], [25], [76], [78]:

$$\mu(t) = 2\mu(0)\,\mu_0(t)\left(\alpha(0)+\gamma_0(t)\right), \tag{2.15}$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.16}$$

$$\beta(t) = -\frac{\beta(0)\,\beta_0(t)}{2\left(\alpha(0)+\gamma_0(t)\right)} = \frac{\beta(0)\,\mu(0)}{\mu(t)}\lambda(t), \tag{2.17}$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4\left(\alpha(0)+\gamma_0(t)\right)} \tag{2.18}$$

and

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)\left(\delta(0)+\varepsilon_0(t)\right)}{2\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.19}$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)\left(\delta(0)+\varepsilon_0(t)\right)}{2\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.20}$$

$$\kappa(t) = \kappa(0)+\kappa_0(t) - \frac{\left(\delta(0)+\varepsilon_0(t)\right)^2}{4\left(\alpha(0)+\gamma_0(t)\right)}, \tag{2.21}$$

where

$$\alpha_0(t) = \frac{1}{4a(t)}\frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{2.22}$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \qquad \lambda(t) = \exp\left(-\int_0^t \left(c(s)-2d(s)\right)\,ds\right), \tag{2.23}$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)}\frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \tag{2.24}$$

and

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)}\int_0^t \left[\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\mu_0(s) + \frac{g(s)}{2a(s)}\mu_0'(s)\right]\frac{ds}{\lambda(s)}, \tag{2.25}$$

$$\begin{aligned}\varepsilon_0(t) &= -\frac{2a(t)\,\lambda(t)}{\mu_0'(t)}\delta_0(t) + 8\int_0^t \frac{a(s)\,\sigma(s)\,\lambda(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\,\delta_0(s)\right)\,ds \\ &\quad +2\int_0^t \frac{a(s)\,\lambda(s)}{\mu_0'(s)}\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\,ds,\end{aligned} \tag{2.26}$$

$$\begin{aligned}\kappa_0(t) &= \frac{a(t)\,\mu_0(t)}{\mu_0'(t)}\delta_0^2(t) - 4\int_0^t \frac{a(s)\,\sigma(s)}{\left(\mu_0'(s)\right)^2}\left(\mu_0(s)\,\delta_0(s)\right)^2\,ds \\ &\quad -2\int_0^t \frac{a(s)}{\mu_0'(s)}\left(\mu_0(s)\,\delta_0(s)\right)\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\,ds\end{aligned} \tag{2.27}$$

($\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$ and $\kappa_0(0) = 0$) provided that μ_0 and μ_1 are the standard solutions of equation (2.13) corresponding to the following initial conditions $\mu_0(0) = 0$, $\mu_0'(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu_1'(0) = 0$ (proofs are outlined in Refs. [23], [27] and [76]). (Formulas (2.22)–(2.27) correspond to Green's function of generalized harmonic oscillators; see, for example, [23], [25], [34], [52], [76], [78] and references therein for more details.)

Proof. Differentiate $\psi = \mu^{-1/2}(t)\, e^{iS(x,t)} \chi(\xi, \tau)$ with $S = \alpha(t)\, x^2 + \delta(t)\, x + \kappa(t)$ and $\xi = \beta(t)\, x + \varepsilon(t)$, $\tau = \gamma(t)$:

$$ie^{-iS}\psi_t = \frac{1}{\sqrt{\mu}}\left[-\left(\alpha' x^2 + \delta' x + \kappa'\right)\chi + i\left(\left(\beta' x + \varepsilon'\right)\chi_\xi + \gamma'\chi_\tau - \frac{\mu'}{2\mu}\chi\right)\right], \tag{2.28}$$

$$e^{-iS}\psi_x = \frac{1}{\sqrt{\mu}}\left[i\left(2\alpha x + \delta\right)\chi + \beta\chi_\xi\right] \tag{2.29}$$

and

$$e^{-iS}\psi_{xx} = \frac{1}{\sqrt{\mu}}\left[\left(2i\alpha - \left(2\alpha x + \delta\right)^2\right)\chi + 2i\left(2\alpha x + \delta\right)\beta\chi_\xi + \beta^2\chi_{\xi\xi}\right]. \tag{2.30}$$

Substitution into (2.2), with the help of the integrability condition (2.3) and the system (2.15)–(2.21), results in (2.5). □

This observation gives a new interpretation of the system (2.15)–(2.21), which has been originally derived in Ref. [23] during integration of the corresponding linear equation.

3. Integration of the Nonautonomous Linear System

The transformation (2.4) reduces the linear Schrödinger equation of generalized harmonic oscillators, namely, equation (2.1) with $h = 0$, to the Schrödinger equation for a free particle $i\chi_\tau = \chi_{\xi\xi}$ with a familiar Green function given by

$$G(\xi, \eta, \tau - \tau_0) = \frac{1}{\sqrt{-4\pi i\left(\tau - \tau_0\right)}} \exp\left[-i\frac{\left(\xi - \eta\right)^2}{4\left(\tau - \tau_0\right)}\right], \tag{3.1}$$

where $\xi = \beta(t)\, x + \varepsilon(t)$, $\eta = \beta(0)\, x + \varepsilon(0)$ and $\tau = \gamma(t)$, $\tau_0 = \gamma(0)$. One can verify directly that Green's functions of generalized harmonic oscillators [23],

$$G(x, y, t) = \frac{1}{\sqrt{2\pi i \mu_0(t)}} \exp\left[i\left(\alpha_0(t)\, x^2 + \beta_0(t)\, xy + \gamma_0(t)\, y^2 + \delta_0(t)\, x + \varepsilon_0(t)\, y + \kappa_0(t)\right)\right], \tag{3.2}$$

are derived from the simplest free particle propagator (3.1) with the help of our transformations (2.15)–(2.21). In is worth noting, though, that the transformation (2.4) requires a knowledge of the functions μ, α, β, γ, δ, ε and κ, which allows to determine the Green's function for the generalized harmonic oscillators directly from (2.2).

Then the superposition principle allows to solve the corresponding Cauchy initial value problem:

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t)\, \psi(y, 0)\ dy \tag{3.3}$$

for suitable initial data $\psi(x, 0) = \varphi(x)$ (see Refs. [23], [78] and [76] for details).

As shown in [76], the following asymptotics hold

$$\begin{aligned}
\alpha_0(t) &= \frac{1}{4a(0)t} - \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \\
\beta_0(t) &= -\frac{1}{2a(0)t} + \frac{a'(0)}{4a^2(0)} + \mathcal{O}(t), \\
\gamma_0(t) &= \frac{1}{4a(0)t} + \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \\
\delta_0(t) &= -\varepsilon_0(t) = \frac{g(0)}{2a(0)} + \mathcal{O}(t), \qquad \kappa_0(t) = \mathcal{O}(t)
\end{aligned} \tag{3.4}$$

as $t \to 0$ for sufficiently smooth coefficients. Then

$$\begin{aligned}
G(x,y,t) \quad &\sim \quad \frac{1}{\sqrt{2\pi i a(0)t}} \exp\left[i\frac{(x-y)^2}{4a(0)t}\right] \\
&\times \exp\left[-i\left(\frac{a'(0)}{8a^2(0)}(x-y)^2 + \frac{c(0)}{4a(0)}\left(x^2-y^2\right) - \frac{g(0)}{2a(0)}(x-y)\right)\right]
\end{aligned} \tag{3.5}$$

as $t \to 0$, which corrects a typo in Ref. [23].

Another form of solution is provided by an eigenfunction expansion [78].

4. One Soliton Solution

As well-known, equation (2.5) has a travelling wave solution of the form

$$\chi(\xi,\tau) = e^{i\left(\xi y + \tau\left(y^2 - g_0\right) + \phi\right)} F(\xi + 2\tau y) \tag{4.1}$$

provided

$$\left(\frac{dF}{dz}\right)^2 = C_0 + g_0 F^2 + \frac{1}{2}h_0 F^4 \qquad (C_0 \text{ is a constant of integration}). \tag{4.2}$$

Examples include bright and dark solitons, and Jacobi elliptic transcendental solutions for solitary wave profiles [2], [47], [63], [70], [77]. Setting $C_0 = y = 0$, gives the stationary breather, which is located about $\xi = 0$ and oscillates at a frequency equal to g_0 [69], [70].

By (2.4), the nonautonomous Schrödinger equation (2.2) under the integrability condition (2.3) has the following solution:

$$\begin{aligned}
\psi(x,t) \quad &= \quad \frac{e^{i\phi}}{\sqrt{\mu}} \exp\left(i\left(\alpha x^2 + \beta xy + \gamma\left(y^2 - g_0\right) + \delta x + \varepsilon y + \kappa\right)\right) \\
&\times F(\beta x + 2\gamma y + \varepsilon),
\end{aligned} \tag{4.3}$$

where the elliptic function F satisfies equation (4.2) and ϕ, y, g_0 and h_0 are real parameters (see also Ref. [77] for a direct derivation of this solution).

5. Integrability of Nonautonomous Nonlinear Schrödinger Equation

The substitution $\Psi\left(X,T\right)=\sqrt{h_0}\chi\left(\sqrt{2}X,-2T\right)$ transforms equation (2.5) into standard forms

$$i\Psi_T+\Psi_{XX}\pm 2\left|\Psi\right|^2\Psi=0, \tag{5.1}$$

which, as well-known, can be obtained as the flatness condition:

$$U_T-V_X+UV-VU=0 \tag{5.2}$$

for the Lax–(Zakharov–Shabat) pair:

$$\begin{aligned} U &= -i\lambda\sigma_3+\Psi\sigma_+\mp\Psi^*\sigma_- \\ &= \left(\begin{array}{cc} -i\lambda & \Psi \\ \mp\Psi^* & i\lambda \end{array}\right) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} V &= i\left(-2\lambda^2\pm\left|\Psi\right|^2\right)\sigma_3+\left(2\lambda\Psi+i\Psi_X\right)\sigma_+\pm\left(-2\lambda\Psi^*+i\Psi_X^*\right)\sigma_- \\ &= \left(\begin{array}{cc} i\left(-2\lambda^2\pm\left|\Psi\right|^2\right) & 2\lambda\Psi+i\Psi_X \\ \mp 2\lambda\Psi^*\pm i\Psi_X^* & i\left(2\lambda^2\mp\left|\Psi\right|^2\right) \end{array}\right). \end{aligned} \tag{5.4}$$

Here, $\sigma_\pm=\left(\sigma_1\pm i\sigma_2\right)/2$ and σ_1, σ_2, σ_3 are the Pauli matrices:

$$\sigma_1=\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \qquad \sigma_2=\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right), \qquad \sigma_3=\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right). \tag{5.5}$$

Since the Lax pair guarantees complete integrability and can alone derive all its associated properties, this consideration trivially explains the integrability features of the nonautonomous nonlinear Schrödinger equation (2.1), including N-soliton solutions, infinite conservative properties, etc., (see Refs. [48], [63] and [70] for more details).

Solution of the Cauchy initial value problem through the inverse scattering method is discussed in [3], [4], [47], [63], [70], [90], [91] and [92]. In the focusing case,

$$i\Psi_T+\Psi_{XX}+2\left|\Psi\right|^2\Psi=0, \tag{5.6}$$

the Zakharov–Shabat system contains four equations for an auxiliary two-component wave function $\mathbf{\Phi}=\left(\varphi,\upsilon\right)^T$:

$$\mathbf{\Phi}_X=U\mathbf{\Phi}, \qquad \mathbf{\Phi}_T=V\mathbf{\Phi}, \tag{5.7}$$

namely,

$$\begin{aligned} \varphi_X &= -i\lambda\varphi+\Psi\upsilon, \\ \upsilon_X &= -\Psi^*\varphi+i\lambda\upsilon \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} \varphi_T &= i\left(-2\lambda^2+\left|\Psi\right|^2\right)\varphi+\left(2\lambda\Psi+i\Psi_X\right)\upsilon, \\ \upsilon_T &= \left(-2\lambda\Psi^*+i\Psi_X^*\right)\varphi+i\left(2\lambda^2-\left|\Psi\right|^2\right)\upsilon. \end{aligned} \tag{5.9}$$

Assuming that $\Psi\left(X,T\right)\to 0$ as $X\to\pm\infty$ implies

$$\varphi_T\to -2i\lambda^2\varphi, \qquad \upsilon_T\to 2i\lambda^2\upsilon \qquad \left(X\to\pm\infty\right) \tag{5.10}$$

so the scattering data for the problem

$$L\begin{pmatrix} \varphi \\ \upsilon \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \upsilon \end{pmatrix}, \tag{5.11}$$

$$L = i\sigma_3 \frac{\partial}{\partial X} - i\Psi\sigma_+ - i\Psi^*\sigma_- = i\begin{pmatrix} \partial_X & -\Psi \\ -\Psi^* & -\partial_X \end{pmatrix} \tag{5.12}$$

evolve with time as

$$b(\lambda, T) = b(\lambda, T_0)\, e^{4i\lambda^2(T-T_0)}, \qquad r_n(T) = r_n(T_0)\, e^{4i\lambda(T-T_0)}. \tag{5.13}$$

Then Cauchy initial value problem for the nonlinear Schrödinger equation (5.1) can be solved as follows [47], [70]:

$$\Psi(X,T) = -2K(X,X,T), \tag{5.14}$$

where $K(X,Y,T)$ satisfies the linear integral equation

$$\begin{aligned} K(X,Y,T) &= B^*(X+Y,T) \\ &\quad - \int_X^\infty \int_X^\infty K(X,Z,T)\, B(Z+W,T)\, B^*(Y+W,T)\ dZdW \end{aligned} \tag{5.15}$$

and $B(X,T)$ can be obtained in terms of the scattering data:

$$B(X,T) = -i\sum_{n=1}^{N} r_n(T_0)\, e^{i\left(\lambda_n X + 4\lambda_n^2(T-T_0)\right)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\lambda, T_0)\, e^{i\left(\lambda X + 4\lambda^2(T-T_0)\right)}\ d\lambda \tag{5.16}$$

(see Refs. [3], [2], [4], [63], [47], [69], [70], [80], [90], [91], [92] for more details).

As a result, the transformation

$$\psi(x,t) = \frac{1}{\sqrt{h_0\mu(t)}} e^{i\left(\alpha(t)x^2 + \delta(t)x + \kappa(t)\right)}\ \Psi\left(\frac{1}{\sqrt{2}}\left(\beta(t)\,x + \varepsilon(t)\right), -\frac{1}{2}\gamma(t)\right) \tag{5.17}$$

allows to solve Cauchy initial value problem for the nonautonomous nonlinear Schrödinger equation (2.2) with the help of the standard inverse scattering technique.

Two well-known solutions of (5.6) are given by [69], [70]:

$$\Psi_1(X,T) = \frac{e^{iT}}{\cosh X} \tag{5.18}$$

and

$$\Psi_2(X,T) = 4e^{iT}\frac{\cosh 3X + 3e^{8iT}\cosh X}{\cosh 4X + 4\cosh 2X + 3\cos 8T}. \tag{5.19}$$

Use of the transformation (5.17), results in one and two soliton solutions for the nonautonomous nonlinear Schrödinger equation (2.2), respectively. (See [69], [70] for more details.)

6. Transformation of the Lax Pair and Zakharov–Shabat System

If needed, an equivalent (nonisospectral) Lax pair for the nonautonomous Schrödinger equation (2.2), which is discussed in [8], [15], [16], [18], [74], [75] for important special cases, can be derived, in general, from (5.3)–(5.4) by inverting our transformation (5.17) (see [3], [48] for more details). The required integrability condition (2.3) has been already incorporated into this transformation. Computational details are left to the reader.

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References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *Nonlinear-evolution equations of physical significance*, Phys. Rev. Lett. **31** (1973) #2, 125–127.
[2] M. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge Univ. Press, Cambridge, 1991.
[3] M. Ablowitz, B. Prinari and A. D. Trubatch, *Discrete and Continuous* Schrödinger Systems, Cambridge Univ. Press, Cambridge, 2004.
[4] M. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.
[5] G. P. Agrawal, *Nonlinear Fiber Optics*, Forth Edition, Academic Press, New York, 2007.
[6] U. Al Khawaja, *A comparative analysis of Painlevé, Lax pair, and similarity transformation methods in obtaining the integrability conditions of nonlinear Schrödinger equations*, J. Math. Phys. **51** (2010), 053506 (11 pages).
[7] R. Atre, P. K. Panigrahi and G. S. Agarwal, *Class of solitary wave solutions of the one-dimensional Gross–Pitaevskii equation*, Phys. Rev. E **73** (2006), 056611.
[8] R. Balakrishnan, *Soliton propagation in nonunivorm media*, Phys. Rev. A **32** (1985) #2, 1144–1149.
[9] M. V. Berry, *Classical adiabatic angles and quantum adiabatic phase*, J. Phys. A: Math. Gen. **18** (1985) # 1, 15–27.
[10] M. Boiti and T. Guizhang, *Bäcklund transformations via gauge transformations*, Nuovo Cimento B **71** (1982) # 2, 253–264.
[11] M. Boiti and F. Pempinelli, *Nonlinear Schrödinger equation, Bäcklund transformations and Painlevé transcendents*, Nuovo Cimento B **59** (1980) # 1, 40–58.
[12] K. Bongs and K. Sengstock, *Physics with coherent matter waves*, Rep. Prog. Phys. **67** (2004), 907–963.
[13] T. Brugarino and M. Sciacca, *Integrability of an inhomogeneous nonlinear Schrödinger equation in Bose–Einstein condensates and fiber optics*, J. Math. Phys. **51** (2010), 093503 (18 pages).
[14] H.-H. Chen, *General deriviation of Bäcklund transformations from inverse scattering problems*, Phys. Rev. Lett. **33** (1974) #15, 925–928.
[15] H.-H. Chen and Ch.-Sh. Liu, *Solitons in nonuniform media*, Phys. Rev. Lett. **37** (1976) #11, 693–697.
[16] H.-H. Chen and Ch.-Sh. Liu, *Nonlinear wave and soliton propagation in media with arbitrary inhomogeneities*, Phys. Fluids **21** (1978) #3, 377–380.
[17] Sh. Chen and L. Yi, *Chirped self-similar solitons of a generalized Schrödinger equation model*, Phys. Rev. E **61** (2005), 016606 (4 pages).
[18] P. A. Clarkson, *Painlevé analysis for the damped, driven nonlinear Schrödinger equation*, Proc. Roy. Soc. Edin., **109A** (1988), 109–126.
[19] R. Conte, *Invariant Painlevé analysis of partial differential equations*, Phys. Lett. A **140** (1989) #7–8, 383–390.
[20] R. Conte, *The Painlevé approach to nonlinear ordinary differential equations*, in: *The Painlevé Property, One Century Later*, CRM Series in Mathematical Physics (R. Conte, Ed.), Springer–Verlag, New York, 1991. pp. 77–180.
[21] R. Conte, A. P. Fordy and A. Pickering, *A perturbative Painlevé approach to nonlinear differential equations*, Physica D **69** (1993), 33–58.
[22] R. Conte and M. Musette, *Elliptic general analytic solutions*, Studies in Applied Mathematics **123** (2009) #1, 63–81.

[23] R. Cordero-Soto, R. M. Lopez, E. Suazo and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. **84** (2008) #2–3, 159–178.

[24] R. Cordero-Soto, E. Suazo and S. K. Suslov, *Models of damped oscillators in quantum mechanics*, Journal of Physical Mathematics, **1** (2009), S090603 (16 pages).

[25] R. Cordero-Soto, E. Suazo and S. K. Suslov, *Quantum integrals of motion for variable quadratic Hamiltonians*, Ann. Phys. **325** (2010), 1884–1912; see also arXiv:0912.4900v9 [math-ph] 19 Mar 2010.

[26] R. Cordero-Soto and S. K. Suslov, *Time reversal for modified oscillators*, Theoretical and Mathematical Physics **162** (2010) #3, 286–316; see also arXiv:0808.3149v9 [math-ph] 8 Mar 2009.

[27] R. Cordero-Soto and S. K. Suslov, *The degenerate parametric oscillator and Ince's equation*, J. Phys. A: Math. Theor. **44** (2011) #1, 015101 (9 pages); see also arXiv:1006.3362v3 [math-ph] 2 Jul 2010.

[28] C. M. Cosgrove, *Painlevé classification of all semilinear partial differential equations of the second order. I Hyperbolic equations in two independent variables*, Studies in Applied Mathematics **89** (1993) # 1, 1–61.

[29] C. M. Cosgrove, *Painlevé classification of all semilinear partial differential equations of the second order. II Parabolic and higher dimentional equations*, Studies in Applied Mathematics **89** (1993) # 2, 95–151.

[30] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, *Theory of Bose–Einstein condensation in trapped gases*, Rev. Mod. Phys. **71** (1999), 463–512.

[31] A. Desgasperis, *Resource letter Sol-1: Solitons*, Am. J. Phys. **66** (1998) 6, 486–497.

[32] A. Desgasperis, *Integrable models in nonlinear optics and soliton solutions*, J. Phys. A: Math. Theor. **43** (2010), 434001 (18 pages).

[33] V. V. Dodonov, I. A. Malkin and V. I. Man'ko, *Integrals of motion, Green functions, and coherent states of dynamical systems*, Int. J. Theor. Phys. **14** (1975) # 1, 37–54.

[34] V. V. Dodonov and V. I. Man'ko, *Invariants and correlated states of nonstationary quantum systems*, in: *Invariants and the Evolution of Nonstationary Quantum Systems*, Proceedings of Lebedev Physics Institute, vol. 183, pp. 71-181, Nauka, Moscow, 1987 [in Russian]; English translation published by Nova Science, Commack, New York, 1989, pp. 103-261.

[35] A. Ebaid and S. M. Khaled, *New types of exact solutions for nonlinear Schrödinger equation with cubic nonlinearity*, Journal of Computational and Applied Mathematics (2010), doi:10.1016/j.cam.2010.09.024.

[36] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer–Verlag, Berlin, New York, 1987.

[37] C. S. Gardner, J. M. Green, M. D. Kruskai and R. M. Miura, *Method for solving the Korteweg–deVries equation*, Phys. Rev. Lett. **19** (1967) #19, 1095–1097.

[38] L. Gagnon and P. Winternitz, *Symmetry classes of variable coefficient nonlinear Schrödinger equations*, J. Phys. A: Math. Gen. **26** (1993), 7061–7076.

[39] J. H. Hannay, *Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian*, J. Phys. A: Math. Gen **18** (1985) # 2, 221–230.

[40] A. Hasegawa, *Optical Solitons in Fibers*, Springer-Verlag, Berlin, 1989.

[41] J. He and Y. Li, *Designable integrability of the variable coefficient nonlinear Schrödinger equations*, Studies in Applied Mathematics (2010), (15 pages), doi:10.1111/j.1467-9590. 2010.00495.x.

[42] X-G. He, D. Zhao, L. Lee and H-G. Luo, *Engineering integrable nonautonomous Schrödinger equations*, Phys. Rev. E **79** (2009), 056610 (9 pages).

[43] R. Hirota, *Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett. **27** (1971) #18, 1192–1194.

[44] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, 2004.

[45] Y. I. Kruglov, A. C. Peacock and J. D. Harvey, *Exact self-similar solutions of the generalized nonlinear Schrödinger equation with distributed coefficients*, Phys. Rev. Lett. **90** (2003) #11, 113902 (4 pages).

[46] Y. I. Kruglov, A. C. Peacock and J. D. Harvey, *Exact solutions of the generalized Schrödinger equation with distributed coefficients*, Phys. Rev. E **71** (2005), 1056619 (11 pages).

[47] N. A. Kudryashov, *Methods of Nonlinear Mathematical Physics*, Intellect, Dolgoprudny, 2010 [in Russian].

[48] A. Kundu, *Integrable nonautonomous Schrödinger equations are equivalent to the standard autonomous equation*, Phys. Rev. E **79** (2009), 015601(R) (4 pages).

[49] N. Lanfear and S. K. Suslov, *The time-dependent Schrödinger equation, Riccati equation and Airy functions*, arXiv:0903.3608v5 [math-ph] 22 Apr 2009.

[50] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure Appl. Math. **21** (1968) #5, 467–490.
[51] Z. X. Liang, Z. D. Zhang and W. M. Liu, *Dynamics of a bright solutions in Bose–Einstein condensates with time-dependent atomic scattering length in an expulsive parabolic potential*, Phys. Rev. Lett. **94** (2005) #5, 050402 (4 pages).
[52] I. A. Malkin and V. I. Man'ko, *Dynamical Symmetries and Coherent States of Quantum System*, Nauka, Moscow, 1979 [in Russian].
[53] I. A. Malkin, V. I. Man'ko and D. A. Trifonov, *Linear adiabatic invariants and coherent states*, J. Math. Phys. **14** (1973) #5, 576–582.
[54] V. B. Matveev and M. A. Salle, *Darboux Transformation and Solitons*, Springer–Verlag, Berlin, 1991.
[55] M. Meiler, R. Cordero-Soto, and S. K. Suslov, *Solution of the Cauchy problem for a time-dependent Schrödinger equation*, J. Math. Phys. **49** (2008) #7, 072102: 1–27; see also arXiv: 0711.0559v4 [math-ph] 5 Dec 2007.
[56] R. M. Miura, *Korteweg-de Vries equation and generalizations. I. A Remarkable explicit nonlinear transformation*, J. Math. Phys. **9** (1968) #9, 1202–1204.
[57] R. M. Miura, C. S. Gardner and M. D. Kruskal, *Korteweg-de Vries equation and generalizations. II. Existence of concervation laws and constants of motion*, J. Math. Phys. **9** (1968) #9, 1204–1209.
[58] J. D. Moores, *Nonlinear compression of chirped solitary waves with and without phase modulation*, Optics Letters **21** (1996) #8, 555–557.
[59] J. D. Moores, *Oscilatory solitons in a novel integrable model of asynchronomous mode locking*, Optics Letters **21** (2001) #2, 87–89.
[60] M. Musette, *Painlevé analysis for nonlinear partial differential equations*, in: *The Painlevé Property, One Century Later*, CRM Series in Mathematical Physics (R. Conte, Ed.), Springer–Verlag, New York, 1991. pp. 517–572.
[61] M. Musette and R. Conte, *Analytic solitary waves of nonintegrable equations*, Physica D **181** (2003), 70–79.
[62] A. C. Newell, *Nonlinear tunnelling*, J. Math. Phys. **19** (1978) #5, 1126–1133.
[63] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Kluwer, Dordrecht, 1984.
[64] P. J. Olver, *Applications of Lie Group to Differential Equations*, Springer–Verlag, Berlin, 1991.
[65] A. N. Oraevsky, *Bose condensates from the point of view of laser physics*, Quantum Electronics **31** (2001) #12, 1038–1057.
[66] L. Pitaevskii and S. Stringari, *Bose–Einstein Condensation*, Oxford University Press, Oxford, 2003.
[67] S. Ponomarenko and G. P. Agrawal, *Optical similaritons in nonlinear waveguides*, Optics Letters **32** (2007) #12, 1659–1661.
[68] C. Rogers and W. K. Scheif, *Bäcklund Transformation and Darboux Transformation*, Cambridge University Press, Cambridge, 2002.
[69] J. Satsuma and N. Yajima, *Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media*, Supp. Prog. Theor. Phys. **85** (1974), 284–306.
[70] A. Scott, *Nonlinear Science: Emergence and Dynamics of Coherent Structures*, Second Edition, Oxford University Press, Oxford, 2003.
[71] V. N. Serkin and A. Hasegawa, *Novel soliton solutions of the nonlinear Schrödinger equation model*, Phys. Rev. Lett. **85** (2000) #21, 4502–4505.
[72] V. N. Serkin and A. Hasegawa, *Soliton management in the nonlinear Schrödinger equation model with varying dispertion, nonlinearity, and gain*, JETP Letters **72** (2000) #2, 89–92.
[73] V. N. Serkin, A. Hasegawa and T. L. Belyeva, *Comment on "Exact self-similar solutions of the generalized nonlinear Schrödinger equation with distributed coefficients"*, Phys. Rev. Lett. **92** (2004) #19, 199401 (1 page).
[74] V. N. Serkin, A. Hasegawa and T. L. Belyeva, *Nonautonomous solitons in external potentials*, Phys. Rev. Lett. **98** (2007) #7, 074102 (4 pages).
[75] V. N. Serkin, A. Hasegawa and T. L. Belyeva, *Nonatonomous matter-wave soliton near the Feshbach resonance*, Phys. Rev. A **81** (2010), 023610 (19 pages).
[76] E. Suazo and S. K. Suslov, *Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians*, under preparation.
[77] E. Suazo and S. K. Suslov, *Soliton-like solutions for nonlinear Schrödinger equation with variable quadratic Hamiltonians*, arXiv:1010.2504v4 [math-ph] 24 Nov 2010.

[78] S. K. Suslov, *Dynamical invariants for variable quadratic Hamiltonians*, Physica Scripta **81** (2010) #5, 055006 (11 pp); see also arXiv:1002.0144v6 [math-ph] 11 Mar 2010.

[79] M. Tajiri, *Similarity reduction of the one and two dimensional nonlinear Schrödinger equations*, J. Phys. Soc. Japan **52** (1983) #2, 1908–1917.

[80] T. Tao, *Why are solitons stable?*, Bull. Amer. Math. Soc. **46** (2009) #1, 1–33.

[81] F. D. Tappert and N. J. Zabusky, *Gradient-induced fission of solitons*, Phys. Rev. Lett. **26** (1971) #26, 1774–1776.

[82] C. Trallero-Giner, J. Drake, V. Lopez-Richard, C. Trallero-Herrero and J. L. Birman, *Bose–Eistein condensates: Analytical methods for the Gross–Pitaevskii equation*, Phys. Lett. A **354** (2006), 115–118.

[83] J. Weiss, M. Tabor and G. Carnevalle, *The Painlevé property for partial differential equation*, J. Math. Phys. **24** (1983) #3, 522–526.

[84] K. B. Wolf, *On time-dependent quadratic Hamiltonians*, SIAM J. Appl. Math. **40** (1981) #3, 419–431.

[85] Z. Yan, *The new extended Jacobian elliptic function expansion algorithm and its applications in nonlinear mathematical physics equations*, Computer Physics Communications **153** (2003), 145–154.

[86] Z. Yan, *An improved algebra method and its applications in nonlinear wave equations*, Chaos, Solitons & Fractals **21** (2004), 1013–1021.

[87] Z. Yan, *Exact analytical solutions for the generalized non-integrable nonlinear Schrödinger equation with varying coefficients*, Phys. Lett. A, doi:10.1016/j.physleta. 2010.09.070.

[88] Z. Yan and V. V. Konotop, *Exact solutions to three-dimensional generalized nonlinear Schrödinger equation with varying potential and nonlinearities*, Phys. Rev. E **80** (2009), 036607 (9 pages).

[89] K-H. Yeon, K-K. Lee, Ch-I. Um, T. F. George and L. N. Pandey, *Exact quantum theory of a time-dependent bound Hamiltonian systems*, Phys. Rev. A **48** (1993) # 4, 2716–2720.

[90] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Zh. Eksp. Teor. Fiz. **61** (1971), 118–134 [Sov. Phys. JETP **34** (1972) #1, 62–69.]

[91] V. E. Zakharov and A. B. Shabat, *A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I*, Functional Analysis and Its Applications **8** (1974) #3, 226–235.

[92] V. E. Zakharov and A. B. Shabat, *Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II*, Functional Analysis and Its Applications **13** (1979) #3, 166–174.

[93] X.-F. Zhang, Q. Yang, J.-F. Zhang, X. Z. Chen and W. M. Liu, *Controlling soliton interactions in Bose–Einstein condensates by synchronizing the Feshbach resonance and harmonic trap*, Phys. Rev. A **77** (2008), 023613 (7 pages).

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