ON CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we give some definitions on quasi-convex functions and we prove inequalities contain J-quasi-convex and W-quasi-convex functions. We give also some inclusions.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well-known in the literature as Hadamard's inequality. We recall some definitions;

In [25], Pecaric et al. defined quasi-convex functions as following

Definition 1. *A function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \qquad (QC)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex.

Definition 2. *(See [6], [12]) We say that $f : I \to \mathbb{R}$ is a Wright-convex function or that f belongs to the class $W(I)$, if for all x, $y + \delta \in I$ with $x < y$ and $\delta > 0$, we have*

$$f(x + \delta) + f(y) \leq f(y + \delta) + f(x)$$

Definition 3. *(See [6]) For $I \subseteq \mathbb{R}$, the mapping $f : I \to \mathbb{R}$ is wright-quasi-convex function if, for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality*

$$\frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right] \leq \max\{f(x), f(y)\}, \qquad (WQC)$$

or equivalently

$$\frac{1}{2}\left[f(y) + f(x+\delta)\right] \leq \max\{f(x), f(y+\delta)\}$$

for every x, $y + \delta \in I$, $x < y$ and $\delta > 0$.

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Definition 4. *(See [6]) The mapping $f : I \to \mathbb{R}$ is Jensen- or J-quasi-convex if*

$$f\left(\frac{x+y}{2}\right) \leq \max\left\{f(x), f(y)\right\}, \qquad (JQC)$$

for all $x, y \in I$.

Note that the class $JQC(I)$ of J-quasi-convex functions on I contains the class $J(I)$ of J-convex functions on I, that is, functions satisfying the condition

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \qquad (J)$$

for all $x, y \in I$.

In [6], Dragomir and Pearce proved following theorems containing J-quasi-convex and Wright-quasi-convex functions.

Theorem 1. *Suppose $a, b \in I \subseteq \mathbb{R}$ and $a < b$. If $f \in JQC(I) \cap L_1[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx + I(a,b) \tag{1.2}$$

where

$$I(a,b) = \frac{1}{2}\int_0^1 \left|f(ta+(1-t)b) - f((1-t)a+tb)\right| dt.$$

Theorem 2. *Let $f : I \to \mathbb{R}$ be a Wright-quasi-convex map on I and suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f \in L_1[a,b]$, one has the inequality*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \max\left\{f(a), f(b)\right\}. \tag{1.3}$$

In [6], Dragomir and Pearce also gave the following theorems involving some inclusions.

Theorem 3. *Let $WQC(I)$ denote the class of Wright-quasi-convex functions on $I \subseteq \mathbb{R}$, then*

$$QC(I) \subset WQC(I) \subset JQC(I). \tag{1.4}$$

Both inclusions are proper.

Theorem 4. *We have the inlusions*

$$W(I) \subset WQC(I), \quad C(I) \subset QC(I), \quad J(I) \subset JQC(I). \tag{1.5}$$

Each inclusion is proper.

For recent results related to quasi-convex functions see the papers [1]-[11] and books [23], [24]. In [19], Dragomir defined co-ordinated convex functions and proved following inequalities.

Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y)$$

and

$$f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$.

Recall that the mapping $f:\Delta\rightarrow\mathbb{R}$ is convex on Δ, if the following inequality;

$$f\left(\lambda x+\left(1-\lambda\right)z,\lambda y+\left(1-\lambda\right)w\right)\leq\lambda f\left(x,y\right)+\left(1-\lambda\right)f\left(z,w\right) \tag{1.6}$$

holds for all $(x,y)\,,\,(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Theorem 5. *(see* [19]*, Theorem 1) Suppose that $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (1.7)\\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&\qquad \left.\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}
\end{aligned}$$

The above inequalities are sharp.

Similar results can be found in [13]-[22].

This paper is arranged as follows. Firstly, we will give some definitions on quasi-convex functions and lemmas belong to this definitions. Secondly, we will prove several inequalities contain co-ordinated quasi-convex functions. Also, we will discuss the inclusions a connection with some different classes of co-ordinated convex functions.

2. DEFINITIONS AND MAIN RESULTS

We will start the following definitions and lemmas;

Definition 5. *A function $f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ is said quasi-convex function on the co-ordinates on Δ if the following inequality*

$$f\left(\lambda x+\left(1-\lambda\right)z,\lambda y+\left(1-\lambda\right)w\right)\leq\max\left\{f\left(x,y\right),f\left(z,w\right)\right\}$$

holds for all $(x,y)\,,\,(z,w)\in\Delta$ and $\lambda\in[0,1]$

$f:\Delta\rightarrow\mathbb{R}$ will be called co-ordinated quasi-convex on the co-ordinates if the partial mappings

$$f_y:[a,b]\rightarrow\mathbb{R},\quad f_y(u)=f(u,y)$$

and

$$f_x:[c,d]\rightarrow\mathbb{R},\quad f_x(v)=f(x,v)$$

are convex where defined for all $y\in[c,d]$ and $x\in[a,b]$. We denote by $QC(\Delta)$ the classes of quasi-convex functions on the co-ordinates on Δ. The following lemma holds.

Lemma 1. *Every quasi-convex mapping $f:\Delta\rightarrow\mathbb{R}$ is quasi-convex on the co-ordinates.*

Proof. Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is quasi-convex on Δ. Then the partial mappings

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y), \quad y \in [c,d]$$

and

$$f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v), \quad x \in [a,b]$$

are convex on Δ. For $\lambda \in [0,1]$ and $v_1, v_2 \in [c,d]$, one has

$$\begin{aligned} f_x(\lambda v_1 + (1-\lambda) v_2) &= f(x, \lambda v_1 + (1-\lambda) v_2) \\ &= f(\lambda x + (1-\lambda) x, \lambda v_1 + (1-\lambda) v_2) \\ &\leq \max\{f(x,v_1), f(x,v_2)\} \\ &= \max\{f_x(v_1), f_x(v_2)\} \end{aligned}$$

which completes the proof of quasi-convexity of f_x on $[c,d]$. Therefore $f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y)$ is also quasi-convex on $[a,b]$ for all $y \in [c,d]$, goes likewise and we shall omit the details. □

Definition 6. *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said J-convex function on the co-ordinates on Δ if the following inequality*

$$f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) \leq \frac{f(x,y) + f(z,w)}{2}$$

holds for all $(x,y), (z,w) \in \Delta$. We denote by $J(\Delta)$ the classes of J-convex functions on the co-ordinates on Δ

Lemma 2. *Every J-convex mapping defined $f : \Delta \to \mathbb{R}$ is J-convex on the co-ordinates.*

Proof. By the partial mappings, we can write for $v_1, v_2 \in [c,d]$,

$$\begin{aligned} f_x\left(\frac{v_1+v_2}{2}\right) &= f\left(x, \frac{v_1+v_2}{2}\right) \\ &= f\left(\frac{x+x}{2}, \frac{v_1+v_2}{2}\right) \\ &\leq \frac{f(x,v_1) + f(x,v_2)}{2} \\ &= \frac{f_x(v_1) + f_x(v_2)}{2} \end{aligned}$$

which completes the proof of J-convexity of f_x on $[c,d]$. Similarly, we can prove J-convexity of f_y on $[a,b]$. □

Definition 7. *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said J-quasi-convex function on the co-ordinates on Δ if the following inequality*

$$f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) \leq \max\{f(x,y), f(z,w)\}$$

holds for all $(x,y), (z,w) \in \Delta$. We denote by $JQC(\Delta)$ the classes of J-quasi-convex functions on the co-ordinates on Δ

Lemma 3. *Every J-quasi-convex mapping defined $f : \Delta \to \mathbb{R}$ is J-quasi-convex on the co-ordinates.*

Proof. By a similar way to proof of Lemma 1, we can write for $v_1, v_2 \in [c,d]$,

$$\begin{aligned} f_x\left(\frac{v_1+v_2}{2}\right) &= f\left(x, \frac{v_1+v_2}{2}\right) \\ &= f\left(\frac{x+x}{2}, \frac{v_1+v_2}{2}\right) \\ &\leq \max\{f(x,v_1), f(x,v_2)\} \\ &= \max\{f_x(v_1), f_x(v_2)\} \end{aligned}$$

which completes the proof of J-quasi-convexity of f_x on $[c,d]$. We can also prove J-quasi-convexity of f_y on $[a,b]$. □

Definition 8. *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said Wright-convex function on the co-ordinates on Δ if the following inequality*

$$f((1-t)a+tb, (1-s)c+sd) + f(ta+(1-t)b, sc+(1-s)d) \leq f(a,c) + f(b,d)$$

holds for all (a,c), $(b,d) \in \Delta$ and $t, s \in [0,1]$. We denote by $W(\Delta)$ the classes of Wright-convex functions on the co-ordinates on Δ

Lemma 4. *Every Wright-convex mapping defined $f : \Delta \to \mathbb{R}$ is Wright-convex on the co-ordinates.*

Proof. Suppose that $f : \Delta \to \mathbb{R}$ is Wright-convex on Δ. Then by partial mapping, for $v_1, v_2 \in [c,d]$, $x \in [a,b]$,

$$\begin{aligned} & f_x((1-t)v_1 + tv_2) + f_x(tv_1 + (1-t)v_2) \\ = \; & f(x, (1-t)v_1 + tv_2) + f(x, tv_1 + (1-t)v_2) \\ = \; & f((1-t)x + tx, (1-t)v_1 + tv_2) + f(tx + (1-t)x, tv_1 + (1-t)v_2) \\ \leq \; & f(x,v_1) + f(x,v_2) \\ = \; & f_x(v_1) + f_x(v_2) \end{aligned}$$

which shows that f_x is Wright-convex on $[c,d]$. Similarly one can see that f_y is Wright-convex on $[a,b]$. □

Definition 9. *A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said Wright-quasi-convex function on the co-ordinates on Δ if the following inequality*

$$\frac{1}{2}\left[f(tx+(1-t)z, ty+(1-t)w) + f((1-t)x+tz, (1-t)y+tw)\right] \leq \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w) \in \Delta$ and $t \in [0,1]$. We denote by $WQC(\Delta)$ the classes of Wright-quasi-convex functions on the co-ordinates on Δ

Lemma 5. *Every Wright-quasi-convex mapping defined $f : \Delta \to \mathbb{R}$ is Wright-quasi-convex on the co-ordinates.*

Proof. Suppose that $f:\Delta\to\mathbb{R}$ is Wright-quasi-convex on Δ. Then by partial mapping, for $v_1,v_2\in[c,d]$,

$$\begin{aligned}
& \frac{1}{2}\left[f_x\left(tv_1+(1-t)v_2\right)+f_x\left((1-t)v_1+tv_2\right)\right] \\
= & \frac{1}{2}\left[f\left(x,tv_1+(1-t)v_2\right)+f\left(x,(1-t)v_1+tv_2\right)\right] \\
= & \frac{1}{2}\left[f\left(tx+(1-t)x,tv_1+(1-t)v_2\right)+f\left((1-t)x+tx,(1-t)v_1+tv_2\right)\right] \\
\leq & \max\left\{f\left(x,v_1\right),f\left(x,v_2\right)\right\} \\
= & \max\left\{f_x\left(v_1\right),f_x\left(v_2\right)\right\}
\end{aligned}$$

which shows that f_x is Wright-quasi-convex on $[c,d]$. Similarly one can see that f_y is Wright-quasi-convex on $[a,b]$. □

Theorem 6. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is J-quasi-convex on the co-ordinates on Δ. If $f_x\in L_1[c,d]$ and $f_y\in L_1[a,b]$, then we have the inequality;*

$$\begin{aligned}
& \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq & \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy+H(x,y)
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
H(x,y) = & \frac{1}{4(d-c)}\int_c^d\int_0^1\left|f\left(ta+(1-t)b,y\right)-f\left((1-t)a+tb,y\right)\right|dtdy \\
& +\frac{1}{4(b-a)}\int_a^b\int_0^1\left|f\left(x,tc+(1-t)d\right)-f\left(x,(1-t)c+td\right)\right|dtdx.
\end{aligned}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is J-quasi-convex on the co-ordinates on Δ. We can write the partial mappings

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f(u,y),\quad y\in[c,d]$$

and

$$f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f(x,v),\quad x\in[a,b]$$

are J-quasi-convex on Δ. Then by the inequality (1.2), we have

$$f_y\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f_y(x)dx+\frac{1}{2}\int_0^1\left|f_y\left(ta+(1-t)b\right)-f_y\left((1-t)a+tb\right)\right|dt.$$

That is

$$f\left(\frac{a+b}{2},y\right)\leq\frac{1}{b-a}\int_a^b f(x,y)dx+\frac{1}{2}\int_0^1\left|f\left(ta+(1-t)b,y\right)-f\left((1-t)a+tb,y\right)\right|dt.$$

Integrating the resulting inequality with respect to y over $[c,d]$ and dividing both sides of inequality with $(d-c)$, we get

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy \\
&+\frac{1}{2(d-c)}\int_c^d\int_0^1 \left|f\left(ta+(1-t)b,y\right)-f\left((1-t)a+tb,y\right)\right|dtdy.
\end{aligned}\tag{2.2}$$

By a similar argument, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&+\frac{1}{2(b-a)}\int_a^b\int_0^1 \left|f\left(x,tc+(1-t)d\right)-f\left(x,(1-t)c+td\right)\right|dtdx.
\end{aligned}\tag{2.3}$$

Summing (2.2) and (2.3), we get the required result. □

Theorem 7. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is Wright-quasi-convex on the co-ordinates on Δ. If $f_x\in L_1[c,d]$ and $f_y\in L_1[a,b]$, then we have the inequality;*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy \\
\leq\ & \frac{1}{2}\left[\max\left\{\frac{1}{(b-a)}\int_a^b f(x,c)dx,\frac{1}{(b-a)}\int_a^b f(x,d)dx\right\}\right. \\
&\left.+\max\left\{\frac{1}{(d-c)}\int_c^d f(a,y)dy,\frac{1}{(d-c)}\int_c^d f(b,y)dy\right\}\right].
\end{aligned}\tag{2.4}$$

Proof. Since $f:\Delta\to\mathbb{R}$ is Wright-quasi-convex on the co-ordinates on Δ. We can write the partial mappings

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f(u,y),\quad y\in[c,d]$$

and

$$f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f(x,v),\quad x\in[a,b]$$

are Wright-quasi-convex on Δ. Then by the inequality (1.3), we have

$$\frac{1}{b-a}\int_a^b f_y(x)dx\leq\max\left\{f_y(a),f_y(b)\right\}.$$

That is

$$\frac{1}{b-a}\int_a^b f(x,y)dx\leq\max\left\{f(a,y),f(b,y)\right\}.$$

Dividing both sides of inequality with $(d-c)$ and integrating with respect to y over $[c,d]$, we get

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\leq\max\left\{\frac{1}{(d-c)}\int_c^d f(a,y)dy,\frac{1}{(d-c)}\int_c^d f(b,y)dy\right\}.\tag{2.5}$$

By a similar argument, we can write
(2.6)

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy \le \max\left\{\frac{1}{(b-a)}\int_a^b f(x,c)dx, \frac{1}{(b-a)}\int_a^b f(x,d)dx\right\}.$$

By addition (2.5) and (2.6), we have

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\\ \le\ & \frac{1}{2}\left[\max\left\{\frac{1}{(b-a)}\int_a^b f(x,c)dx, \frac{1}{(b-a)}\int_a^b f(x,d)dx\right\}\right.\\ &\left.+\max\left\{\frac{1}{(d-c)}\int_c^d f(a,y)dy, \frac{1}{(d-c)}\int_c^d f(b,y)dy\right\}\right]\end{aligned}$$

which completes the proof. □

Theorem 8. *Let $C(\Delta)$, $J(\Delta)$, $W(\Delta)$, $QC(\Delta)$, $JQC(\Delta)$, $WQC(\Delta)$ denote the classes of functions co-ordinated convex, co-ordinated J-convex, co-ordinated W-convex, co-ordinated quasi-convex, co-ordinated J-quasi-convex and co-ordinated W-quasi-convex functions on $\Delta = [a,b]\times[c,d]$, respectively, we have following inclusions.*

$$QC(\Delta) \subset WQC(\Delta) \subset JQC(\Delta) \tag{2.7}$$

$$W(\Delta) \subset WQC(\Delta), \quad C(\Delta) \subset J(\Delta), \quad J(\Delta) \subset JQC(\Delta). \tag{2.8}$$

Proof. Let $f \in QC(\Delta)$. Then for all $(x,y), (z,w) \in \Delta$ and $t\in[0,1]$, we have

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \max\{f(x,y), f(z,w)\}$$
$$f((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w) \le \max\{f(x,y), f(z,w)\}.$$

By addition, we obtain

$$\begin{aligned}&\frac{1}{2}\left[f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) + f((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w)\right]\\ \le\ & \max\{f(x,y), f(z,w)\}\end{aligned} \tag{2.9}$$

that is, $f \in WQC(\Delta)$. In (2.9), if we choose $\lambda = \frac{1}{2}$, we obtain $WQC(\Delta) \subset JQC(\Delta)$. Which completes the proof of (2.7).

In order to prove (2.8), taking $f \in W(\Delta)$ and using the definition, we get

$$\frac{1}{2}\left[f((1-t)a + tb, (1-s)c + sd) + f(ta + (1-t)b, sc + (1-s)d)\right] \le \frac{f(a,c) + f(b,d)}{2}$$

for all $(a,c), (b,d) \in \Delta$ and $t\in[0,1]$. Using the fact that

$$\frac{f(a,c) + f(b,d) + |f(a,c) - f(b,d)|}{2} = \max\{f(a,c), f(b,d)\}$$

we can write

$$\frac{f(a,c) + f(b,d)}{2} \le \max\{f(a,c), f(b,d)\}$$

for all $(a,c), (b,d) \in \Delta$, we obtain $W(\Delta) \subset WQC(\Delta)$.

Taking $f \in C(\Delta)$ and, if we choose $t = \frac{1}{2}$ in (1.6), we obtain

$$f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) \le \frac{f(x,y) + f(z,w)}{2}$$

for all $(x,y)\,,\,(z,w)\in\Delta$. One can see that $C(\Delta)\subset J(\Delta)$.

Taking $f\in J(\Delta)$, we can write

$$f\left(\frac{x+z}{2},\frac{y+w}{2}\right)\leq\frac{f(x,y)+f(z,w)}{2}$$

for all $(x,y)\,,\,(z,w)\in\Delta$. Using the fact that

$$\frac{f(x,y)+f(z,w)+|f(x,y)-f(z,w)|}{2}=\max\{f(x,y),f(z,w)\}$$

we can write

$$\frac{f(x,y)+f(z,w)}{2}\leq\max\{f(x,y),f(z,w)\}.$$

Then obviously, we obtain

$$f\left(\frac{x+z}{2},\frac{y+w}{2}\right)\leq\max\{f(x,y),f(z,w)\}$$

which shows that $f\in JQ(\Delta)$. □

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