Normal completely positive maps on the space of quantum operations

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Abstract

We define a class of higher-order linear maps that transform quantum operations into quantum operations and satisfy suitable requirements of normality and complete positivity. For this class of maps we prove two dilation theorems that are the analogues of the Stinespring and Radon-Nikodym theorems for quantum operations. A structure theorem for probability measures with values in this class of higher-order maps is also derived.

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1. Introduction

Quantum operations [19] are the fundamental building block of the theory of open quantum systems [13]. They provide a general input-output description of the possible state changes in quantum theory, encompassing both the unitary evolution of a closed system and the stochastic evolutions associated

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with the possible events in a quantum measurement. Technically, quantum operations are defined in the Schrödinger picture as trace non-increasing completely positive maps sending trace-class operators on the input Hilbert space $\mathcal{K}$ to trace-class operators on the output Hilbert space $\mathcal{H}$. Denoting by $\mathcal{L}(\mathcal{H})$ (resp. $\mathcal{L}(\mathcal{K})$) the von Neumann algebra of bounded linear operators on $\mathcal{H}$ (resp. $\mathcal{K}$), a quantum operation can be described in the dual (or Heisenberg) picture as a normal completely positive map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ satisfying the bound $\mathcal{E}(I_{\mathcal{H}}) \leq I_{\mathcal{K}}$, where $I_{\mathcal{H}}$ (resp. $I_{\mathcal{K}}$) is the identity operator on $\mathcal{H}$ (resp. $\mathcal{K}$). Unital maps, for which $\mathcal{E}(I_{\mathcal{H}}) = I_{\mathcal{K}}$, represent deterministic quantum evolutions, and are usually referred to as *quantum channels* [16]. Quantum operations can be characterized by means of Stinespring's theorem [26], when specialized to the case of normal maps: Precisely, $\mathcal{E}$ is a quantum operation if and only if there exists a Hilbert space $\mathcal{V}$ and a contraction $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{V}$ such that $\mathcal{E}(A) = V^*(A \otimes I_{\mathcal{V}})V \ \forall A \in \mathcal{L}(\mathcal{H})$. In particular, if $\mathcal{E}$ is a quantum channel, then V is an isometry, with $V^*V = I_{\mathcal{K}}$.

Very recently, following the advent of quantum computation and quantum information theory, there has been increasing interest in the study of higher order maps that transform quantum operations into quantum operations, rather then quantum states into quantum states. These maps, called *quantum supermaps*, are particularly relevant to the study of transformations of quantum devices in quantum networks, and have been introduced in the literature in a series of papers by D'Ariano, Perinotti and one of the authors [4, 5, 6]. Quantum supermaps describe all transformations that a quantum device can possibly undergo: For example, a device implementing the quantum operation $\mathcal{E}$ can be connected with other devices, so that the resulting circuit implements a new quantum operation $\mathcal{E}'$. The theory of quantum supermaps, developed in Refs. [5, 6] for the finite dimensional quantum systems, has proven to be a powerful tool for the treatment of many advanced topics in quantum information theory [7, 8, 9, 10, 11, 12, 27, 29]. However, a rigorous definition and characterization of quantum supermaps in infinite dimensions is still lacking. This problem will be the main focus of the present paper.

Before presenting our results, we briefly review the definition and characterization of supermaps in finite dimensions [5, 6]. Quantum supermaps are defined axiomatically as linear completely positive maps transforming quantum operations into quantum operations (see [5, 6] for the physical motivation of linearity and complete positivity). The notion of complete positivity used in this definition is the following: Suppose that $\mathcal{A}$ is a finite dimensional unital C^*-algebra, $L(\mathcal{A})$ is the set of all linear maps on $\mathcal{A}$, and $L(L(\mathcal{A}))$ is the set of all linear maps on $L(\mathcal{A})$. In particular, if $\mathcal{A} = M_n(\mathbb{C})$ is the C^*-algebra of the $n \times n$ complex matrices, we denote by $\mathsf{I}_n : L(M_n(\mathbb{C})) \to L(M_n(\mathbb{C}))$ the identity map on $L(M_n(\mathbb{C}))$. A linear map $\mathsf{S} \in L(L(\mathcal{A}))$ is then called *completely positive* if for all $n \in \mathbb{N}$ the map $\mathsf{I}_n \otimes \mathsf{S} \in L(L(M_n(\mathbb{C}))) \otimes L(L(\mathcal{A})) = L(L(M_n(\mathbb{C}) \otimes \mathcal{A}))$ preserves the subset of completely positive maps on the tensor product C^*-algebra $M_n(\mathbb{C}) \otimes \mathcal{A}$. Following [5, 6], we call the map S *quantum supermap* if it leaves invariant the set of quantum operations. A quantum supermap is *deterministic* if it transforms quantum channels (unital completely positive maps in $L(\mathcal{A})$) into quantum channels.

A dilation theorem for deterministic supermaps was proved in [5, 6] in the case where $\mathcal{A} = \mathcal{L}(\mathcal{H})$ is the C^*-algebra of all linear operators on a finite dimensional Hilbert space $\mathcal{H}$. Precisely, Refs. [5, 6] showed that if $\mathsf{S} : L(\mathcal{L}(\mathcal{H})) \to L(\mathcal{L}(\mathcal{H}))$ is a deterministic supermap, then there exists two finite dimensional Hilbert spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ and two isometries $V_1 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{V}_1$, $V_2 : \mathcal{H} \otimes \mathcal{V}_1 \to \mathcal{H} \otimes \mathcal{V}_2$ such that

$$[\mathsf{S}(\mathcal{E})](A) = V_1^* \left[(\mathcal{E} \otimes \mathcal{I}_1)(V_2^*(A \otimes I_2)V_2) \right] V_1 \quad \forall\, A \in \mathcal{L}(\mathcal{H}),\ \mathcal{E} \in L(\mathcal{L}(\mathcal{H})), \tag{1}$$

where $\mathcal{I}_1$ and I_2 are the identity maps on $\mathcal{L}(\mathcal{V}_1)$ and $\mathcal{V}_2$, respectively. This result can be viewed as an analog of the classical Stinespring theorem for completely positive maps on $\mathcal{L}(\mathcal{H})$: a deterministic supermap is the composition of *two* amplifications followed by *two* dilations. Besides the doubling, the main difference with Stinespring's theorem is that the amplification $\mathcal{E} \otimes \mathcal{I}_1$ here is not a C^*-algebra representation, since quantum operations only generate a Banach algebra.

The first result of our paper will be the proof of the dilation theorem for deterministic supermaps in the infinite dimensional case. Like Refs. [5, 6], we will restrict our analysis to the choice where $\mathcal{A} = \mathcal{L}(\mathcal{H})$ is the von Neumann algebra of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. However, there will be two key differences with respect to the finite dimensional case. The first difference concerns the domain of definition of quantum supermaps. Clearly, the natural domain for a quantum supermap is the linear space spanned by quantum operations. However, while in finite dimensions quantum operations span the whole set $L(\mathcal{L}(\mathcal{H}))$ of linear operators on $\mathcal{L}(\mathcal{H})$, in infinite dimension they only span the proper subset $\mathrm{CB}(\mathcal{H})$ of weak*-continuous completely bounded linear maps on $\mathcal{L}(\mathcal{H})$, which is strictly contained in the space $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ of bounded linear operators on $\mathcal{L}(\mathcal{H})$. The second key difference concerns the necessary and sufficient conditions needed for the proof of the dilation theorem. Indeed, not every deterministic quantum supermap admits a dilation of the form of eq. (1) with all finite dimensional Hilbert spaces replaced by separable Hilbert spaces. We will prove that such a dilation exists if and only if the deterministic supermap S is *normal*, in a suitable sense that will be defined later. Under the normality hypothesis, a natural algebraic construction leads to our dilation theorem (Theorem 5) for deterministic supermaps, which is the main result of the paper.

We then prove a Radon-Nikodym theorem for probabilistic supermaps, namely supermaps that are dominated by deterministic supermaps. The class of probabilistic supermaps is particularly interesting for physical applications, as such maps naturally appear in the description of quantum circuits that are designed to test properties of physical devices [4, 5, 6]. Higher-order quantum measurements are indeed described by *quantum superinstruments*, which are the generalization of the quantum instruments of Davies and Lewis [14]. Given a measurable space Ω, a quantum superinstrument with outcome space Ω is a countably additive measure with values in the set of quantum supermaps, and with the normalization condition that the measure of the whole space Ω is a deterministic supermap. We conclude our paper with the proof of a dilation theorem for quantum superinstruments, in analogy with Ozawa's dilation theorem for ordinary instruments [21].

The paper is organized as follows. In Section 2 we fix the elementary definitions and notations, and recall some basic facts needed in the rest of the paper. In Section 3 we extend the notion of increasing nets from positive operators to normal completely positive maps. Section 4 contains some elementary results about the tensor product of weak*-continuous maps. In Section 5 we define normal completely positive supermaps and provide some examples. In Section 6 we prove the dilation Theorem 5 for deterministic supermaps. Section 7 extends Theorem 5 to probabilistic supermaps, providing a Radon-Nikodym theorem for supermaps. We define quantum superinstruments in Section 8 and use the Radon-Nikodym theorem to prove a dilation theorem for quantum superinstruments, in analogy with Ozawa's result for ordinary instruments (see in particular Proposition 4.2 in [21]). Finally, Appendix A contains the proofs for some standard results on weak*-continuous completely bounded maps on $\mathcal{L}(\mathcal{H})$, which are collected here for the reader's convenience.

2. Preliminaries and notations

2.1. Linear spaces

If $\mathcal{X}$, $\mathcal{Y}$ are lnear spaces (always assumed complex in the paper), we denote by $L(\mathcal{X}, \mathcal{Y})$ the space of linear maps from $\mathcal{X}$ to $\mathcal{Y}$. We abbreviate $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{X})$. $I_{\mathcal{X}}$ (or simply I when no confusion can arise) is the identity map on $\mathcal{X}$. $\mathcal{X} \hat{\otimes} \mathcal{Y}$ is the algebraic tensor product of $\mathcal{X}$ and $\mathcal{Y}$.

A linearly ordered vector space $\mathcal{X}$ is *positively generated* if $\mathcal{X}$ is spanned by the cone $\mathcal{X}_+$ of its positive elements. Cones $\mathcal{X}_+$ and $\mathcal{Y}_+$ in $\mathcal{X}$ and $\mathcal{Y}$, respectively, induce a corresponding cone $L(\mathcal{X}, \mathcal{Y})_+$ in $L(\mathcal{X}, \mathcal{Y})$ in the following way:

$$A \in L(\mathcal{X}, \mathcal{Y})_+ \iff Ax \in \mathcal{Y}_+ \ \forall x \in \mathcal{X}_+.$$

The cone $L(\mathcal{X}, \mathcal{Y})_+$ is the cone of *positive maps* from $\mathcal{X}$ to $\mathcal{Y}$ (again, we set $L(\mathcal{X})_+ := L(\mathcal{X}, \mathcal{X})_+$). The same symbol used to denote the order relation in $\mathcal{X}$ and $\mathcal{Y}$ will be used for this order relation in $L(\mathcal{X}, \mathcal{Y})$.

If $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces, we let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ ($\mathcal{L}(\mathcal{X}) \equiv \mathcal{L}(\mathcal{X}, \mathcal{X})$) be the Banach space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ endowed with the uniform norm $\|\cdot\|_\infty$. As usual, $\mathcal{X}^* := \mathcal{L}(\mathcal{X}, \mathbb{C})$ will be the dual space of $\mathcal{X}$.

If the contrary is not explicitely stated, by *Hilbert space* we will always mean a complex separable Hilbert space, with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$, linear in the first entry. When different Hilbert spaces are taken into consideration, we add a subscript to the norm and scalar product indicating the Hilbert space they refer to. If $\mathcal{H}$, $\mathcal{K}$ are two Hilbert spaces, we let $\mathcal{H} \otimes \mathcal{K}$ be their Hilbert space tensor product. If $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$, the tensor product $A \otimes B$ is a bounded operator on $\mathcal{H} \otimes \mathcal{K}$, and we have the inclusion $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{K}) \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$.

The Banach space $\mathcal{L}(\mathcal{H})$ is ordered in the usual way, its positive cone $\mathcal{L}(\mathcal{H})_+$ consisting of those $A \in \mathcal{L}(\mathcal{H})$ such that $\langle Av, v \rangle \geq 0$ for all $v \in \mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$, then $A^* \in \mathcal{L}(\mathcal{H})$ is the Hilbert space adjoint of A.

We denote by $\mathcal{T}(\mathcal{H})$ the Banach space of trace class operators in $\mathcal{L}(\mathcal{H})$ endowed with the trace class norm. We consider $\mathcal{T}(\mathcal{H})$ as a linealy ordered vector space with respect to the ordering inerhited from $\mathcal{L}(\mathcal{H})$. The trace functional in $\mathcal{T}(\mathcal{H})$ is denoted by tr. The Banach dual $\mathcal{T}(\mathcal{H})^*$ coincides with $\mathcal{L}(\mathcal{H})$, the pairing of $A \in \mathcal{L}(\mathcal{H})$ with $T \in \mathcal{T}(\mathcal{H})$ being given by $\mathrm{tr}\,(AT)$. The weak* topology of $\mathcal{L}(\mathcal{H})$ is thus the topology induced on $\mathcal{L}(\mathcal{H})$ by the family of seminorms $\nu_T(A) = |\mathrm{tr}\,(AT)|$, $T \in \mathcal{T}(\mathcal{H})$. If $\{A_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{L}(\mathcal{H})$ converging to A in the weak* topology, we write $A = \text{wk*-lim}_\lambda A_\lambda$.

2.2. Normal completely positive maps

If a linear map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is weak*-continuous, then it is automatically bounded and has a bounded *preadjoint* $\mathcal{E}_* \in \mathcal{L}(\mathcal{T}(\mathcal{K}), \mathcal{T}(\mathcal{H}))$ defined by

$$\mathrm{tr}\,[A\mathcal{E}_*(T)] := \mathrm{tr}\,[\mathcal{E}(A)T] \quad \forall A \in \mathcal{L}(\mathcal{H}),\, T \in \mathcal{T}(\mathcal{K}).$$

If Λ is a directed set, we say that a net $\{A_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{L}(\mathcal{H})_+$ is

- *increasing* if $A_{\lambda_1} \leq A_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,
- *bounded* if there exists $A \in \mathcal{L}(\mathcal{H})_+$ such that $A_\lambda \leq A$ for all $\lambda \in \Lambda$.

We have the following well known fact (see for example Lemma 1.7.4 in [25] or Lemma 5.1.4 in [17]).

Theorem 1. *If a net $\{A_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{L}(\mathcal{H})_+$ is increasing and bounded, then it converges in the weak* topology. Its limit A has the following property: $A_\lambda \leq A$ for all $\lambda \in \Lambda$, and, if $A' \in \mathcal{L}(\mathcal{H})$ is such that $A_\lambda \leq A'$ for all $\lambda \in \Lambda$, then $A \leq A'$.*

If $\{A_\lambda\}_{\lambda\in\Lambda}$ and A are as in the statement of the theorem, then we write $A_\lambda \uparrow A$.

Definition 1. A positive linear map $\mathcal{E} \in L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))_+$ is *normal* if $\mathcal{E}(A_n) \uparrow \mathcal{E}(A)$ for all sequences $\{A_n\}_{n\in\mathbb{N}}$ such that $A_n \uparrow A$.

It is a standard fact that the sets of normal maps and weak*-continuous maps in $L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))_+$ coincide (see for example §2, Lemma 2.2 in [13]).

We now recall some elementary facts about complete boundedness and complete positivity that will be used in the paper. For every Hilbert space $\mathcal{H}$ and every finite number $n \in \mathbb{N}$ define the Hilbert space $\mathcal{H}^{(n)} := \mathbb{C}^n \otimes \mathcal{H}$. Since n is finite, the space $\mathcal{L}(\mathcal{H}^{(n)})$ is identified with $M_n(\mathbb{C})\hat{\otimes}\mathcal{L}(\mathcal{H})$. Likewise, the space of linear maps $L\left(\mathcal{L}(\mathcal{H}^{(n)}), \mathcal{L}(\mathcal{K}^{(n)})\right)$ is identified with $L(M_n(\mathbb{C}))\,\hat{\otimes}L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$.

Let $\mathcal{I}_n$ denote the identity map on $M_n(\mathbb{C})$. We then have the following

Definition 2. A bounded linear operator $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is *completely bounded (CB)* if there exists $C > 0$ such that $\|\mathcal{I}_n \otimes \mathcal{E}\|_\infty \leq C$ for all $n \in \mathbb{N}$.

Example 1. If $E \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $F \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we denote by $E \odot F$ the element in $L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$ given by

$$(E \odot F)(A) = EAF \quad \forall A \in \mathcal{L}(\mathcal{H}).$$

It is immediate to verify that the map $E \odot F$ is completely bounded, with $C = \|E\|_\infty \|F\|_\infty$.

We will denote by $\mathrm{CB}(\mathcal{H}, \mathcal{K})$ ($\mathrm{CB}(\mathcal{H}) := \mathrm{CB}(\mathcal{H}, \mathcal{H})$) the linear space of completely bounded *and weak*-continuous* maps in $L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$. Note that the linear spaces $\mathrm{CB}(\mathbb{C}^m, \mathbb{C}^n)$ and $L(M_n(\mathbb{C}), M_m(\mathbb{C})) \simeq M_{mn}(\mathbb{C})$ coincide for all $m, n \in \mathbb{N}$ (see e.g. Proposition 3.8 and Exercise 3.11 in [23]). More generally, the linear space $\mathrm{CB}\left(\mathcal{H}^{(m)}, \mathcal{K}^{(n)}\right)$ can be identified with the algebraic tensor product $\mathrm{CB}(\mathbb{C}^m, \mathbb{C}^n) \hat{\otimes} \mathrm{CB}(\mathcal{H}, \mathcal{K})$ for all Hilbert spaces $\mathcal{H}$, $\mathcal{K}$.

Definition 3. A linear map $\mathcal{E} \in L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$ is *completely positive (CP)* if the linear map $\mathcal{I}_n \otimes \mathcal{E} \in L\left(\mathcal{L}\left(\mathcal{H}^{(n)}\right), \mathcal{L}\left(\mathcal{K}^{(n)}\right)\right)$ is positive for all $n \in \mathbb{N}$.

Example 2. If $E \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, then the map $\mathcal{E} = E^* \odot E$ is completely positive.

We will denote by $\mathrm{CP}(\mathcal{H}, \mathcal{K})$ the set of *normal* completely positive maps in $L(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$ ($\mathrm{CP}(\mathcal{H}) := \mathrm{CP}(\mathcal{H}, \mathcal{H})$).

It is well known that CP maps are automatically CB, and that any CB map can be written as a linear combination of four CP maps (see for example Theorem 8.5 in [23]). The corresponding statement for weak*-continuous maps, given in the next theorem, is proved in [15] (see also the Appendix of the present paper).

Theorem 2. *The inclusion* $\mathrm{CP}(\mathcal{H}, \mathcal{K}) \subset \mathrm{CB}(\mathcal{H}, \mathcal{K})$ *holds. Moreover, if* $\mathcal{E} \in \mathrm{CB}(\mathcal{H}, \mathcal{K})$*, then there exists* $\mathcal{E}_k \in \mathrm{CP}(\mathcal{H}, \mathcal{K})$*,* $k = 0, 1, 2, 3$*, such that* $\mathcal{E} = \sum_{k=0}^{3} i^k \mathcal{E}_k$*.*

In other words, Theorem 2 states that $\mathrm{CP}(\mathcal{H}, \mathcal{K})$ is a cone in $\mathrm{CB}(\mathcal{H}, \mathcal{K})$ and that $\mathrm{CB}(\mathcal{H}, \mathcal{K})$ is positively generated. We will denote by $\preceq$ the ordering in $\mathrm{CB}(\mathcal{H}, \mathcal{K})$ induced by the cone $\mathrm{CP}(\mathcal{H}, \mathcal{K})$: given two maps $\mathcal{E}, \mathcal{F} \in \mathrm{CB}(\mathcal{H}, \mathcal{K})$ we will write $\mathcal{E} \preceq \mathcal{F}$ whenever $\mathcal{F} - \mathcal{E}$ is completely positive. We will always consider $\mathrm{CB}(\mathcal{H}, \mathcal{K})$ as an ordered linear space with respect to $\preceq$.

Definition 4. A normal completely positive map $\mathcal{E} \in \mathrm{CP}\,(\mathcal{H},\mathcal{K})$ is a *quantum channel* if it is unital, that is, if $\mathcal{E}(I_{\mathcal{H}}) = I_{\mathcal{K}}$.

The set of quantum channels in $\mathrm{CP}\,(\mathcal{H},\mathcal{K})$ will be denoted by $\mathrm{CP}_1\,(\mathcal{H},\mathcal{K})$.

Definition 5. A normal completely positive map $\mathcal{E} \in \mathrm{CP}\,(\mathcal{H},\mathcal{K})$ is a *quantum operation* if there exists a quantum channel $\mathcal{F} \in \mathrm{CP}\,(\mathcal{H},\mathcal{K})$ such that $\mathcal{E} \preceq \mathcal{F}$.

The set of quantum operations in $\mathrm{CP}\,(\mathcal{H},\mathcal{K})$ will be denoted by $\mathrm{CP}_0\,(\mathcal{H},\mathcal{K})$. Clearly, quantum channels are a particular class of quantum operations, i.e. we have the inclusion $\mathrm{CP}_1\,(\mathcal{H},\mathcal{K}) \subset \mathrm{CP}_0\,(\mathcal{H},\mathcal{K})$.

A simple characterization of quantum operations is given by the following

Proposition 1. *A normal CP map $\mathcal{E} \in \mathrm{CP}\,(\mathcal{H},\mathcal{K})$ is a quantum operation if and only if $\mathcal{E}(I_{\mathcal{H}}) \leq I_{\mathcal{K}}$.*

Proof. If $\mathcal{E}$ is a quantum operation, then by definition there exists a quantum channel $\mathcal{F}$ such that $\mathcal{E} \preceq \mathcal{F}$. Then, $\mathcal{E}(I_{\mathcal{H}}) \leq \mathcal{F}(I_{\mathcal{H}}) = I_{\mathcal{K}}$. Conversely, if $\mathcal{E}(I_{\mathcal{H}}) \leq I_{\mathcal{K}}$ we can define a normal CP map $\mathcal{E}'$ by the relation $\mathcal{E}'(A) := (I_{\mathcal{K}} - \mathcal{E}(I_{\mathcal{H}}))\mathrm{tr}\,(A\rho)$, where $\rho \in \mathcal{T}(\mathcal{H})_+$ is a positive trace-class operator such that $\mathrm{tr}\,(\rho) = 1$. The map $\mathcal{F} := \mathcal{E} + \mathcal{E}'$ is then a quantum channel and $\mathcal{E} \preceq \mathcal{F}$.

3. Increasing nets of normal CP maps

We say that the net $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda} \subset L\,(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{K}))_+$ is

- *increasing* if $\mathcal{E}_{\lambda_1} \leq \mathcal{E}_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,
- *bounded* if there exists $\mathcal{E} \in L\,(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{K}))_+$ such that $\mathcal{E}_\lambda \leq \mathcal{E}$ for all $\lambda \in \Lambda$.

With this definition we have the following

Proposition 2. *If $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ is a net in $L\,(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{K}))_+$ which is increasing and bounded, then there exists an unique $\mathcal{E} \in L\,(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{K}))_+$ such that*

$$\mathcal{E}_\lambda(A) \uparrow \mathcal{E}(A) \quad \forall A \in \mathcal{L}(\mathcal{H})_+. \tag{2}$$

For such $\mathcal{E}$, we have $\mathcal{E}_\lambda \leq \mathcal{E}$ for all $\lambda \in \Lambda$, and, if $\mathcal{E}' \in L\,(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{K}))$ satisfies $\mathcal{E}_\lambda \leq \mathcal{E}'$ for all $\lambda \in \Lambda$, then $\mathcal{E} \leq \mathcal{E}'$.
If $B \in \mathcal{L}(\mathcal{H})$, then $\{\mathcal{E}_\lambda(B)\}_{\lambda\in\Lambda}$ converges to $\mathcal{E}(B)$ in the weak topology.*
If in addition $\mathcal{E}_\lambda$ is normal for all $\lambda \in \Lambda$, then $\mathcal{E}$ is normal.

Proof. By boundedness, there is $\mathcal{F} \in L\left(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K})\right)_+$ such that $\mathcal{E}_\lambda \leq \mathcal{F}$ for all λ. If $A \in \mathcal{L}(\mathcal{H})_+$, then the net $\{\mathcal{E}_\lambda(A)\}_{\lambda\in\Lambda}$ is increasing and bounded by $\mathcal{F}(A) \in \mathcal{L}(\mathcal{K})_+$, hence by Theorem 1 the net has a limit, call it $\mathcal{E}(A)$. This defines the map $\mathcal{E}$ on $\mathcal{L}(\mathcal{H})_+$. It is easy to check that $\mathcal{E}(\alpha A) = \alpha\mathcal{E}(A)$ for all real $\alpha \geq 0$ and $\mathcal{E}(A_1+A_2) = \mathcal{E}(A_1)+\mathcal{E}(A_2)$ for all $A_1, A_2 \in \mathcal{L}(\mathcal{H})_+$. Since the linearly ordered space $\mathcal{L}(\mathcal{H})$ is positively generated, $\mathcal{E}$ then uniquely extends to an element of $L\left(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K})\right)_+$.
For each λ, since $\mathcal{E}_\lambda(A) \leq \mathcal{E}(A)$ for every $A \in \mathcal{L}(\mathcal{H})_+$, we have $\mathcal{E}_\lambda \leq \mathcal{E}$. If $\mathcal{E}_\lambda \leq \mathcal{E}'$, then $\mathcal{E}_\lambda(A) \leq \mathcal{E}'(A)$ for all λ and for all $A \in \mathcal{L}(\mathcal{H})_+$, hence by Theorem 1 we have $\mathcal{E}(A) \leq \mathcal{E}'(A)$, i.e. $\mathcal{E} \leq \mathcal{E}'$.
If $B \in \mathcal{L}(\mathcal{H})$, it can be decomposed as $B = \sum_{k=0}^{3} i^k B_k$, where each B_k is positive. Then $\mathcal{E}(B_k) = \text{wk*-}\lim_\lambda \mathcal{E}_\lambda(B_k)$, and, by linearity $\mathcal{E}(B) = \text{wk*-}\lim_\lambda \mathcal{E}_\lambda(B)$.
Finally, suppose $\mathcal{E}_\lambda$ is normal for all $\lambda \in \Lambda$. If $A_n \uparrow A$ in $\mathcal{L}(\mathcal{H})_+$, then we have $\text{wk*-}\lim_\lambda \mathcal{E}_\lambda(A_n) = \mathcal{E}(A_n)$ for all n by eq. (2), and $\text{wk*-}\lim_n \mathcal{E}_\lambda(A_n) = \mathcal{E}_\lambda(A)$ for all λ by normality. On the other hand, the sequence $\{\mathcal{E}(A_n)\}_{n\in\mathbb{N}}$ is increasing and bounded by $\mathcal{E}(A)$, hence Theorem 1 implies $\mathcal{E}(A_n) \uparrow B$ for some $B \in \mathcal{L}(\mathcal{K})_+$. For all $T \in \mathcal{T}(\mathcal{K})_+$, we have

$$\begin{aligned} \operatorname{tr}[T\mathcal{E}(A)] &= \sup_\lambda \operatorname{tr}[T\mathcal{E}_\lambda(A)] = \sup_\lambda \sup_n \operatorname{tr}[T\mathcal{E}_\lambda(A_n)] = \sup_n \sup_\lambda \operatorname{tr}[T\mathcal{E}_\lambda(A_n)] \\ &= \sup_n \operatorname{tr}[T\mathcal{E}(A_n)] = \operatorname{tr}(TB), \end{aligned}$$

hence $B = \mathcal{E}(A)$. This proves that $\mathcal{E}$ is normal.

If $\mathcal{E}_\lambda$, $\mathcal{E}$ are as in the above proposition, we write $\mathcal{E}_\lambda \uparrow \mathcal{E}$.
We say that a net $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ of elements in $\mathrm{CP}(\mathcal{H}, \mathcal{K})$ is

- *CP-increasing* if $\mathcal{E}_{\lambda_1} \preceq \mathcal{E}_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,
- *CP-bounded* if there exists a map $\mathcal{E} \in \mathrm{CP}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{E}_\lambda \preceq \mathcal{E}$ for all $\lambda \in \Lambda$.

The central result of this section is the following analogue of Theorem 1 for normal CP maps.

Theorem 3. *If* $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ *is a net in* $\mathrm{CP}(\mathcal{H}, \mathcal{K})$ *which is CP-increasing and CP-bounded, then there exists a unique* $\mathcal{E} \in \mathrm{CP}(\mathcal{H}, \mathcal{K})$ *such that* $\mathcal{E}_\lambda \uparrow \mathcal{E}$. *Moreover, one has*

$$\mathcal{I}_n \otimes \mathcal{E}_\lambda \uparrow \mathcal{I}_n \otimes \mathcal{E} \quad \forall n \in \mathbb{N}. \tag{3}$$

$\mathcal{E}$ has the following property: $\mathcal{E}_\lambda \preceq \mathcal{E}$ for all $\lambda \in \Lambda$, and, if $\mathcal{E}' \in \mathrm{CP}(\mathcal{H},\mathcal{K})$ is such that $\mathcal{E}_\lambda \preceq \mathcal{E}'$ for all $\lambda \in \Lambda$, then $\mathcal{E} \preceq \mathcal{E}'$.
If $B \in \mathcal{L}(\mathcal{H})$, then $\{\mathcal{E}_\lambda(B)\}_{\lambda\in\Lambda}$ converges to $\mathcal{E}(B)$ in the weak topology.*

Proof. The existence and uniqueness of the normal positive map $\mathcal{E}$ such that $\mathcal{E}_\lambda \uparrow \mathcal{E}$ was proved in Proposition 2. We now prove that $\mathcal{E}$ is CP and that $\mathcal{I}_n \otimes \mathcal{E}_\lambda \uparrow \mathcal{I}_n \otimes \mathcal{E}$. Let $\mathcal{F} \in \mathrm{CP}(\mathcal{H},\mathcal{K})$ be such that $\mathcal{E}_\lambda \preceq \mathcal{F}$ for all λ. For all $n \in \mathbb{N}$, the net $\{\mathcal{I}_n \otimes \mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ is increasing and bounded by $\mathcal{I}_n \otimes \mathcal{F}$ in $L\left(\mathcal{L}(\mathcal{H}^{(n)}),\mathcal{L}(\mathcal{K}^{(n)})\right)_+$, hence by Proposition 2 there exists a unique $\mathcal{E}^{[n]} \in L\left(\mathcal{L}(\mathcal{H}^{(n)}),\mathcal{L}(\mathcal{K}^{(n)})\right)_+$ such that $(\mathcal{I}_n \otimes \mathcal{E}_\lambda) \uparrow \mathcal{E}^{[n]}$. So one has $\mathcal{E}^{[n]}(\tilde{B}) = \text{wk*-lim}_\lambda(\mathcal{I}_n \otimes \mathcal{E}_\lambda)(\tilde{B})$, $\forall \tilde{B} \in \mathcal{L}(\mathcal{H}^{(n)})$. On the other hand, we clearly have $(\mathcal{I}_n \otimes \mathcal{E})(\tilde{B}) = \text{wk*-lim}_\lambda(\mathcal{I}_n \otimes \mathcal{E}_\lambda)(\tilde{B})$. By comparison we obtain $\mathcal{I}_n \otimes \mathcal{E} = \mathcal{E}^{[n]}$, that is, $\mathcal{I}_n \otimes \mathcal{E}_\lambda \uparrow \mathcal{I}_n \otimes \mathcal{E}$. Moreover, since $\mathcal{E}^{[n]} \geq 0$ for all n, the equation $\mathcal{I}_n \otimes \mathcal{E} = \mathcal{E}^{[n]}$ also shows that $\mathcal{E}$ is completely positive.
Since $\mathcal{I}_n \otimes \mathcal{E}_\lambda \leq \mathcal{I}_n \otimes \mathcal{E}$ for all λ and n, we have $\mathcal{E}_\lambda \preceq \mathcal{E}$ for all λ. If $\mathcal{E}_\lambda \preceq \mathcal{E}'$, then $\mathcal{I}_n \otimes \mathcal{E}_\lambda \leq \mathcal{I}_n \otimes \mathcal{E}'$ for all λ and n, hence $\mathcal{I}_n \otimes \mathcal{E} \leq \mathcal{I}_n \otimes \mathcal{E}'$ by eq. (3) and Proposition 2. This means $\mathcal{E} \preceq \mathcal{E}'$.
The last claim of the theorem is a consequence of the analogue statement in Proposition 2.

If $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{E}$ are as in the statement of the above theorem, then we write $\mathcal{E}_\lambda \Uparrow \mathcal{E}$.

The next theorem collects a number of facts proved by Kraus in [19], stated in the language of Theorem 3. Recall that, if I is a generic index set, then the collection Λ_I of finite subsets of I is a directed set ordered by the inclusion relation. If $\{A_i\}_{i\in I}$ is a set of elements in $\mathcal{L}(\mathcal{H})_+$ and the net $\{\sum_{i\in J} A_i\}_{J\in\Lambda_I}$ converges in the sense of Theorem 1, then we denote its limit by $\sum_{i\in I} A_i$. Likewise, if $\{\mathcal{E}_i\}_{i\in I}$ are elements in $\mathrm{CP}(\mathcal{H},\mathcal{K})$ and the net $\{\sum_{i\in J} \mathcal{E}_i\}_{J\in\Lambda_I}$ converges in the sense of Theorem 3, then we denote its limit by $\sum_{i\in I} \mathcal{E}_i$.

Theorem 4 (Kraus form). *We have the following facts.*

1. *If I is a finite or countable set and $\{E_i\}_{i\in I}$ are elements in $\mathcal{L}(\mathcal{K},\mathcal{H})$ such that the sum $\{\sum_{i\in J} E_i^* E_i\}_{J\in\Lambda_I}$ converges in $\mathcal{L}(\mathcal{K})_+$ (in the sense of Theorem 1), then the sum $\{\sum_{i\in J} E_i^* \odot E_i\}_{J\in\Lambda_I}$ converges in $\mathrm{CP}(\mathcal{H},\mathcal{K})$ (in the sense of Theorem 3).*
2. *If $\mathcal{E} \in \mathrm{CP}(\mathcal{H},\mathcal{K})$, then there exists a countable set I and a sequence $\{E_i\}_{i\in I}$ such that $\mathcal{E}$ can be written in the* Kraus form $\mathcal{E} = \sum_{i\in I} E_i^* \odot E_i$.

In this case, $\mathcal{E} \in \mathrm{CP}_0(\mathcal{H},\mathcal{K})$ *[resp.* $\mathcal{E} \in \mathrm{CP}_1(\mathcal{H},\mathcal{K})$*] if and only if* $\sum_{i\in I} E_i^* E_i \le I_{\mathcal{K}}$ *[resp.* $\sum_{i\in I} E_i^* E_i = I_{\mathcal{K}}$*].*

4. Tensor product of weak*-continuous CB maps

This section contains some elementary facts about the tensor product of weak*-continuous CB maps.

Lemma 1. *If* $\mathcal{A}, \mathcal{A}' \in \mathrm{CB}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ *are equal on product operators, namely* $\mathcal{A}(A \otimes B) = \mathcal{A}'(A \otimes B)$ $\forall A \in \mathcal{L}(\mathcal{H}_1), B \in \mathcal{L}(\mathcal{H}_2)$*, then* $\mathcal{A} = \mathcal{A}'$*.*

Proof. By linearity, $\mathcal{A}$ and $\mathcal{A}'$ are equal on the algebraic tensor product $\mathcal{L}(\mathcal{H}_1) \hat{\otimes} \mathcal{L}(\mathcal{H}_2)$, which is a weak*-dense subset of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (see for example eq. (10) p. 185 in [28] or the last equation of Example 11.2.2 in [18]). By weak*-continuity, this implies $\mathcal{A} = \mathcal{A}'$.

Proposition 3. *Given two maps* $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $\mathcal{F} \in \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ *there exists a unique map* $\mathcal{E} \otimes \mathcal{F} \in \mathrm{CB}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ *such that*

$$(\mathcal{E} \otimes \mathcal{F})(A \otimes B) = \mathcal{E}(A) \otimes \mathcal{F}(B) \qquad \forall A \in \mathcal{L}(\mathcal{H}_1), B \in \mathcal{L}(\mathcal{H}_2). \tag{4}$$

If $\mathcal{E}$ *and* $\mathcal{F}$ *are CP, then* $\mathcal{E} \otimes \mathcal{F} \in \mathrm{CP}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$*.*

Proof. Uniqueness is immediate from Lemma 1. To prove existence we first consider the case where the maps $\mathcal{E}$ and $\mathcal{F}$ are completely positive. Given the Kraus forms $\mathcal{E} = \sum_{i\in I} E_i^* \odot E_i$ and $\mathcal{F} = \sum_{j\in J} F_j^* \odot F_j$ it is easy to verify that the map $\mathcal{E} \otimes \mathcal{F} := \sum_{(i,j)\in I\times J} (E_i \otimes F_j)^* \odot (E_i \otimes F_j)$ is an element in $\mathrm{CP}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ which satisfies eq. (4). When $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ and $\mathcal{F} \in \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ are generic, the existence of the map $\mathcal{E} \otimes \mathcal{F}$ is proved using the decompositions $\mathcal{E} = \sum_{k=0}^3 i^k \mathcal{E}_k$ and $\mathcal{F} = \sum_{l=0}^3 i^l \mathcal{F}_l$, where $\mathcal{E}_k \in \mathrm{CP}(\mathcal{H}_1, \mathcal{K}_1)$ and $\mathcal{F}_l \in \mathrm{CP}(\mathcal{H}_2, \mathcal{K}_2)$ (Theorem 2), and defining $\mathcal{E} \otimes \mathcal{F} := \sum_{k,l=0}^3 i^{k+l} (\mathcal{E}_k \otimes \mathcal{F}_l)$.

When $\mathcal{H}_1$ and $\mathcal{K}_1$ are both finite dimensional, note that the definition of $\mathcal{E} \otimes \mathcal{F}$ given in Proposition 3 coincides with the algebraic tensor product of the two maps $\mathcal{E}$ and $\mathcal{F}$, which was considered in the previous sections.

Proposition 4. *For* $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ *one has* $\mathcal{E}_1\mathcal{E}_2 \otimes \mathcal{F}_1\mathcal{F}_2 = (\mathcal{E}_1 \otimes \mathcal{F}_1)(\mathcal{E}_2 \otimes \mathcal{F}_2)$ *and* $(\mathcal{E}_1 + \mathcal{E}_2) \otimes (\mathcal{F}_1 + \mathcal{F}_2) = \sum_{i,j=1}^2 \mathcal{E}_i \otimes \mathcal{F}_j$*. Moreover, if* $\mathcal{E}_1 \preceq \mathcal{E}_2$ *and* $\mathcal{F}_1 \preceq \mathcal{F}_2$*, then* $\mathcal{E}_1 \otimes \mathcal{F}_1 \preceq \mathcal{E}_2 \otimes \mathcal{F}_2$*.*

Proof. It is immediate to check the two equalities on product operators $A\otimes B$, $A\in\mathcal{L}(\mathcal{H}_1)$, $B\in\mathcal{L}(\mathcal{H}_2)$. The equality on the whole $\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ then follows from Lemma 1. The second claim in the proposition is a consequence of the elementary inequalities $\mathcal{E}_1\otimes\mathcal{F}_1\preceq\mathcal{E}_1\otimes\mathcal{F}_2\preceq\mathcal{E}_2\otimes\mathcal{F}_2$.

If Λ and Σ are directed sets, then the cartesian product $\Lambda\times\Sigma$ can be viewed as a directed set endowed with the product order $(\lambda,\sigma)\leq(\lambda',\sigma')$ if and only if $\lambda\leq\lambda'$ and $\sigma\leq\sigma'$. We then have the following

Proposition 5. *If $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ is a net in* $\mathrm{CP}(\mathcal{H}_1,\mathcal{K}_1)$ *such that $\mathcal{E}_\lambda\Uparrow\mathcal{E}$ and $\{\mathcal{F}_\sigma\}_{\sigma\in\Sigma}$ is a net in* $\mathrm{CP}(\mathcal{H}_2,\mathcal{K}_2)$ *such that $\mathcal{F}_\sigma\Uparrow\mathcal{F}$, then $\mathcal{E}_\lambda\otimes\mathcal{F}_\sigma\Uparrow\mathcal{E}\otimes\mathcal{F}$.*

Proof. By Proposition 4, the net $\{\mathcal{E}_\lambda\otimes\mathcal{F}_\sigma\}_{(\lambda,\sigma)\in\Lambda\times\Sigma}$ is CP-increasing and CP-bounded by $\mathcal{E}\otimes\mathcal{F}$ in $\mathrm{CP}(\mathcal{H}\otimes\mathcal{K})$. Hence, by Theorem 3 there is a map $\mathcal{A}\in\mathrm{CP}(\mathcal{H}\otimes\mathcal{K})$ such that $\mathcal{E}_\lambda\otimes\mathcal{F}_\sigma\Uparrow\mathcal{A}$. In particular, one has $\mathcal{A}(A\otimes B)=\text{wk*-}\lim_{(\lambda,\sigma)}\mathcal{E}_\lambda(A)\otimes\mathcal{F}_\sigma(B)=\mathcal{E}(A)\otimes\mathcal{F}(B)$ for all $A\in\mathcal{L}(\mathcal{H}), B\in\mathcal{L}(\mathcal{K})$, which implies $\mathcal{A}=\mathcal{E}\otimes\mathcal{F}$ by Lemma 1.

5. Quantum supermaps: definition and examples

In this section we introduce the definition of quantum supermaps in the infinite dimensional case. The main difference with the finite-dimensional case of Refs. [5, 6] is the role of normality, which will be crucial for our dilation theorem (see Theorem 5 of the next section).

We recall that we call a linear map $\mathsf{S}:\mathrm{CB}(\mathcal{H}_1,\mathcal{K}_1)\to\mathrm{CB}(\mathcal{H}_2,\mathcal{K}_2)$ positive if $\mathsf{S}(\mathcal{E})\succeq 0$ for all $\mathcal{E}\succeq 0$, and in this case we write $\mathsf{S}\succeq 0$.

Definition 6. A positive map $\mathsf{S}:\mathrm{CB}(\mathcal{H}_1,\mathcal{K}_1)\to\mathrm{CB}(\mathcal{H}_2,\mathcal{K}_2)$ is *normal* if $\mathsf{S}(\mathcal{E}_n)\Uparrow\mathsf{S}(\mathcal{E})$ for all sequences $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in $\mathrm{CP}(\mathcal{H}_1,\mathcal{K}_1)$ such that $\mathcal{E}_n\Uparrow\mathcal{E}$.

Remark 1. Note that not every positive map S is normal, even though, by definition, S transforms normal maps into normal maps. An example of non-normal positive map is the following: consider a singular state $\rho:\mathcal{L}(\mathcal{H})\to\mathbb{C}$, i.e. a positive functional such that $\rho(K)=0$ for every compact operator $K\in\mathcal{L}(\mathcal{H})$ and $\rho(I_\mathcal{H})=1$. Define the linear map $\mathsf{S}:\mathrm{CB}(\mathcal{H})\to\mathrm{CB}(\mathcal{K})$ given by $\mathsf{S}(\mathcal{E})=\rho(\mathcal{E}(I_\mathcal{H}))\mathcal{F}$, where $\mathcal{F}\in\mathrm{CP}(\mathcal{K})$. Clearly, S is positive. However, S is not normal: consider for example a Hilbert basis $\{e_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ and define the sequence of maps $\mathcal{E}_n=\sum_{i=1}^n Q_i\odot Q_i$, where Q_i is the projector on e_i. In this way, one has $\mathcal{E}_n\Uparrow\mathcal{E}=\sum_{i\in\mathbb{N}}Q_i\odot Q_i$, whereas $\mathsf{S}(\mathcal{E}_n)=0$ and $\mathsf{S}(\mathcal{E})=\mathcal{F}$. Hence, S is not normal.

We now discuss the requirement of complete positivity for higher-order maps. Except for the domain of definition of the maps, here there is no significant difference with the finite-dimensional case discussed in Refs. [5, 6].

Since we identify $\mathrm{CB}\left(\mathcal{H}^{(n)}, \mathcal{K}^{(n)}\right)$ with $\mathrm{CB}\left(\mathbb{C}^n\right) \hat{\otimes} \mathrm{CB}\left(\mathcal{H}, \mathcal{K}\right)$, we also make the identification of the linear space $L\left(\mathrm{CB}\left(\mathcal{H}_1^{(n)}, \mathcal{K}_1^{(n)}\right), \mathrm{CB}\left(\mathcal{H}_2^{(n)}, \mathcal{K}_2^{(n)}\right)\right)$ with the tensor product $L\left(\mathrm{CB}\left(\mathbb{C}^n\right)\right) \hat{\otimes} L\left(\mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right), \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)\right)$. Let I_n be the identity map on $\mathrm{CB}\left(\mathbb{C}^n\right)$. Then we have the following

Definition 7. A linear map $\mathsf{S} : \mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right) \to \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)$ is *completely positive* if the map $\mathsf{I}_n \otimes \mathsf{S} : \mathrm{CB}\left(\mathcal{H}_1^{(n)}, \mathcal{K}_1^{(n)}\right) \to \mathrm{CB}\left(\mathcal{H}_2^{(n)}, \mathcal{K}_2^{(n)}\right)$ is positive for every $n \in \mathbb{N}$.

Remark 2. In order for the linear map $\mathsf{S} : \mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right) \to \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)$ to be completely positive, it is enough that, for every $n \in \mathbb{N}$, the map $\mathsf{I}_n \otimes \mathsf{S}$ sends completely positive normal maps to *positive* normal maps. Indeed, in the latter case the complete positivity of the map $(\mathsf{I}_n \otimes \mathsf{S})(\tilde{\mathcal{E}})$ for $\tilde{\mathcal{E}} \in \mathrm{CP}\left(\mathcal{H}_1^{(n)}, \mathcal{K}_1^{(n)}\right)$ follows automatically from the equalities $\mathcal{I}_m \otimes (\mathsf{I}_n \otimes \mathsf{S})(\tilde{\mathcal{E}}) = (\mathsf{I}_m \otimes \mathsf{I}_n \otimes \mathsf{S})(\mathcal{I}_m \otimes \tilde{\mathcal{E}}) = (\mathsf{I}_{mn} \otimes \mathsf{S})(\mathcal{I}_m \otimes \tilde{\mathcal{E}})$. Since $(\mathsf{I}_{mn} \otimes \mathsf{S})(\mathcal{I}_m \otimes \tilde{\mathcal{E}})$ is positive and normal by assumption, this implies that $(\mathsf{I}_n \otimes \mathsf{S})(\tilde{\mathcal{E}}) \in \mathrm{CP}\left(\mathcal{H}_2^{(n)}, \mathcal{K}_2^{(n)}\right)$, hence S is completely positive.

We are now ready to define quantum supermaps in infinite dimensions:

Definition 8. A *quantum supermap* (or simply, *supermap*) is a normal completely positive linear map $\mathsf{S} : \mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right) \to \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)$.

The set of quantum supermaps in $L\left(\mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right), \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)\right)$ will be denoted by $\mathrm{SCP}\left(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2\right)$ ($\mathrm{SCP}\left(\mathcal{H}; \mathcal{K}\right)$ if $\mathcal{H}_1 = \mathcal{K}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{K}_2 = \mathcal{K}$).

The set of quantum supermaps $\mathrm{SCP}\left(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2\right)$ is clearly a cone, and hence defines a partial order in $L\left(\mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right), \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)\right)$. Such a partial order will be denoted by $\ll$: given maps $\mathsf{S}, \mathsf{T} \in L\left(\mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right), \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right)\right)$ we will write $\mathsf{T} \ll \mathsf{S}$ if $\mathsf{S} - \mathsf{T} \in \mathrm{SCP}\left(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2\right)$.

Definition 9. A quantum supermap $\mathsf{S} \in \mathrm{SCP}\left(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2\right)$ is *deterministic* if it maps quantum channels to quantum channels, that is, if the inclusion $\mathsf{S}[\mathrm{CP}_1\left(\mathcal{H}_1, \mathcal{K}_1\right)] \subseteq \mathrm{CP}_1\left(\mathcal{H}_2, \mathcal{K}_2\right)$ holds.

The set of deterministic supermaps in $\mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ will be denoted by $\mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ ($\mathrm{SCP}_1(\mathcal{H}; \mathcal{K})$ if $\mathcal{H}_1 = \mathcal{K}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{K}_2 = \mathcal{K}$).

Definition 10. A quantum supermap $\mathsf{S} \in \mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ is *probabilistic* if there exists a deterministic supermap $\mathsf{T} \in \mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ such that $\mathsf{S} \ll \mathsf{T}$.

The set of probabilistic supermaps in $\mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ will be denoted by $\mathrm{SCP}_0(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ ($\mathrm{SCP}_0(\mathcal{H}; \mathcal{K})$ if $\mathcal{H}_1 = \mathcal{K}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{K}_2 = \mathcal{K}$). Clearly, deterministic supermaps are a special case of probabilistic supermaps, i.e. the inclusion $\mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2) \subset \mathrm{SCP}_0(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ holds.

Obviously, the composition of two quantum supermaps is a supermap: for every $\mathsf{S}_1 \in \mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ and $\mathsf{S}_2 \in \mathrm{SCP}(\mathcal{H}_2, \mathcal{K}_2; \mathcal{H}_3, \mathcal{K}_3)$ one has the composition $\mathsf{S}_2\mathsf{S}_1 \in \mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_3, \mathcal{K}_3)$. Similarly, the composition of two probabilistic [resp. deterministic] supermaps is a probabilistic [resp. deterministic] supermap.

We conclude this section with three examples of deterministic supermaps which will play an important role in the next section. The first example is given by the composition with two quantum channels (normal unital CP maps):

Proposition 6. *If $\mathcal{A} \in \mathrm{CP}_1(\mathcal{K}_1, \mathcal{K}_2)$ and $\mathcal{B} \in \mathrm{CP}_1(\mathcal{H}_2, \mathcal{H}_1)$, then the map $\mathsf{S} : \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1) \to \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ defined by $\mathsf{S}(\mathcal{E}) = \mathcal{A}\mathcal{E}\mathcal{B}$ is a deterministic supermap in $\mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$.*

Proof. The map S is normal: if $\mathcal{E}_n \Uparrow \mathcal{E}$, then the sequence $\{\mathcal{A}\mathcal{E}_n\mathcal{B}\}_{n\in\mathbb{N}}$ is CP-increasing and CP-bounded by $\mathcal{A}\mathcal{E}\mathcal{B}$. Using Theorem 3, we have wk*-$\lim_n \mathcal{A}\mathcal{E}_n\mathcal{B}(A) = \mathcal{A}\mathcal{E}\mathcal{B}(A)$, hence $\mathcal{A}\mathcal{E}_n\mathcal{B} \Uparrow \mathcal{A}\mathcal{E}\mathcal{B}$, i.e. S is normal. To prove complete positivity, note that for every map $\tilde{\mathcal{E}} \in \mathrm{CB}\left(\mathcal{H}_1^{(n)}, \mathcal{K}_1^{(n)}\right)$ one has $(\mathsf{I}_n \otimes \mathsf{S})(\tilde{\mathcal{E}}) = (\mathcal{I}_n \otimes \mathcal{A})\tilde{\mathcal{E}}(\mathcal{I}_n \otimes \mathcal{B})$. Therefore, if $\tilde{\mathcal{E}} \succeq 0$, then also $(\mathsf{I}_n \otimes \mathsf{S})(\tilde{\mathcal{E}}) \succeq 0$, hence $\mathsf{I}_n \otimes \mathsf{S} \succeq 0$ and S is completely positive. Finally, if $\mathcal{E}$ is unital, $\mathsf{S}(\mathcal{E}) = \mathcal{A}\mathcal{E}\mathcal{B}$ is unital as well.

Our second example of deterministic supermap is as follows: Suppose $\mathcal{V}$ is a Hilbert space and define the map $\hat{\pi}_{\mathcal{V}} : \mathrm{CB}(\mathcal{H}, \mathcal{K}) \to \mathrm{CB}(\mathcal{H} \otimes \mathcal{V}, \mathcal{K} \otimes \mathcal{V})$ by $\hat{\pi}_{\mathcal{V}}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}$, where $\mathcal{I}_{\mathcal{V}}$ is the identity map in $\mathrm{CB}(\mathcal{V})$ (cf. Proposition 3 for the definition of the tensor product). We then have the following

Proposition 7. *The map* $\hat{\pi}_{\mathcal{V}}$ *is a deterministic supermap, that is,* $\hat{\pi}_{\mathcal{V}} \in \mathrm{SCP}_1(\mathcal{H}, \mathcal{K}; \mathcal{H} \otimes \mathcal{V}, \mathcal{K} \otimes \mathcal{V})$.

Proof. If $\mathcal{E}_n \Uparrow \mathcal{E}$, then by Proposition 5 one has $\hat{\pi}_{\mathcal{V}}(\mathcal{E}_n) = \mathcal{E}_n \otimes \mathcal{I}_{\mathcal{V}} \Uparrow \mathcal{E} \otimes \mathcal{I}_{\mathcal{V}} = \hat{\pi}_{\mathcal{V}}(\mathcal{E})$. Hence, $\hat{\pi}_{\mathcal{V}}$ is normal. Clearly, if $\mathcal{E}$ is unital, so is $\hat{\pi}_{\mathcal{V}}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}$. To prove complete positivity, note that for every $\tilde{\mathcal{E}} \in \mathrm{CP}\left(\mathcal{H}^{(n)}, \mathcal{K}^{(n)}\right)$ we have $(\mathsf{I}_n \otimes \hat{\pi}_{\mathcal{V}})(\tilde{\mathcal{E}}) = \tilde{\mathcal{E}} \otimes \mathcal{I}_{\mathcal{V}} \succeq 0$, hence $\mathsf{I}_n \otimes \hat{\pi}_{\mathcal{V}} \succeq 0$ and $\hat{\pi}_{\mathcal{V}}$ is completely positive.

Combining the previous examples we obtain a third example of deterministic supermap. The example is important because, as we will show in the next section, every deterministic supermap can be expressed in this form.

Proposition 8. *Let* $\mathcal{V}_1$, $\mathcal{V}_2$ *be Hilbert spaces and* $V_1 : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}_1$, $V_2 : \mathcal{H}_1 \otimes \mathcal{V}_1 \to \mathcal{H}_2 \otimes \mathcal{V}_2$ *be two isometries. Then the linear map* $\mathsf{S} : \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1) \to \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ *defined by*

$$[\mathsf{S}(\mathcal{E})](A) := V_1^* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1}) \left(V_2^*(A \otimes I_{\mathcal{V}_2})V_2\right)\right] V_1$$

for all $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $A \in \mathcal{L}(\mathcal{H}_2)$ *is a deterministic supermap in* $\mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$.

Proof. Let us define the map $\mathsf{S}_1 : \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1}$, which is a deterministic supermap by Proposition 7. Moreover, define the maps $\mathcal{A} = V_1^* \odot V_1 \in \mathrm{CP}_1(\mathcal{K}_1 \otimes \mathcal{V}_1, \mathcal{K}_2)$, $\pi_{\mathcal{V}_2} \in \mathrm{CP}(\mathcal{H}_2, \mathcal{H}_2 \otimes \mathcal{V}_2), \pi_{\mathcal{V}_2}(A) = A \otimes I_{\mathcal{V}_2}$, and $\mathcal{B} = (V_2^* \odot V_2)\pi_{\mathcal{V}_2} \in \mathrm{CP}_1(\mathcal{H}_2, \mathcal{H}_1 \otimes \mathcal{V}_1)$. Finally, define the map $\mathsf{S}_2 : \mathcal{F} \mapsto \mathcal{A}\mathcal{F}\mathcal{B}$, which is a deterministic supermap by Proposition 6. With these definitions we have $\mathsf{S} = \mathsf{S}_2\mathsf{S}_1$. Hence, S is a deterministic supermap.

6. Dilation of deterministic supermaps

This section contains the central result of our paper, namely the dilation theorem for deterministic supermaps.

Theorem 5 (Dilation of deterministic supermaps). *A linear map* $\mathsf{S} : \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1) \to \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ *is a deterministic supermap if and only if there exist two Hilbert spaces* $\mathcal{V}_1$, $\mathcal{V}_2$ *and two isometries* $V_1 : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}_1$, $V_2 : \mathcal{H}_1 \otimes \mathcal{V}_1 \to \mathcal{H}_2 \otimes \mathcal{V}_2$ *such that for all* $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $A \in \mathcal{L}(\mathcal{H}_2)$,

$$[\mathsf{S}(\mathcal{E})](A) = V_1^* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes I_{\mathcal{V}_2})V_2)\right] V_1, \tag{5}$$

and

$$\mathcal{H}_1 \otimes \mathcal{V}_1 = \overline{\mathrm{span}}\,\{(E \otimes I_{\mathcal{V}_1})V_1 v \mid E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1)\,,\, v \in \mathcal{K}_2\} \qquad (6)$$

$$\mathcal{H}_2 \otimes \mathcal{V}_2 = \overline{\mathrm{span}}\,\{(A \otimes I_{\mathcal{V}_2})V_2 w \mid A \in \mathcal{L}(\mathcal{H}_2)\,,\, w \in \mathcal{H}_1 \otimes \mathcal{V}_1\}\,. \qquad (7)$$

Definition 11. If $\mathcal{V}_1$, $\mathcal{V}_2$ are Hilbert spaces and $V_1 : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}_1$, $V_2 : \mathcal{H}_1 \otimes \mathcal{V}_1 \to \mathcal{H}_2 \otimes \mathcal{V}_2$ are isometries such that eq. (5) holds, then we say that the quadruple $(\mathcal{V}_1, \mathcal{V}_2, V_1, V_2)$ is a *dilation* of the supermap S. If also eqs. (6) and (7) hold, then we say that the dilation is *minimal*.

The importance of the minimality property is highlighted by the following

Proposition 9. *Let* $(\mathcal{V}_1, \mathcal{V}_2, V_1, V_2)$ *and* $(\mathcal{V}_1', \mathcal{V}_2', V_1', V_2')$ *be two dilations of the supermap* $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$*. If* $(\mathcal{V}_1, \mathcal{V}_2, V_1, V_2)$ *is minimal, then there exist two isometries* $W_1 : \mathcal{V}_1 \to \mathcal{V}_1'$, $W_2 : \mathcal{V}_2 \to \mathcal{V}_2'$ *such that* $V_1' = (I_{\mathcal{K}_1} \otimes W_1)V_1$ *and* $V_2'(I_{\mathcal{H}_1} \otimes W_1) = (I_{\mathcal{H}_2} \otimes W_2)V_2$.

The proofs of Theorem 5 and Proposition 9 will be given in the end of this section.

Remark 3. In Proposition 9, if also the dilation $(\mathcal{V}_1', \mathcal{V}_2', V_1', V_2')$ is minimal, then the isometries W_1 and W_2 are actually unitaries as a simple consequence of minimality. Thus, the minimal dilation is unique up to unitary isomorphism.

Remark 4. As anticipated in the introduction, eq. (5) is the desired generalization of the finite dimensional result in Refs. [5, 6]. The physical interpretation of the dilation of quantum superinstruments is clear in the Schrödinger picture: indeed, turning eq. (5) into its predual, we then obtain

$$[\mathsf{S}(\mathcal{E})]_*(T) = \mathrm{tr}_{\mathcal{V}_2}\{V_2\,[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})_*(V_1 T V_1^*)]\,V_2^*\}$$

for all $T \in \mathcal{T}(\mathcal{K}_2)$ and $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$, where $\mathrm{tr}_{\mathcal{V}_2}$ denotes the partial trace on $\mathcal{V}_2$. If T is a quantum state (i.e. $T \geq 0$ and $\mathrm{tr}(T) = 1$), the above equation means that the quantum system with Hilbert space $\mathcal{K}_2$ first undergoes the invertible evolution V_1, then the quantum channel $(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})_*$ and the invertible evolution V_2, after which the ancillary system with Hilbert space $\mathcal{V}_2$ is discarded. It is interesting to note that the same kind of sequential composition of invertible evolutions also appears in a very different context: the reconstruction of quantum stochastic processes from correlation kernels [2, 20, 22]. That context is very different from the present context of higher-order maps, and it is a remarkable feature of Theorem 5 that any deterministic supermap on the space of quantum operations can be achieved through a two-step sequence of invertible evolutions.

Theorem 5 contains as a special case the Stinespring dilation of quantum channels (normal unital CP maps). This fact is illustrated in the following two examples:

Example 3. Suppose that $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$. In this case we have the identification $\mathrm{CB}\,(\mathbb{C}, \mathcal{K}_i) \simeq \mathcal{L}\,(\mathcal{K}_i)$. Precisely, the element $\mathcal{E} \in \mathrm{CB}\,(\mathbb{C}, \mathcal{K}_i)$ is identified with the operator $A_{\mathcal{E}} = \mathcal{E}(1) \in \mathcal{L}\,(\mathcal{K}_i)$. Eq. (5) then reads

$$[\mathsf{S}(\mathcal{E})](1) = V_1^*(A_{\mathcal{E}} \otimes I_{\mathcal{V}_1})V_1,$$

which is just Stinespring's dilation theorem for normal CP maps. A linear map $\mathsf{S} : \mathcal{L}\,(\mathcal{K}_1) \to \mathcal{L}\,(\mathcal{K}_2)$ is thus in $\mathrm{SCP}_1\,(\mathbb{C}, \mathcal{K}_1; \mathbb{C}, \mathcal{K}_2)$ if and only if it is a unital normal CP map, i.e. a quantum channel.

Example 4. Suppose now that $\mathcal{K}_1 = \mathcal{K}_2 = \mathbb{C}$. In this case we have the identification $\mathrm{CB}\,(\mathcal{H}_i, \mathbb{C}) \simeq \mathcal{T}\,(\mathcal{H}_i)$ (see e.g. Proposition 3.8 in [23]). Precisely, the element $\mathcal{E} \in \mathrm{CB}\,(\mathcal{H}_i, \mathbb{C})$ is identified with the trace class operator $T_{\mathcal{E}}$ given by $\mathcal{E}(A) = \mathrm{tr}\,(AT_{\mathcal{E}})\ \ \forall A \in \mathcal{L}\,(\mathcal{H}_i)$. In this case the isometry $V_1 : \mathbb{C} \to \mathbb{C} \otimes \mathcal{V}_1 = \mathcal{V}_1$ is identified with a vector $v_1 \in \mathcal{V}_1$, $\|v_1\| = 1$, and Eq. (5) becomes

$$\begin{aligned}[\mathsf{S}(\mathcal{E})](A) &= \langle (\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes I_{\mathcal{V}_2})V_2)v_1, v_1\rangle \\ &= \mathrm{tr}\,[(T_{\mathcal{E}} \otimes P_{v_1})V_2^*(A \otimes I_{\mathcal{V}_2})V_2] \\ &= \mathrm{tr}\,\{A\, \mathrm{tr}_{\mathcal{V}_2}[V_2(T_{\mathcal{E}} \otimes P_{v_1})V_2^*]\}\,,\end{aligned}$$

where $P_{v_1} \in \mathcal{L}\,(\mathcal{V}_1)$ is the orthogonal projector on v_1. Defining the isometry $W : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathcal{V}_2$, $Wu = V_2(u \otimes v_1)$, we thus obtain

$$\mathsf{S}(\mathcal{E}) = \mathrm{tr}_{\mathcal{V}_2}(WT_{\mathcal{E}}W^*),$$

which is a Stinespring dilation in the Schrödinger picture. We thus find that a linear map $\mathsf{S} : \mathcal{T}\,(\mathcal{H}_1) \to \mathcal{T}\,(\mathcal{H}_2)$ is in $\mathrm{SCP}_1\,(\mathcal{H}_1, \mathbb{C}; \mathcal{H}_2, \mathbb{C})$ if and only if it is completely positive and trace-preserving, that is a quantum channel in the Schrödinger picture.

The rest of this section is devoted to the proof of Theorem 5, which first requires some auxiliary lemmas.

Lemma 2. *Suppose that* $\mathsf{S} \in L(\mathrm{CB}(\mathcal{H}), \mathrm{CB}(\mathcal{K}))$ *is such that* $\mathsf{S}(\mathrm{CP}_1(\mathcal{H})) \subset \mathrm{CP}_1(\mathcal{K})$. *If* $\mathcal{E}, \mathcal{F} \in \mathrm{CP}(\mathcal{H})$ *and* $\mathcal{E}(I) = \mathcal{F}(I)$, *then* $[\mathsf{S}(\mathcal{E})](I) = [\mathsf{S}(\mathcal{F})](I)$

Proof. By linearity, it is enough to prove the claim for $\mathcal{E}(I) = \mathcal{F}(I) \leq I$. Let $A := I - \mathcal{E}(I)$, $\mathcal{A} := A^{1/2} \odot A^{1/2}$, $\mathcal{E}' := \mathcal{E} + \mathcal{A}$, and $\mathcal{F}' := \mathcal{F} + \mathcal{A}$. With this definition, $\mathcal{E}', \mathcal{F}' \in \mathrm{CP}_1(\mathcal{H})$. Hence, one has

$$\begin{aligned} I &= [\mathsf{S}(\mathcal{E}')](I) &&= [\mathsf{S}(\mathcal{E})](I) + [\mathsf{S}(\mathcal{A})](I) \\ I &= [\mathsf{S}(\mathcal{F}')](I) &&= [\mathsf{S}(\mathcal{F})](I) + [\mathsf{S}(\mathcal{A})](I). \end{aligned}$$

By comparison, this implies that $[\mathsf{S}(\mathcal{E})](I) = [\mathsf{S}(\mathcal{F})](I)$.

Lemma 3. *Suppose* $\mathsf{S} \in L(\mathrm{CB}(\mathcal{H}), \mathrm{CB}(\mathcal{K}))$ *is such that* $\mathsf{S}(\mathrm{CP}_1(\mathcal{H})) \subset \mathrm{CP}_1(\mathcal{K})$. *If* $\mathcal{E} \in \mathrm{CP}(\mathcal{H})$, *then*

$$[\mathsf{S}(\mathcal{E}\mathcal{F})](I) = [\mathsf{S}(\mathcal{E})](I) \quad \forall \mathcal{F} \in \mathrm{CP}_1(\mathcal{H}).$$

Proof. Since $\mathcal{E}\mathcal{F}(I) = \mathcal{E}(I)$, this is an immediate consequence of Lemma 2.

Lemma 4. *If* $\mathsf{S} \in L(\mathrm{CB}(\mathcal{H}), \mathrm{CB}(\mathcal{K}))$ *is such that* $\mathsf{S}(\mathrm{CP}_1(\mathcal{H})) \subset \mathrm{CP}_1(\mathcal{K})$, *then*

$$[\mathsf{S}(\mathcal{E}(I \odot A))](I) = [\mathsf{S}(\mathcal{E}(A \odot I))](I) \quad \forall \mathcal{E} \in \mathrm{CB}(\mathcal{H}), \forall A \in \mathcal{L}(\mathcal{H}).$$

In particular,

$$[\mathsf{S}(E \odot AF)](I) = [\mathsf{S}(EA \odot F)](I) \quad \forall E, F, A \in \mathcal{L}(\mathcal{H}).$$

Proof. By linearity, it is enough to prove the claim for $A^* = A$ and for $\mathcal{E} \in \mathrm{CP}(\mathcal{H})$. One has $A \odot I - I \odot A = \frac{1}{2i}(\mathcal{E}_+ - \mathcal{E}_-)$, where $\mathcal{E}_\pm = (A \pm iI)^* \odot (A \pm iI)$. Now, $\mathcal{E}_+$ and $\mathcal{E}_-$ are CP and $\mathcal{E}_+(I) = \mathcal{E}_-(I)$. Applying Lemma 2 to the maps $\mathcal{E}\mathcal{E}_+$ and $\mathcal{E}\mathcal{E}_-$ we then obtain $[\mathsf{S}(\mathcal{E}(A \odot I))](I) - [\mathsf{S}(\mathcal{E}(I \odot A))](I) = \frac{1}{2i}\{[\mathsf{S}(\mathcal{E}\mathcal{E}_+)](I) - [\mathsf{S}(\mathcal{E}\mathcal{E}_-)](I)\} = 0$.

Let $\mathsf{S} \in L(\mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1), \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2))$ be a linear supermap. Then, we can define an associated sesquilinear form $\langle \cdot, \cdot \rangle_{\mathsf{S}}$ on the algebraic tensor product $\mathcal{L}(\mathcal{K}_1, \mathcal{H}_1) \hat{\otimes} \mathcal{L}(\mathcal{H}_2) \hat{\otimes} \mathcal{K}_2$ as follows

$$\langle E_1 \otimes A_1 \otimes v_1, E_2 \otimes A_2 \otimes v_2 \rangle_{\mathsf{S}} = \langle [\mathsf{S}(E_2^* \odot E_1)](A_2^* A_1) v_1, v_2 \rangle.$$

We have the following

Lemma 5. *If a linear map* $\mathsf{S} : \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1) \to \mathrm{CB}(\mathcal{H}_2, \mathcal{K}_2)$ *is completely positive, then the associated sesquilinear form* $\langle \cdot, \cdot \rangle_{\mathsf{S}}$ *is positive semidefinite.*

Proof. Let $\phi = \sum_{i=1}^{n} E_i \otimes A_i \otimes v_i$ be a generic element in the linear space $\mathcal{L}(\mathcal{K}_1, \mathcal{H}_1) \hat{\otimes} \mathcal{L}(\mathcal{H}_2) \hat{\otimes} \mathcal{K}_2$. Let $\{e_i\}_{i=1}^{n}$ be the standard basis for $\mathbb{C}^n$, and $\{e_{ij}\}_{i,j=1}^{n}$ be the standard basis of $M_n(\mathbb{C})$, given by $e_{ij}(e_k) = \delta_{jk} e_i$. Define

$$\tilde{v} := \sum_{i=1}^{n} e_i \otimes v_i \in \mathcal{K}_2^{(n)}$$
$$\tilde{A} := \sum_{i=1}^{n} e_{1i} \otimes A_i \in \mathcal{L}\left(\mathcal{H}_2^{(n)}\right)$$
$$\tilde{E} := \sum_{i=1}^{n} e_{ii} \otimes E_i \in \mathcal{L}\left(\mathcal{K}_1^{(n)}, \mathcal{H}_1^{(n)}\right)$$

With this definition we have $\tilde{E}^* \odot \tilde{E} = \sum_{i,j=1}^{n} (e_{ii} \odot e_{jj}) \otimes (E_i^* \odot E_j)$ and $\tilde{A}^* \tilde{A} = \sum_{i,j=1}^{n} e_{ij} \otimes A_i^* A_j$. Hence, we obtain

$$(\mathsf{I}_n \otimes \mathsf{S})(\tilde{E}^* \odot \tilde{E}) = \sum_{i,j} (e_{ii} \odot e_{jj}) \otimes \mathsf{S}(E_i^* \odot E_j)$$

and

$$[(\mathsf{I}_n \otimes \mathsf{S})(\tilde{E}^* \odot \tilde{E})](\tilde{A}^* \tilde{A}) = \sum_{i,j} e_{ij} \otimes [\mathsf{S}(E_i^* \odot E_j)](A_i^* A_j).$$

Finally, this implies

$$\begin{aligned} 0 &\leq \left\langle [(\mathsf{I}_n \otimes \mathsf{S})(\tilde{E}^* \odot \tilde{E})](\tilde{A}^* \tilde{A}) \tilde{v}, \tilde{v} \right\rangle \\ &= \sum_{i,j} \langle [\mathsf{S}(E_i^* \odot E_j)](A_i^* A_j) v_j, v_i \rangle \\ &= \langle \phi, \phi \rangle_{\mathsf{S}}, \end{aligned}$$

the inequality coming from the fact that $(\mathsf{I}_n \otimes \mathsf{S})(\tilde{E}^* \odot \tilde{E})$ is positive.

We are now in position to prove the existence of the dilation of Theorem 5 in the special case $\mathcal{H}_1 = \mathcal{K}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{K}_2 = \mathcal{K}$ and $\dim \mathcal{H} = \dim \mathcal{K} = \infty$.

Proposition 10. *Let* $\dim \mathcal{H} = \dim \mathcal{K} = \infty$*. A linear map* $\mathsf{S} : \mathrm{CB}(\mathcal{H}) \to \mathrm{CB}(\mathcal{K})$ *is a deterministic supermap if and only if there exist two Hilbert spaces* $\mathcal{W}_1, \mathcal{W}_2$ *and two isometries* $W_1 : \mathcal{K} \to \mathcal{H} \otimes \mathcal{W}_1$, $W_2 : \mathcal{H} \otimes \mathcal{W}_1 \to \mathcal{K} \otimes \mathcal{W}_2$ *such that*

$$[\mathsf{S}(\mathcal{E})](A) = W_1^* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{W}_1})(W_2^*(A \otimes I_{\mathcal{W}_2}) W_2) \right] W_1 \tag{8}$$

for all $\mathcal{E} \in \mathrm{CB}(\mathcal{H})$ *and* $A \in \mathcal{L}(\mathcal{K})$*.*

Proof. The 'if' part of the statement is proved in Proposition 8.
Conversely, suppose $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{H}; \mathcal{K})$. Let $\langle \cdot, \cdot \rangle_1$ be the positive sesquilinear form in $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{K}$ defined by

$$\langle E_1 \otimes v_1, E_2 \otimes v_2 \rangle_1 := \langle E_1 \otimes I_{\mathcal{K}} \otimes v_1, E_2 \otimes I_{\mathcal{K}} \otimes v_2 \rangle_{\mathsf{S}} .$$

Let $\mathcal{N}_1$ be its kernel and $\hat{\mathcal{H}}_1$ be the Hilbert space completion of the quotient space $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{K} / \mathcal{N}_1$ (not assumed separable). We denote by $\langle \cdot, \cdot \rangle_1$ and $\|\cdot\|_1$ the scalar product and norm in $\hat{\mathcal{H}}_1$.
Moreover, let $\mathcal{N}_2$ be the kernel of the positive sesquilinear form $\langle \cdot, \cdot \rangle_{\mathsf{S}}$, and let $\hat{\mathcal{H}}_2$ be the Hilbert space completion (not assumed separable) of the quotient space $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{K}) \hat{\otimes} \mathcal{K} / \mathcal{N}_2$ with respect to such form. We denote by $\langle \cdot, \cdot \rangle_2$ and $\|\cdot\|_2$ the resulting scalar product and norm in $\hat{\mathcal{H}}_2$.
We define two linear maps

$$\begin{array}{ll} W_1 : \mathcal{K} \to \mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{K} & W_1 v = I_{\mathcal{H}} \otimes v \\ W_2 : \mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{K} \to \mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{K}) \hat{\otimes} \mathcal{K} & W_2(E \otimes v) = E \otimes I_{\mathcal{K}} \otimes v . \end{array}$$

It is easy to verify that W_1 and W_2 extend to isometries $W_1 : \mathcal{K} \to \hat{\mathcal{H}}_1$ and $W_2 : \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_2$, respectively. Indeed, for W_1 we have the equality

$$\begin{aligned} \langle W_1 v, W_1 v \rangle_1 &= \langle I_{\mathcal{H}} \otimes I_{\mathcal{K}} \otimes v, I_{\mathcal{H}} \otimes I_{\mathcal{K}} \otimes v \rangle_{\mathsf{S}} \\ &= \langle [\mathcal{S}(I_{\mathcal{H}} \odot I_{\mathcal{H}})](I_{\mathcal{K}}) v, v \rangle \\ &= \langle v, v \rangle , \end{aligned}$$

where we used the fact that S is deterministic, and, therefore, $[\mathsf{S}(\mathcal{I}_{\mathcal{H}})](I_{\mathcal{K}}) = I_{\mathcal{K}}$. For W_2, taking $\phi = \sum_{i=1}^n E_i \otimes v_i$ we have the equality

$$\begin{aligned} \langle W_2 \phi, W_2 \phi \rangle_2 &= \sum_{i,j} \langle E_i \otimes I_{\mathcal{K}} \otimes v_i, E_j \otimes I_{\mathcal{K}} \otimes v_j \rangle_{\mathsf{S}} \\ &= \sum_{i,j} \langle E_i \otimes v_i, E_j \otimes v_j \rangle_1 \\ &= \langle \phi, \phi \rangle_1 . \end{aligned}$$

For $B_1 \in \mathcal{L}(\mathcal{H})$, $B_2 \in \mathcal{L}(\mathcal{K})$, we define the linear operators $\pi_1(B_1) \in L\left(\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}\right)$ and $\pi_2(B_2) \in L\left(\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{L}(\mathcal{K})\hat{\otimes}\mathcal{K}\right)$ as

$$[\pi_1(B_1)](E \otimes v) = B_1 E \otimes v$$
$$[\pi_2(B_2)](E \otimes A \otimes v) = E \otimes B_2 A \otimes v.$$

We claim that, for $i = 1, 2$, $\pi_i(B_i)$ extends to a bounded linear operator on $\hat{\mathcal{H}}_i$, that $\pi_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}\left(\hat{\mathcal{H}}_1\right)$ and $\pi_2 : \mathcal{L}(\mathcal{K}) \to \mathcal{L}\left(\hat{\mathcal{H}}_2\right)$ are normal unital $*$-representations, and that $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_2$ are separable.
Let us start from the case $i = 1$. For every $\phi = \sum_{r=1}^n E_r \otimes v_r$, $\psi = \sum_{s=1}^n F_s \otimes w_s$ and $B \in \mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned}
\langle \pi_1(B)\phi, \psi \rangle_1 &= \sum_{r,s} \langle [\mathsf{S}(F_s^* \odot BE_r)]\,(I_{\mathcal{K}})v_r, w_s \rangle \\
&= \sum_{r,s} \langle [\mathsf{S}(F_s^* B \odot E_r)]\,(I_{\mathcal{K}})v_r, w_s \rangle \\
&= \langle \phi, \pi_1(B^*)\psi \rangle_1 ,
\end{aligned}$$

where we used Lemma 4. Note that $\pi_1(I_{\mathcal{K}}) = I_{\hat{\mathcal{H}}_1}$, and

$$\pi_1(B)\pi_1(B') = \pi_1(BB') \quad \forall B, B' \in \mathcal{L}(\mathcal{H}).$$

It follows that, for all $\phi \in \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}$, the map $\omega_\phi : B \mapsto \langle \pi_1(B)\phi, \phi \rangle_1$ is a positive linear functional on $\mathcal{L}(\mathcal{H})$, hence

$$\|\pi_1(B)\phi\|_1^2 = \omega_\phi(B^*B) \leq \|B^*B\|_\infty \omega_\phi(I_{\mathcal{H}}) = \|B\|_\infty^2 \|\phi\|_1^2 .$$

Therefore, $\pi_1(B)$ extends to a bounded operator on $\hat{\mathcal{H}}_1$, and π_1 is a unital $*$-representation of $\mathcal{L}(\mathcal{H})$ in $\hat{\mathcal{H}}_1$. We now prove that π_1 is normal. Let $\{e_i\}_{i\in\mathbb{N}}$ be a Hilbert basis for $\mathcal{H}$, Q_i be the projector on e_i, and P_n be the projector on span $\{e_i \mid i \leq n\}$. By Proposition 11 of the Appendix, to prove that π_1 is normal it is enough to prove that $\pi_1(P_n) \uparrow I_{\hat{\mathcal{H}}_1}$. For every $\phi = E \otimes v$, $\psi = F \otimes w$ we have

$$\begin{aligned}
\langle \pi_1(P_n)\phi, \psi \rangle_1 &= \sum_{i=1}^n \langle \pi_1(Q_i)\phi, \pi_1(Q_i)\psi \rangle_1 \\
&= \sum_{i=1}^n \langle [\mathsf{S}(F^* Q_i \odot Q_i E)](I_{\mathcal{K}})v, w \rangle \\
&= \langle [\mathsf{S}((F^* \odot E)\mathcal{F}_n)](I_{\mathcal{K}})v, w \rangle ,
\end{aligned}$$

where $\mathcal{F}_n = \sum_{i=1}^n Q_i \odot Q_i$. Let $\mathcal{F} \in \mathrm{CP}_1(\mathcal{H})$ be the unital map defined by $\mathcal{F}_n \Uparrow \mathcal{F}$. Using the polarization identity $F^* \odot E = \frac{1}{4}\sum_{k=0}^3 i^k (i^k F + E)^* \odot (i^k F + E)$, the normality of S and Lemma 3 we then obtain

$$
\begin{aligned}
\lim_n \langle \pi_1(P_n)\phi, \psi\rangle_1 &= \lim_n \langle [\mathsf{S}((F^* \odot E)\mathcal{F}_n)](I_{\mathcal{K}})v, w\rangle \\
&= \frac{1}{4}\sum_{k=0}^3 i^k \lim_n \left\langle [\mathsf{S}(((i^k F + E)^* \odot (i^k F + E))\mathcal{F}_n)](I_{\mathcal{K}})v, w\right\rangle \\
&= \frac{1}{4}\sum_{k=0}^3 i^k \left\langle [\mathsf{S}(((i^k F + E)^* \odot (i^k F + E))\mathcal{F})](I_{\mathcal{K}})v, w\right\rangle \\
&= \frac{1}{4}\sum_{k=0}^3 i^k \left\langle [\mathsf{S}((i^k F + E)^* \odot (i^k F + E))](I_{\mathcal{K}})v, w\right\rangle \\
&= \langle [\mathsf{S}(F^* \odot E)](I_{\mathcal{K}})v, w\rangle \\
&= \langle \phi, \psi\rangle_1 .
\end{aligned}
$$

This relation extends by linearity to all $\phi, \psi \in \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}$, and, since the sequence $\{\pi_1(P_n)\}_{n\in\mathbb{N}}$ is norm bounded, by density to all $\phi, \psi \in \hat{\mathcal{H}}_1$. Therefore, we obtain wk*-$\lim_n \pi_1(P_n) = I_{\hat{\mathcal{H}}_1}$, thus concluding the proof of normality of π_1. Note that by definition span $\{E \otimes v = \pi_1(E)W_1 v \mid E \in \mathcal{L}(\mathcal{H}),\, v \in \mathcal{K}\}$ is a dense linear subspace of $\hat{\mathcal{H}}_1$, hence, using Lemma 6 of the Appendix with $\mathcal{S} = W_1\mathcal{K}$, we obtain that $\hat{\mathcal{H}}_1$ is separable.
We now prove that π_2 is a normal unital $*$-representation and $\hat{\mathcal{H}}_2$ is separable. For every $\phi = \sum_{r=1}^n E_r \otimes A_r \otimes v_r$, $\psi = \sum_{s=1}^n F_s \otimes B_s \otimes w_s$ and $C \in \mathcal{L}(\mathcal{K})$ we have

$$
\begin{aligned}
\langle \pi_2(C)\phi, \psi\rangle_2 &= \sum_{r,s} \langle [\mathsf{S}(F_s^* \odot E_r)]\,(B_s^* C A_r)v_r, w_s\rangle \\
&= \langle \phi, \pi_2(C^*)\psi\rangle_2 .
\end{aligned}
$$

Clearly, $\pi_2(I_{\mathcal{K}}) = I_{\hat{\mathcal{H}}_2}$ and $\pi_2(C)\pi_2(C') = \pi_2(CC')$. The same argument used for π_1 then shows that π_2 extends to a unital $*$-representation of $\mathcal{L}(\mathcal{K})$ in $\hat{\mathcal{H}}_2$. To prove normality of π_2, we proceed as in the case of π_1 by choosing a Hilbert basis $\{e_i\}_{i\in\mathbb{N}}$ for $\mathcal{K}$ and letting P_n be the projector on span $\{e_i \mid i \leq n\}$. It is again enough to prove that $\pi_2(P_n) \uparrow I_{\hat{\mathcal{H}}_2}$, and this follows as before from the relation

$$\lim_n \langle \pi_2(P_n)(E\otimes A\otimes v), F\otimes B\otimes w\rangle_2 = \lim_n \langle [\mathsf{S}(F^*\odot E)](B^*P_nA)v, w\rangle$$
$$= \langle [\mathsf{S}(F^*\odot E)](B^*A)v, w\rangle$$
$$= \langle E\otimes A\otimes v, F\otimes B\otimes w\rangle_2 \,,$$

due to the weak*-continuity of $\mathsf{S}(F^*\odot E)$. Note that

$$\begin{aligned} \hat{\mathcal{H}}_2 &= \overline{\mathrm{span}}\,\{E\otimes A\otimes v \mid E\in\mathcal{L}(\mathcal{H}),\, A\in\mathcal{L}(\mathcal{K}),\, v\in\mathcal{K}\} \\ &= \overline{\mathrm{span}}\,\{\pi_2(A)W_2\pi_1(E)W_1v \mid E\in\mathcal{L}(\mathcal{H}),\, A\in\mathcal{L}(\mathcal{K}),\, v\in\mathcal{K}\} \\ &= \overline{\mathrm{span}}\,\left\{\pi_2(A)W_2w \mid A\in\mathcal{L}(\mathcal{K}), w\in\hat{\mathcal{H}}_1\right\}. \end{aligned}$$

Separability of $\hat{\mathcal{H}}_2$ then follows from Lemma 6 of the Appendix, with $\mathcal{S} = W_2\hat{\mathcal{H}}_1$, using the fact that $\hat{\mathcal{H}}_1$ is separable.
By Corollary 10.4.14 in [18], there exist Hilbert spaces $\mathcal{W}_1$, $\mathcal{W}_2$ such that $\hat{\mathcal{H}}_i = \mathcal{H}\otimes\mathcal{W}_i$ and $\pi_i(B_i) = B_i\otimes I_{\mathcal{W}_i}$ for $i=1,2$ (separability of $\mathcal{W}_i$ clearly follows from the analogue property of $\hat{\mathcal{H}}_i$).
We conclude with the proof of eq. (8). If we take $\mathcal{E} = E^*\odot E$ with $E\in\mathcal{L}(\mathcal{H})$, $A = B^*B$ with $B\in\mathcal{L}(\mathcal{K})$ and $v\in\mathcal{K}$, then we have

$$\begin{aligned} \langle [\mathsf{S}(\mathcal{E})]\,(A)v, v\rangle &= \langle E\otimes B\otimes v, E\otimes B\otimes v\rangle_{\mathsf{S}} \\ &= \|\pi_2(B)W_2\pi_1(E)W_1v\|_2^2 \\ &= \|(B\otimes I_{\mathcal{W}_2})W_2(E\otimes I_{\mathcal{W}_1})W_1v\|_2^2 \\ &= \langle W_1^*\,\{[(E^*\otimes I_{\mathcal{W}_1})\odot(E\otimes I_{\mathcal{W}_1})](W_2^*(B^*B\otimes I_{\mathcal{W}_2})W_2)\}\,W_1v, v\rangle \\ &= \langle W_1^*\,[(\mathcal{E}\otimes\mathcal{I}_{\mathcal{W}_1})(W_2^*(A\otimes I_{\mathcal{W}_2})W_2)]\,W_1v, v\rangle \end{aligned}$$

thus proving eq. (8) in the special case $\mathcal{E} = E^*\odot E$, $A = B^*B$. Since the equality holds for every positive A, by linearity it holds for every operator $A\in\mathcal{L}(\mathcal{K})$. The equality for generic $\mathcal{E}\in\mathrm{CP}\,(\mathcal{H})$ then follows by Kraus Theorem 4 using normality of S and of the supermap $\hat{\pi}_{\mathcal{W}_1} : \mathcal{E}\mapsto\mathcal{E}\otimes\mathcal{I}_{\mathcal{W}_1}$. Finally, by linearity and using Theorem 2 it is immediate to show the equality for arbitrary maps $\mathcal{E}\in\mathrm{CB}\,(\mathcal{H})$. This concludes the proof of eq. (8).

We are now in position to prove Theorem 5:

Proof (Proof of Theorem 5). If $\mathcal{V}_1$, $\mathcal{V}_2$ and V_1, V_2 are as in the statement of the theorem, then eq. (5) defines a deterministic supermap by Proposition 8.

Conversely, suppose $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$. Let ℓ^2 denote the Hilbert space of square-summable sequences and define the two isometries S_1, S_2 as follows

$$\begin{aligned} S_1 &: \mathcal{K}_1 \to \mathcal{K}_1 \otimes \ell^2 \qquad S_1 v_1 = v_1 \otimes e_0 \\ S_2 &: \mathcal{K}_2 \to \mathcal{K}_2 \otimes \ell^2 \qquad S_2 v_2 = v_2 \otimes e_0, \end{aligned}$$

where $e_0 \in \ell^2$ is such that $\|e_0\| = 1$.
Then, define three supermaps

$$\begin{aligned} \mathsf{S}_1 &: \mathrm{CB}\left(\mathcal{H}_1 \otimes \ell^2, \mathcal{K}_1 \otimes \ell^2\right) \to \mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right) \\ \mathsf{S}_2 &: \mathrm{CB}\left(\mathcal{H}_2 \otimes \ell^2, \mathcal{K}_2 \otimes \ell^2\right) \to \mathrm{CB}\left(\mathcal{H}_2, \mathcal{K}_2\right) \\ \tilde{\mathsf{S}} &: \mathrm{CB}\left(\mathcal{H}_1 \otimes \ell^2, \mathcal{K}_1 \otimes \ell^2\right) \to \mathrm{CB}\left(\mathcal{H}_2 \otimes \ell^2, \mathcal{K}_2 \otimes \ell^2\right) \end{aligned}$$

given by

$$\begin{aligned} [\mathsf{S}_1(\tilde{\mathcal{A}})](A) &= S_1^* \tilde{\mathcal{A}}(A \otimes I_{\ell^2}) S_1 \\ [\mathsf{S}_2(\tilde{\mathcal{B}})](B) &= S_2^* \tilde{\mathcal{B}}(B \otimes I_{\ell^2}) S_2 \\ \tilde{\mathsf{S}}(\tilde{\mathcal{A}}) &= \mathsf{S}\mathsf{S}_1(\tilde{\mathcal{A}}) \otimes \mathcal{I}_{\ell^2} \end{aligned}$$

Since the input and output spaces of the supermap $\tilde{\mathsf{S}}$ are all isomorphic, we can apply Proposition 10 and obtain that there exist two separable Hilbert spaces $\mathcal{W}_1$, $\mathcal{W}_2$ and two isometries

$$\begin{aligned} W_1 &: \mathcal{K}_2 \otimes \ell^2 \to \mathcal{K}_1 \otimes \ell^2 \otimes \mathcal{W}_1 \\ W_2 &: \mathcal{H}_1 \otimes \ell^2 \otimes \mathcal{W}_1 \to \mathcal{H}_2 \otimes \ell^2 \otimes \mathcal{W}_2. \end{aligned}$$

such that

$$[\tilde{\mathsf{S}}(\tilde{\mathcal{A}})](\tilde{A}) = W_1^*(\tilde{\mathcal{A}} \otimes \mathcal{I}_{\mathcal{W}_1})(W_2^*(\tilde{A} \otimes I_{\mathcal{W}_2})W_2)W_1$$

for every $\tilde{\mathcal{A}}$ in $\mathrm{CB}\left(\mathcal{H}_1 \otimes \ell^2, \mathcal{K}_1 \otimes \ell^2\right)$ and $\tilde{A}$ in $\mathcal{L}\left(\mathcal{H}_2 \otimes \ell^2\right)$. On the other hand we have the relations

$$\begin{aligned} &\mathsf{S}_1(\mathcal{E} \otimes \mathcal{I}_{\ell^2}) = \mathcal{E} && \forall \mathcal{E} \in \mathrm{CB}\left(\mathcal{H}_1, \mathcal{K}_1\right) \\ &\mathsf{S}_2\tilde{\mathsf{S}}(\tilde{\mathcal{A}}) = \mathsf{S}\mathsf{S}_1(\tilde{\mathcal{A}}) && \forall \tilde{\mathcal{A}} \in \mathrm{CB}\left(\mathcal{H}_1 \otimes \ell^2, \mathcal{K}_1 \otimes \ell^2\right) \end{aligned}$$

which together imply $\mathsf{S}_2\tilde{\mathsf{S}}(\mathcal{E} \otimes \mathcal{I}_{\ell^2}) = \mathsf{S}(\mathcal{E})$. Therefore, we obtain

$$\begin{aligned} [\mathsf{S}(\mathcal{E})](A) &= [\mathsf{S}_2\tilde{\mathsf{S}}(\mathcal{E} \otimes \mathcal{I}_{\ell^2})](A) \\ &= S_2^*[\tilde{\mathsf{S}}(\mathcal{E} \otimes \mathcal{I}_{\ell^2})](A \otimes I_{\ell^2})S_2 \\ &= S_2^* W_1^*[(\mathcal{E} \otimes \mathcal{I}_{\ell^2} \otimes \mathcal{I}_{\mathcal{W}_1})(W_2^*(A \otimes I_{\ell^2} \otimes I_{\mathcal{W}_2})W_2)]W_1 S_2. \end{aligned} \tag{9}$$

For $i = 1, 2$ we define the subspaces

$$\begin{aligned}\hat{\mathcal{H}}_1 = \overline{\text{span}} \left\{ (E \otimes I_{\ell^2 \otimes \mathcal{W}_1}) W_1 S_2 v \mid E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1), \, v \in \mathcal{K}_2 \right\} \\ \subseteq \mathcal{H}_1 \otimes \ell^2 \otimes \mathcal{W}_1\end{aligned} \tag{10}$$

and

$$\begin{aligned}\hat{\mathcal{H}}_2 = \overline{\text{span}} \left\{ (A \otimes I_{\ell^2 \otimes \mathcal{W}_2}) W_2 u \mid A \in \mathcal{L}(\mathcal{H}_2), \, u \in \hat{\mathcal{H}}_1 \right\} \\ \subseteq \mathcal{H}_2 \otimes \ell^2 \otimes \mathcal{W}_2.\end{aligned} \tag{11}$$

Let P_1 and P_2 be the projectors onto $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$, respectively. Note that by definition of P_1 and P_2 we have $W_2 P_1 = P_2 W_2 P_1$. Moreover, using the relation $(\mathcal{L}(\mathcal{H}_i) \otimes I_{\ell^2 \otimes \mathcal{W}_i}) \hat{\mathcal{H}}_i \subset \hat{\mathcal{H}}_i$, we obtain that the projector P_i must have the form $P_i = I_{\mathcal{H}_i} \otimes Q_i$ for some projector $Q_i \in \mathcal{L}(\ell^2 \otimes \mathcal{W}_i)$. Define $\mathcal{V}_i := Q_i(\ell^2 \otimes \mathcal{W}_i)$, so that we have $\hat{\mathcal{H}}_i = \mathcal{H}_i \otimes \mathcal{V}_i$ (note that since $\ell^2 \otimes \mathcal{W}_i$ is separable, then also $\mathcal{V}_i$ must be separable). Then, define the operators

$$\begin{aligned}
&V_1 : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}_1 && V_1 = (I_{\mathcal{K}_1} \otimes Q_1) W_1 S_2 \\
&V_2 : \mathcal{H}_1 \otimes \mathcal{V}_1 \to \mathcal{H}_2 \otimes \mathcal{V}_2 && V_2 = (I_{\mathcal{H}_2} \otimes Q_2) W_2 (I_{\mathcal{H}_1} \otimes Q_1) = P_2 W_2 P_1.
\end{aligned}$$

Now, if $\mathcal{E} = E^* \odot E$ with $E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1)$ and $A = B^* B$ with $B \in \mathcal{L}(\mathcal{H}_2)$, from eq. (9) we have

$$[\mathsf{S}(\mathcal{E})](A) = C^* C \qquad C := (B \otimes I_{\ell^2 \otimes \mathcal{W}_2}) W_2 (E \otimes I_{\ell^2 \otimes \mathcal{W}_1}) W_1 S_2. \tag{12}$$

On the other hand, for every $v \in \mathcal{K}_2$ we have

$$\begin{aligned}
Cv &= (B \otimes I_{\ell^2 \otimes \mathcal{W}_2}) W_2 (E \otimes I_{\ell^2 \otimes \mathcal{W}_1}) W_1 S_2 v \\
&= (B \otimes I_{\ell^2 \otimes \mathcal{W}_2}) W_2 P_1 (E \otimes I_{\ell^2 \otimes \mathcal{W}_1}) W_1 S_2 v \\
&= (B \otimes I_{\ell^2 \otimes \mathcal{W}_2}) P_2 W_2 P_1 (E \otimes I_{\ell^2 \otimes \mathcal{W}_1}) W_1 S_2 v \\
&= (B \otimes I_{\mathcal{V}_2}) V_2 (E \otimes Q_1) W_1 S_2 v \\
&= (B \otimes I_{\mathcal{V}_2}) V_2 (E \otimes I_{\mathcal{V}_1}) (I_{\mathcal{K}_1} \otimes Q_1) W_1 S_2 v \\
&= (B \otimes I_{\mathcal{V}_2}) V_2 (E \otimes I_{\mathcal{V}_1}) V_1 v,
\end{aligned}$$

that is, $C = (B \otimes I_{\mathcal{V}_2}) V_2 (E \otimes I_{\mathcal{V}_1}) V_1$. Inserting the last expression in eq. (12) we then obtain $[\mathsf{S}(\mathcal{E})](B^* B) = V_1^* (\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^* (B^* B \otimes I_{\mathcal{V}_2}) V_2) V_1$. Since B is arbitrary, the relation holds for every positive $A = B^* B$, and hence, by linearity, for every $A \in \mathcal{L}(\mathcal{H}_2)$. The relation extends to arbitrary maps

$\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ by the usual continuity and linearity argument. This proves eq. (5).
We now prove the density conditions of eqs. (6) and (7). We have

$$(E \otimes I_{\mathcal{V}_1})V_1 v = (E \otimes Q_1)W_1 S_2 v = P_1(E \otimes I_{\ell^2 \otimes \mathcal{W}_1})W_1 S_2 v = (E \otimes I_{\ell^2 \otimes \mathcal{W}_1})W_1 S_2 v$$

for all $E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1)$, $v \in \mathcal{K}_2$, hence eq. (6) follows from eq. (10). On the other hand, eq. (7) is a direct consequence of eq. (11) and the fact that

$$(A \otimes I_{\mathcal{V}_2})V_2 w = (A \otimes I_{\mathcal{V}_2})P_2 W_2 P_1 w = (A \otimes I_{\ell^2 \otimes \mathcal{V}_2})W_2 w$$

for all $A \in \mathcal{L}(\mathcal{H}_2)$, $w \in \mathcal{H}_1 \otimes \mathcal{V}_1 = \hat{\mathcal{H}}_1$.
It is now easy to see that V_1 and V_2 are isometries. For V_2, this follows from the relation

$$V_2^* V_2 = (P_2 W_2 P_1)^*(P_2 W_2 P_1) = (W_2 P_1)^*(W_2 P_1) = P_1(W_2^* W_2)P_1 = P_1 = I_{\mathcal{H}_1 \otimes \mathcal{V}_1}.$$

For V_1, choosing $\mathcal{E} \in \mathrm{CP}_1(\mathcal{H}_1, \mathcal{K}_1)$, we have

$$V_1^* V_1 = V_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(I_{\mathcal{H}_2} \otimes I_{\mathcal{V}_2})V_2)]V_1 = [\mathsf{S}(\mathcal{E})](I_{\mathcal{H}_2}) = I_{\mathcal{K}_2}.$$

We conclude the section with the proof of Proposition 9:

Proof (Proof of Proposition 9). Define an isometry $U_1 : \mathcal{H}_1 \otimes \mathcal{V}_1 \to \mathcal{H}_1 \otimes \mathcal{V}_1'$ by means of the following construction: first, U_1 is defined on the linear space $\mathrm{span}\{(E \otimes I_{\mathcal{V}_1})V_1 v \mid E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1), v \in \mathcal{K}_2\}$ through the relation

$$U_1 \sum_{k=1}^{n}(E_k \otimes I_{\mathcal{V}_1})V_1 v_k := \sum_{k=1}^{n}(E_k \otimes I_{\mathcal{V}_1'})V_1' v_k. \tag{13}$$

The definition is well posed: indeed for every vector $\phi = \sum_{k=1}^{n}(E_k \otimes I_{\mathcal{V}_1})V_1 v_k$

we have

$$\begin{aligned}
\|U_1\phi\|^2 &= \left\|\sum_k (E_k \otimes I_{\mathcal{V}_1'})V_1' v_k\right\|^2 \\
&= \sum_{k,l} \left\langle V_1'^{*}[(E_l^* \odot E_k) \otimes \mathcal{I}_{\mathcal{V}_1'}](I_{\mathcal{H}_1 \otimes \mathcal{V}_1'})V_1' v_k, v_l\right\rangle \\
&= \sum_{k,l} \langle [\mathsf{S}(E_l^* \odot E_k)](I_{\mathcal{H}_2})v_k, v_l\rangle \\
&= \sum_{k,l} \langle V_1^*[(E_l^* \odot E_k) \otimes \mathcal{I}_{\mathcal{V}_1}](I_{\mathcal{H}_1 \otimes \mathcal{V}_1})V_1 v_k, v_l\rangle \\
&= \left\|\sum_k (E_k \otimes I_{\mathcal{V}_1})V_1 v_k\right\|^2 \\
&= \|\phi\|^2,
\end{aligned}$$

where the third equality comes from eq. (5). Since U_1 is defined on a dense subspace of $\mathcal{H}_1 \otimes \mathcal{V}_1$ (see eq. (6)), this also means that U_1 extends to an isometry from $\mathcal{H}_1 \otimes \mathcal{V}_1$ to $\mathcal{H}_1 \otimes \mathcal{V}_1'$, as claimed.
Similarly, we define an isometry $U_2 : \mathcal{H}_2 \otimes \mathcal{V}_2 \to \mathcal{H}_2 \otimes \mathcal{V}_2'$ through the relation

$$U_2 \sum_{h=1}^{m} (A_h \otimes I_{\mathcal{V}_2})V_2 w_h := \sum_{h=1}^{m} (A_h \otimes I_{\mathcal{V}_2'})V_2' U_1 w_h \tag{14}$$

for all $m \in \mathbb{N}$, $A_h \in \mathcal{L}(\mathcal{H}_2)$, $w_h \in \mathcal{H}_1 \otimes \mathcal{V}_1$. Again, the definition is well posed due to eq. (5): indeed, for every vector $\psi = \sum_{h=1}^{m} (A_h \otimes I_{\mathcal{V}_2})V_2 w_h$ with

$w_h = \sum_{k=1}^{n_h}(E_{hk} \otimes I_{\mathcal{V}_1})V_1 v_{hk}$, $E_{hk} \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1)$, $v_{hk} \in \mathcal{K}_2$, we have

$$\begin{aligned}
\|U_2\psi\|^2 &= \left\| \sum_h (A_h \otimes I_{\mathcal{V}_2'})V_2' U_1 w_h \right\|^2 \\
&= \sum_{h,k,l,m} \langle V_1'^* \left\{ \left[(E_{lm}^* \odot E_{hk}) \otimes \mathcal{I}_{\mathcal{V}_1} \right] (V_2'^* (A_l^* A_h \otimes I_{\mathcal{V}_2})V_2') \right\} V_1' v_{hk}, v_{lm} \rangle \\
&= \sum_{h,k,l,m} \langle [\mathsf{S}(E_{lm}^* \odot E_{hk})](A_l^* A_h) v_{hk}, v_{lm} \rangle \\
&= \sum_{h,k,l,m} \langle V_1^* \left\{ \left[(E_{lm}^* \odot E_{hk}) \otimes \mathcal{I}_{\mathcal{V}_1} \right] (V_2^* (A_l^* A_h \otimes I_{\mathcal{V}_2})V_2) \right\} V_1 v_{hk}, v_{lm} \rangle \\
&= \left\| \sum_h (A_h \otimes I_{\mathcal{V}_2})V_2 w_h \right\|^2 \\
&= \|\psi\|^2 ,
\end{aligned}$$

Since the vectors of the form $w_h = \sum_{k=1}^{n_h}(E_{hk} \otimes I_{\mathcal{V}_1})V_1 v_{hk}$ are dense in $\mathcal{H}_1 \otimes \mathcal{V}_1$, the above equality holds for every $\psi = \sum_{h=1}^{m}(A_h \otimes I_{\mathcal{V}_2})V_2 w_h$ with $w_h \in \mathcal{H}_1 \otimes \mathcal{V}_1$. Moreover, since U_2 preserves the norm on a dense subspace of $\mathcal{H}_2 \otimes \mathcal{V}_2$ (see eq. (7)), it can be extended to an isometry from $\mathcal{H}_2 \otimes \mathcal{V}_2$ into $\mathcal{H}_2 \otimes \mathcal{V}_2'$.

The isometries U_1 and U_2 have a simple tensor product structure. For $i = 1, 2$ it follows immediately from the definition that $(B_i \otimes I_{\mathcal{V}_i'})U_i = U_i(B_i \otimes I_{\mathcal{V}_i})$ $\forall B_i \in \mathcal{L}(\mathcal{H}_i)$, and therefore $U_i = I_{\mathcal{H}_i} \otimes W_i$ for some isometry $W_i : \mathcal{V}_i \to \mathcal{V}_i'$.

Finally, we prove the relations $V_1' = (I_{\mathcal{K}_1} \otimes W_1)V_1$ and $V_2'(I_{\mathcal{H}_1} \otimes W_1) = (I_{\mathcal{H}_2} \otimes W_2)V_2$. From eq. (13) we have

$$\begin{aligned}
(E \otimes I_{\mathcal{V}_1'})V_1' &= U_1(E \otimes I_{\mathcal{V}_1})V_1 \\
&= (I_{\mathcal{H}_1} \otimes W_1)(E \otimes I_{\mathcal{V}_1})V_1 \\
&= (E \otimes I_{\mathcal{V}_1'})(I_{\mathcal{K}_1} \otimes W_1)V_1 \quad \forall E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1) ,
\end{aligned}$$

that is, $V_1' = (I_{\mathcal{K}_1} \otimes W_1)V_1$. On the other hand, taking $m = 1$ and $A_1 = I_{\mathcal{H}_2}$ in eq. (14) we obtain $U_2 V_2 = V_2' U_1$, i.e. $(I_{\mathcal{H}_2} \otimes W_2)V_2 = V_2'(I_{\mathcal{H}_1} \otimes W_1)$.

7. Radon-Nikodym derivatives of supermaps

The dilation theorem for deterministic supermaps will be generalized here to the case of probabilistic supermaps. In this case, the following theorem provides an analog of the Radon-Nikodym theorem for completely positive maps [1, 3] (see also [24] for the particular case of quantum operations).

Theorem 6 (Radon-Nikodym theorem for supermaps). *Suppose* $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ *and let* $(\mathcal{V}_1, \mathcal{V}_2, V_1, V_2)$ *be its minimal dilation. If* $\mathsf{T} \in \mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ *is such that* $\mathsf{T} \ll \mathsf{S}$*, then there exists a unique positive contraction* $P \in \mathcal{L}(\mathcal{V}_2)$ *such that*

$$[\mathsf{T}(\mathcal{E})](A) = V_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes P)V_2)]V_1 \tag{15}$$

for all $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $A \in \mathcal{L}(\mathcal{H}_2)$*.*

PROOF. In the dense linear subspace of $\mathcal{H}_2 \otimes \mathcal{V}_2$

$$\hat{\mathcal{H}} = \mathrm{span}\,\{(A \otimes I_{\mathcal{V}_2})V_2(E \otimes I_{\mathcal{V}_1})V_1 v \mid A \in \mathcal{L}(\mathcal{H}_2), \, E \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1), \, v \in \mathcal{K}_2\}$$

we define the sesquilinear form $\langle \cdot, \cdot \rangle_0$ as

$$\begin{aligned}\left\langle \sum_{i=1}^n (A_i \otimes I_{\mathcal{V}_2})V_2(E_i \otimes I_{\mathcal{V}_1})V_1 v_i, \sum_{j=1}^n (B_j \otimes I_{\mathcal{V}_2})V_2(F_j \otimes I_{\mathcal{V}_1})V_1 w_j \right\rangle_0 &:= \\ = \sum_{i,j} \left\langle [\mathsf{T}(F_j^* \odot E_i)](B_j^* A_i) v_i, w_j \right\rangle \\ \equiv \left\langle \sum_i E_i \otimes A_i \otimes v_i, \sum_j F_j \otimes B_j \otimes w_j \right\rangle_{\mathsf{T}} .\end{aligned}$$

Note that, due to Lemma 5 and the condition $\mathsf{T} \ll \mathsf{S}$, for every vector $\phi = \sum_{i=1}^n (A_i \otimes I_{\mathcal{V}_2})V_2(E_i \otimes I_{\mathcal{V}_1})V_1 v_i$ we have

$$\begin{aligned}0 \leq \langle \phi, \phi \rangle_0 &= \left\langle \sum_i E_i \otimes A_i \otimes v_i, \sum_j E_j \otimes A_j \otimes v_j \right\rangle_{\mathsf{T}} \\ &\leq \left\langle \sum_i E_i \otimes A_i \otimes v_i, \sum_j E_j \otimes A_j \otimes v_j \right\rangle_{\mathsf{S}}\end{aligned}$$

and, by eq. (5),

$$\left\langle \sum_i E_i \otimes A_i \otimes v_i, \sum_j E_j \otimes A_j \otimes v_j \right\rangle_{\mathsf{S}} = \sum_{i,j} \left\langle [\mathsf{S}(E_j^* \odot E_i)](A_j^* A_i) v_i, v_j \right\rangle$$
$$= \left\langle \sum_i (A_i \otimes I_{\mathcal{V}_2}) V_2 (E_i \otimes I_{\mathcal{V}_1}) V_1 v_i, \sum_j (A_j \otimes I_{\mathcal{V}_2}) V_2 (E_j \otimes I_{\mathcal{V}_1}) V_1 v_j \right\rangle$$
$$= \|\phi\|^2 .$$

This shows that the sesquilinear form $\langle \cdot, \cdot \rangle_0$ is well defined on $\hat{\mathcal{H}}$ and extends to a bounded sesquilinear form on the whole $\mathcal{H}_2 \otimes \mathcal{V}_2$, with $0 \leq \langle \phi, \phi \rangle_0 \leq \langle \phi, \phi \rangle$ for all $\phi \in \mathcal{H}_2 \otimes \mathcal{V}_2$. Let $P_0 \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{V}_2)$ be the bounded operator such that $\langle \phi, \phi \rangle_0 = \langle P_0 \phi, \phi \rangle$. Clearly, P_0 satisfies $0 \leq P_0 \leq I_{\mathcal{H}_2 \otimes \mathcal{V}_2}$. Note that P_0 is uniquely identified by $\langle \cdot, \cdot \rangle_0$, which in turn is uniquely identified by T. Moreover, for every vector $\phi = \sum_{i=1}^n (A_i \otimes I_{\mathcal{V}_2}) V_2 (E_i \otimes I_{\mathcal{V}_2}) V_1 v_i$ we have

$$\begin{aligned} \langle P_0 (B \otimes I_{\mathcal{V}_2}) \phi, \phi \rangle &= \langle (B \otimes I_{\mathcal{V}_2}) \phi, \phi \rangle_0 \\ &= \sum_{i,j} \left\langle [\mathsf{T}(E_j^* \odot E_i)](A_j^* B A_i) v_i, v_j \right\rangle \\ &= \langle \phi, (B^* \otimes I_{\mathcal{V}_2}) \phi \rangle_0 \\ &= \langle P_0 \phi, (B^* \otimes I_{\mathcal{V}_2}) \phi \rangle \\ &= \langle (B \otimes I_{\mathcal{V}_2}) P_0 \phi, \phi \rangle , \end{aligned}$$

which implies $P_0 (B \otimes I_{\mathcal{V}_2}) = (B \otimes I_{\mathcal{V}_2}) P_0 \ \forall B \in \mathcal{L}(\mathcal{H}_2)$. Therefore, $P_0 = I_{\mathcal{H}_2} \otimes P$ for some operator $P \in \mathcal{L}(\mathcal{V}_2)$ with $0 \leq P \leq I_{\mathcal{V}_2}$. Finally, for $\mathcal{E} = F^* \odot E$, $E, F \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1)$, and $A \in \mathcal{L}(\mathcal{H}_2)$ we have

$$\begin{aligned} \langle [\mathsf{T}(F^* \odot E)](A) v, w \rangle &= \langle (A \otimes I_{\mathcal{V}_2}) V_2 (E \otimes I_{\mathcal{V}_1}) V_1 v, V_2 (F \otimes I_{\mathcal{V}_1}) V_1 w \rangle_0 \\ &= \langle (A \otimes P) V_2 (E \otimes I_{\mathcal{V}_1}) V_1 v, V_2 (F \otimes I_{\mathcal{V}_1}) V_1 w \rangle \\ &= \langle V_1^* [(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^* (A \otimes P) V_2)] V_1 v, w \rangle , \end{aligned}$$

from which eq. (15) for all $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ follows by the usual density argument.

Definition 12. With the notations of the last theorem, the operator $P \in \mathcal{L}(\mathcal{V}_2)$ defined by eq. (15) is the *Radon-Nikodym derivative* of the supermap T with respect to S.

Remark 5. Note that the validity of Theorem 6 can be trivially extended to quantum supermaps that are bounded by positive multiples of deterministic supermaps, i.e. to supermaps T such that $\mathsf{T} \ll \lambda \mathsf{S}$ for some positive $\lambda \in \mathbb{R}$ and some deterministic supermap S.

8. Superinstruments

Here we apply the Radon-Nikodym theorem proved in the previous section to the study of *quantum superinstruments*. Quantum superinstruments describe measurement processes where the measured object is not a quantum system, as in ordinary instruments, but rather a quantum device. While ordinary quantum instruments are defined as measures with values in the set of quantum operations [14] (see also [13] for a more complete exposition), quantum superinstruments are defined as probability measures with values in the set of quantum supermaps:

Definition 13. Let Ω be a measurable space with σ-algebra $\sigma(\Omega)$ and let S be a map from $\sigma(\Omega)$ to $\mathrm{SCP}\,(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$, sending the measurable subset $B \in \sigma(\Omega)$ to the quantum supermap $\mathsf{S}_B \in \mathrm{SCP}\,(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$. We say that S is a *quantum superinstrument* if it satisfies the following properties:

(i) $\mathsf{S}_\Omega \in \mathrm{SCP}_1\,(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$
(ii) if $B = \bigcup_{i=1}^\infty B_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, then $\mathsf{S}_B = \sum_{i=1}^\infty \mathsf{S}_{B_i}$, where the series converges in the following weak* sense: $\mathrm{tr}\,\{T[\mathsf{S}_B(\mathcal{E})](A)\} = \sum_{i=1}^\infty \mathrm{tr}\,\{T[\mathsf{S}_{B_i}(\mathcal{E})](A)\}$ for all $\mathcal{E} \in \mathrm{CB}\,(\mathcal{H}_1, \mathcal{K}_1)$, all $A \in \mathcal{L}\,(\mathcal{H}_2)$, and all $T \in \mathcal{T}\,(\mathcal{K}_2)$.

We recall the notion of (normalized) positive operator-valued measure, which is central in the statistical description of quantum measurements:

Definition 14. A map $P : \sigma(\Omega) \to \mathcal{L}\,(\mathcal{H})_+$ is a *positive operator-valued measure (POVM)* if it satisfies the following properties:

(i) $P_\Omega = I_\mathcal{H}$
(ii) if $B = \bigcup_{i=1}^\infty B_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, then $P_B = \sum_{i=1}^\infty P_{B_i}$, where the series converges in the weak* sense.

We then have the following dilation theorem for quantum superinstruments:

Theorem 7 (Dilation of quantum superinstruments). *Suppose that* $\mathsf{S} : \sigma(\Omega) \to \mathrm{SCP}(\mathcal{H}_1, \mathcal{K}_1; \mathcal{H}_2, \mathcal{K}_2)$ *is a quantum superinstrument and let the quadruple* $(\mathcal{V}_1, \mathcal{V}_2, V_1, V_2)$ *be the minimal dilation of* S_Ω*. Then there exists a unique positive-operator valued measure* $P : \sigma(\Omega) \to \mathcal{L}(\mathcal{V}_2)$ *such that*

$$[\mathsf{S}_B(\mathcal{E})](A) = V_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes P_B)V_2)]V_1 \tag{16}$$

for all $B \in \sigma(\Omega)$*,* $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$ *and* $A \in \mathcal{L}(\mathcal{H}_2)$*.*

Proof. Let $B \in \sigma(\Omega)$ be an arbitrary measurable set. By additivity of the measure S, we have $\mathsf{S}_\Omega = \mathsf{S}_B + \mathsf{S}_{\Omega \setminus B}$, that is, $\mathsf{S}_B \ll \mathsf{S}_\Omega$. By Theorem 6, this implies eq. (16) with some uniquely defined positive contraction $P_B \in \mathcal{L}(\mathcal{V}_2)$. Clearly, for $B = \Omega$ one has $P_\Omega = I$. Now, suppose that $B = \bigcup_{i=1}^\infty B_i$, with $B_i \cap B_j = \emptyset$ for $i \neq j$. The sequence of positive operators $S_n = \sum_{i=1}^n P_{B_i}$ is bounded and increasing, hence $S_n \uparrow S_\infty$ for some $S_\infty \in \mathcal{L}(\mathcal{V}_2)_+$. Using σ-additivity of the superinstrument and uniqueness of the Radon-Nikodym derivative it is immediate to see that $S_\infty = P_B$. Indeed, for every $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$, $A \in \mathcal{L}(\mathcal{H}_2)$, and $T \in \mathcal{T}(\mathcal{K}_2)$ we have

$$\begin{aligned}
&\mathrm{tr}\{TV_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes S_\infty)V_2)]V_1\} = \\
&\qquad = \sum_{i=1}^\infty \mathrm{tr}\{TV_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes P_{B_i})V_2)]V_1\} \\
&\qquad = \sum_{i=1}^\infty \mathrm{tr}\{T[\mathsf{S}_{B_i}(\mathcal{E})](A)\} \\
&\qquad = \mathrm{tr}\{T[\mathsf{S}_B(\mathcal{E})](A)\},
\end{aligned}$$

which implies $[\mathsf{S}_B(\mathcal{E})](A) = V_1^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})(V_2^*(A \otimes S_\infty)V_2)]V_1$. By uniqueness, we then conclude $P_B = S_\infty$.

The physical interpretation of the dilation of quantum superinstruments is clear in the Schrödinger picture. Indeed, taking the predual of eq. (16), we have for all $T \in \mathcal{T}(\mathcal{K}_2)$, $\mathcal{E} \in \mathrm{CB}(\mathcal{H}_1, \mathcal{K}_1)$

$$[\mathsf{S}_B(\mathcal{E})]_*(T) = \mathrm{tr}_{\mathcal{V}_2}\{(I_\mathcal{K} \otimes P_B)V_2\,[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})_*(V_1 T V_1^*)]\,V_2^*\},$$

where $\mathrm{tr}_{\mathcal{V}_2}$ denotes the partial trace on $\mathcal{V}_2$. This means that the quantum state T first undergoes the invertible evolution V_1, then the quantum channel

$(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}_1})_*$, the invertible evolution V_2, and finally a quantum measurement is performed on the ancillary system with Hilbert space $\mathcal{V}_2$.

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Appendix A. Weak*-continuous completely bounded maps and normal $*$-representations of $\mathcal{L}(\mathcal{H})$

In this section, we *do not* assume separability as a part in the definition of Hilbert spaces.

Proposition 11. *Let $\mathcal{H}$ be separable, $\{e_i\}_{i\in\mathbb{N}}$ be a Hilbert basis for $\mathcal{H}$, and P_n be the projector onto* span $\{e_i \mid i \leq n\}$*. A unital $*$-representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is normal if and only if $\pi(P_n) \uparrow I_{\mathcal{K}}$.*

Proof. Since $P_n \uparrow I_{\mathcal{H}}$, if π is normal one must necessarily have $\pi(P_n) \uparrow \pi(I_{\mathcal{H}}) = I_{\mathcal{K}}$. Conversely, assume that $\pi(P_n) \uparrow I_{\mathcal{K}}$. Let us decompose π into the direct sum of $*$-representations $\pi = \pi_{\text{nor}} \oplus \pi_{\text{sin}}$, where π_{nor} is normal and π_{sin} is singular, that is $\pi_{\text{sin}}(K) = 0$ for every compact operator $K \in \mathcal{L}(\mathcal{H})$ (see e.g. Proposition 10.4.13, p. 757 of [18]). We then have $\pi(P_n) = \pi_{\text{nor}}(P_n) \uparrow \pi_{\text{nor}}(I_{\mathcal{H}})$ by normality, hence $\pi_{\text{nor}}(I_{\mathcal{H}}) = I_{\mathcal{K}}$. On the other hand, $I_{\mathcal{K}} = \pi_{\text{nor}}(I_{\mathcal{H}}) \oplus \pi_{sin}(I_{\mathcal{H}})$. This implies $\pi_{\text{sin}}(I_{\mathcal{H}}) = 0$, and, therefore, $\pi_{\text{sin}} = 0$.

The next lemma gives a criterion for estabilishing the separability of the Hilbert space $\mathcal{K}$ in the case π is a normal $*$-representation of $\mathcal{L}(\mathcal{H})$ in $\mathcal{K}$, and is used in the proofs of Proposition 10 and Theorem 8 below.

Lemma 6. *Let $\mathcal{H}$ be separable and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ be a normal $*$-representation. If there exists a separable subset $\mathcal{S} \subset \mathcal{K}$ such that the linear space*

$$\text{span}\,\{\pi(A)v \mid A \in \mathcal{L}(\mathcal{H}), v \in \mathcal{S}\} \tag{A.1}$$

is dense in $\mathcal{K}$, then $\mathcal{K}$ is separable.

Proof. Since the Hilbert space $\mathcal{H}$ is separable, the Banach subspace $\mathcal{L}_0(\mathcal{H})$ of the compact operators in $\mathcal{L}(\mathcal{H})$ is separable. Let P_n be defined as in the previous proposition. By normality of π, we have $\lim_n \|\pi(P_n)v - v\| = 0$ for all $v \in \mathcal{K}$ (Lemma 5.1.4 in [17]). Therefore, $\pi(A)v = \lim_n \pi(AP_n)v$ for all $A \in \mathcal{L}(\mathcal{H})$ and $v \in \mathcal{K}$, where $AP_n \in \mathcal{L}_0(\mathcal{H})$ because P_n has finite rank. Therefore, the closure of the linear space defined in (A.1) coincides with the closure of the linear space spanned by the set $\{\pi(A)v \mid A \in \mathcal{L}_0(\mathcal{H}), v \in \mathcal{S}\}$, which is separable by separability of $\mathcal{L}_0(\mathcal{H})$ and $\mathcal{S}$ and by continuity of the mapping $\mathcal{L}_0(\mathcal{H}) \times \mathcal{S} \ni (A, v) \mapsto \pi(A)v \in \mathcal{K}$. Separability of $\mathcal{K}$ then follows.

Theorem 8. *Suppose $\mathcal{H}$, $\mathcal{K}$ separable Hilbert spaces, and let $\mathcal{E} \in \mathrm{CB}(\mathcal{H}, \mathcal{K})$. Then there exists a separable Hilbert space $\mathcal{V}$ and two operators $V, W \in \mathcal{L}(\mathcal{K}, \mathcal{H} \otimes \mathcal{V})$ such that*

$$\mathcal{E}(A) = V^*(A \otimes I_{\mathcal{V}})W \quad \forall A \in \mathcal{L}(\mathcal{H}). \tag{A.2}$$

Proof. By Theorem 8.4 in [23], there exist a (not necessarily separable) Hilbert space $\hat{\mathcal{H}}$, a $*$-representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}\left(\hat{\mathcal{H}}\right)$ and operators $V, W \in \mathcal{L}\left(\mathcal{K}, \hat{\mathcal{H}}\right)$ such that $\mathcal{E}(A) = V^*\pi(A)W \quad \forall A \in \mathcal{L}(\mathcal{H})$. Writing $\pi = \pi_{\mathrm{nor}} \oplus \pi_{\mathrm{sin}}$ and continuing with the notations of the two previous proofs, we have

$$\begin{aligned}\mathcal{E}(A) &= \text{wk*-}\lim_n \mathcal{E}(AP_n) = \text{wk*-}\lim_n V^*\pi_{\mathrm{nor}}(AP_n)W \\ &= V^*\pi_{\mathrm{nor}}(A)W.\end{aligned}$$

Therefore, π can be chosen to be normal. We now prove that $\hat{\mathcal{H}}$ can be chosen to be separable. Since the subspace $\mathcal{S} = \overline{\mathrm{span}}\,\{Wv \mid v \in \mathcal{H}\}$ is separable, by Lemma 6 the π-invariant subspace $\hat{\mathcal{H}}' = \overline{\mathrm{span}}\,\{\pi(A)Wv \mid v \in \mathcal{H},\, A \in \mathcal{L}(\mathcal{H})\}$ is separable. Denoting by Q the projector onto $\hat{\mathcal{H}}'$ and defining $V' := QV$, $W' := QW$, $\pi'(A) = Q\pi(A)Q$ we then have $\mathcal{E}(A) = V'^*\pi'(A)W'$. Since π' is a representation of $\mathcal{L}(\mathcal{H})$ on the separable Hilbert space $\hat{\mathcal{H}}'$, up to unitary equivalence we have $\hat{\mathcal{H}}' = \mathcal{H} \otimes \mathcal{V}$ and $\pi'(A) = A \otimes I_{\mathcal{V}}$ $\forall A \in \mathcal{L}(\mathcal{H})$ for some separable Hilbert space $\mathcal{V}$ (see e.g. Corollary 10.4.14, p. 747 of [18]).

It is now easy to prove that the set of weak*-continuous completely bounded maps is the linear span of the cone of normal completely positive maps. The proof is the obvious adaptation of the proof of the analogous statement for completely bounded maps (see Theorem 8.5 in [23]). This provides the proof of Theorem 2:

Proof (Proof of Theorem 2). The inclusion $\mathrm{CP}(\mathcal{H}, \mathcal{K}) \subset \mathrm{CB}(\mathcal{H}, \mathcal{K})$ is an immediate consequence of the fact that every normal positive map is weak*-continuous and that every completely positive map is completely bounded (see e.g. Proposition 3.6 in [23]). Moreover, if $\mathcal{E} \in \mathrm{CB}\,(\mathcal{H}, \mathcal{K})$, let $\mathcal{V}$, V and W be as in the previous theorem. Eq. (A.2) can be rewritten $\mathcal{E} = (V^* \odot W)\pi$, where $\pi(A) = A \otimes I_{\mathcal{V}}$. By polarization, we have $\mathcal{E} = \sum_{k=0}^{3} i^k \mathcal{E}_k$, where $\mathcal{E}_k = \frac{1}{4}[(i^k V + W)^* \odot (i^k V + W)]\pi \in \mathrm{CP}\,(\mathcal{H}, \mathcal{K})$.

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