NON ABELIAN TENSOR SQUARE OF NON ABELIAN PRIME POWER GROUPS

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ABSTRACT. For every p-group of order p^n with the derived subgroup of order p^m, Rocco in [7] has shown that the order of tensor square of G is at most $p^{n(n-m)}$. In the present paper not only we improve his bound for non-abelian p-groups but also we describe the structure of all non-abelian p-groups when the bound is attained for a special case. Moreover, our results give as well an upper bound for the order of $\pi_3(SK(G,1))$.

1. Introduction and Preliminaries

The tensor square $G \otimes G$ of a group G is a group generated by the symbols $g \otimes h$ subject to the relations

$$gg^{'} \otimes h = ({}^{g}g^{'} \otimes {}^{g} h)(g \otimes h) \text{ and } g \otimes hh^{'} = (g \otimes h)\ ({}^{h}g \otimes h^{'})$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. The non abelian tensor square is a special case of non abelian tensor product, which was introduced by R. Brown and J.-L. Loday in [3].

There exists a homomorphism of groups $\kappa : G \otimes G \to G^{'}$ sending $g \otimes h$ to $[g, h] = ghg^{-1}h^{-1}$. The kernel of κ is denoted by $J_2(G)$; its topological interest is in the formula $\pi_3 SK(G, 1) = J_2(G)$ (see [3]).

According to the formula $\pi_3 SK(G, 1) = J_2(G)$ computing the order of $G \otimes G$ has interests in topology in addition to its interpretation as a problem in the group theory.

Rocco in [7] and later Ellis in [4] have shown that the order of tensor square of G is at most $p^{n(n-m)}$ for every p-group of order p^n with the derived subgroup of order p^m.

The purpose of this paper is a further investigation on the order of tensor square of non abelian p-groups. We focus on non abelian p-groups because in abelian case the non abelian tensor square coincides with the usual abelian tensor square of abelian groups. To be precise, for a non abelian p-group of order p^n and the derived subgroup of order p^m, we prove that $|G\otimes G| \leq p^{(n-1)(n-m)+2}$ and also we obtain the explicit structure of G when $|G \otimes G| = p^{(n-1)^2+2}$. It easily seen that the bound is less than of Rocco's bound, unless that $G \cong Q_8$ or $G \cong E_1$, which causes two bounds to be equal. As a corollary by using the fact $\pi_3 SK(G, 1) \cong \mathrm{Ker}(G \otimes G \xrightarrow{\kappa} G^{'})$, we can see that $|\pi_3 SK(G, 1)| = |J_2(G)| \leq p^{n(n-m-1)+2}$.

Thorough the paper, D_8, Q_8 denote the dihedral and quaternion group of order 8, E_1 and E_2 denote the extra-special p-groups of order p^3 of exponent p and p^2,

Key words and phrases. Tensor square, non abelian p-groups.
Mathematics Subject Classification 2010. 20D15.

respectively. Also $C_{p^t}^{(k)}$ and $\nabla(G)$ denote the direct product of k copies of the cyclic group of order p^t and the subgroup generated by $g\otimes g$ for all g in G, respectively.

2. Main Results

The aim of this section is finding an upper bound for the order tensor square of non abelian p-groups of order p^n in terms of the order of $G^{'}$. Also in the case for which $|G^{'}|=p$, the structure of groups is obtained when $|G\otimes G|$ reaches the upper bound.

Proposition 2.1. [2, Proposition 9]. *Given a central extension*

$$1\rightarrow Z\longrightarrow H\longrightarrow G\longrightarrow 1$$

there is an exact sequence

$$(Z\otimes H)\times(H\otimes Z)\stackrel{l}{\longrightarrow} H\otimes H\longrightarrow G\otimes G\longrightarrow 1$$

in which Im l *is central.*

Proposition 2.2. [2, Proposition 13, 14] *The tensor square of D_8 and Q_8 is isomorphic to*

$$C_2^{(3)}\times C_4 \text{ and } C_2^{(2)}\times C_4^{(2)},$$

respectively.

Recall that [2, 3] the order of tensor square of G is equal to $|\nabla(G)||\mathcal{M}(G)||G^{'}|$, where $\mathcal{M}(G)$ is the Schur multiplier of G.

Put $G^{ab}=G/G^{'}$. In analogy with the above proposition the following lemma is characterized the tensor square of extra-special p-groups of order $p^3 (p\neq 2)$.

Lemma 2.3. *The tensor square of E_1 and E_2 are isomorphic to $C_p^{(6)}$ and $C_p^{(4)}$, respectively.*

Proof. It can be proved from [4, Theorem 2] that $E_1\otimes E_1$ is elementary abelian. Now, by invoking [1, Proposition 2.2 (iii)], $\nabla(E_1)\cong\nabla(E_1^{ab})$ and hence $|\nabla(E_1)|=p^3$. On the other hand, [5, Theorem 3.3.6] implies that the Schur multiplier of E_1 is of order p^2, and so $|E_1\otimes E_1|=p^6$.

In the case $G=E_2$ in a similar fashion, we can prove that $E_2\otimes E_2\cong E_2^{ab}\otimes E_2^{ab}$. □

Lemma 2.4. [6, Corollary 2.3] *The tensor square of an extra-special p-group H of order p^{2m+1} is elementary abelian of order p^{4m^2}, for $m\geq 2$.*

Proposition 2.5. *Let G be a p-group of order p^n and $|G^{'}|=p$. If one of the following conditions holds, then the order of tensor square is less than $p^{(n-1)^2+2}$.*

(i) *G^{ab} is not elementary abelian;*
(ii) *G^{ab} is elementary abelian and $Z(G)$ is not elementary abelian.*

Proof (i). The proof is an upstanding result of Proposition 2.1 while $Z=G^{'}$. Let $G^{ab}=C_{p^{m_1}}\times C_{p^{m_2}}\times\ldots\times C_{p^{m_k}}$ where $\sum_{i=1}^k m_i=n-1$ and $m_i\leq m_{i+1}$ for all i

$1 \leq i \leq k-1$. Then

$$\begin{aligned}
|G \otimes G| &\leq |G^{'} \otimes G^{ab}||G^{ab} \otimes G^{ab}| \\
&= |C_p \otimes C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}| \\
&\quad |C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}} \otimes C_{p^{m_1}} \times C_{p^{m_2}} \times \ldots \times C_{p^{m_k}}| \\
&= p^{m_k+\ldots+m_1+2(m_{k-1}+\ldots+m_1+m_{k-2}+\ldots+m_1+\ldots+m_1)+k} \\
&\leq p^{n-1+2(n-3+n-4+\ldots+n-2k+3)+k} \\
&< p^{(n-1)^2+2},
\end{aligned}$$

as required.

(ii). Since G^{ab} is a vector space on C_p, let $H/G^{'}$ be the complement of $Z(G)/G^{'}$ in G^{ab}. Moreover H is extra-special and $G = HZ(G)$. There is an epimorphism $H \times Z(G) \otimes H \times Z(G) \longrightarrow G \otimes G$, so

$$|G \otimes G| \leq |H \times Z(G) \otimes H \times Z(G)|.$$

Let $|Z(G)| = p^k$ and $|H| = p^{2m+1}$, we can suppose that $k \geq 2$ by using Proposition 2.2. Now the following two cases can be considered.

Case (i). First suppose that $m \geq 2$.

Let $Z(G) \cong C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}$ and $\sum_{i=1}^{t} k_i = n-2m$. Applying Lemma 2.4 and [2, Proposition 11], we have

$$\begin{aligned}
|G \otimes G| &\leq |H \otimes H||H \otimes Z(G)|^2|Z(G) \otimes Z(G)| \\
&= p^{4m^2}|C_p^{(m)} \otimes C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}|^2|C_{p^{k_1}} \times \ldots \times C_{p^{k_t}} \otimes C_{p^{k_1}} \times \ldots \times C_{p^{k_t}}| \\
&= p^{4m^2}p^{2mt}p^{(2t-1)k_1+(2t-3)k_2+\ldots+k_t} \\
&\leq p^{4m^2}p^{2mt}p^{n-2m+2(n-2m-2+\ldots+n-2m-t)} \\
&< p^{(n-1)^2+2},
\end{aligned}$$

as required.

Case (ii). Without loss of generality, we can suppose that $Z(G) \cong C_{p^2}$. Now the result is obtained by using Proposition 2.1 and the fact that $|Iml| \geq p$. □

Theorem 2.6. *Let G be a non abelian p-group of order p^n. If $|G^{'}| = p$, then*

$$|G \otimes G| \leq p^{(n-1)^2+2},$$

and the equality holds if and only if G is isomorphic to $H \times E$, where $H \cong E_1$ or $H \cong Q_8$ and E is an elementary abelian p-group.

Proof. One can assume that G^{ab} and $Z(G)$ are elementary abelian and $|Z(G)| \geq p^2$ by Proposition 2.3. Let E be the complement of $G^{'}$ in $Z(G)$. Thus there exists an extra-special p-group H of order p^{2m+1} such that $G \cong H \times E$.

In the case $m \geq 2$, it is easily seen that $|G \otimes G| < p^{(n-1)^2+2}$. For $m = 1$,

$$|G \otimes G| = |H \otimes H||E \otimes E||E \otimes H|^2$$

where $|E \otimes E||E \otimes H|^2 = p^{(n-1)(n-3)}$.

Now Proposition 2.2 and Lemma 2.3 imply that $|G \otimes G| = p^{(n-1)^2+2}$ when $H \cong Q_8$ or H has exponent p. $\square$

Theorem 2.7. *Let G be a non abelian p-group of order p^n. If $|G^{'}| = p^m$, then*

$$|G \otimes G| \leq p^{(n-1)(n-m)+2}.$$

Proof. We prove theorem by induction on m. For $m = 1$ the result is obtained by Theorem 2.6.

Let $m \geq 2$ and K be a central subgroup of order p contained in $G^{'}$. Induction hypothesis and Proposition 2.2 yield

$$\begin{aligned} |G \otimes G| &\leq |K \otimes G^{ab}||G/K \otimes G/K| \\ &\leq p^{n-m}p^{(n-m)(n-2)+2} = p^{(n-1)(n-m)+2}. \end{aligned}$$

$\square$

Corollary 2.8. *Let G be a non abelian p-group of order p^n. If $|G^{'}| = p^m$, then*

$$|\pi_3 SK(G,1)| \leq p^{n(n-m-1)+2}.$$

In particular when $m = 1$, then

$$|\pi_3 SK(G,1)| \leq p^{n(n-2)+2},$$

and the equality holds if and only if G is isomorphic to $H \times E$, in which H is extra-special of order p^3 of exponent p or $H \cong Q_8$ and E is an elementary abelian p-group.

Corollary 2.9. *If the order of tensor square of G is equal to $p^{(n-1)^2+2}$, then*

$$G \otimes G \cong {C_p}^{((n-1)^2+2)}\ (p \neq 2) \text{ or } G \otimes G \cong C_4^{(2)} \times C_2^{((n-1)^2-2)}.$$

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