SCHRÖDINGER OPERATORS AND THE DISTRIBUTION OF RESONANCES IN SECTORS

T.J. CHRISTIANSEN

Abstract. The purpose of this paper is to give some refined results about the distribution of resonances in potential scattering. We use techniques and results from one and several complex variables, including properties of functions of completely regular growth. This enables us to find asymptotics for the distribution of resonances in sectors for certain potentials and for certain families of potentials.

1. Introduction

The purpose of this paper is to prove some results about the distribution of resonances in potential scattering. In particular, we study the distribution of resonances in sectors and give asymptotics of the "expected value" of the number of resonances in certain settings.

More precisely, we consider the operator $-\Delta + V$, where $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and Δ is the (non-positive) Laplacian. Then, with a finite number of exceptions, $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$, $\operatorname{Im}\lambda > 0$, is a bounded operator on $L^2(\mathbb{R}^d)$ for λ in the upper half plane. When d is odd and $\chi \in C_c^\infty$ satisfies $\chi V = V$, $\chi R_V(\lambda)\chi$ has a meromorphic continuation to the lower half plane. The poles of $\chi R_V(\lambda)\chi$ are called *resonances*, and are independent of choice of χ satisfying these hypotheses. Resonances are analogous to eigenvalues not only in their appearance as poles of the resolvent, but also because they appear in trace formulas much as eigenvalues do [1, 9, 12]. Physically, they may be thought of as corresponding to decaying waves.

Let $n_V(r)$ denote the number of resonances of $-\Delta + V$, counted with multiplicity, with norm at most r. When $d = 1$, asymptotics of $n_V(r)$ are known:

$$\lim_{r\to\infty} \frac{n_{-\Delta+V}(r)}{r} = \frac{2}{\pi}\operatorname{diam}(\operatorname{supp}(V))$$

[19]; see also [5, 15, 17]. Moreover, "most" of the resonances occur in sectors about the real axis, in the sense that for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{\#\{\lambda_j \text{ pole of } R_V(\lambda) : |\arg\lambda_j - \pi| < \epsilon \text{ or } |\arg\lambda_j - 2\pi| < \epsilon\}}{r} = \frac{2}{\pi}\operatorname{diam}(\operatorname{supp}(V))$$

[5]. These results are valid for complex-valued V. The case $d = 1$ is exceptional, though: in higher dimensions much less is known.

Now we turn to $d \geq 3$ odd, where the question is more subtle. If $V \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0,a) = \{x \in \mathbb{R}^d : \ |x| \leq a\}$, then

$$d \int_0^r \frac{n_V(t) - n_V(0)}{t} dt \leq c_d a^d r^d + o(r^d). \tag{1.1}$$

where c_d is defined in (3.5) and depends only on the dimension. Zworski [21] showed that such a bound holds, and Stefanov [18] identified the optimal constant. There are examples for which equality holds in (1.1), [20, 18]. Lower bounds have proved more elusive. The current best known general quantitative lower bound is for non-trivial real-valued $V \in C_c^\infty(\mathbb{R})$

$$\limsup_{r \to \infty} \frac{n_V(r)}{r} > 0 \tag{1.2}$$

[16]. On the other hand, there are nontrivial complex-valued potentials V for which $\chi R_V(\lambda)\chi$ has no poles, [3].

We wish to single out the set for which asymptotics actually hold in (1.1). This is the set defined, for $a > 0$, as

(1.3)

$$\mathfrak{M}_a = \{V \in L^\infty(\mathbb{R}^d) : \ \operatorname{supp} V \subset \overline{B}(0,a) \text{ and } n_V(r) = c_d a^d r^d + o(r^d) \text{ as } r \to \infty\}.$$

This set contains infinitely many radial potentials. By results of [18, 20], this set contains any potential of the form $V(x) = v(|x|)$, where $v \in C^2([0,a])$ is real-valued, $v(a) \neq 0$, and $V(x) = 0$ for $|x| > a$. Additionally, it contains infinitely many complex-valued potentials which are isoresonant with these real-valued radial potentials [4].

We now can state some results. For the first, we set, for $\varphi < \theta$, $n_V(r, \varphi, \theta)$ to be the set of poles of $R_V(\lambda)$, counted with multiplicity, with norm at most r and with argument between φ and θ.

Proposition 1.1. *Let $V \in \mathfrak{M}_a$. Then, if $0 < \varphi < \theta < \pi$,*

$$n_V(r, \pi + \varphi, \pi + \theta) = \frac{1}{2\pi d} \tilde{s}_d(\varphi, \theta) r^d a^d + o(r^d)$$

where

$$\tilde{s}_d(\varphi, \theta) = h_d'(\theta) - h_d'(\varphi) + d^2 \int_\varphi^\theta h_d(s) ds,$$

and $h_d(\theta)$ is as defined in (3.4).

If V is real-valued, then λ_0 is a resonance of $-\Delta + V$ if and only if $-\overline{\lambda_0}$ is a resonance. In this case for $V \in \mathfrak{M}_a$ and $0 < \theta < \pi$

$$n_V(r, \pi, \pi + \theta) = \left[h_d'(\theta) + d^2 \int_0^\theta h_d(s) ds \right] a^d r^d + o(r^d).$$

Corollary 1.4 shows this holds for any $V \in \mathfrak{M}_a$. These results show that any potential in $\mathfrak{M}_a$ must have resonances distributed regularly in sectors, as well as being distributed regularly in balls centered at the origin. A result like this lemma and Corollary 1.4 is, for the special potentials of the form $V(x) = v(|x|)$ mentioned earlier, implicit in the papers of Zworski [20] and Stefanov [18]. Here we derive it as a corollary of some complex-analytic results, and note that it holds for *any* potential $V \in \mathfrak{M}_a$. We note that this proposition could, in fact, follow as a corollary to Theorem 1.3. However, we prefer to give a separate proof that uses standard results for functions of completely regular growth. In the following theorem we use the notation $N_V(r) = \int_0^r \frac{1}{t}(n_V(t) - n_V(0))dt$.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^p$ be an open connected set. Suppose $V(z) = V(z,x)$ is holomorphic in $z \in \Omega$ and, for each $z \in \Omega$, $V(z,x) \in L^\infty(\mathbb{R}^d)$, and $V(z,x) = 0$ if $|x| > a$. Suppose in addition that for some $z_0 \in \Omega$, $V(z_0) \in \mathfrak{M}_a$. Then there is a pluripolar set $E \subset \Omega$ so that*

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r)}{r^d} = dc_d a^d \text{ for all } z \in \Omega \setminus E.$$

Moreover, for any $\theta > 0$, $\theta < \pi$, there is a pluripolar set E_θ so that

$$\limsup_{r\to\infty} \frac{N_{V(z)}(r, \pi, \pi + \theta)}{r^d} \geq \lim_{\epsilon \downarrow 0} \frac{a^d}{4\pi d^2} h_d'(\epsilon)$$

for all $z \in \Omega \setminus E_\theta$.

For example, one may take, for $z \in \mathbb{C}$, $V(z) = zV_1 + (1 - z)V_0$, where $V_0 \in \mathfrak{M}_a$ and $V_1 \in L^\infty(\mathbb{R}^d)$ has support in $\overline{B}(0, a)$. Since $h_d'(0+) = \lim_{\epsilon\downarrow 0} h_d'(\epsilon) > 0$, see Lemma 3.3, the second statement of the theorem is meaningful. This result is of particular interest since resonances near the real axis are considered the more physically relevant ones.

We recall the definition of a pluripolar set in Section 2. Here we mention that a pluripolar set is small. A pluripolar set $E \subset \mathbb{C}^p$ has $\mathbb{R}^{2p}$ Lebesque measure 0, and if $E \subset \mathbb{C}$ is pluripolar, $E \cap \mathbb{R}$ has one-dimensional Lebesque measure 0 (see, for example, [10, 14]). Thus the statements of Theorem 1.2 hold for "most" values of $z \subset \Omega$.

If we take a weighted average over a family of potentials, a kind of expected value, we are able to find asymptotics analogous to those which hold for a potential in $\mathfrak{M}_a$. In the statement of the next theorem and later in the paper, we use the notation $d\mathcal{L}(z) = d\,\mathrm{Re}\,z_1 d\,\mathrm{Im}\,z_1 \cdots d\,\mathrm{Re}\,z_p d\,\mathrm{Im}\,z_p$.

Theorem 1.3. *Suppose the hypotheses of Theorem 1.2 are satisfied. Then for any $\psi \in C_c(\Omega)$,*

$$\int_\Omega \psi(z) n_{V(z)}(r) d\mathcal{L}(z) = c_d a^d r^d \int_\Omega \psi(z) d\mathcal{L}(z) + o(r^d).$$

Additionally, we have, for $0<\varphi<\theta<\pi$*,*

$$\int_\Omega \psi(z) n_{V(z)}(r,\varphi+\pi,\theta+\pi)dxdy = \frac{1}{2\pi d}\tilde{s}_d(\varphi,\theta) r^d a^d \int_\Omega \psi(z) d\mathcal{L}(z) + o(r^d)$$

where $\tilde{s}_d$ *is as defined in Proposition 1.1. Moreover, for* $0<\theta<\pi$*,*

$$\int_\Omega \psi(z) n_{V(z)}(r,\pi,\theta+\pi)dxdy = \left[h_d'(\theta) + d^2\int_0^\theta h_d(s)ds\right] a^d r^d \int_\Omega \psi(z)\mathcal{L}(z) + o(r^d). \tag{1.4}$$

Corollary 1.4. *Let* $V\in\mathfrak{M}_a$*. For any* $0<\theta<\pi$*,*

$$n_V(r,\pi,\theta+\pi) = \left[h_d'(\theta) + d^2\int_0^\theta h_d(s)ds\right] a^d r^d + o(r^d) \tag{1.5}$$

and, for any $0<\varphi<\pi$*,*

$$n_V(r,\varphi+\pi,2\pi) = \left[-h_d'(\varphi) + d^2\int_\varphi^\pi h_d(s)ds\right] a^d r^d + o(r^d). \tag{1.6}$$

This corollary follows from Theorem 1.3 by taking $V(z)$ equal to the constant (in z) potential V. We could instead give a more direct proof by, essentially, simplifying the proof of Proposition 5.3 and then applying Lemma 5.4.

It is worth noting that the coefficients of r^d in (1.5) and (1.6) are positive, so that in any sector in the lower half plane which touches the real axis, the number of resonances grows like r^d.

The proofs of the results here are possible because of the optimal upper bounds on $\limsup_{r\to\infty} r^{-d}\log|\det S_V(re^{i\theta})|$, $0<\theta<\pi$, proved in [18], see Theorem 3.2 here. These, combined with some one-dimensional complex analysis, are used to prove Proposition 1.1, and could be used to give a direct proof of Corollary 1.4. The proofs of the other theorems use, in addition to one-dimensional complex analysis, some facts about plurisubharmonic functions. Most of the complex-analytic results which we shall use are recalled in Section 2.

Acknowledgments. It is a pleasure to thank Plamen Stefanov and Maciej Zworski for helpful conversations during the writing of this paper. The author gratefully acknowledges the partial support of the NSF under grant DMS 1001156.

2. Some complex analysis

In this section we recall some definitions and results from complex analysis in one and several variables. We will mostly follow the notation and conventions of [11] and [10]. We also prove a result, Proposition 2.2, for which we are unaware of a proof in the literature.

The *upper relative measure* of a set $E\subset\mathbb{R}_+$ is

$$\limsup_{r\to\infty}\frac{\text{meas}(E\cap(0,r))}{r}.$$

A set $E \subset \mathbb{R}_+$ is said to have *zero relative measure* if it has upper relative measure 0.

If f is a function holomorphic in the sector $\varphi < \arg z < \theta$, we shall say f is of order ρ if $\limsup_{r\to\infty} \frac{\log\log(\max_{\varphi<\phi<\theta}|f(re^{i\phi})|)}{\log r} = \rho$. We shall further restrict ourselves to functions of *finite type*, so that

$$\limsup_{r\to\infty} \frac{\log(\max_{\varphi<\phi<\theta}|f(re^{i\phi})|)}{r^\rho} < \infty.$$

In this section only, we shall, following Levin [11], use the notation h_f for the *indicator* (or *indicator function*) of a function f of order ρ: $h_f(\theta) = \limsup_{r\to\infty}\left(r^{-\rho}\ln|f(re^{i\theta})|\right)$. Suppose f is a function analytic in the angle (θ_1, θ_2) and of order ρ and finite type there. The function f is of *completely regular growth* on some set of rays $R_{\mathfrak{M}}$ ($\mathfrak{M}$ is the set of values of θ) if the function

$$h_{f,r}(\theta) = \frac{\ln|f(re^{i\theta})|}{r^\rho}$$

converges uniformly to $h_f(\theta)$ for $\theta \in \mathfrak{M}$ when r goes to infinity taking on all positive values except possibly for a set $E_{\mathfrak{M}}$ of zero relative measure. The function f is of *completely regular growth in the angle* (θ_1, θ_2) if it is of completely regular growth on every closed interior angle.

Functions of completely regular growth have zeros that are rather regularly distributed. For a function f holomorphic in $\{z : \theta_1 < \arg z < \theta_2\}$ we define, for $\theta_1 < \varphi < \theta < \theta_2$, $m_f(r, \varphi, \theta)$ to be the number of zeros of $f(z)$ in the sector $\varphi \le \arg z \le \theta$, $|z| \le r$. [1] We recall the following theorem from [11].

Theorem 2.1. [11, Chapter III, Theorem 3] *If a holomorphic function $f(z)$ of order d and finite type has completely regular growth within an angle (θ_1, θ_2), then for all values of φ and θ, ($\theta_1 < \varphi < \theta < \theta_2$) except possibly for a denumerable set, the following limit will exist:*

$$\lim_{r\to 0} \frac{m_f(r, \varphi, \theta)}{r^d} = \frac{1}{2\pi d}\tilde{s}_f(\varphi, \theta)$$

where

$$\tilde{s}_f(\varphi, \theta) = \left[h'_f(\theta) - h'_f(\varphi) + d^2 \int_\varphi^\theta h_f(s)ds\right].$$

The exceptional denumerable set can only consist of points for which $h'_f(\theta + 0) \ne h'_f(\theta - 0)$.

In the following proposition we use the notation $m_f(r)$ to denote the number of zeros of a function f, counted with multiplicity, with norm at most r. It is likely that some of the hypotheses included here could be relaxed. However, when we apply this

[1] More standard notation would be $n(r, \varphi, \theta)$, but we have already defined $n_V(r, \varphi, \theta)$ to be something else.

proposition, f will be the determinant of the scattering matrix, perhaps multiplied by a rational function, and many of these hypotheses are natural in such applications.

Proposition 2.2. *Let f be a function meromorphic in the complex plane, with neither zeros nor poles on the real line. Suppose all the zeros of f lie in the open upper half plane, and all the poles in the open lower half plane. Furthermore, assume f is of order $d>1$ in the upper half plane, h_f is finite for $0\leq\theta\leq\pi$, and $h_f(\theta_0)\neq 0$ for some θ_0, $0<\theta_0<\pi$. Suppose in addition*

$$\int_0^r \frac{f'(t)}{f(t)}dt = o(r^d) \text{ as } r\to\pm\infty, \tag{2.1}$$

and the number of poles of f with norm at most r is of order at most d. If

$$\liminf_{r\to\infty}\frac{m_f(r)}{r^d} = \frac{d}{2\pi}\int_0^\pi h_f(\theta)d\theta,$$

then f is of completely regular growth in the angle $(0,\pi)$.

Before proving the proposition, we note that Govorov [7, 8] has studied the issue of completely regular growth of functions holomorphic in an angle. This is discussed in [11, Appendix VIII, section 2]. This is somewhat different than what we consider, since we use the assumption that f is meromorphic on the plane. Thus Govorov uses different restrictions on the distribution of the zeros of f.

Proof. The proof of this proposition follows in outline the proof of the analogous theorem for entire functions in the plane, [11, Chapter IV, Theorem 3]. Rather than using Jensen's theorem, though, it uses the equality

$$\int_0^r \frac{m_f(t)}{t}dt = \frac{1}{2\pi}\operatorname{Im}\int_0^r \frac{1}{t}\int_{-t}^t \frac{f'(s)}{f(s)}dsdt + \frac{1}{2\pi}\int_0^\pi \log|f(re^{i\theta})|d\theta \tag{2.2}$$

if $f(0)=1$, which follows using the proof of [6, Lemma 6.1].

By [11, Property (4), Chapter I, section 12],

$$\liminf_{r\to\infty}\frac{m_f(r)}{r^d} \leq \liminf_{r\to\infty} dr^{-d}\int_0^r \frac{m_f(t)}{t}dt. \tag{2.3}$$

We note that for any $\epsilon>0$ there is an $R>0$ so that

$$r^{-d}\ln|f(re^{i\theta})| \leq h_f(\theta)+\epsilon, \text{ for } r>R,\ 0\leq\theta\leq\pi. \tag{2.4}$$

Using this, (2.2), and our assumptions on the behavior of f on the real axis, we see that

$$\limsup_{r\to\infty} r^{-d}\int_0^r \frac{m_f(t)}{t}dt \leq \frac{1}{2\pi}\int_0^\pi h_f(\theta)d\theta.$$

Combining this with (2.3) and using our assumptions on $m_f(r)$, we get

$$\lim_{r\to\infty} r^{-d}\int_0^r \frac{m_f(t)}{t}dt = \frac{1}{2\pi}\int_0^\pi h_f(\theta)d\theta.$$

Thus we have

$$\lim_{r\to\infty}\int_0^\pi [h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|]d\theta = 0,$$

and, using (2.4),

$$\lim_{r\to\infty}\int_0^\pi \left|h_f(\theta) - r^{-d}\ln|f(re^{i\theta})|\right| d\theta = 0.$$

We note that our assumptions on f imply that it can be represented as the quotient of two entire functions, each of order at most d. Then we may apply [11, Chapter 2, Theorem 7] to find that for every $\eta > 0$ there is a set E_η of positive numbers of upper relative measure less than η so that if $r \notin E_\eta$, the family of functions of θ,

$$h_{f,r}(\theta) \stackrel{\text{def}}{=} r^{-d}\ln|f(re^{i\theta})|,$$

is equicontinuous in the angle $0 < \epsilon_0 \le \theta \le \pi - \epsilon_0$.

Given $\eta > 0$ and $\epsilon > 0$ we can, by the above result, find a $\delta > 0$ with $(\theta_1 - \delta, \theta_2 + \delta) \subset (0, \pi)$ and a set E_η of upper relative measure at most η so that for $\theta \in (\theta_1, \theta_2)$, $r \notin E_\eta$, $|\varphi - \theta| < \delta$, $|h_{f,r}(\theta) - h_{f,r}(\varphi)| < \epsilon/4$, and $|h_f(\theta) - h_f(\varphi)| < \epsilon/4$. Then for $0 < |k| < \delta$, $r \notin E_\eta$,

$$\begin{aligned}|h_{f,r}(\theta) - h_f(\theta)| &< \epsilon/2 + \frac{1}{k}\int_\theta^{\theta+k} |h_{f,r}(\varphi) - h_f(\varphi)|d\varphi \\ &\le \epsilon/2 + \frac{1}{k}\int_0^\pi |h_{f,r}(\varphi) - h_f(\varphi)|d\varphi.\end{aligned}$$

Since the integral goes to 0 as $r \to \infty$, we have shown that for $r > r_\epsilon$, $r \notin E_\eta$, $|h_{f,r}(\theta) - h_f(\theta)| < \epsilon$. Since $\eta > 0$ and $\epsilon > 0$ are arbitrary, we have, by [11, Chapter III, Lemma 1], f is of completely regular growth in (θ_1, θ_2). □

We shall also need some basics about plurisubharmonic functions and pluripolar sets. We use notation as in [10] and refer the reader to this reference for more details.

Let $\Omega \subset \mathbb{C}^p$ be an open connected set. A function $\psi : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if $\psi \not\equiv -\infty$, ψ is upper semi-continuous, and

$$\psi(z) \le \frac{1}{2\pi}\int_0^{2\pi} \psi(z + wre^{i\theta})d\theta$$

for all w, r such that $z + uw \subset \Omega$ for $u \in \mathbb{C}$, $|u| \le r$. The classic example of a plurisubharmonic function is $\ln|f(z)|$, where $f(z)$ is holomorphic. A subset $E \subset \Omega \subset \mathbb{C}^p$ is said to be *pluripolar* if there is a function ψ plurisubharmonic on Ω so that $E \subset \{z : \psi(z) = -\infty\}$.

For the convenience of the reader, we recall [10, Proposition 1.39], which is the main additional fact from several complex variables which we shall need.

Proposition 2.3. *(*[10, Prop. 1.39]*) Let $\{\psi_q\}$ be a sequence of plurisubharmonic functions uniformly bounded above on compact subsets in an open connected set*

$\Omega \subset \mathbb{C}^p$, with $\limsup_{q\to\infty}\psi_q \leq 0$ and suppose that there exist $\xi \in \Omega$ such that $\limsup_{q\to\infty}\psi_q(\xi) = 0$. Then $A = \{z \in \Omega : \limsup_{q\to\infty}\psi_q(z) < 0\}$ is pluripolar in Ω.

3. The functions $s_V(\lambda) = \det S_V(\lambda)$ and $h_d(\theta)$

For $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$, let $S_V(\lambda)$ be the associated scattering matrix and $s_V(\lambda) = \det S_V(\lambda)$. We recall [2, Lemma 3.1]:

Lemma 3.1. *Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^d;\mathbb{C})$. For $\lambda \in \mathbb{R}$, there is a C_V so that*

$$\left|\frac{d}{d\lambda}\log s_V(\lambda)\right| \leq C_V|\lambda|^{d-2}$$

whenever $|\lambda|$ is sufficiently large.

In fact, there is a constant α_d so that it suffices to take $|\lambda| \geq 2\alpha_d\|V\|_\infty$ for such a bound to hold. We note that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2\alpha_d\|V\|_\infty$ under these same assumptions on V,

$$\|S_V(\lambda) - I\| \leq C|\lambda|^{-1}. \tag{3.1}$$

This is relatively easy to see from an explicit representation of the scattering matrix; see, for example, the proof of [2, Lemma 3.1]. The constants in the statement of [2, Lemma 3.1] and in (3.1) can be chosen to depend only on the dimension, $\|V\|_\infty$ and the support of V. We note that it follows from Lemma 3.1, (3.1), and (2.2) that as $r \to \infty$

$$\int_0^r \frac{n_V(t)}{t}dt = \int_0^\pi \log|\det S_V(re^{i\theta})|d\theta + O(r^{d-1}). \tag{3.2}$$

Let

$$\rho(z) = \log\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2},\ 0 < \arg z < \pi. \tag{3.3}$$

This is a function which arises in studying the asymptotics of Bessel functions; see [13]. To define the square root which appears here, take the branch cut on the negative real axis and define ρ to be a continuous function in $\{0 < \arg z < \pi\} \cup (0,1)$ and use the principal branches of the logarithm and the square root when $z \in (0,1)$.

We use some notation of [18]. Set, for $0 < \theta < \pi$,

$$h_d(\theta) = \frac{4}{(d-2)!}\int_0^\infty \frac{[-\operatorname{Re}\rho]_+(te^{i\theta})}{t^{d+1}}dt \tag{3.4}$$

and set $h_d(0) = 0$, $h_d(\pi) = 0$. Now set

$$c_d \stackrel{\text{def}}{=} \frac{d}{2\pi}\int_0^\pi h_d(\theta)d\theta = \frac{2d}{\pi(d-2)!}\int_{\operatorname{Im} z>0}\frac{[-\operatorname{Re}\rho]_+(z)}{|z|^{d+2}}dxdy. \tag{3.5}$$

This is the constant c_d which appears in (1.1).

We recall the following result of [18, Theorem 5], which we paraphrase to suit our setting; [18, Theorem 5] actually covers a much larger class of operators.

Theorem 3.2. *(from* [18, Theorem 5]*) Let $V \in L^\infty(\mathbb{R}^d)$ be supported in $\overline{B}(0,a)$.*
(a) For any $\theta \in [0,\pi]$,

$$\log |s_V(re^{i\theta})| \leq h_d(\theta) a^d r^d + o(r^d) \text{ as } r \to \infty, \tag{3.6}$$

and the remainder term depends on V, and is uniform for $0 < \delta \leq \theta \leq \pi - \delta$ for any $\delta \in (0,\pi)$.
(b) For any $\delta > 0$,

$$\log |s_V(re^{i\theta})| \leq (h_d(\theta) a^d + \delta) r^d + o(r^d)$$

uniformly in $\theta \in [0,\pi]$.

It is important to note several things about the bounds in this theorem. One is that although Stefanov's theorem is stated only for self-adjoint operators (hence V real) it is equally valid when we allow complex-valued potentials. In fact, the proof of (a) in [18, Theorem 5] uses self-adjointness only to obtain a bound on the resolvent for λ in the upper half plane. A similar bound is true for the operator $-\Delta + V$ when V is complex-valued. The proof of (b) uses the fact that for real V, if $\lambda \in \mathbb{R}$, $\ln |s_V(\lambda)| = 1$. For complex-valued V, the proof in [18] of (b) can be adapted by using (3.1) and Lemma 3.1 to show that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2\alpha_d \|V\|_\infty$, $|\ln s_V(\lambda)| \leq C(1+|\lambda|)^{d-1}$. Here C can be chosen to depend only on d, $\|V\|_\infty$ and the diameter of the support of V.

Likewise, the particulars of the operator enter only through the diameter of the support of the perturbation (for us, the diameter of the support of V, which is $2a$) and the afore-mentioned bound on the resolvent in the good half plane. Thus, it is easy to see that the estimates of Theorem 3.2 are uniform in V as long as $\operatorname{supp} V \subset \overline{B}(0,a)$, $\|V\|_\infty \leq M$, and $r \geq 2\alpha_d M$.

We note that the upper bound (1.1) on the integrated resonance-counting function holds with the constant c_d defined in (3.5) even if V is complex-valued. This follows from the proof in [18]. In fact, the proof uses the bounds recalled in Theorem 3.2 and the identity (2.2). Together with the bounds in Lemma 3.1 and (3.1), these prove (1.1), even when V is complex-valued.

We shall want to understand the function $h_d(\theta)$ better. Note that for $0 < \theta \leq \pi/2$,

$$h_d(\pi/2 + \theta) = h_d(\pi/2 - \theta).$$

This can be seen directly using the definition of h_d and ρ.

Lemma 3.3. *The function $h_d(\theta)$, defined in (3.4), is C^1 on $(0,\pi)$. Moreover,*

$$h_d(0+) \stackrel{\text{def}}{=} \lim_{\epsilon \downarrow 0} h_d(\epsilon) = \sqrt{\pi} \frac{\Gamma\left(\frac{d-1}{2}\right)}{(d-2)!\,\Gamma\left(1+\frac{d}{2}\right)}.$$

Proof. We note [13, Section 4] that $\operatorname{Re}\rho(z) < 0$ if $0 < \arg z < \pi$ and $|z| > |z_0(\arg z)|$, where $z_0(\theta)$ is the unique point in $\mathbb{C}$ with argument θ and which lies on the curve given by

$$\pm(s\coth s - s^2)^{1/2} + i(s^2 - s\tanh s)^{1/2}, 0 \leq s \leq s_0.$$

Here s_0 is the positive solution of $\coth s = s$. Furthermore, $\operatorname{Re}\rho(z) > 0$ if z is in the upper half plane but $|z| < |z_0(\arg z)|$. Hence, recalling the definition of h_d, we have

$$h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{[-\operatorname{Re}\rho](te^{i\theta})}{t^{d+1}}dt.$$

Using the definition of ρ (3.3) and the following comments, we see that ρ is in fact a smooth function of z with $0 < \arg z < \pi$, $|z| > 0$. Since $|\rho(z)|/|z| \to 1$ when $|z| \to \infty$ in this region, the integral defining h_d is absolutely convergent. Likewise, since

$$\frac{\partial}{\partial\theta}\rho(te^{i\theta}) = -i\sqrt{1-(te^{i\theta})^2}$$

the integral

$$\int_{|z_0(\theta)|}^{\infty} \frac{-\operatorname{Re}\left[\frac{\partial}{\partial\theta}\rho(te^{i\theta})\right]}{t^{d+1}}dt$$

converges absolutely. A computation shows that $|z_0|$ is a C^1 function of θ for θ in $(0,\pi)$, and $\lim_{\epsilon\downarrow 0}\frac{\partial}{\partial\theta}|z_0|$ is finite. Thus, using that $\operatorname{Re}\rho(z_0(\theta)) = 0$ and the regularity of the derivative of $|z_0|(\theta)$, we get

$$\frac{d}{d\theta}h_d(\theta) = \frac{4}{(d-2)!}\int_{|z_0(\theta)|}^{\infty} \frac{\operatorname{Re} i\sqrt{1-(te^{i\theta})^2}}{t^{d+1}}dt.$$

Thus h_d is C^1 on $(0,\pi)$, $h_d'(0+) = \frac{4}{(d-2)!}\int_1^\infty \frac{\sqrt{t^2-1}}{t^{d+1}}dt$, and a computation now finishes the proof of the lemma. □

If $d = 3$, we can compute that

$$h_3(\theta) = \frac{4}{9}\left(\sin(3\theta) + \operatorname{Re}\frac{(1-z_0^2(\theta))^{3/2}}{|z_0(\theta)|^3}\right)$$

where $z_0(\theta)$ is as in the proof of the lemma. We comment that the $\sin(3\theta)$ term is missing from the first remark following the statement of [18, Theorem 5].

4. Proof of Proposition 1.1

We can now give the proof of Proposition 1.1, which follows by combining Theorem 2.1, Proposition 2.2, and [18, Theorem 5].

Recall that $S_V(\lambda)$ is the scattering matrix associated with the operator $-\Delta + V$, and $s_V(\lambda) = \det S_V(\lambda)$. Then s_V has a pole at λ if and only if s_V has a zero at $-\lambda$, and the multiplicities coincide. Moreover, with at most a finite number of exceptions, the poles of $s_V(\lambda)$ coincide, with multiplicity, with the zeros of $R_V(\lambda)$.

If $s_V(\lambda)$ has poles in the upper half plane, it has only finitely many, say $\lambda_1, ..., \lambda_m$. Set $f(\lambda) = \prod_{j=1}^{m}(\lambda - \lambda_j)s_V(\lambda)$. Then from [18, Theorem 5], for $0 \leq \theta \leq \pi$ and large r,

$$r^{-d} \log |f(re^{i\theta})| \leq a^d h_d(\theta) + o(1).$$

Using the equation (2.2), the fact that $V \in \mathfrak{M}_a$, and Lemma 3.1, we see that we must have

$$\limsup_{r\to\infty} r^{-d} \log |f(re^{i\theta})| = a^d h_d(\theta) \text{ for } \theta \in (0, \pi).$$

Applying Proposition 2.2 to $f(\lambda)$, we see that $f(\lambda)$ is a function of completely regular growth in the upper half plane. Since $h_d(\theta)$ is a C^1 function of θ for $\theta \in (0, \pi)$, we get the first part of the proposition from Theorem 2.1.

5. Proof of Theorem 1.3

We shall need an identity related to both (2.2) and to what Levin calls a generalized formula of Jensen [11, Chapter 3, section 2]. We define, for a function f meromorphic in a neighborhood of $\arg z = \theta$ and with $|f(0)| = 1$,

$$J_f^r(\theta) = \int_0^r \frac{\ln |f(te^{i\theta})|}{t} dt. \tag{5.1}$$

Lemma 5.1. *Let f be holomorphic in $\varphi \leq \arg z \leq \theta$, $f(0) = 1$, f have no zeros with argument φ or θ and with norm less than r, and let $m(r, \varphi, \theta)$ be the number of zeros of f in the sector $\varphi < \arg z < \theta$, $|z| \leq r$. Then*

$$\int_0^r \frac{m(t, \varphi, \theta)}{t} dt$$
$$= \frac{1}{2\pi} \int_0^r \frac{d}{d\theta} J_f^t(\theta) \frac{dt}{t} + \frac{1}{2\pi} \int_0^r \frac{1}{t} \int_0^t \frac{d}{ds} \arg f(se^{i\varphi}) ds\, dt + \frac{1}{2\pi} \int_\varphi^\theta \ln |f(re^{i\omega})| d\omega.$$

Proof. Using the argument principle and the Cauchy-Riemann equations just as in [11, Chapter 3, section 2] we see that

$$2\pi m(r, \varphi, \theta) = \int_0^r \frac{\partial}{\partial t} \arg f(te^{i\varphi}) dt + \int_0^r \frac{1}{t} \frac{\partial}{\partial \theta} \ln |f(te^{i\theta})| dt + r \int_\varphi^\theta \frac{\partial}{\partial t} \ln |f(te^{i\omega})| d\omega$$

when there are no zeros on the boundary of the sector. As in [11], by dividing by $2\pi r$ and integrating from 0 to r we obtain the lemma. □

For $0 < \varphi < \theta < 2\pi$, recall the notation $n_V(r, \varphi, \theta)$ for the number of poles of $R_V(\lambda)$ in the sector $\{z : |z| \leq rr, \varphi < \arg z < \theta\}$. Moreover, we set

$$N_V(r, \varphi, \theta) = \int_0^r \frac{1}{t}(n_V(t, \varphi, \theta) - n_V(0, \varphi, \theta)) dt.$$

We note that $|s_V(0)| = 1$, since $s_V(\lambda)s_V(-\lambda) = 1$.

Lemma 5.2. *Suppose $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$. Then*

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)d\theta' = \frac{1}{2\pi}\int_0^r J^t_{s_V}(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|d\omega d\theta' + O(r^{d-1}).$$

The error can be bounded by $c\langle r^{d-1}\rangle$ where the constant depends only on $\|V\|$, the support of V, and d.

Proof. Using the relationship between the poles of $R_V(\lambda)$ and the zeros of $s_V = \det S_V$, we find from Lemma 5.1 that

$$N_V(t,\pi,\theta'+\pi) = \frac{1}{2\pi}\int_0^r \frac{\partial}{\partial\theta'}J^t_{s_V}(\theta')\frac{dt}{t} + \frac{1}{2\pi}\int_0^r \frac{1}{t}\int_0^t \frac{d}{dt'}\arg s_V(t')dt'dt + \frac{1}{2\pi}\int_0^{\theta'} \ln|s_V(re^{i\omega})|d\omega + O(\log r). \tag{5.2}$$

Integrating in θ' from 0 to θ, and using the fact that both sides are continuous functions of θ', we get

$$\int_0^\theta N_V(r,\pi,\theta'+\pi)d\theta' = \frac{1}{2\pi}\int_0^r J^t_{s_V}(\theta)\frac{dt}{t} - \frac{1}{2\pi}\int_0^r J^t_{s_V}(0)\frac{dt}{t} + \frac{\theta}{2\pi}\int_0^r \frac{1}{t}\int_0^t \frac{d}{dt'}\arg s_V(t')dt'dt + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_V(re^{i\omega})|d\omega d\theta' + O(\log r)$$

The bounds of Lemma 3.1 and (3.1) mean that as $r\to\infty$

$$\frac{1}{2\pi}\int_0^r J^t_{s_V}(0)\frac{dt}{t} = O(r^{d-1})$$

and

$$\frac{\theta}{2\pi}\int_0^r \frac{1}{t}\int_0^t \frac{d}{dt'}\arg s_V(t')dt'dt = O(r^{d-1})$$

where the bounds can be made uniform in V within a fixed compact set. □

We shall need some notation for the results which follow. Let $\Omega \subset \mathbb{C}^{d'}$ be an open set containing a point z_0. For $\rho > 0$ small enough that $B(z_0,\rho)\subset\Omega$ we define Ω_ρ to be the connected component of $\{z\in\Omega : \mathrm{dist}(z,\Omega^c)\geq\rho\}$ which contains z_0.

Proposition 5.3. *Let V, z_0, Ω satisfy the assumptions of Theorem 1.2, let $\rho>0$ be small enough that $B(z_0,\rho)\subset\Omega$, and let Ω_ρ be as defined above. Then for $z\in\Omega_\rho$, $0<\theta<\pi$,*

$$\psi(z,r,\rho) \stackrel{\mathrm{def}}{=} \frac{1}{\mathrm{vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta N_{V(z')}(r,\pi,\theta'+\pi)d\theta' d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right) + o(r^d).$$

Proof. First note that since $0 \le N_{V(z)}(z,\pi,\theta+\pi) \le c_d r^d a^d + o(r^d)$, and the bound is uniform on compact sets of z, we get that holding ρ fixed, $r^{-d}\psi(\bullet, r, \rho)$ is a family uniformly continuous in z for z in compact sets of Ω_ρ.

We shall use Lemma 5.2. Note that by Stefanov's results recalled in Theorem 3.2,

$$\frac{1}{2\pi}\int_0^r J^t_{s_{V(z)}}(\theta)\frac{dt}{t} \le \frac{1}{2\pi}\frac{1}{d^2}h_d(\theta)a^d r^d + o(r^d)$$

where the term $o(r^d)$ can be bounded uniformly in z in compact sets of Ω_ρ. By the same argument,

$$\int_0^\theta \int_0^{\theta'} \ln|s_{V(z)}(re^{i\omega})|d\omega d\theta' \le \int_0^\theta \int_0^{\theta'} h_d(\omega)d\omega d\theta' a^d r^d + o(r^d).$$

Using Lemma 5.2, we find that

$$\begin{aligned}\psi(z,r,\rho) = {} & \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}d\mathcal{L}(z') \\ & + \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z')}(re^{i\omega})|d\omega d\theta' d\mathcal{L}(z') + O(r^{d-1}).\end{aligned}$$

Let $M = 2\max_{z\in\overline{\Omega_\rho}}\|V\|_\infty$ and set, for $r > M$,

$$\begin{aligned}\Psi_1(z,r,\rho) = {} & \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_M^r J^t_{s_{V(z')}}(\theta)\frac{dt}{t}d\mathcal{L}(z') \\ & + \frac{1}{2\pi \operatorname{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z')}(re^{i\omega})|d\omega d\theta' d\mathcal{L}(z')\end{aligned}$$

and note that

$$\psi(z,r,\rho) = \psi_1(z,r,\rho) + O(r^{d-1}).$$

By the bounds above,

$$\psi_1(z,r,\rho) \le \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d r^d + o(r^d). \tag{5.3}$$

Using [10, Proposition I.14] and the fact that $\ln|s_{V(z)}(\lambda)|$ is a plurisubharmonic function of $z \in \Omega$ when $\|\lambda\| > \frac{3}{2}\|V(z)\|_\infty$ and λ lies in the upper half plane, we see that $\Psi_1(z,r,\rho)$ is a plurisubharmonic function of $z \in \Omega_\rho$. Since by Proposition 2.2 $s_{V(z_0)}(\lambda)$ is of completely regular growth in $0 < \arg\lambda < \pi$, using Lemma 5.2 and [11, Chapter III, Sec. 2, Lemma 2],

$$\lim_{r\to\infty} r^{-d}\int_0^{\theta'} N_{V(z_0)}(r,\pi,\theta'+\pi)d\theta' = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d.$$

By the most basic property of plurisubharmonic functions,

$$\psi_1(z_0,r,\rho) \geq \frac{1}{2\pi}\int_M^r J^t_{s_{V(z_0)}}(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^\theta\int_0^{\theta'} \ln|s_{V(z_0)}(re^{i\omega})|d\omega d\theta'.$$

But the right hand side of this equation is $\int_0^\theta N_{V(z_0)}(r,0,\theta')d\theta' + O(r^{d-1})$, so we see that

$$\liminf_{r\to\infty} r^{-d}\psi_1(z_0,r,\rho) \geq \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d.$$

Combining this with (5.3), we find

$$\lim_{r\to\infty} r^{-d}\psi_1(z_0,r,\rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d. \tag{5.4}$$

Using this and the upper bound (5.3) on ψ_1, since ψ_1 is plurisubharmonic in z it follows from [10, Proposition 1.39] (recalled here in Proposition 2.3) that for any sequence $\{r_j\}$, $r_j\to\infty$ there is a pluripolar set $E\subset\Omega_\rho$ (which may depend on the sequence) so that

$$\limsup_{j\to\infty} r_j^{-d}\psi_1(z,r_j,\rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d$$

for all $z\in\Omega_\rho\setminus E$. Since $\lim_{j\to\infty} r_j^{-d}(\psi_1(z,r,\rho)-\psi(z,r,\rho))=0$, the same conclusion holds for ψ in place of ψ_1.

Suppose there is some $z_1\in\Omega_\rho$ and some sequence $r_j\to\infty$ so that

$$\lim_{j\to\infty} r_j^{-d}\psi(z_1,r_j,\rho) < \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d.$$

Then, using the uniform continuity of $r^{-d}\psi(z,r,\rho)$ in z, we find there must be an $\epsilon>0$ so that

$$\limsup_{j\to\infty} r_j^{-d}\psi(z,r_j,\rho) < \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d$$

for all $z\in B(z_1,\epsilon)$. But since $B(z_1,\epsilon)$ is not contained in a pluripolar set, we have a contradiction. Thus

$$\lim_{r\to\infty} r^{-d}\psi(z_1,r,\rho) = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^\theta\int_0^{\theta'} h_d(\omega)d\omega d\theta'\right)a^d.$$

□

Lemma 5.4. *Let $M(r,\theta)$ be a function so that for any fixed positive r_0, $M(r_0,\theta)$ is a non-decreasing function of θ, and suppose*

$$\lim_{r\to\infty} r^{-d}\int_0^\theta M(r,\theta')d\theta' = \alpha(\theta)$$

for $\theta_1 < \theta < \theta_2$*. Then if* α *is differentiable at* θ*, then*

$$\lim_{r\to\infty} r^{-d} M(r,\theta) = \alpha'(\theta).$$

Proof. Let $\epsilon > 0$. Then, since $M(r,\theta)$ is non-decreasing in θ,

$$\int_0^{\theta+\epsilon} M(r,\theta')d\theta' - \int_0^{\theta} M(r,\theta')d\theta' \geq \epsilon M(r,\theta)$$

which, under rearrangement, yields

$$r^{-d}M(r,\theta) \leq r^{-d}\frac{\int_0^{\theta+\epsilon} M(r,\theta')d\theta' - \int_0^{\theta} M(r,\theta')d\theta'}{\epsilon}.$$

Thus

$$\limsup_{r\to\infty} r^{-d}M(r,\theta) \leq \frac{\alpha(\theta+\epsilon)-\alpha(\theta)}{\epsilon}.$$

Likewise, we find

$$\liminf_{r\to\infty} r^{-d}M(r,\theta) \geq \frac{\alpha(\theta)-\alpha(\theta-\epsilon)}{\epsilon}.$$

Since both these equalities must hold for all $\epsilon > 0$, the lemma follows from the assumption that α is differentiable at θ. □

The following proposition follows from Proposition 5.3, but is stronger as it does not require averaging in the θ' variable.

Proposition 5.5. *Let* V*,* z_0*,* Ω *satisfy the assumptions of Theorem 1.2, and* $\rho > 0$*,* Ω_ρ *be as in Proposition 5.3. Then for* $z \in \Omega_\rho$*,* $0 < \theta < \pi$*,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r,\pi,\theta+\pi)d\mathcal{L}(z') \\ = \frac{1}{2\pi}a^d r^d\left(\frac{1}{d^2}h_d'(\theta) + \int_0^{\theta} h_d(\omega)d\omega\right) + o(r^d).$$

Proof. This follows from applying Lemmas 5.4 and 3.3 to the results of Proposition 5.3. □

Proposition 5.5 does not give results for the counting function for all the resonances (note that we cannot have $\theta = \pi$). The following fills this gap.

Proposition 5.6. *Let* V*,* z_0*,* Ω *satisfy the assumptions of Theorem 1.2, and* $\rho > 0$*,* Ω_ρ *as in Proposition 5.3. Then for* $z \in \Omega_\rho$*,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d \int_0^{\theta} h_d(\omega)d\omega + o(r^d).$$

Proof. The proof of this is very similar to that of Proposition 5.3. In fact, the main difference is the use of (2.2), which together with Lemma 3.1 and (3.1) gives us

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} N_{V(z')}(r)d\mathcal{L}(z') = \psi_1(z,r,\rho)+O(r^{d-1})$$

where

$$\psi_1(z,r,\rho)=\frac{1}{\mathrm{Vol}(B(z,\rho))}\frac{1}{2\pi}\int_{z'\in B(z,\rho)}\int_0^\pi \log|f(re^{i\theta})|d\theta d\mathcal{L}(z').$$

Using that ψ_1 is plurisubharmonic in z, the proof now follows just as in Proposition 5.3. □

Proposition 5.7. *Let V, Ω, z_0 satisfy the conditions of Theorem 1.2, and let ρ, Ω_ρ be as in Proposition 5.3. Then for $0<\theta<\pi$,*

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r,\pi,\theta+\pi)d\mathcal{L}(z') = \frac{1}{2\pi}a^d r^d\left(\frac{1}{d}h_d'(\theta)+d\int_0^\theta h_d(\theta)d\theta\right)+o(r^d)$$

and

$$\frac{1}{\mathrm{Vol}(B(z,\rho))}\int_{z'\in B(z,\rho)} n_{V(z')}(r)d\mathcal{L}(z') = \frac{d}{2\pi}a^d r^d\int_0^\pi h_d(\theta)d\theta+o(r^d).$$

Proof. This proof follows from Propositions 5.5 and 5.6, using, in addition, a result like that of [18, Lemma 1] or 5.4. □

Proof of Theorem 1.3. Let $M=\max(1+|\psi(z)|)$, and for $\rho>0$ small enough that $B(z_0,\rho)\subset\Omega$, set Ω_ρ to be the connected component of $\{z\in\Omega : \mathrm{dist}(z,\Omega^c)>\rho\}$ which contains z_0. Given $\epsilon>0$, choose $\rho_0>0$ such that $B(z_0,\rho_0)\subset\Omega$ and so that

$$\mathrm{vol}(\mathrm{supp}\,\psi\cap(\Omega\setminus\Omega_\rho))<\frac{\epsilon}{5Me(c_d a^d+1)}. \tag{5.5}$$

Since ψ is continuous with compact support, we can find a $\delta_1>0$ so that if $|z-z'|<\delta_1$, then $|\psi(z)-\psi(z')|<\frac{\epsilon}{5e(a^d+1)c_d}$. We may find a finite number J of disjoint balls $B(z_j,\epsilon_j)$ so that $\epsilon_j<\delta_1$, $B(z_j,\epsilon_j)\subset\Omega_\rho$, and

$$\mathrm{vol}(\mathrm{supp}\,\psi\setminus(\cup_1^J B(z_j,\epsilon_j))+\mathrm{vol}(\cup_1^J B(z_j,\epsilon_j)\setminus\mathrm{supp}\,\psi)<\frac{\epsilon}{4Me(a^d c_d+1)}.$$

Now

$$\begin{aligned}&\int\psi(z)n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z)\\&=\sum_{j=1}^J\int_{B(z_j,\epsilon_j)}\psi(z)n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z)+\int_{\mathrm{supp}\,\psi\setminus(\cup B(z_j,\epsilon_j))}\psi(z)n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z).\end{aligned}$$

We will use that the bound (1.1) implies that $n_V(z) \leq ec_d a^d r^d + o(r^d)$. By our choice of $B(z_j,\epsilon_j)$,

$$\left|\int_{\operatorname{supp}\psi\setminus(\cup B(z_j,\epsilon_j))} \psi(z) n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z)\right| \leq \frac{\epsilon}{4}(r^d+o(r^d)).$$

By our choice of δ_1 and the assumption that $\epsilon_j<\delta_1$, we have

$$\left|\sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z) n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z) - \sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r,\varphi,\theta)d\mathcal{L}(z)\right| \leq \frac{\epsilon}{5}(r^d+o(r^d)).$$

By Proposition 5.7, if $\theta'<\pi$,

$$\sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r,\pi,\pi+\theta')d\mathcal{L}(z) = \left(\sum_{j=1}^J \psi(z_j)\operatorname{vol}(B(z_j,\epsilon_j))\right)\frac{1}{2\pi}a^d r^d\left(\frac{1}{d}h_d'(\theta')+d\int_0^{\theta'} h_d(\omega)d\omega\right)+o(r^d),$$

and

$$\sum_{j=1}^J \int_{B(z_j,\epsilon_j)} \psi(z_j) n_{V(z)}(r)d\mathcal{L}(z) = \left(\sum_{j=1}^J \psi(z_j)\operatorname{vol}(B(z_j,\epsilon_j))\right)\frac{d}{2\pi}a^d r^d\int_0^{\pi} h_d(\omega)d\omega+o(r^d).$$

Again using our choice of δ_1 and ϵ_j, we have

$$\left|\sum_{j=1}^J \psi(z_j)\operatorname{vol}(B(z_j,\epsilon_j)) - \int\psi(z)d\mathcal{L}(z)\right| < \frac{2\epsilon}{5(c_d a^d+1)}.$$

Thus we have shown that given $\epsilon>0$, if $0<\theta'<\pi$,

(5.6)
$$\left|\int\psi(z)n_{V(z)}(r,\pi,\theta'+\pi)d\mathcal{L}(z) - \int\psi(z)d\mathcal{L}(z)\frac{1}{2\pi}a^d r^d\left(\frac{1}{d}h_d'(\theta')+d\int_0^{\theta} h_d(\omega)d\omega\right)\right| \leq \epsilon r^d+o(r^d)$$

and

(5.7)
$$\left|\int\psi(z)n_{V(z)}(r)d\mathcal{L}(z) - c_d a^d r^d\int\psi(z)d\mathcal{L}(z)\right| \leq \epsilon r^d+o(r^d).$$

Thus we have proved the first and third statements of the theorem. The second statement of the theorem follows from the other two.

6. Proof of Theorem 1.2

This proof uses some ideas similar to those used in the proofs of Propositions 5.3 and 5.6. In fact, because the proofs are so similar we shall only give an outline.

Note that by (2.2), (3.1), and Lemma 3.1,

$$N_{V(z)}(r) = \psi(z,r) + o(r^{d-1})$$

where

$$\psi(z,r) = \frac{1}{2\pi}\int_0^{\pi} \ln |s_{V(z)}(re^{i\theta})| d\theta$$

is, for fixed (large) r a plurisubharmonic function of $z \in \tilde{\Omega} \Subset \Omega$. Since

$$\limsup_{r\to\infty} r^{-d}\psi_z(z,r) \leq \frac{1}{2\pi}\int_0^{\theta} h_d(\theta)d\theta$$

and this maximum is achieved at $z = z_0 \in \Omega$, we get the first part of the Theorem by applying [10, Proposition 1.39], recalled in Proposition 2.3.

To obtain the second part, note that as in the proof of Proposition 5.3, for $0 < \theta < \pi$,

$$\int_0^{\theta} N_{V(z)}(r,\pi,\theta'+\pi)d\theta' = \psi_2(z,r,\theta) + o(r^d)$$

where

$$\psi_2(z,r,\theta) = \frac{1}{2\pi}\int_M^r J^t_{s_{V(z)}}(\theta)\frac{dt}{t} + \frac{1}{2\pi}\int_0^{\theta}\int_0^{\theta'} \ln|s_{V(z)}(re^{i\omega})|d\omega d\theta'.$$

Since this is a plurisubharmonic function of $z \in \tilde{\Omega}$, $\tilde{\Omega} \Subset \Omega$, if M is chosen so that $M \geq 2\alpha_d \max_{z\in\overline{\tilde{\Omega}}} \|V\|_\infty$, a similar argument as in the proof of Proposition 5.3 shows that there exists a pluripolar set $E_\theta \subset \Omega$ so that

$$2\pi \limsup_{r\to\infty} r^{-d}\psi_2(z,r,\theta) = \frac{1}{d^2}h_d(\theta) + \int_0^{\theta} h_d(\theta')d\theta'$$

for all $z \in \Omega \setminus E_\theta$. Note that if the second part of the theorem can be proved for a small θ_0, it is proved for all θ with $\theta \geq \theta_0$. Thus, it is most interesting for small θ. Choose $\theta > 0$ sufficiently small that $h_d(\theta) \geq \theta h_d'(0+)/2$, where we denote $\lim_{\epsilon\downarrow 0} h_d(\epsilon) = h_d(0+)$. Now, if

$$\limsup_{r\to 0} r^{-d}\int_0^{\theta} N_V(r,\pi,\pi+\theta')d\theta' = \frac{1}{2\pi}\left(\frac{1}{d^2}h_d(\theta) + \int_0^{\theta} h_d(\theta')d\theta'\right) \geq \frac{1}{4\pi d^2}h_d'(0+)\theta,$$

then since $N_V(r,\pi,\pi) = O(1)$, we must have

$$\limsup_{r\to 0} r^{-d}N_V(r,\pi,\pi+\theta) \geq \frac{1}{4\pi d^2}h_d'(0+).$$

References

[1] C. Bardos, J.-C. Guillot and J. V. Ralston, *La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion,* Comm. in PDE **7** (1982), 905-958.

[2] T. Christiansen, *Several complex variables and the distribution of resonances for potential scattering*, Commun. Math. Phys **259** (2005), 711-728.

[3] T. Christiansen, *Schrödinger operators with complex-valued potentials and no resonances*, Duke Math Journal **133**, no. 2 (2006), 313-323.

[4] T. Christiansen, *Isophasal, isopolar, and isospectral Schrödinger operators and elementary complex analysis.* Amer. J. Math. **130** (2008), no. 1, 49–58.

[5] R. Froese, *Asymptotic distribution of resonances in one dimension*, J. Differential Equations **137** (1997), no. 2, 251–272.

[6] R. Froese, *Upper bounds for the resonance counting function of Schrödinger operators in odd dimensions*, Canad. J. Math. **50** (1998), no. 3, 538–546.

[7] N.V. Govorov, *On the indicator of functions of non-integral order, analytic and of completely regular growth in a half-plane.* (Russian) Dokl. Akad. Nauk SSSR **162** (1965) 495–498.

[8] N. V. Govorov, *On the indicator of functions of integral order, analytic and of completely regular growth in a half-plane.* (Russian) Dokl. Akad. Nauk SSSR **172** (1967) 763–766.

[9] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Annals of Mathematics **145** (1997), 597-660.

[10] P. Lelong and L. Gruman, *Entire functions of several complex variables*, Springer Verlag, Berlin, 1986.

[11] B. Ja. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, R.I. 1964.

[12] R.B. Melrose, *Scattering theory and the trace of the wave group*, J. Funct. Anal. **45** (1982), 29-40.

[13] F.W.J. Olver, *The asymptotic solution of linear differential equations of the second order for large values of a parameter.* Philos. Trans. Roy. Soc. London. Ser. A. **247**, (1954). 307–327.

[14] T. Ransford, *Potential theory in the complex plane*, Cambridge University Press, Cambridge, 1995.

[15] T. Regge, *Analytic properties of the scattering matrix*, Nuovo Cimento **8** (5), (1958), 671–679.

[16] A. Sá Barreto, *Remarks on the distribution of resonances in odd dimensional Euclidean scattering*, Asymptot. Anal. **27** (2001), no. 2, 161–170.

[17] B. Simon, *Resonances in one dimension and Fredholm determinants*, J. Funct. Anal. **178** (2000), no. 2, 396–420.

[18] P. Stefanov, *Sharp upper bounds on the number of the scattering poles*, J. Funct. Anal. **231** (2006), no. 1, 111–142.

[19] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73** (2) (1987), 277-296.

[20] M. Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials*, J. Funct. Anal. **82** (1989), 370-403.

[21] M. Zworski, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), no. 2, 311–323.

Department of Mathematics, University of Missouri, Columbia, Missouri 65211, USA

E-mail address: christiansent@missouri.edu