RATIONAL CURVES ON K3 SURFACES

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Abstract. We show that projective K3 surfaces with odd Picard rank contain infinitely many rational curves. Our proof extends the Bogomolov-Hassett-Tschinkel approach, i.e., uses moduli spaces of stable maps and reduction to positive characteristic.

Introduction

For a complex variety of general type, Lang's conjecture [La] predicts that all rational curves are contained in a proper algebraic set. On the other extreme, varieties of negative Kodaira dimension are conjecturally uniruled, and these contain moving families of rational curves through every general point. In between these two extremes lie varieties with trivial canonical sheaves, e.g., K3 surfaces, Calabi-Yau manifolds and Abelian varieties. Abelian varieties contain no rational curves at all. Although complex K3 surfaces contain no moving families of rational curves, we have the well-known

Conjecture. Every projective K3 surface over an algebraically closed field contains infinitely many integral rational curves.

Bogomolov and Mumford [MM] showed that every complex projective K3 surface contains at least one rational curve. Next, Chen [Ch] established existence of infinitely many rational curves on very general complex projective K3 surfaces. Since then, infinitely many rational curves have been established on polarized K3 surfaces of degree 2 and Picard rank $\rho = 1$ [BHT], elliptic K3 surfaces [BT], and K3 surfaces with infinite automorphism groups. In particular, this includes all K3 surfaces with $\rho \geq 5$, as well as "most" K3 surfaces with $\rho \geq 3$, see [BT]. In this article, we prove

Theorem. *A complex projective K3 surface with odd Picard rank contains infinitely many integral rational curves.*

Our proof uses the approach of Bogomolov, Tschinkel, and Hassett from [BHT], i.e., reduction to positive characteristic and moduli spaces of stable maps. Our techniques also yield the following result in positive characteristic:

Theorem. *A non-supersingular K3 surface with odd Picard rank over an algebraically closed field of characteristic $p \geq 5$ contains infinitely many integral rational curves.*

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The article is organized as follows:

In Section 1, we recall a couple of results about rational curves on K3 surfaces. Also, we extend them to characteristic p, which is probably known to the experts.

In Section 2, we discuss rigid genus zero stable maps and introduce the notion of *rigidifiers*. Rigidifiers are genus zero stable maps that have the property that any sum of rational curves on a K3 surface can be represented by a rigid genus zero stable map after adding sufficiently many rigidifiers. In this article, *rigid* means that we allow infinitesimal deformations but no one-dimensional non-trivial families.

In Section 3, we prove our main result. By [BHT], it suffices to establish it for K3 surfaces over number fields. For such surfaces, we find rational curves of arbitrary high degree on reductions modulo p. Next, we deform the surface and its high degree curve to a nearby surface that contains rigdifiers. Then, we use these rigidifiers to deform, as well as to lift to characteristic zero.

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1. Generalities

In this section we review general results about rational curves on K3 surfaces. On our way, we extend these to characteristic p, which is probably known to the experts.

Theorem 1.1 (Bogomolov-Mumford $+\varepsilon$)**.** *Let X be a projective K3 surface over an algebraically closed field k. Let $\mathcal{L}$ be a non-trivial, effective and invertible sheaf. Then there exists a divisor in $|\mathcal{L}|$ that is a sum of rational curves.*

Proof. Since X is a K3 surface, $\mathcal{L}$ is isomorphic to $\mathcal{L}'\otimes\mathcal{O}_X(\sum_i a_iC_i)$, where the a_i are positive integers, the C_i are smooth rational curves, and $\mathcal{L}'$ is a nef invertible sheaf. Replacing $\mathcal{L}$ by $\mathcal{L}'$ we may assume that $\mathcal{L}$ is non-trivial, nef and satisfies $\mathcal{L}^2\geq 0$.

If $\mathcal{L}^2=0$ then $|\mathcal{L}|$ defines a genus-one fibration $X\to\mathbb{P}^1$. Since X is K3, not all fibers are smooth. In particular, there exists a fiber, i.e., a divisor in $|\mathcal{L}|$, that is a sum of rational curves.

Next, we assume $\mathcal{L}^2>0$, i.e., $\mathcal{L}$ is big and nef. In characteristic zero, our assertion is shown in [BT, Proposition 2.5]. If $\mathrm{char}(k)=p>0$, then there exists a possibly ramified extension R of the Witt ring $W(k)$ such that the pair $(X,\mathcal{L})$ lifts to a formal scheme over $\mathrm{Spf}\ R$ by [Del, Corollaire 1.8]. We write this as a limit of schemes $X_n\to\mathrm{Spec}\,R_n$. Then, for all $n\geq 0$

$$X'_n\ :=\ \mathrm{Proj}\ \bigoplus_{k\geq 0}H^0(X_n,\mathcal{L}_n^{\otimes k})\ \longrightarrow\ \mathrm{Spec}\,R_n$$

is a projective surface, whose special fiber $X'=X'_0\to\mathrm{Spec}\,k$ has at worst Du Val singularities. Since each $\mathcal{O}_{X'_n}(1)$ is ample on X'_n we obtain an ample invertible sheaf on the limit, which is algebraizable by Grothendieck's existence theorem. We thus obtain a scheme $\mathcal{X}'\to\mathrm{Spec}\,R$ lifting X'. By [Ar2], there exists a possibly ramified extension $R\subseteq R'$ and a smooth algebraic space $\widetilde{\mathcal{X}}\to\mathrm{Spec}\,R'$ with special fiber X.

The ample invertible sheaf on $\mathcal{X}'$ pulls back to an invertible sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}$, which lifts $\mathcal{L}$. Applying [BT, Proposition 2.5] to $\widetilde{\mathcal{L}}$ and reducing modulo p, we find a divisor in $|\mathcal{L}|$ that is a sum of rational curves. $\square$

As corollary, we obtain a result of Mori and Mukai [MM] that they attribute to Bogomolov and Mumford, see also [BHT, Corollary 18].

Theorem 1.2 (Bogomolov-Mori-Mukai-Mumford $+\varepsilon$)**.** *A projective K3 surface over an algebraically closed field contains a rational curve.* $\square$

Moreover, it is commonly believed that

Conjecture 1.3. Every projective K3 surface over an algebraically closed field contains infinitely many integral rational curves.

Chen [Ch] has shown that this is true for very general complex projective K3 surfaces, but see also the discussion in [BHT, Section 3]. Moreover, let us mention the following reduction to the case of number fields:

Theorem 1.4 (Bogomolov-Hassett-Tschinkel [BHT, Theorem 3])**.** *Assume that for every K3 surface X defined over a number field K, there are infinitely many rational curves on*

$$X_{\overline{\mathbb{Q}}} := X \otimes_K \overline{\mathbb{Q}}\,.$$

Then, Conjecture 1.3 holds for algebraically closed fields of characteristic zero. $\square$

Remark 1.5. Moreover, the proof of [BHT, Theorem 3] shows that Conjecture 1.3 holds for projective K3 surfaces with Picard rank ρ_0 over algebraically closed fields of characteristic zero if it holds for K3 surfaces with Picard rank ρ_0 over $\overline{\mathbb{Q}}$.

2. Rigid stable maps and rigidifiers

In this section we first give a criterion for a genus zero stable map to a projective K3 surface to be rigid. Then, we introduce a class of stable maps, called *rigidifiers*, that has the property that given any sum of rational curves on a K3 surface, this sum can be represented by a rigid stable map after adding sufficiently many rigidifiers. Finally, we show that surfaces with rigidifiers form an open and dense subset inside the moduli space of polarized K3 surfaces.

Definition 2.1. A morphism $f : C \to X$, where C is a proper and connected curve with at worst nodal singularities, is called *stable*, if $\mathrm{Aut}(f)$ is finite. Here, $\mathrm{Aut}(f)$ denotes the automorphism group scheme of all automorphisms of C that commute with f.

For a projective K3 surface (X, H), there exists a moduli space of stable maps [Kon]. Various algebraic constructions are discussed in [AV], and a formal Artin stack over the formal deformation space of K3 surfaces is constructed in the proof of [BHT, Theorem 19]. In this paper, without further mentioning, all domain curves of stable maps will have arithmetic genus zero. For an integer β, we denote by $\overline{\mathcal{M}}_0(X, \beta)$ the moduli stack of genus zero stable maps $[f, C]$ of degree β, i.e., stable genus zero maps $f : C \to X$ with $\deg f_*[C] = \beta$.

Definition 2.2. Let $[f, C]$ be a stable map and D a sum of rational curves.

(1) We call $[f, C]$ *rigid* if $\overline{\mathcal{M}}_0(X, \beta)$ is zero-dimensional at $[f]$.
(2) We say that the rational curve $D = \sum_i D_i$ on X has a *rigid representative* (by a stable map) if there exists a rigid stable map $[f, C] \in \overline{\mathcal{M}}_0(X, \beta)$ such that $f_*[C] = [D]$.

Remark 2.3. Here, we use the word "rigid" in the most liberal manner, i.e., for a rigid stable map $[f, C]$ the morphism f may admit infinitesimal but no one-dimensional deformations in $\overline{\mathcal{M}}_0(X, \beta)$.

For example, any integral rational curve D on a non-supersingular K3 surface X has a rigid representative, namely via its normalization $\nu : \widetilde{D} \to D \subseteq X$. On the other hand, no multiple of an irreducible smooth rational curve has a rigid representative, as any representative $[f]$ must involve multiple covers that deform and give rise to a positive dimensional component of $\overline{\mathcal{M}}_0(X, \beta)$ through $[f]$.

Let us now introduce some useful notions for genus zero stable maps.

Definition 2.4. Let $[f, C]$ be a stable map.

(1) An irreducible component $\Sigma \subseteq C$ is a *ghost-component* if $f(\Sigma)$ is a point.
(2) Two irreducible components $\Sigma_1, \Sigma_2 \subseteq C$ are *adjacent* if either $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ or if they are connected by a chain of ghost-components of C.
(3) For two adjacent components Σ_1 and Σ_2 we call $p_i \in \Sigma_i$ their *intersection points* if $p_i \in \Sigma_i$ is the intersection $\Sigma_1 \cap \Sigma_2$ if non-empty, or if $p_i = \Sigma_i \cap B$ for some chain $B \subseteq C$ of ghost-components.

Let $\Sigma_1, \Sigma_2 \subseteq C$ be two non-ghost adjacent components, and let $p_i \in \Sigma_i$ be their intersection points. Let $\widehat{\Sigma}_i$ be the formal completion of Σ_i at p_i and denote by $\widehat{f}_i : \widehat{\Sigma}_i \to X$ the induced morphism.

Definition 2.5. Two adjacent non-ghost components Σ_1 and Σ_2 intersect *properly at their intersection in* X if for $p_i \in \Sigma_i$ and $[\widehat{f}_i, \widehat{\Sigma}_i]$ just mentioned, $\widehat{f}_1^{-1}(\widehat{f}_2(\widehat{\Sigma}_2)) \subseteq \widehat{\Sigma}_1$ is non-trivial and zero-dimensional.

Next, we establish a criterion for a stable map to be rigid. Although we will apply it later only to genus zero stable maps that contain no ghost-components, we prove a more general result, which may be useful in future applications.

Lemma 2.6. *Let X be a non-supersingular K3 surface. Let $[f, C]$ be a genus zero stable map to X. Then $[f, C]$ is rigid if the following conditions hold:*

(1) *for every non-ghost-component $\Sigma \subseteq C$, $f|_\Sigma : \Sigma \to f(\Sigma)$ is birational,*
(2) *ghost-components of $[f, C]$ are disjoint, and every ghost-component contains exactly three nodal points of C, and*
(3) *any two adjacent non-ghost-components intersect properly at their intersection in X.*

Proof. Let $f : \mathcal{C} \to X$ with $\mathcal{C} \to S$ be an S-family of stable maps over a smooth and irreducible curve S. Assume that the special fiber $f_0 : \mathcal{C}_0 \to X$ over $0 \in S$ satisfies the conditions (1) - (3). We will show that f is a constant family of stable maps over an open and dense neighborhood of $0 \in S$.

We denote by $\mathcal{C}_a$, $a \in A$ the irreducible components of $\mathcal{C}$. For a general closed point $s \in S$ and $a \in A$ we consider the fiber $\mathcal{C}_{a,s} := \mathcal{C}_a \times_S s$. Since $\mathcal{C}_{a,0}$ has arithmetic genus 0, each $\mathcal{C}_{a,s}$ is isomorphic to $\mathbb{P}^1$.

We distinguish two cases: first, assume that $f(\mathcal{C}_{a,s})$ is a curve, which is necessarily irreducible. Since the image of f is a rational curve, and X is a non-supersingular K3 surface, the image does not move, i.e., does not depend on s for all $s \neq 0$. We denote this image by R_a. We claim that $f : \mathcal{C}_{a,s} \to R_a$ is birational.

Indeed, since $\mathcal{C}_a$ is proper and flat over S, we conclude $f(\mathcal{C}_{a,0}) = R_a$. We next show that $\mathcal{C}_{a,0}$ contains only one non-ghost-component: let $\Sigma_1, \ldots, \Sigma_r$ be the non-ghost-components of $\mathcal{C}_{a,0}$. We have $R_a = f(\mathcal{C}_a)$ and denote by $\widetilde{R}_a$ its normalization. Then $f : \Sigma_i \to R_a$ lifts uniquely to $h_i : \Sigma_i \to \widetilde{R}_a$. After possibly reindexing, we may suppose that Σ_1 and Σ_2 are adjacent. By condition (3), they intersect properly at $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ under f. Let $q = f(p_1) = f(p_2) \in R_a$, and let $\widehat{R}_a$ be the formal completion of R_a at q. By the proper intersection assumption, the images $h_1(\widehat{\Sigma}_1)$ and $h(\widehat{\Sigma}_2)$ do not lie on the same branch of $\widehat{R}_a$, which shows that $h_1(p_1)$ and $h_2(p_2)$ are distinct.

On the other hand, since S is smooth, $\mathcal{C}_a$ is normal. Thus, $f : \mathcal{C}_a \to R_a$ uniquely lifts to $\widetilde{f}_a : \mathcal{C}_a \to \widetilde{R}_a$. Since $\widetilde{f}_a|_{\Sigma_i}$ coincides with $h_i|_{\Sigma_i}$ at general points of Σ_i, they are identical. Therefore, $h_1(p_1) = \widetilde{f}_a(p_1) = \widetilde{f}_a(p_2) = h_2(p_2)$, which contradicts $h_1(p_1) \neq h_2(p_2)$. This proves that $\mathcal{C}_a$ contains precisely one non-ghost-component. Therefore, by assumption (1), $f : \mathcal{C}_{a,0} \to R_a$ is generically bijective. Since f is flat over S, $f : \mathcal{C}_{a,s} \to R_a$ is also generically bijective, and we conclude birationality.

The second case is when $f|_{\mathcal{C}_{a,s}}$ is a constant map. Since f is flat, $f|_{\mathcal{C}_{a,0}}$ is a constant map, too. By assumption (2), $\mathcal{C}_{a,0}$ is irreducible, and therefore contains exactly three nodal points of $\mathcal{C}_0$. This proves that for general $s \in S$, $\mathcal{C}_{a,s}$ contains three nodes of $\mathcal{C}_a$.

We now show that for a Zariski open and dense subset $U \subseteq S$, $f|_{\mathcal{C}_U}$ is a constant family of stable maps, where $\mathcal{C}_U := \mathcal{C} \times_S U$. By the previous discussion, we find a dense open subset $U \subseteq S$ so that $\mathcal{C}_{a,U} = \mathcal{C}_a \times_S U \cong \mathbb{P}^1 \times U$. We next study the nodal points of $\mathcal{C}_s$. Let $T_{ab} = \mathcal{C}_a \cap \mathcal{C}_b$. Since $\mathcal{C}$ is a family of arithmetic genus zero curves, T_{ab} is either a section of $\mathcal{C} \to S$ or empty. Let $\pi : \mathcal{C} \to S$ be the projection.

Suppose $T_{ab} \neq \emptyset$ and that both, $\mathcal{C}_a$ and $\mathcal{C}_b$, are not families of ghost-components. We set $\mathcal{C}_{ab} := \mathcal{C}_a \cup \mathcal{C}_b \subseteq \mathcal{C}$, and conclude that $(f, \pi) : \mathcal{C}_{ab} \to (R_a \cup R_b) \times S$ is generically finite. We denote by $\mathcal{C}_{ab}^{\mathrm{st}}$ the contraction of the exceptional divisor of $(f, \pi) : \mathcal{C}_{ab} \to X \times S$. Then $\mathcal{C}_{ab}^{\mathrm{st}} = (\mathbb{P}^1 \sqcup \mathbb{P}^1) \times S$, where $\mathbb{P}^1 \sqcup \mathbb{P}^1$ denotes the union of two $\mathbb{P}^1$'s intersecting at one point. Applying assumption (3), we see that the two irreducible components of $\mathcal{C}_{ab}^{\mathrm{st}} \times_S 0$ intersect properly at their intersection in X. By the same argument as in proving that each $\mathcal{C}_a$ contains at most one non-ghost component, we conclude that $f(T_{ab}) \subseteq \mathrm{Sing}(R)$, where $R = \cup_a R_a$ with the reduced structure. Using $\mathcal{C}_{a,U} \cong \widetilde{R}_a \times U$ from above, we see that

$$T_{ab} \times_S U \subseteq \mathcal{C}_{a,U} \cong \widetilde{R}_a \times U$$

is a constant section.

In case $T_{ab} \neq \emptyset$ and $\mathcal{C}_b$ is a family of ghost components, (by assumption (2), $\mathcal{C}_a$ cannot be a family of ghost-components,) there must be a third component $\mathcal{C}_c$ so that $T_{bc} \neq \emptyset$. We set $\mathcal{C}_{ac} := (\mathcal{C}_a \cup \mathcal{C}_c)/\sim$, where $\sim$ means that we identify $T_{ab} \subset \mathcal{C}_a$ with $T_{bc} \subset \mathcal{C}_c$. The morphism f restricted to $\mathcal{C}_a$ and $\mathcal{C}_c$ defines an S-family of stable maps $f_{ac} : \mathcal{C}_{ac} \to X$. By the same arguments as before, we conclude that $T_{ab,U} \subset \mathcal{C}_{a,U} \cong \widetilde{R}_a \times U$ is a constant family.

We conclude that the restricted family $f : \mathcal{C}|_U \to X$ is a constant family of stable maps over U. Finally, for all $s \in U \cup \{0\}$, the restriction of f_s to a non-ghost-component is birational onto its image, and so f_s is tame. Since the moduli space of tame stable maps is separated, the family $f : \mathcal{C} \to X$ is constant over an open and dense neighborhood of $0 \in S$. □

We now come to the main definition of this section, which will be motivated by Theorem 2.9 below.

Definition 2.7. A *rigidifier* is a morphism $f : \mathbb{P}^1 \to X$ to a surface, where

(1) $f : \mathbb{P}^1 \to D := f_*\mathbb{P}^1$ is the normalization morphism,
(2) D is an integral rational curve with only simple nodes as singularities, and
(3) the class $f_*[\mathbb{P}^1]$ is ample.

In particular, $[f, \mathbb{P}^1]$ is a genus zero stable map.

We recall that the moduli space $\mathcal{M}_{2d}$ of polarized K3 surfaces of degree $2d$ exists as separated Deligne-Mumford stack of finite type over $\operatorname{Spec}\mathbb{Z}$, which is even smooth over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2d}]$, see [Riz]. For every integer $d \geq 1$, we define

$$U_{2d} := \left\{ (X, H) \in \mathcal{M}_{2d} \;\middle|\; \begin{array}{l} \text{there exists an } n \in \mathbb{N} \text{ such that } |nH| \text{ contains} \\ \text{a curve that can be represented by a rigidifier} \end{array} \right\}.$$

Before coming to the main result of this section, we establish existence and openness of these stable maps.

Proposition 2.8. *For every $d \geq 1$, the set U_{2d} is Zariski-open and dense in $\mathcal{M}_{2d}$. Moreover, U_{2d} is of finite type over* $\operatorname{Spec}\mathbb{Z}$.

Proof. Non-emptiness of U_{2d} for all $d \geq 1$ follows from [Ch, Theorem 1.2].

Given a family $(\mathcal{X}, \mathcal{H}) \to S$ of polarized K3 surfaces and a rigid stable map of genus zero $f_b : C \to \mathcal{X}_b$ with $f_{b*}C \in |nH_b|$ for some $n \in \mathbb{N}$, it follows from the proof of [BHT, Theorem 19] that the component of $\overline{\mathcal{M}}_0(\mathcal{X}/S, n\mathcal{H})$ that contains $[f_b]$ is proper and surjective over S. (In the analytic setting this follows from [Ra].)

By definition, the image $D \subset X$ of a rigidifier is an irreducible curve with only simple nodes as singularities. Clearly, if X is not unirational, then there is no one-dimensional and non-trivial family of rational curves containing D. But even if X is unirational, which may happen in positive characteristic, the generic member of such a family must have unibranch singularities by [Ta], and thus cannot contain D as special member. We conclude that rigidifiers are rigid stable maps and thus extend over $\mathcal{M}_{2d}$.

The image of a rigidifier stable map is an integral curve, which is an open property. Next, having only simple nodes as singularities is an open property: the arithmetic genus of the image curve is constant in flat families, and so singularities cannot smoothen out. This shows that U is open.

Being open in a Noetherian Deligne-Mumford stack of finite type over $\mathbb{Z}$, also U is of finite type over $\mathbb{Z}$. $\square$

Theorem 2.9. *Let X be a non-supersingular K3 surface over an algebraically closed field. Let $D_1, ..., D_m$ be integral rational curves on X, not necessarily distinct, and let $[f, \mathbb{P}^1]$ be a rigidifier. Then, for some $k \leq m$*

$$D_1 \, + \, ... \, + \, D_m \, + \, k \cdot f_* \mathbb{P}^1$$

has a rigid representative.

Proof. Let us explicitly construct the rigid representative:

First, we choose $q_1, q_2 \in \mathbb{P}^1$ such that q_1, q_2 are distinct points mapping to the same node $f(q_1) = f(q_2)$. Then, we take m copies $[f_i, C_i]$, $i = 1, ..., m$ of $[f, \mathbb{P}^1]$ and construct a new stable map $[\widetilde{f}, \widetilde{C}]$ by connecting $q_1 \in C_i$ to $q_2 \in C_{i+1}$ for all $i = 1, ..., m-1$. We note that $q_1 \in C_i$ and $q_2 \in C_{i+1}$ intersect properly in X.

For all i, we represent D_i via the stable map $[\nu_i, \mathbb{P}^1]$ coming from the normalization $\nu_i : \mathbb{P}^1 \to D_i$. Next, we insert $[\nu_i, \mathbb{P}^1]$ into $[\widetilde{f}, \widetilde{C}]$ by attaching it to $[f_i, C_i]$, as follows: since $f_* \mathbb{P}^1$ is ample, it intersects every D_i. Then, at least one of the following cases is fulfilled, which gives us a recipe to build $\nu_i : \mathbb{P}^1 \to D_i$ into the rigid stable map we want to construct.

(1) Assume that $D_i = \nu_{i,*} \mathbb{P}^1$ is not equal to $f_* \mathbb{P}^1$ and that $\nu_i(\mathbb{P}^1)$ intersects $\nu_i(C_i)$ in $\nu_i(p) = f_i(q)$, say. If $q \in C_i$ is a smooth point of $\widetilde{C}$, then we simply add $[\nu_i, \mathbb{P}^1]$ to $[f_i, C_i]$ by connecting q to p.

(2) Assume that $D_i = \nu_{i,*} \mathbb{P}^1$ is not equal to $f_* \mathbb{P}^1$, and that $\nu_i(\mathbb{P}^1)$ intersects $\nu_i(C_i)$ in $\nu_i(p) = f_i(q)$. But now, assume that $q \in C_i$ is a node of $\widetilde{C}$.

- (a) If $q = C_{i-1} \cap C_i$ then we let q' be the other point of C_i mapping to the node $f_i(q)$. If q' is not a node of $\widetilde{C}$, we add $[\nu_i, \mathbb{P}^1]$ to $[f_i, C_i]$ by connecting q' to p. However, if q' is a node of $\widetilde{C}$ then it connects C_i with C_{i+1}. In this case, we disconnect C_i and C_{i+1}, and connect $[\nu_i, \mathbb{P}^1]$ to $[f_i, C_i]$ by connecting q' to p. Since $f_* \mathbb{P}^1$ is a nodal rational curve of arithmetic genus $p_a = 1 + H^2/2 \geq 2$, it has at least two nodes. We use this second node to connect C_i again to C_{i+1}.
- (b) If $q = C_i \cap C_{i+1}$ then we disconnect C_i and C_{i+1}, connect $[\nu_i, \mathbb{P}^1]$ to $[f_i, C_i]$ by connecting q to p, and use another node of $f_* \mathbb{P}^1$ to connect C_i to C_{i+1}.

(3) Finally, assume that $D_i = \nu_{i,*} \mathbb{P}^1$ is equal to $f_* \mathbb{P}^1$. Then, we simply leave out $[\nu_i, \mathbb{P}^1]$ as it is already included in $[\widetilde{f}, \widetilde{C}]$ via $[f_i, C_i]$.

Inspecting the previous construction, we see that the conditions of Lemma 2.6 are fulfilled. In particular, $[\widetilde{f}, \widetilde{C}]$ is a rigid stable map. Moreover, by construction $\widetilde{f}_* \widetilde{C}$ equals $\sum_{i=1}^m D_i + k \cdot f_* \mathbb{P}^1$ for some $k \leq m$. $\square$

3. Infinitely Many Rational Curves

In this section we prove that complex projective K3 surfaces with odd Picard rank contain infinitely many rational curves, which is our main result. We also establish it for non-supersingular K3 surfaces in characteristic $p \geq 5$.

By [AM], the formal Brauer group of a K3 surface is a smooth and 1-dimensional formal group. In positive characteristic, its height h satisfies $1 \leq h \leq 10$ or $h = \infty$. Surfaces with $h = 1$ are called *ordinary* and this property is open in families of equal characteristic, whereas surfaces with $h = \infty$ are called *supersingular*. If a K3 surface contains a moving family of rational curves, then it is uniruled, and in particular, supersingular.

Theorem 3.1 (Bogomolov-Zarhin, Nygaard-Ogus). *Let X be a K3 surface over a number field K. Then,*

1. *for all but finitely many places $\mathfrak{p}$ of K, the reduction $X_{\mathfrak{p}}$ is smooth,*
2. *there exists a finite extension L/K and a set S of places of L of density 1, such that the reduction $(X_L)_{\mathfrak{q}}$ is ordinary for all $\mathfrak{q} \in S$,*
3. *for all places $\mathfrak{p}$ of characteristic $p \geq 5$, where X has good and non-supersingular reduction, the geometric Picard rank $\rho((X_{\mathfrak{p}})_{\overline{\mathbb{F}}_p})$ is even.*

Proof. The first assertion follows from openness of smoothness. The second statement is shown in [BZ]. The final assertion follows from the Weil conjectures and the results on the Tate conjecture in [NO], see also [BHT, Theorem 15]. □

The following result shows that we can find rational curves of arbitrary high degree when reducing a surface with odd Picard rank modulo p:

Proposition 3.2. *Let (X, H) be a polarized K3 surface over a number field K, such that $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\overline{\mathbb{Q}}})$ and such that the Picard rank is odd. Then, there is a finite extension L/K such that for every $N \geq 0$ there exists a set S_N of places of L of density 1 such that for all $\mathfrak{q} \in S_N$*

1. *the reduction $(X_L)_{\mathfrak{q}}$ is a smooth and non-supersingular K3 surface,*
2. *the reduction $H_{\mathfrak{q}}$ is ample,*
3. *there exists an integral rational curve $D_{\mathfrak{q}}$ on $((X_L)_{\mathfrak{q}})_{\overline{\mathbb{F}}_p}$ such that*
   (a) *the class of $D_{\mathfrak{q}}$ does not lie in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, where we view $\mathrm{Pic}(X)$ as subgroup of $\mathrm{Pic}(((X_L)_{\mathfrak{q}})_{\overline{\mathbb{F}}_p})$ via the specialization homomorphism, and*
   (b) *$D_{\mathfrak{q}} \cdot H_{\mathfrak{q}} \geq N$.*

Proof. By Theorem 3.1, there exists a finite extension L/K and set S of primes of density 1, such that the reduction $(X_L)_{\mathfrak{q}}$ for all $\mathfrak{q} \in S$ is smooth and not supersingular. Since ampleness is an open property, we may assume - after possibly removing a finite number of places from S - that the reduction $H_{\mathfrak{q}}$ is ample.

By Theorem 3.1, we may choose for every $\mathfrak{q} \in S_N$ an invertible sheaf $\mathcal{L}_{\mathfrak{q}}$ in $\mathrm{Pic}(((X_L)_{\mathfrak{q}})_{\overline{\mathbb{F}}_p})$ that does not lie in $\mathrm{Pic}(X_L) \otimes_{\mathbb{Z}} \mathbb{Q}$. Here, we view $\mathrm{Pic}(X) = \mathrm{Pic}(X_L) = \mathrm{Pic}(X_{\overline{\mathbb{Q}}})$ as subgroup of $\mathrm{Pic}(((X_L)_{\mathfrak{q}})_{\overline{\mathbb{F}}_p})$ via specialization. Without loss of generality, these $\mathcal{L}_{\mathfrak{q}}$ are effective, and thus, by Theorem 1.1, every $|\mathcal{L}_{\mathfrak{q}}|$ contains a sum $D_{\mathfrak{q}}$ of rational curves. Passing to an appropriate subdivisor, we may assume that $D_{\mathfrak{q}}$ is a geometrically integral rational curve, whose class does not lie in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Seeking a contradiction, we assume that $D_{\mathfrak{q}} \cdot H_{\mathfrak{q}} < N$ for infinitely many $\mathfrak{q} \in S$. Let $\mathcal{X} \to \mathrm{Spec}\, \mathcal{O}_{L,T}$ be a smooth model of X_L, where $\mathcal{O}_{L,T}$ is the ring of integers of L localized at some set of places T. The scheme

$$\mathrm{Mor}_{<N}(\mathbb{P}^1, \mathcal{X}),$$

which parametrizes morphisms $f : \mathbb{P}^1 \to \mathcal{X}$ with $f_*(\mathbb{P}^1) \cdot H < N$, is of finite type over $\mathbb{Z}$. It has $\overline{\mathbb{F}}_p$-rational points for infinitely many p, corresponding to all the $D_\mathfrak{q}$ with $D_\mathfrak{q} \cdot H_\mathfrak{q} < N$. Thus, it has a $\overline{\mathbb{Q}}$-rational point, and we may even assume that the corresponding rational curve $\widetilde{D}$ on $X_{\overline{\mathbb{Q}}}$ specializes to a $D_{\mathfrak{q}'}$ with $D_{\mathfrak{q}'} \cdot H_{\mathfrak{q}'} < N$. Now, $\widetilde{D}$ gives a class in $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) = \mathrm{Pic}(X)$, whereas, by construction, none of the $D_\mathfrak{q}$ gives a class that lies inside the image of the specialization homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(((X_L)_\mathfrak{q})_{\overline{\mathbb{F}}_p})$, a contradiction. Thus, after removing finitely many places from S we arrive at a set S_N such that $D_\mathfrak{q} \cdot H_\mathfrak{q} \geq N$ for all $\mathfrak{q} \in S_N$. $\square$

After this preparation, we now come to our main result:

Theorem 3.3. *A projective K3 surface X with odd Picard rank over an algebraically closed field of characteristic zero contains infinitely many rational curves.*

Proof. By [BHT, Theorem 3] and Remark 1.5, we may and will assume that X is defined over $\overline{\mathbb{Q}}$. We choose a number field K such that X and every class of $\mathrm{Pic}(X)$ is already defined over K. Then, we replace X by this model over K and choose a polarization H, which has some degree $2d := H^2$, say. Let $\mathcal{M}_{2d}$ be the corresponding moduli space of polarized K3 surfaces, which exists as a separated Deligne-Mumford stack over $\mathbb{Z}$ by [Riz].

We choose an arbitrary positive integer N. Our theorem follows, if we find an integral rational curve $D \subset X_{\overline{\mathbb{Q}}}$ with $D \cdot H \geq N$.

Let L/K and S_N be as in Proposition 3.2 and replace X by X_L. After possibly removing a finite set of places from S_N, we find a model $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_{L,S_N}$ of X over the ring of integers of L localized at S_N. After passing to a Zariski-open subset of $\mathcal{M}_{2d}$ and taking an appropriate finite and étale cover, we arrive at a polarized family of K3 surfaces $(\mathcal{Y}, \mathcal{H}) \to U$ containing $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_{L,S_N}$, and where U is a scheme of finite type over $\mathbb{Z}$. We note that there exists a proper algebraic stack of stable maps $\overline{\mathcal{M}}_0(\mathcal{Y}/U) \to U$, see [AV]. (The reason for passing to the scheme U rather than working with $\mathcal{M}_{2d}$ is to avoid certain technicalities when working with moduli of stable maps over a base that is a Deligne-Mumford stack, see [AV, Section (1.3)].)

Let $f_\mathfrak{q} : \mathbb{P}^1 \to \mathcal{X}_\mathfrak{q}$ be the stable map representing $D_\mathfrak{q}$, which is rigid, as $X_\mathfrak{q}$ is not supersingular. Thus, the relative dimension of $\overline{\mathcal{M}}_0(\mathcal{Y}/U) \to U$ at $[f_\mathfrak{q}]$ is zero. We choose a component of $\overline{\mathcal{M}}_0(\mathcal{Y}/U)$ through $[f_\mathfrak{q}]$ and denote its image in U by $B_\mathfrak{q}$. Deformation theory of morphisms implies that $\dim B_\mathfrak{q} \geq \dim U - 1$. However, the invertible sheaf $\mathcal{O}_{X_\mathfrak{q}}(D_\mathfrak{q})$ cannot extend over all of U, which implies that $B_\mathfrak{q}$ is a divisor in U. To keep things simpler, we replace $B_\mathfrak{q}$ by an irreducible divisor through $X_\mathfrak{q}$. If $B_\mathfrak{q}$ were not flat over $\mathbb{Z}$, it would be contained completely in characteristic p, in which case $\mathcal{O}_{X_\mathfrak{q}}(D_\mathfrak{q})$ would extend to $\mathcal{M}_{2d} \otimes_\mathbb{Z} \mathbb{F}_p$. However, the formal divisor inside the formal deformation space $\mathrm{Spf}\, W(k)[[x_1, ..., x_{20}]]$ of $X_\mathfrak{q}$ along which $\mathcal{O}_{X_\mathfrak{q}}(D_\mathfrak{q})$ extends, is flat over $W(k)$ by [Del, Corollaire 1.8]. This implies that $\mathcal{O}_{X_\mathfrak{q}}(D_\mathfrak{q})$ cannot extend over $\mathcal{M}_{2d} \otimes_\mathbb{Z} \mathbb{F}_p$. Thus, $B_\mathfrak{q}$ is a divisor in U that is flat over $\mathbb{Z}$.

We claim that for every $\mathfrak{q} \in S_N$, there are only finitely many $\mathfrak{q}' \in S_N$ such that $B_\mathfrak{q} = B_{\mathfrak{q}'}$: if not, $\mathcal{X} \to \mathrm{Spec}\,\mathcal{O}_{L,S_N}$ would specialize into $B_\mathfrak{q}$ for infinitely many places $\mathfrak{q} \in S_N$. This would imply that also the generic fiber X is a point of $B_\mathfrak{q}$, i.e., $D_\mathfrak{q}$ and the invertible sheaf $\mathcal{O}_{X_\mathfrak{q}}(D_\mathfrak{q})$ would lift from $X_\mathfrak{q}$ to X, a contradiction. Thus, the $B_{\mathfrak{q}}$'s form a set with infinitely many distinct divisors in U. By the openness result Proposition 2.8, almost all of these $B_{\mathfrak{q}}$'s contain surfaces with rigidifiers in some multiple of their polarization. Let $\mathfrak{q} \in S_N$ be such a place.

The curve $D_{\mathfrak{q}} \subset X_{\mathfrak{q}}$ extends to a rational curve along $B_{\mathfrak{q}}$ and on an open dense subset this extension will be an integral curve. Let Z be a non-supersingular K3 surface on $B_{\mathfrak{q}}$ such that $D_{\mathfrak{q}}$ extends to some integral rational curve D on Z and such that Z contains a rigidifier $\mathbb{P}^1 \to R \subset Z$ with $R \in |rH|$ for some $r \in \mathbb{N}$. Next, for a sufficiently large integer m, the linear system $|mH - D|$ is effective. By Theorem 1.1, there exist integral rational curves D_i, such that $\sum_i D_i$ lies in $|mH - D|$. By Theorem 2.9, there exists an integer k and a *rigid* stable map $[f_Z] \in \overline{\mathcal{M}}_0(Z, (m+kr)H)$ representing $D + \sum_i D_i + kR$.

Now, $\overline{\mathcal{M}}_0(\mathcal{Y}/U, (m+kr)\mathcal{H})$ is at least 19-dimensional, as shown in the proof of [BHT, Theorem 19]. It is proper over U, which is also 19-dimensional. Since the fiber above $Z \in U$ at $[f_Z]$ is zero-dimensional, we can extend $[f_Z]$ over the whole of U.

There exists a component $\mathcal{M}$ of $\overline{\mathcal{M}}_0(\mathcal{Y}/U, (m+kr)\mathcal{H}) \to U$ containing $[f_Z]$. Also, there exists a connected family $[f_t]$ of stable maps in $\mathcal{M}$, containing $[f_Z]$ and whose limit $f_{X_{\mathfrak{q}}} : C_{\mathfrak{q}} \to X_{\mathfrak{q}}$ over $X_{\mathfrak{q}} \in U$ contains $D_{\mathfrak{q}}$ in its image. We pass to the Stein factorization of $\overline{\mathcal{M}}_0(\mathcal{Y}/U, (m+kr)\mathcal{H}) \to U$, and let $\mathcal{M}'$ be the component that contains this family $[f_t]$. Then, we choose some point $[f_X] \in \mathcal{M}'$ lying above the surface X. The corresponding stable map $f_X : C_X \to X$ specializes to some stable map on $X_{\mathfrak{q}}$. Now, since $\mathcal{M}'$ has connected fibers, the specialization of $[f_X]$ modulo $\mathfrak{q}$ is a deformation of $[f_{X_{\mathfrak{q}}}]$. But $X_{\mathfrak{q}}$ is not supersingular, and so the two stable maps have the same image curve. In particular, the image $f_X(C_X)$ contains an integral rational curve $\widetilde{D}$ that contains $D_{\mathfrak{q}}$ in its specialization. We compute $\widetilde{D} \cdot H \geq D_{\mathfrak{q}} \cdot H_{\mathfrak{q}} \geq N$, which establishes existence of an integral rational curve of degree $\geq N$. $\square$

We finish with a characteristic p version of Theorem 3.3 for polarized K3 surfaces.

Theorem 3.4. *A non-supersingular K3 surface with odd Picard rank over an algebraically closed field of characteristic $p \geq 5$ contains infinitely many rational curves.*

Proof. As explained in [BHT, Theorem 15], X cannot be defined over a finite field. Thus, we may assume that X is defined over the function field of a variety B with $\dim B \geq 1$ over some finite field $\mathbb{F}_q \supseteq \mathbb{F}_p$. After possibly shrinking B, we find a smooth fibration of K3 surfaces $\mathcal{X} \to B$ with generic fiber X. By closedness of supersingularity [Ar1], we may, after possibly shrinking B further, assume that no fiber is supersingular.

By [BHT, Theorem 15], the Picard rank of $(\mathcal{X}_{\mathfrak{q}})_{\overline{\mathbb{F}}_p}$ for every closed point $\mathfrak{q}$ of B is even. Thus, given N, a straight forward adaption of Proposition 3.2 shows that for almost all closed points $\mathfrak{q} \in B$ there exists an integral rational curve $D_{\mathfrak{q}} \subset (\mathcal{X}_{\mathfrak{q}})_{\overline{\mathbb{F}}_p}$ with $D_{\mathfrak{q}} \cdot H \geq N$.

From here we argue as in the proof of Theorem 3.3 to produce an integral rational curve D with $D \cdot H \geq N$ on X. We leave the details to the reader. $\square$

Remark 3.5. Let us comment on the assumptions:

(1) According to conjectures of Artin, Mazur and Tate, supersingular K3 surfaces should satisfy $\rho = b_2 = 22$, see, for example, [Ar1]. In particular, there should exist no supersingular K3 surfaces with odd Picard rank.
(2) Using [Ny] in the proof of Proposition 3.2, and openness of ordinarity in equal characteristic, we see that Theorem 3.4 also holds for ordinary K3 surfaces with odd Picard rank in characteristic $p = 2, 3$.

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