

An elementary construction of Khovanov-Rozansky type link homology

KENJI ARAGANE

Abstract

In this article, we give an elementary construction of $sl(n)$ -homological invariants of links presented by braid forms. The Euler characteristic of this complex is equal to $sl(n)$ quantum polynomial invariant of link.

1 Introduction

We present here a new method of categorifying quantum polynomial invariants.

M. Khovanov and L. Rozansky defined homological invariants of links whose graded Euler classes are quantum polynomial invariants. They used the notion of matrix factorization and resolutions of diagrams. In this paper, we will define another homology theory which is similar to the Khovanov-Rozansky theory and related to the computational method of quantum polynomial invariant.

In Khovanov-Rozansky theory, graded vector spaces are associated to oriented trivalent graphs. We will construct a graded module whose generators are sets of colorings on graphs. These colored graphs have representation theoretic meanings. One can easily confirm that if we take a field coefficient, our graded vector space is isomorphic to the one Khovanov and Rozansky defined using the matrix factorization. So, our complex is isomorphic to the Khovanov-Rozansky complex as vector spaces. Similar relations have been already discussed and developed. Our theory can be defined for the integer coefficient though Khovanov-Rozansky theory are defined for fields of characteristic zero. We first review quantum polynomial link invariants associated to the vector representation of $U_q(sl(n))$ and its graphical calculus. Next,

we introduce the graded module and construct complexes. We prove the invariance of its homology under the Reidemeister moves.

Our method may suggest the importance of representation theory of quantum groups in categorical link theory and the possibility of categorification of quantum polynomial invariants associated with other representations of quantum groups.

2 Review

2.1 Quantum invariants

In this chapter we review the state sum construction of the quantum $sl(n)$ link invariant associated with the vector representation.

Definition 1. *Let L be an oriented link in S^3 and D be an oriented link diagram of L . A resolution of the diagram D means a locally oriented trivalent graph which is obtained by resolutions of all crossings of D by either one of the two ways depicted in Fig. 1. \square*

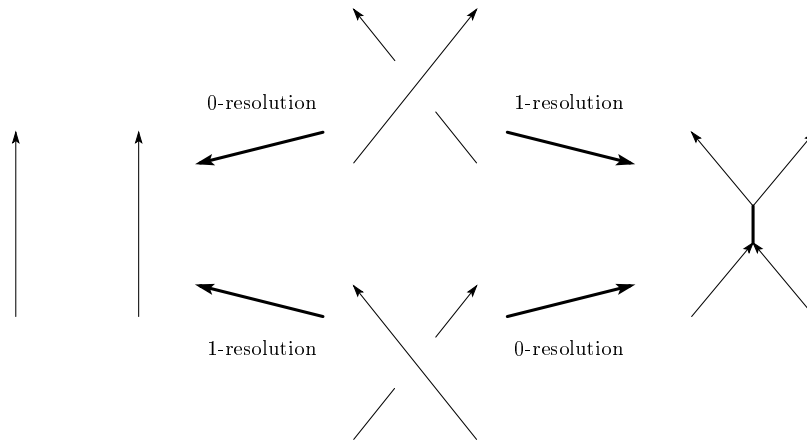


Figure 1: 0, 1-resolutions of crossings

We call the wide edge obtained by the 1-resolution of positive crossing and the 0-resolution of negative crossing the singular edge. Edges of a resolution of the diagram D except singular edges will be called the normal edges. We

call the two edges coming to the singular edge the legs and two edges going out the singular edge the heads.

Fix a nonnegative integer n and denote by N the set of n elements $\{1, \dots, n\}$. For two different elements a and b of N we define a number $\pi(a, b) := 1$ if $a > b$, $\pi(a, b) := 0$ if $a < b$. Let G be an oriented, trivalent, planar graph obtained by the resolution of D . A state σ is an assignment of an element of N to each normal edge e . It should satisfy the conditions that for each singular edge the set of the elements of N attached to the legs of the singular edge is equal to the set of the elements of N attached to the heads of the singular edge and the elements attached to the heads (or legs) of singular edge must be different from each other.

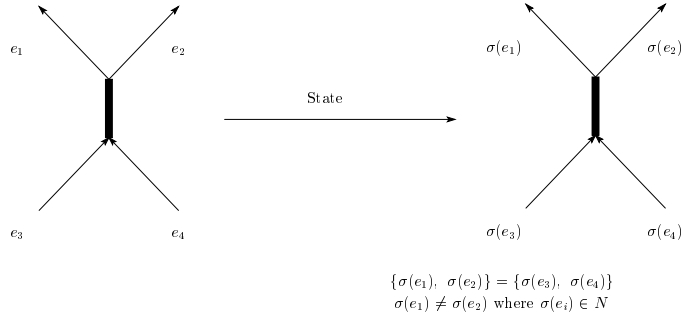


Figure 2: Assign an element of N to each normal edge

Given a state σ of G , we define the weight $\text{wt}(v, \sigma)$ of a vertex v of a singular edge to be

$$(1) \quad \text{wt}(v, \sigma) = q^{1/2 - \pi(\sigma(e_1), \sigma(e_2))},$$

where q is an indeterminate, and e_1 and e_2 are the left and right legs (resp. heads) respectively with respect to the orientation of G .

Let E be a singular edge of G and v_1, v_2 be vertices of E as in the figure 4. We define a weight of a singular edge as $\text{wt}(E) := \text{wt}(v_1)\text{wt}(v_2)$. The singular edge can take three values according to the state around the singular edge.

$$(2) \quad \text{wt}(E) = \begin{cases} q & \sigma(e_1) = \sigma(e_3) < \sigma(e_2) = \sigma(e_4), \\ 1 & \sigma(e_1) = \sigma(e_4), \sigma(e_2) = \sigma(e_3), \\ q^{-1} & \sigma(e_1) = \sigma(e_3) > \sigma(e_2) = \sigma(e_4). \end{cases}$$

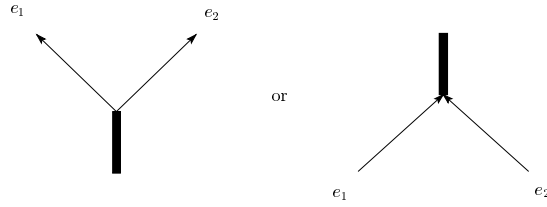


Figure 3: Leg and Head of a vertex

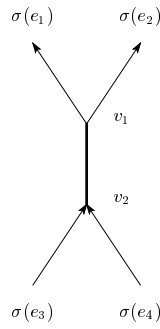


Figure 4: Singular edge and vertices

If we delete every singular edge of G and identify the heads and the legs of the singular edges which were connected by singular edges (in other words, collapse the singular edges to points), we obtain an union of oriented closed curves each of which is equipped with the element of N . Then we define the rotation number of the state to be

$$\text{rot}(\sigma) = \sum_c (2\sigma(C) - n - 1)\text{rot}(C)$$

where the sum is over all closed curves C equipped with $\sigma(C) \in N$ and $\text{rot}(C)$ is the rotation number of C . (It is 1 if C is counterclockwise and -1 otherwise.) Now we define a Laurent polynomial $\langle G \rangle_n$ as follows.

$$\langle G \rangle_n = \sum_{\text{states } \sigma} \left\{ \prod_{\text{vertices } v} \text{wt}(v, \sigma) q^{\text{rot}(\sigma)} \right\}$$

Denote by Cr the set of crossings of D . Given a crossing of a diagram, we can resolve it in two possible ways. A *resolution* of D is a resolution

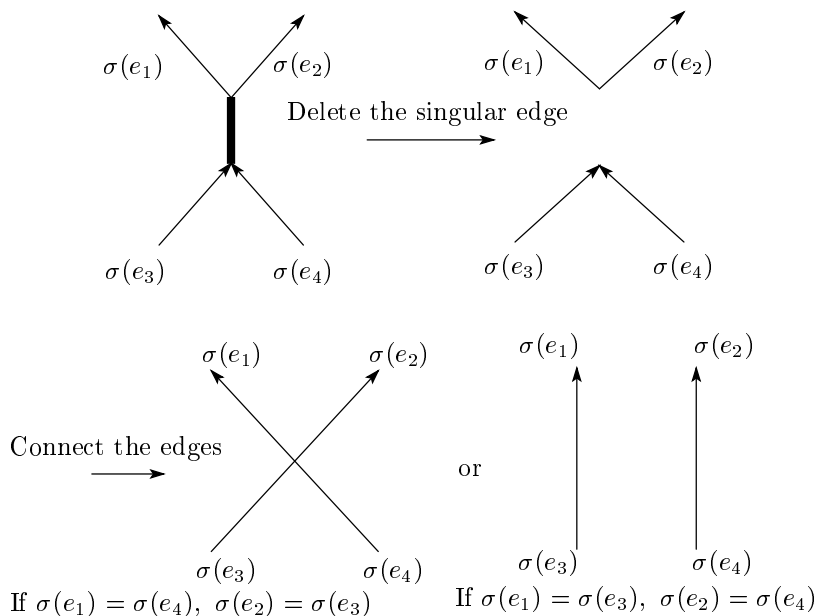


Figure 5: Delete and connect the edge

of each crossing of D . Thus, D admits $2^{\#\{Cr\}}$ resolutions. There is a one-to-one correspondence between resolutions of D and subsets of the set Cr of crossings. Namely, to $cr \subset Cr$ we associate a resolution of D , denoted $D(cr)$, by taking a 1-resolution of each crossing that belongs to $cr \subset Cr$ and a 0-resolution if the crossing does not lie in cr . Let N_0^+, N_1^+ be the number of 0-resolutions, 1-resolutions of positive crossings respectively, and N_0^-, N_1^- be the number of 0-resolutions, 1-resolutions of negative crossings respectively. We denote the writhe number of D as $wr(D)$. Following these notations, we can state the theorem [5]. These definitions are just based on the representation theory of quantum groups.

Theorem 2.1 (Murakami, Ohtsuki, Yamada). *Let L be an oriented link and D be its diagram. Associate a polynomial to D as follows*

$$\langle D \rangle_n := q^{n(-wr(D))} \sum_{cr \subset Cr} (-1)^{\#\{cr\}} q^{N_0^+ - N_1^-} \langle D(cr) \rangle_n,$$

then $\langle D \rangle_n$ is invariant under the Reidemeister moves I, II, and III.

3 Homology

3.1 Cube structure

We review the cube structure according to [1]. Let A be a finite set. Denote by $r(A)$ the set of all pairs (B, a) where B is a subset of A and a an element of A that does not belong to B . To simplify notation we often denote

- (a) a one element set $\{a\}$ by a ,
- (b) a finite set $\{a, b \cdots c\}$ by $ab \cdots c$,
- (c) the disjoint union $A \sqcup B$ of two sets A, B by AB ; for example, we denote by Aa the disjoint union of a set A and a one element set $\{a\}$; similarly, Aab means $A \sqcup \{a\} \sqcup \{b\}$, and so on.

Definition 2. Let A be a finite set and C an additive category. A commutative A -cube V over C is a collection of objects $V(X) \in \text{Ob}(C)$ for each subset X of A and morphisms for each $(X, a) \in r(A)$,

$$\xi_a^V(X) : V(X) \longrightarrow V(Xa)$$

such that for each triple (X, a, b) , where X is a subset of A and $a, b, a \neq b$ are two elements of A that do not lie in X , there is an equality of morphisms $\xi_b^V(Xa)\xi_a^V(X) = \xi_a^V(Xb)\xi_b^V(X)$

We say a commutative A -cube is an A -cube. Maps ξ_a^V are called structure maps of V .

Definition 3. Let A be a finite set and C an additive category. A skew-commutative A -cube V over C is a collection of objects $V(X) \in \text{Ob}(C)$ for each subset X of A , and morphisms

$$\xi_a^V(X) : V(X) \longrightarrow V(Xa)$$

such that for each triple (X, a, b) , where X is a subset of A and $a, b, a \neq b$ are two elements of A that do not lie in X , there is an equality of morphisms $\xi_b^V(Xa)\xi_a^V(X) + \xi_a^V(Xb)\xi_b^V(X) = 0$.

Let R be an arbitrary commutative ring and we will consider only cube structure over an additive category $R - \text{mod}$ unless stated otherwise. Given

A -cubes or skew A -cubes V and W over R -mod, their tensor product is defined to be an A -cube (if V and W are both cubes or skew cubes) or a skew cube (if one of V , W is a cube and the other is a skew cube), denoted $V \otimes W$, given by

$$(V \otimes W)(X) = V(X) \otimes W(X)$$

$$\xi_a^{V \otimes W}(X) = \xi_a^V(X) \otimes \xi_a^W(X)$$

where tensor products are taken over R .

For a finite set \mathcal{L} , denote by $o(\mathcal{L})$ the set of complete orderings or elements of \mathcal{L} . For $x, y \in o(\mathcal{L})$ let $p(x, y)$ be the parity function. $p(x, y) = 0$ if y can be obtained by x via an even number of transpositions of two neighboring elements in the ordering, otherwise, $p(x, y) = 1$. To a finite set \mathcal{L} , associate a R -module $E(\mathcal{L})$ defined as the quotient of the R -module, freely generated by elements x for all $x \in o(\mathcal{L})$, by relations $x = (-1)^{p(x, y)}y$ for all pairs $x, y \in o(\mathcal{L})$. The module $E(\mathcal{L})$ is a free R -module of rank 1. For $a \notin \mathcal{L}$ there is a canonical isomorphism of graded R -modules $E(\mathcal{L}) \rightarrow E(\mathcal{L}a)$ induced by the map $o(\mathcal{L}) \rightarrow o(\mathcal{L}a)$ that takes $x \in o(\mathcal{L})$ to $xa \in o(\mathcal{L}a)$. Moreover, for $a, b, a \neq b$, the diagram below anticommutes

$$\begin{array}{ccc} E(\mathcal{L}) & \longrightarrow & E(\mathcal{L}a) \\ \downarrow & & \downarrow \\ E(\mathcal{L}b) & \longrightarrow & E(\mathcal{L}ab) \end{array}$$

Denote by E_I the skew I -cube with $E_I(\mathcal{L}) = E(\mathcal{L})$ for $\mathcal{L} \subset I$ and the structure map $E_I(\mathcal{L}) \rightarrow E_I(\mathcal{L}a)$ being canonical isomorphism $E(\mathcal{L}) \rightarrow E(\mathcal{L}a)$.

We will use E_I to pass from I -cubes over R -mod to skew I -cubes over R -mod by tensoring an I -cube with E_I .

Let V be a skew I -cube over an abelian category C . To V we associate a complex $C(V) = (C^i(V), d^i)$, $i \in Z$ of objects of C by

$$(3) \quad C^i(V) = \bigoplus_{\mathcal{L} \subset I, |\mathcal{L}|=i} V(\mathcal{L})$$

The differential $d^i : C^i(V) \rightarrow C^{i+1}(V)$ is given on an element $x \in V(\mathcal{L})$, $|\mathcal{L}| = i$ by

$$(4) \quad d^i(x) = \sum_{a \in I \setminus \mathcal{L}} \xi_a^V(\mathcal{L})x.$$

In practice we shall habitually drop $E(\mathcal{L})$ from the notations of skew cube

and cube complex which are derived from commutative cube because they are only a choice of sign.

3.2 Colored graph spaces and constructions of structure morphisms

We give a definition of our chain complex of link diagram using cube structure. From now on, we shall always assume that L means a link and D mean a link diagram of the closure of a clockwise oriented braid representing L . By restricting our attention to braid closure diagram, we will prove the invariance under the Markov-moves instead of the Reidemeister-moves.

Definition 4. *Let G be an oriented trivalent graph obtained by a resolution of a link diagram. We define a graded \mathbb{Z} -module $C(G)$ as follows. As a generator of $C(G)$ we take all the states of graph G and call them as colored graphs. Colors of edges are represented by integers from 1 to n . Fix a colored graph $\sigma \in C(G)$, we will define a grade of a state around a singular edge as follows. Let E be a singular edge of G as in figure 4.*

$$(5) \quad \deg(E) = \begin{cases} 1 & \sigma(e_1) = \sigma(e_3) < \sigma(e_2) = \sigma(e_4), \\ 0 & \sigma(e_1) = \sigma(e_4), \sigma(e_2) = \sigma(e_3), \\ -1 & \sigma(e_1) = \sigma(e_3) > \sigma(e_2) = \sigma(e_4). \end{cases}$$

Similarly, we define a grade of parallel edges obtained by 0-resolution of positive crossing as 1 and also define the degree of parallel edges obtained by 1-resolution of negative crossing as -1 .

We define a grading of a colored graph by

$$\text{gr}(\sigma) := N_0^+ - N_1^- - \text{wr}(D) \cdot n + \sum_{\text{singular edges } E} \deg(E) + \sum_{\text{curves } C} (2\sigma(C) - n - 1)\text{rot}(C)$$

where the sums are over all singular edges in G and curves as in the definition of the rotation number of the state, N_0^+ , N_1^- are the number of 0-resolutions of positive crossings, 1-resolutions of negative crossings respectively. We call this graded module $C(G)$ as colored graph space associated to G .

We will depict the colored graph around the singular edge whose grading is 0 as in the left hand side of the figure 6 and the colored graph whose grading is not 0 as in the right hand side of figure 6. In the figure 6, we

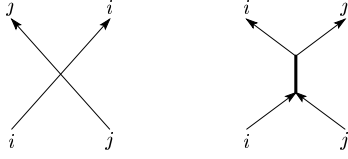


Figure 6: States around the singular edge whose grade=0 or $\neq 0$

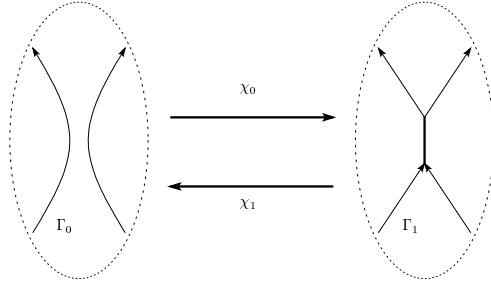


Figure 7: Morphisms between Γ^0 and Γ^1

depict the singular edge whose grade is equal to 0 as a simple crossing but we will treat as an singular edge.

We will associate a Cr -cube to D as follows. Given a crossing of a diagram D , we can resolve it in two possible ways as in Section2. A *resolution* of D is a resolution of each crossing of D . Thus, D admits $2^{\#\{Cr\}}$ -resolutions. There is a one-to-one correspondence between resolutions of D and subsets cr of the set of crossings Cr . Namely, to $cr \subset Cr$ we associate a resolution of D , denoted $D(cr)$, by taking a 1-resolution of each crossing that belongs to cr and a 0-resolution if the crossing does not lie in cr and assign the colored graph space $C(D(cr))$ to $D(cr)$.

Take a crossing a of D and a subset A of Cr which does not contain a . Resolve crossings of D by the 1-resolutions of the crossings A and the 0-resolutions of the complement of A except a . If we take a resolution of a which replace the crossing a to parallel normal edges, we will denote this resolution of D as Γ^0 . If we take a resolution of a which replace the crossing a to a singular edge, we will denote this resolution of D as Γ^1 . The difference between Γ^0 and Γ^1 is depicted in the figure 7. To define the cube structure, we must define morphisms between them. We define degree 0 morphisms χ_0 , χ_1 between colored graph spaces as follows. Let E be a singular edge of Γ^1

depicted in the figure7. We will call colorings of singular edges as positive type (resp. negative) if the degree of the singular edge is equal to 1 (resp. -1). We define the morphisms χ_0, χ_1 as follows.

First we will define χ_1 morphism. χ_1 morphisms are morphisms from 0-resolution to 1-resolution of a negative crossing. Take a colored graph $\sigma \in C(\Gamma^1)$ and depict σ as a colored oriented trivalent planar graph. If we delete all the singular edges of σ and identify the vertices of heads and legs of same singular edge as in the Fig. 8 , we will obtain a new colored oriented

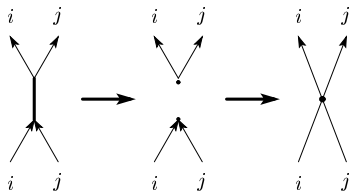


Figure 8: Delete the singular edge and identify the vertices

graph. We will call this new graph as colored circles of σ and consider them as simple closed oriented curves with colorings. We can also regard the colored circles as the union of oriented colored normal edges of σ because we deleted only the singular edges of σ . We will say that a colored circle C goes through a singular edge S (or a singular edge S is on a colored circle C), if a normal edge of the colored circle is the head or the leg of the singular edge S .

Take a colored circle C and singular edges S, S' on C . We can take an oriented path on C which is outgoing from S and incoming to S' . We will call this subgraph of C as colored path from S to S' and call these singular edges a source singular edge S and a target singular edge S' . We can regard a colored path on C as a subset of normal edges of C . (We do not assume $S \neq S'$. If $S = S'$, the colored path from S to S' is equal to the colored circle itself.)

If there exist two distinct colored paths which start from the same singular edge and end at the same singular edge, they consist a circle and can be seen as an oriented colored planar subgraph. If this subgraph does not have self-intersections and does not have a path which crosses this subgraph whose intersection points with this subgraph are positive singular edges or negative singular edges, we call them as distinguished circle. By definition, it is trivial that the normal edges of distinguished circle have just 2 distinct

colorings. Exchanging the colorings of colored paths of distinguished circle will introduce a new distinguished circle and a new colored oriented graph. We will call this coloring exchanging procedure as exchanging colorings of distinguished circle (or short, color exchange).

Suppose the singular edge E of Γ^1 in the Fig. 7 has degree= 1 and there exist distinguished circles which contain the leg or the head of E and its source and target singular edges are not positive types. If color exchanging of the distinguished circle shifts the degree of the source and the target vertices +1 and does not change the degrees of singular edges on the distinguished circle except 3 singular edges "the source singular edge" "the target singular edge" and E , we will exchange the colorings of colored paths of the distinguished circle then replace the singular edge E to the parallel normal edges as in the Fig. 11. We can consider it as an element of $C(\Gamma^0)$.

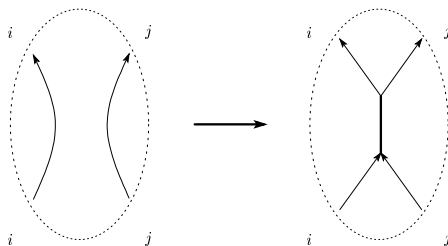


Figure 9: Replace

We will modify and add other term as the special case. We must add and consider exceptional terms to χ_1 which is related to the Reidemeister1-move. If the normal edge which is the head or leg of E does not intersect with other normal edges (i.e., the head and the leg of singular edge E consist a simple loop), delete the singular edge and replace to parallel normal edges and change the coloring of simple closed curve as to preserve the grade as in the Fig. 10.

If the subgraph does not permit these procedure, we send them to 0. If there are some possibilities of taking distinguished circles and exchanging the colorings, sum up all of them. If there are no possibilities of taking distinguished circles and exchanging the colorings which satisfies the conditions, we will send to 0.

In the case when the singular edge E has degree= -1 , replace the singular edge E to parallel normal edges and we can naturally consider colorings of

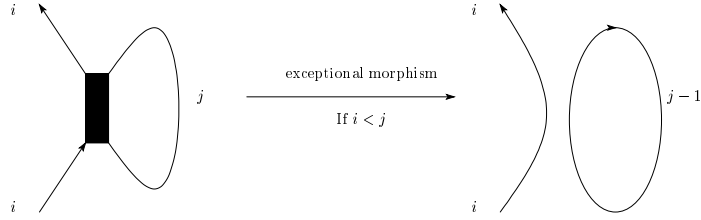


Figure 10: Exceptional term

them as in the Fig. 11. We will consider them as an element of $C(\Gamma^0)$. We will denote it as $\tilde{\sigma}$ and consider as an element of $C(\Gamma^0)$. The relation between σ and $\tilde{\sigma}$ is $\text{gr}(\sigma) = \text{gr}(\tilde{\sigma})$.

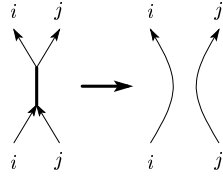


Figure 11: Replace the singular edge E

If the singular edge E has degree = 0, take two colored paths outgoing from E and incoming to the same singular edge (or incoming to E and outgoing from the same singular edge). If the degrees of the singular edges on the colored paths except the source and target vertices are not changed by exchanging the colorings of these colored paths, exchange the colorings of the colored paths. These conditions implies that the source or the target singular edge which is not E must not be positive type and the degree of its singular edge must be shifted +1, and the degree of the singular edge E is changed to -1 . So we can replace the singular edge E to the parallel normal edges and consider it as an element of $C(\Gamma^0)$. We denote it as $\sigma' \in C(\Gamma^0)$. We only permit color exchanging which satisfies $\text{gr}(\sigma) = \text{gr}(\sigma')$. If there are some possibilities of exchanging the colorings, sum up all of them. If there are no possibilities of changing which satisfy the conditions, we consider the value as 0.

In the χ_0 case, we would like to construct a morphism $C(\Gamma^0) \longrightarrow C(\Gamma^1)$

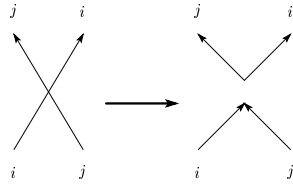


Figure 12: Delete the singular edge whose grade=0

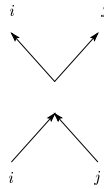


Figure 13: Changing must be these type

which is similar to χ_1 . Fix an element $\sigma \in C(\Gamma^0)$ and consider σ as colored oriented trivalent planar graph.

Assume that the parallel normal edges of σ have colorings $i \geq j$ depicted as in the Fig. 15 and put a dotted circle around the parallel normal edges. This dotted circle can be seen as a dotted line as in the left-hand side of the figure {fig. We may treat a dotted circle like a singular edge. Take two colored paths which start from same singular edge (or the dotted circle around the parallel edges) and end at same singular edge (or the dotted circle around the parallel normal edges). We assume that one of the colored paths contains the normal edge of parallel edges colored i , or j . These two path can be seen as a colored subgraph of σ and we impose them not to have

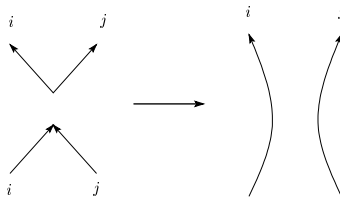


Figure 14: Parallelize the edges which were connected to E

self-intersections and not to have a path which crosses this subgraph whose intersection points with this subgraph are positive singular edges or negative singular edges. So paths which traverse the subgraph are allowed to have only degree zero intersection points. We also call them as distinguished subgraph. (By considering the dotted circle as singular edge, a distinguished subgraph can be seen as a distinguished circle which is used in the χ_1 case.) Exchanging colorings of colored paths of a distinguished subgraph will introduce a new distinguished subgraph and a new colored oriented graph.

Take a distinguished circle of σ whose source and terminal singular edges are not positive. If exchanging colorings of the distinguished subgraph does not change the degrees of the singular edges on the distinguished subgraph except the source and target singular edges (or the dotted circle as source or target singular edge of distinguished subgraph), we will exchange the colorings of colored paths of the distinguished subgraph and replace the parallel normal edges to the singular normal edge E by the natural way as in the Fig. 11. We can consider it as an element of $C(\Gamma^1)$. We will modify and add other term as the special case. We must add and consider exceptional terms to χ_0 which is related to the Reidemeister 1-move. If the normal edges of parallel edges colored i or j consist a simple loop, replace the parallel normal edges to singular edges and change the coloring of simple closed curve as to preserve the grade as in the χ_1 case.

If the subgraph does not permit these procedure (or there are no such subgraph satisfying the conditions), we send them to 0. If there are some possibilities of taking distinguished subgraphs and exchanging the colorings, sum up all of them.

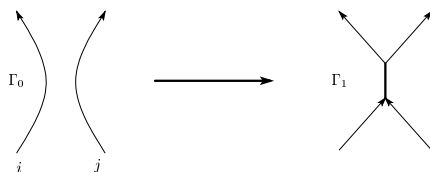


Figure 15: χ_0 morphism

Suppose the parallel edges of σ colored i , j depicted in the Fig. 15 has a relation $i < j$, replace the parallel normal edges to a singular edge E and naturally extend its colorings. This singular edge E has the degree 1 and we will consider them as an element of $C(\Gamma^1)$.

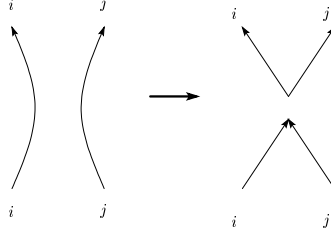


Figure 16: Replace the parallel edges obtained by the 0-resolution

So, χ_0 and χ_1 change some gradings of singular edges and colorings of normal edges. They are degree= 0 morphisms by definitions.

Proposition 1. *The colored modules and the modules $(C(D(cr)), \chi_0, \chi_1)$ defined above admit the cube structure.*

Proof. To prove they have the cube structure, it is sufficient to confirm that the structure morphisms commute. Let cr be a subset of Cr , and take two distinct elements $a \neq b \in (Cr \setminus cr)$. We denote the structure morphisms as follows.

$$\begin{array}{ccc}
 C(D(cr)) & \xrightarrow{\chi_a} & C(D(cr \sqcup a)) \\
 \chi_b \downarrow & & \downarrow \chi_{ab} \\
 C(D(cr \sqcup b)) & \xrightarrow{\chi_{ba}} & C(D(cr \sqcup a \sqcup b))
 \end{array}$$

Let a and b be negative crossings and E_a and E_b be the singular edges appearing in the 0-resolutions of a and b . Then the structure morphisms χ_a , χ_b , χ_{ab} and χ_{ba} are all χ_1 type morphisms. If the distinguished circles of E_a and E_b do not intersect each other, χ_a and χ_b do not affect to each other because the χ_1 -type structure morphism changes only the local colorings of edges of distinguished colored circles or changes the colorings of simple loops. If the singular edges E_a and E_b have degree= -1 , the morphisms χ_a and χ_b are trivial. Thus in these cases, $\chi_{ab} \circ \chi_a = \chi_{ba} \circ \chi_b$. So, we assume that the distinguished circles have intersections.

By the definitions and the assumptions of the distinguished circle and the structure morphism, the distinguished circles can have common singular edges. But it is assumed that the structure morphism does not change the degree of singular edges except source and target singular edges and preserve

the degree. If the source or the target singular edge of the distinguished circle of a is coincide to the singular edge E_b , they have common distinguished circle and the structure morphisms also commute. Therefore exchanging procedures do not depend on the order. Thus, they commute to each other. So we can conclude that the compositions are commutative for χ_1 -type morphisms. Suppose a, b be positive crossings and E_a, E_b be the singular edges appearing in the 1-resolutions of a, b . In this case, we also exchange colorings of distinguished subgraph but this procedure is also commutative by the definitions and the assumptions of the distinguished subgraph and structure morphism. So, in this case, It is similarly confirmed that the structure morphisms commutes. The other cases can be checked by similar way. So, we can conclude that the structure morphisms commute to each other. \square

Let k be an integer. We denote by $\{k\}$ the grading shift up by k . (i.e. Let $\sigma \in C(G)$ be a colored graph and $\text{gr}(\sigma) = n$, then $\sigma\{k\} \in C(G)\{k\}$ has the grading $\text{gr}(\sigma) = n + k$.) We denote the some propositions which are called MOY relations developed by Murakami, Ohtsuki and Yamada [5].

Proposition 2. *Let Γ and Γ_1 be the graphs depicted in the figure 17. There is an isomorphisms as graded modules $C(\Gamma) \cong \bigoplus_{i=0}^{n-2} C(\Gamma_1)\{2 - n + 2i\}$.*

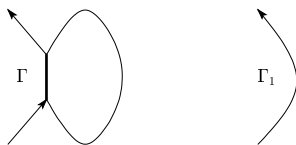


Figure 17:

Proposition 3. *Let Γ and Γ' be the graphs depicted in the figure 18. There is an isomorphism as graded modules $C(\Gamma) \cong C(\Gamma')\{1\} \oplus C(\Gamma')\{-1\}$.*

Proposition 4. *Consider graphs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ depicted in the figure 19. There is an isomorphism as graded modules $C(\Gamma_1) \oplus C(\Gamma_2) \cong C(\Gamma_3) \oplus C(\Gamma_4)$.*

These propositions are the consequence of [5], derived from the representation of quantum group and its R -matrix, also have close connection to the theory of Kazhdan-Lusztig basis of Hecke algebras but we shall not pursue this line of thought. We omit the proofs.

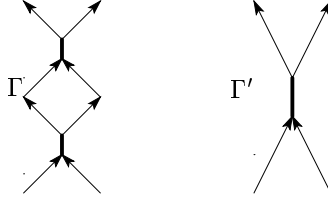


Figure 18:

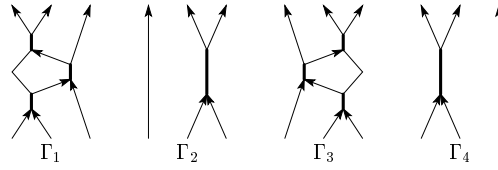


Figure 19:

For a link diagram D and $Cr \supset cr$ be a crossing set, we have constructed a complex associated with $C(D(cr))$, χ_0, χ_1 . We assume to locate $C(D(cr))$ at the cohomological degree $= N_1^+ - N_0^-$, and we will denote by $C^*(D)$ the complex associated with the diagram constructed by the shifted cube structure. We state the main theorem whose proof is stated in the next section.

Theorem 3.1. • *Let D be an oriented link diagram and $H\langle D \rangle$ be the homology groups of $C^*(D)$. Then $H\langle D \rangle$ is invariant under the Reidemeister moves.*

- *The graded Euler characteristic of this homology is equal to the quantum polynomial invariant.*

$$\langle D \rangle_n = \sum_{i, j} (-1)^i q^j \text{rank}(H^{i, j}(D))$$

where i is the cohomological grading and j is the grading of colored graphs.

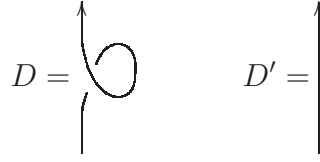
Proof. The second statement follows from the above propositions and [5]. \square

4 Invariance under the Reidemeister moves

We give the proof of the main theorem stated above. These proofs are similar to the one in [2].

4.1 R1-move

Consider the type-I Reidemeister move.



Proposition 5. *For diagrams D and D' depicted above, $C(D)$ is quasi-isomorphic to $C(D')$.*

Proof. Let $a \in Cr(D)$ be a crossing of D depicted above. We can consider $\oplus_{A \subset Cr(D), a \notin A} C(D(A))$ and $\oplus_{B \subset Cr(D), a \in B} C(D(B))$ as subcomplexes of $C(D)$ whose differentials are induced from the differential of $C(D)$. Then $C(D)$ can be written as the total complex of the bicomplex

$$0 \longrightarrow \oplus_{A \subset Cr(D), a \notin A} C(D(A)) \xrightarrow{\chi} \oplus_{B \subset Cr(D), a \in B} C(D(B)) \longrightarrow 0$$

where the map denoted χ is induced by the structure morphisms of cubes $C(D(A)) \longrightarrow C(D(a \sqcup A))$, $A \subset Cr(D)$, $a \notin A$.

We will define morphisms $f : C(D') \longrightarrow \oplus_{B \subset Cr(D), a \in B} C(D(B))$ and $\epsilon : \oplus_{B \subset Cr(D), a \in B} C(D(B)) \longrightarrow C(D')$ as follows.

Define $\epsilon : \oplus_{B \subset Cr(D), a \in B} C(D(B)) \longrightarrow C(D')$ to be a map which simply delete the simple closed curve depicted in the Fig. 20 if its coloring is equal to n , otherwise (if its coloring j is not equal to n), multiple $(-1)^{n-j}$ and take $n - j$ distinguished circles which contain the normal edge colored i depicted in the Fig. 20, then exchange colorings of each distinguished circles and delete the simple closed curves depicted in the figure above and consider as an element of $C(D')$. Exchanging colorings of $(n - j)$ distinguished circles shift down its quantum degree $(n - j)$. So the map ϵ is a quantum degree preserving morphism.

If we add a simple circle which is colored n to an element of $C(D')$ and

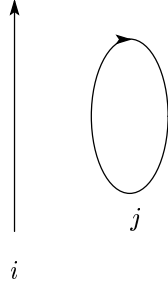


Figure 20: An element of 1-resolution of the crossing $a \in Cr(D)$

consider it as an element of 1-resolution of D , we can consider this procedure as a morphism from $C(D')$ to $\oplus_{A \subset Cr(D), a \notin A} C(D(A))$ and we will denote f .

By the definitions of morphisms, we can check the formulas $\epsilon \circ f = Id$, $\epsilon \circ \chi = 0$ and we have a following formula.

$$C(D) \cong f(C(D')) \oplus \{0 \longrightarrow \oplus_{A \subset Cr(D), a \notin A} C(D(B)) \longrightarrow ker(\epsilon) \longrightarrow 0\}$$

Then we can say that the complex is isomorphic to the direct sum of the contractible subcomplex and the nontrivial part which is isomorphic to the $C(D')$ by the formula $\epsilon f = Id$. This establishes a homotopy equivalence between $C(D)$ and $C(D')$. The invariance under other cases of Reidemeister-1 move can be verified similarly. \square

4.2 Reidemeister-2 move

Consider diagrams D and D' depicted in the figure21.

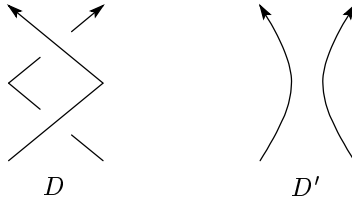
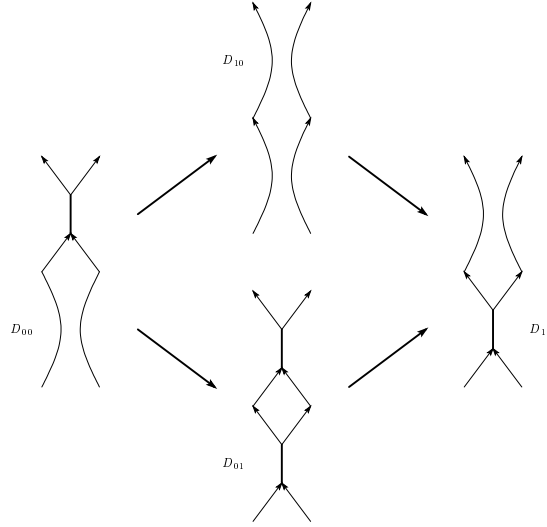


Figure 21: Type IIa move

Proposition 6. *The complex $C(D)$ is quasi-isomorphic to $C(D')$ for D and D' depicted in the Fig. 21.*



Proof.

Figure 22: Resolutions of D

Fig. 22 depicts the four resolutions of D and morphisms between them. Let ψ be a morphism $\psi : C(D_{10}) \rightarrow C(D_{00})$ which is defined by the inverse procedure of the structure morphism $\chi_1 : C(D_{00}) \rightarrow C(D_{10})$ (i.e, take distinguished subgraphs and exchanging its colorings as to preserve the degree and sum up all of its possibilities and the inverse of extra term.) We define α to be a composition of ψ and the structure morphism $\chi_0 : C(D) \rightarrow C(D_{01})$. Let $Y_1 = \{(x, \alpha(x)) \in C(D_{10}) \oplus C(D_{01}), x \in C(D_{10})\}$. We can induce a differential on Y_1 by the differential on $C(D)$ and direct calculations shows that Y_1 is stable under the induced differentials. By the construction, it is obvious that $\chi_0(x) + \chi_1(\alpha(x)) = 0$.

Let Y_2 be the subcomplex of $C(D)$ generated by $C(D_{00})$.

Let $Y_3 = \{(\beta(x), y) \in C(D_{01}) \oplus C(D_{11}), x, y \in C(D_{11})\}$ where β is a morphism $C(D_{11}) \rightarrow C(D_{01})$ defined as the inverse procedure of the structure morphism $\chi_1 : C(D_{11}) \rightarrow C(D_{01})$. As their construction, we can consider Y_1, Y_2 and Y_3 as the subcomplexes of $C(D)$. Then we can confirm directly that $C(D)$ is isomorphic to the direct sum $Y_1 \oplus Y_2 \oplus Y_3$. The direct calculations of the differentials can also show that Y_2 and Y_3 are contractible complex by their construction and we can naturally calculate that Y_1 is a subcomplex of

$C(D)$ and there are natural morphism to $C(D')$ which is a quasi-isomorphism to $C(D')$. Then the complex $C(D)$ can be described as $C(D) \cong Y_1 \oplus Y_2 \oplus Y_3$. This finishes the proof of this proposition. □

4.3 Reidemeister-3

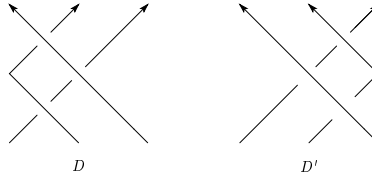


Figure 23: Type III move

Consider diagrams D, D' which is almost same but different only in the small area depicted in the Fig. 23.

We will prove the following proposition.

Proposition 7. *The complex $C(D)$ is quasi-isomorphic to $C(D')$ for D, D' depicted in the Fig. 23.*

Proof. The complex $C(D)$ is the total complex of the cube of 8-subcomplexes which is shown as in the Fig. 24 and we denote the colored graphs of the complex $C(D)$ as $C(\Gamma_{ijk})$, for $i, j, k \in \{0, 1\}$ in the figure 24.

We decompose $C(\Gamma_{000})$ into a direct sum of two sub-modules. Let W be a submodule of Γ_{000} generated by

- $\Gamma_{\bullet\bullet\bullet}$ with $(k > j > i)$ and $(i > j > k)$.
- $\Gamma_{\times\bullet\times}$ with $(j > i > k)$ and $(k > i > j)$.
- $\Gamma_{\bullet\bullet\bullet} + \Gamma_{\times\bullet\times}$ with $(j < k < i)$ and $(i < k < j)$.
- $\Gamma_{\times\bullet\bullet}$ with $(i > j > k)$ and $(k > j > i)$.
- $\Gamma_{\bullet\bullet\times}$ with $(j > i > k)$ and $(k > i > j)$.
- $\Gamma_{\times\bullet\bullet} + \Gamma_{\bullet\bullet\times}$ with $(j > k > i)$ and $(i > k > j)$.

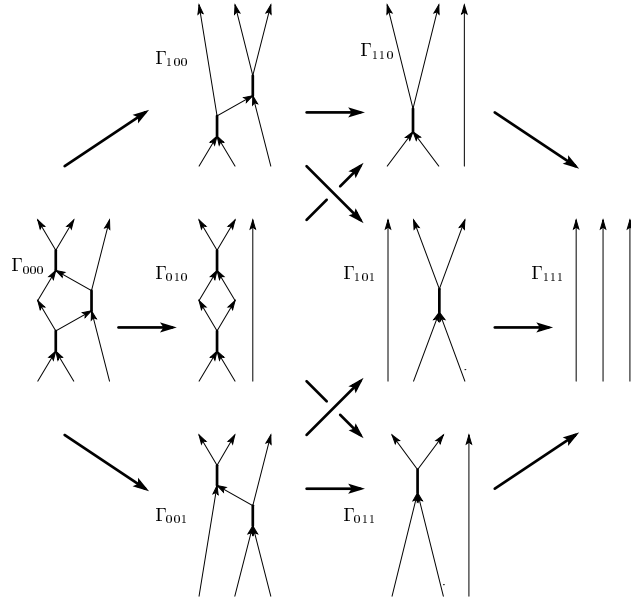


Figure 24: Resolution cube of D

- $\Gamma_{\bullet\times\bullet}$, $\Gamma_{\bullet\times\times}$, $\Gamma_{\times\times\bullet}$ and $\Gamma_{\times\times\times}$ with all colorings.

where $\Gamma_{\bullet\bullet\bullet}$, $\Gamma_{\bullet\bullet\times}$, $\Gamma_{\bullet\times\bullet}$, $\Gamma_{\times\bullet\bullet}$, $\Gamma_{\bullet\times\times}$, $\Gamma_{\times\bullet\times}$, $\Gamma_{\times\times\bullet}$ and $\Gamma_{\times\times\times}$ are colored graphs depicted in the Fig. 25. Singular edges whose degree are not equal to 0 will be depicted by black thick line and degree 0 singular edges will be depicted by simple crossing as in the Fig. 25.

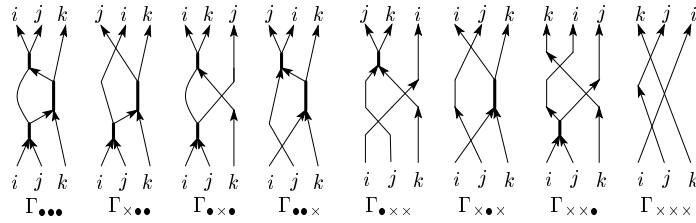


Figure 25: colored graphs of $C(\Gamma_{000})$

Let ψ be a morphism $C(\Gamma_{001}) \rightarrow C(\Gamma_{000})$ which is defined by the inverse procedure of the structure morphism χ_1 and define $\alpha : C(\Gamma_{001}) \rightarrow C(\Gamma_{010})$ to be the composition of ψ and structure morphism $\chi_1 : C(\Gamma_{000}) \rightarrow C(\Gamma_{010})$.

Using the submodule $W \subset C(\Gamma_{000})$ and the morphism α , we can decompose $C(D)$ as follows.

Let Y_1 be a submodule of $C(D)$ which consists of

- $C(\Gamma_{1ij})$, for $i, j \in \{0, 1\}$,
- submodule $W \subset C(\Gamma_{000})$ defined above,
- $(x, \alpha(x))$, for $x \in C(\Gamma_{001})$ where the morphism α is defined above.

We can induce a differential on Y_1 by the differential on $C(D)$ and can check straightforward that they are stable under the induced differential. So Y_1 become a subcomplex of $C(D)$.

Let Y_2 be the subcomplex of $C(D)$ which is generated by

- $\Gamma_{\bullet\bullet\bullet}$ with $(k < j < i < j)$ and $(j < i, j < k)$,
- $\Gamma_{\times\bullet\times}$ with $(k < j < i)$ and $(i < j < k)$,
- $\Gamma_{\bullet\bullet\times}$ with $(j < i, k < i)$ and $(i < j, i < k)$.
- $\Gamma_{\times\bullet\bullet}$ with $(j < i \leq k)$ and $(k \leq i < j)$.

Let Y_3 be a subcomplex of $C(D)$ which is generated by

- $\Gamma_{\bullet\bullet\bullet}$ with $(i > j)$,
- $\Gamma_{\bullet\bullet\times}$ with $(i < j)$,
- $\Gamma_{\times\bullet\bullet}$ with $(i > j)$,
- $\Gamma_{\times\bullet\times}$ with $(i < j)$

where $\Gamma_{\bullet\bullet\bullet}$, $\Gamma_{\bullet\bullet\times}$, $\Gamma_{\times\bullet\bullet}$ and $\Gamma_{\times\bullet\times}$ are colored graphs depicted in the Fig. 26.

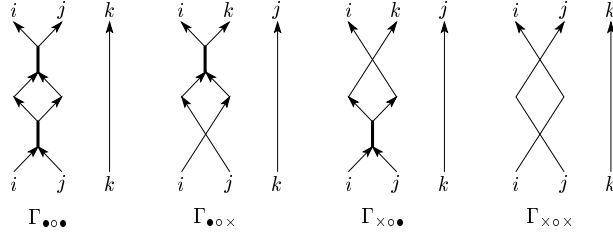


Figure 26: generators of Γ_{010}

We can induce differentials on Y_1 , Y_2 , Y_3 by a differential on $C(D)$ and consider them as subcomplexes of $C(D)$. There is a natural isomorphism between $Y_1 \oplus Y_2 \oplus Y_3$ and $C(D)$ as abelian groups. The direct calculation shows that $\Gamma_{\bullet\bullet\bullet}$ with $(i < j, k < j)$ $(i > j, k > j)$, $\Gamma_{\times\bullet\times}$ with $(i < j, i < k)$ $(i > j, i > k)$, $\Gamma_{\bullet\bullet\times}$ with $(i < j, i < k)$ $(i > j, i > k)$ and

$\Gamma_{\times\bullet\bullet}$ with $(i < j, k < j)$ ($i > j, k > j$) are mapped to Γ_{101} injectively and this shows that Y_2 is acyclic complex. It is also confirmed directly that the $\Gamma_{\bullet\bullet\bullet}$ with $(i > j)$, $\Gamma_{\times\circ\times}$ with $(i < j)$, $\Gamma_{\bullet\circ\times}$ with $(i < j)$ and $\Gamma_{\times\circ\bullet}$ with $(i > j)$ are mapped injectively to Γ_{110} and this shows that Y_3 is acyclic complex.

Next, we will establish a similar decomposition. The complex $C(D')$ also has 8 subcomplexes and denoted it as Γ'_{ijk} for $i, j, k \in \{0, 1\}$ in the figure 27

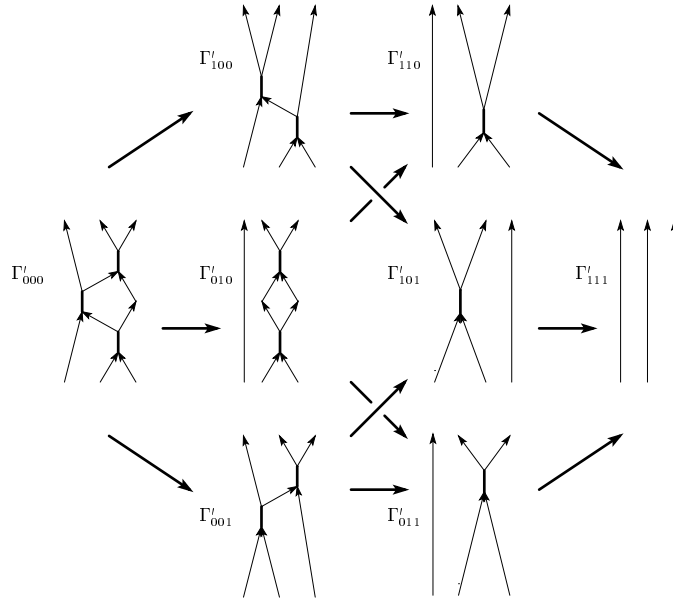


Figure 27: Resolution cube of D'

A similar decomposition for $C(D')$ will also defined as follows.

Let W' be a submodule of Γ'_{000} generated by

- $\Gamma'_{\bullet\bullet\bullet}$ with $(i > j > k)$ and $(k > j > i)$.
- $\Gamma'_{\times\bullet\times}$ with $(j > k > i)$ and $(i > k > j)$.
- $\Gamma'_{\bullet\bullet\bullet} + \Gamma'_{\times\bullet\times}$ with $(j > i > k)$ and $(k > i > j)$.
- $\Gamma'_{\times\bullet\bullet}$ with $(i > j > k)$ and $(k > j > i)$.

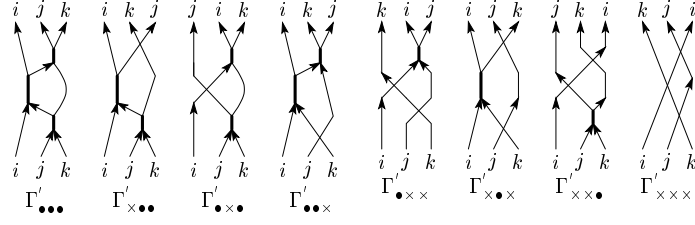


Figure 28: colored graphs $C(\Gamma'_{000})$

- Γ'_{\dots} with $(i > k > j)$ and $(j > k > i)$.
- $\Gamma'_{x..} + \Gamma'_{..x}$ with $(j > i > k)$ and $(k > i > j)$.
- $\Gamma'_{.x.}, \Gamma'_{.xx}, \Gamma'_{x.x.}$ and Γ'_{xxx} with all colorings.

where $\Gamma'_{\dots}, \Gamma'_{x..}, \Gamma'_{.x.}, \Gamma'_{..x}, \Gamma'_{.xx}, \Gamma'_{x.x.}, \Gamma'_{x.x.}$ and Γ'_{xxx} are colored graphs depicted in the Fig. 28.

Let α' be a morphism $\Gamma'_{100} \longrightarrow \Gamma'_{010}$ defined similarly as α by a composition of an inverse procedure of structure morphism $\chi_1 : \Gamma'_{000} \longrightarrow \Gamma'_{100}$ and the structure morphism $\chi_1 : \Gamma'_{000} \longrightarrow \Gamma'_{010}$.

Let Y'_1 be a subset of $C(D')$ which consists of

- Γ'_{1ij} , for $i, j \in \{0, 1\}$,
- submodule $W' \subset \Gamma'_{000}$ defined above,
- $(x, \alpha'(x))$, for $x \in \Gamma'_{100}$ where the morphism α' is defined above.

We can induce a differential by the differential on $C(D')$ and can check straightforward that Y'_1 are stable under induced differential. So Y_1 become a subcomplex of $C(D')$.

Let Y'_2 be a submodule of $C(D')$ which is generated by

- Γ'_{\dots} with $(k < j, i < j)$ and $(j < i, j < k)$,
- $\Gamma'_{x..}$ with $(k < j < i)$ and $(i < j < k)$,
- $\Gamma'_{..x}$ with $(k < i, k < j)$ and $(i < k, j < k)$.
- $\Gamma'_{x..}$ with $(i \leq k < j)$ and $(j \leq k < i)$.

Let Y'_3 be a subcomplex of $C(D')$ which is generated by

- Γ'_{\dots} with $(j > k)$,
- Γ'_{\dots} with $(j < k)$,
- $\Gamma'_{x..}$ with $(j > k)$,

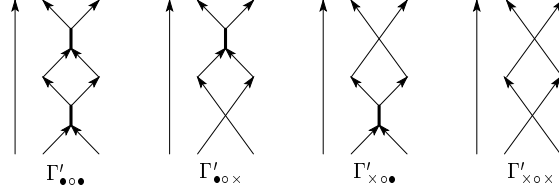


Figure 29: generators of Γ'_{010}

• $\Gamma'_{\times \circ \times}$ with $(j < k)$

where $\Gamma'_{\bullet \bullet \bullet}$, $\Gamma'_{\bullet \bullet \times}$, $\Gamma'_{\times \bullet \bullet}$ and $\Gamma'_{\times \circ \times}$ are colored graphs depicted in the Fig. 29.

As in the $C(D)$ case, direct calculations of the differentials show that Y'_2 and Y'_3 are acyclic complexes.

Thus, we can twist out the acyclic complexes from $C(D)$ and $C(D')$. Therefore we can obtain a reduced complex Y_1 (resp. Y'_1) from $C(D)$ (resp. $C(D')$).

Next, we will make a correspondence graphically between Y_1 and Y'_1 which induces a quasi-isomorphism between them.

For $\Gamma_{\bullet \bullet \bullet}$ ($k > j > i$), we associate to $\Gamma'_{\bullet \bullet \bullet}$ ($k > j > i$). We associate for $\Gamma_{\bullet \bullet \bullet}$ ($i > j > k$) to $\Gamma'_{\bullet \bullet \bullet}$ ($i > j > k$).

For $\Gamma_{\times \bullet \times}$ ($j > i > k$), we associate to $\Gamma'_{\bullet \bullet \bullet} + \Gamma'_{\times \bullet \times}$ ($j > i > k$). We associate for $\Gamma_{\times \bullet \times}$ ($k > i > j$) to $\Gamma_{\bullet \bullet \bullet} + \Gamma_{\times \bullet \times}$ ($k > i > j$).

For $\Gamma_{\bullet \bullet \bullet} + \Gamma_{\times \bullet \times}$ ($j < k < i$), we associate $\Gamma'_{\times \bullet \times}$ ($j < k < i$). We associate for $\Gamma_{\bullet \bullet \bullet} + \Gamma_{\times \bullet \times}$ ($i < k < j$) to $\Gamma'_{\times \bullet \times}$ ($i < k < j$).

For $\Gamma_{\times \bullet \bullet}$ ($i > j > k$), we associate $\Gamma'_{\bullet \bullet \times}$ ($i > j > k$). We associate for $\Gamma_{\times \bullet \bullet}$ ($k > j > i$) to $\Gamma'_{\bullet \bullet \times}$ ($k > j > i$).

For $\Gamma_{\bullet \bullet \times}$ ($j > i > k$), we associate $\Gamma'_{\bullet \bullet \times}$ ($j > i > k$). We associate for $\Gamma_{\bullet \bullet \times}$ ($k > i > j$) to $\Gamma'_{\bullet \bullet \times}$ ($k > i > j$).

For $\Gamma_{\times \bullet \bullet} + \Gamma_{\bullet \bullet \times}$ ($j > k > i$), we associate $\Gamma'_{\bullet \bullet \times}$ ($j > k > i$). We associate for $\Gamma_{\times \bullet \bullet} + \Gamma_{\bullet \bullet \times}$ to $\Gamma'_{\bullet \times \bullet}$ ($i > k > j$).

For $\Gamma_{\bullet \times \bullet}$ ($j > k > i$), we associate $\Gamma'_{\bullet \bullet \times}$ ($j > k > i$).

For $\Gamma_{\bullet \times \bullet}$ ($i > k > j$), we associate $\Gamma'_{\bullet \bullet \times}$ ($i > k > j$).

For $\Gamma_{\bullet \times \bullet}$ ($k > j > i$), we associate $\Gamma'_{\times \bullet \bullet}$ ($k > j > i$).

For $\Gamma_{\bullet \times \bullet}$ ($i > j > k$), we associate $\Gamma'_{\times \bullet \bullet}$ ($i > j > k$).

For $\Gamma_{\bullet \times \bullet}$ ($k > i > j$), we associate $\Gamma'_{\bullet \bullet \times} + \Gamma'_{\times \bullet \bullet}$ ($k > i > j$).

For $\Gamma_{\bullet \times \bullet}$ ($j > i > k$), we associate $\Gamma'_{\bullet \bullet \times} + \Gamma'_{\times \bullet \bullet}$ ($j > i > k$).

For $\Gamma_{\bullet \times \times}$, we can associate naturally $\Gamma'_{\times \times \bullet}$.

For $\Gamma_{\times \times \bullet}$, we can associate naturally $\Gamma'_{\bullet \times \times}$.

For $\Gamma_{\times\times\times}$, we can associate naturally $\Gamma'_{\times\times\times}$.

We can construct a natural correspondence between Γ_{1ij} and Γ'_{1ij} , and also between $\{(x, \alpha(x)) \mid x \in \Gamma_{001}\}$ and $\{(y, \alpha'(y)) \mid y \in \Gamma'_{100}\}$.

We can directly confirm that the above correspondence can be seen as chain homomorphisms $Y_1 \longrightarrow Y'_1$ and $Y'_1 \longrightarrow Y_1$ and this chain homomorphisms give a homotopy equivalence. Therefore we conclude that our cohomology is invariant under the Reidemeister-move III \square

This completes the proof that $H\langle G \rangle$ is invariant under all the Reidemeister moves.

5 Applications and generalizations

We sketch here an application and generalize to a graphically defined infinite dimensional complex. One application of these graphical cohomology is to establish dualities between links by constructing non-degenerate pairings.

Theorem 5.1. *Let L be a link. Denote by $-L$ the link obtained from L by reversing the orientation of every component of L and \bar{L} the link obtained from L by reversing the orientation of every component of L and switching the upper- and lower-branches at each crossing. Then there are a non-degenerate pairing $H(L, k) \otimes_k H(\bar{L}, k) \implies k$ where k means a field and a isomorphism $H(L, k) \cong H(-L, k)$.*

Proof. Consider the chains $C(L)$, $C(-L)$. Denote D be a diagram of L which is presented by a braid form. By reversing the orientation of L , we can naturally make an isomorphism between $C(D)$ and $C(-D)$ as a graded module because reversing orientations does not change the crossing-type of D and $-D$. But our structure morphism of cube does not depend on the orientations of underlying graphs. So the differentials also unchanged by reversing the orientations. Thus we have an isomorphism $H(L, k) \cong H(-L, k)$.

$C(\bar{L}, k)$ can be naturally identified to $\text{Hom}(C(L), k)$, where k is a coefficient ring. By the standard homological algebra, we can identify the cohomologies of $C(L, k)$ and $\text{Hom}(C(L), k)$. Thus we have a non-degenerate pairings $H(L, k) \otimes H(\bar{L}, k) \implies k$. \square

As a consequence we can conclude that the following formula.

Proposition 8. *Let L be a link and L' be a link obtained from L by switching the upper- and lower-branches at each crossing. If $L = \bar{L}$ or $L = L'$, we have a non-degenerate pairing $H(L, k) \otimes_k H(L, k) \implies k$*

We can consider a generalization of this graphically defined cohomology to infinite dimensional cohomology as the *HOMFLY* polynomial can be obtained by generalizing the quantum $\mathfrak{sl}(n)$ polynomial invariant motivated by the skein theory. The key observation is that the *HOMFLY* polynomial is obtained by generalizing the quantum polynomial $\mathfrak{sl}(\mathbf{n})$ invariant based on the skein theory and that the propositions 3.6 – 3.7, the [MOY]-relations, and the proof of the invariance under Reidemeister moves *II*, *III* almost unaffected by the choice of n which is the size of colorings (or the index of quantum group $\mathfrak{sl}(n)$). We state here a non-modified definition of generalized complex but to construct a link homology, we must modify the definition.

Let B_n be a n -strands braid group and $b_i \in B_n$ with $(1 \leq i \leq n-1)$ be the standard generators and $b = (b_{i_1}^{j_1} \cdots b_{i_k}^{j_k}) \in B_n$ be an element. We use graphical braid presentations as usual. We will assign a complex to b which is invariant under the Reidemeister move *II*, *III*. We resolve the crossing of b by the same rule of the 0, 1 resolutions of crossings of a braid closure diagram. (See the Fig. 1.) Let G be a oriented trivalent planar diagram obtained by the resolutions of each crossing of b . A state of G is a function from the set of normal edges of G to \mathbb{N} . It should satisfy the following conditions that for each singular edge, the set of elements of \mathbb{N} attached to the legs of the singular edge is equal to the set of elements of \mathbb{N} attached to the heads of the singular edge and the elements attached to the legs (resp. heads) must be different from each other. Furthermore the states must have same colorings at the initial and the terminal points of the braid diagram (i.e., the state can be extended to the closure of braid diagram). Given a state of G , we can take the n -colored paths by considering the braid as the n -paths. If the strand has been attached i by the state, we define the quantum degree of colored strand as $2i$. We define the quantum degree of a singular edge. Let E be a singular edge of G appearing in the 1-resolution of positive crossing as in figure 4.

$$(6) \quad \deg(E) = \begin{cases} 0 & \sigma(e_1) = \sigma(e_3) < \sigma(e_2) = \sigma(e_4), \\ 1 & \sigma(e_1) = \sigma(e_4), \sigma(e_2) = \sigma(e_3), \\ 2 & \sigma(e_1) = \sigma(e_3) > \sigma(e_2) = \sigma(e_4). \end{cases}$$

Let E' be a singular edge of G appearing in the 0-resolution of negative crossing as in figure 4.

$$(7) \quad \deg(E') = \begin{cases} -2 & \sigma(e_1) = \sigma(e_3) < \sigma(e_2) = \sigma(e_4), \\ 1 & \sigma(e_1) = \sigma(e_4), \sigma(e_2) = \sigma(e_3), \\ 0 & \sigma(e_1) = \sigma(e_3) > \sigma(e_2) = \sigma(e_4). \end{cases}$$

This definition of quantum degree is the modified one defined in the definition 3.3. We also put the quantum degree on the parallel normal edges appearing in the 0-resolutions of positive crossings and the 1-resolutions of negative crossings. Let P_0, N_1 be the number of 0-resolutions of positive crossings and 1-resolutions of negative crossings respectively. Then we shift the quantum degree $2P_0 - 2N_1$. This shift can be thought that the 0-resolutions of positive crossings have quantum degree $+2$ and the 1-resolutions of negative crossings have quantum degree -2 . Then we define the quantum degree of σ as

$$\deg(\sigma) = (2P_0 - 2N_1) + \sum_{\text{singular edges}} \deg(E) + \sum_{i; \text{colorings of strands}} 2i$$

Let $C(\tilde{G})$ be a graded \mathbb{Z} -module whose generators are the state of G and the degrees are induced by the above formula. We also define the structure morphisms between them. In this infinite dimensional graphical complex, we can also define the distinguished circles, the distinguished subgraphs and the coloring exchange of distinguished circle or distinguished subgraph of G . The morphism corresponding to χ_1 is defined to coloring exchanging of the distinguished circle (without modification term) and the morphism corresponding to χ_0 is defined to coloring exchanging of the distinguished subgraph. It is confirmed similarly that these graded modules and the morphisms consist a cube. The propositions 3.6 – 3.7 and the Reidemeister move II, III also hold by the almost same arguments in the previous section. Thus we obtain a categorical braid representation.

Theorem 5.2. *$C(\tilde{G})$ and the structure morphisms define a chain complex which is invariant under the Reidemeister-moves II, III.*

If we modify the $C(\tilde{G})$, we can obtain a link homology which is invariant under the Reidemeister-moves I, II, III. But we do not pursue these things in this paper.

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OSAKA CITY UNIVERSITY

E-mail address: d07sag0702@ex.media.osaka-cu.ac.jp