ON THE PLURICANONICAL MAPS OF VARIETIES OF INTERMEDIATE KODAIRA DIMENSION

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Abstract. In this paper we will prove a uniformity result for the Iitaka fibration $f : X \to Y$, provided that the generic fiber has a good minimal model and the variation of f is zero.

1. Introduction

One of the main problems in complex projective algebraic geometry is to understand the structure of pluricanonical maps. Recently, Hacon and M^{c}Kernan [HM06], Takayama [Tak06] and Tsuji [Tsu06] have proved a beautiful result stating that there is a universal constant r_n such that if X is a smooth projective variety of general type and dimension n, then the pluricanonical map

$$\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rK_X)))$$

is birational for all $r \geq r_n$. In [HM06], Hacon and M^{c}Kernan also proposed a related conjecture for the Iitaka fibration in the case $\dim X > \kappa(X) \geq 0$.

Conjecture 1.1 ([HM06, Conjecture 1.7]). *Fix $n \in \mathbb{Z}_{>0}$. There is positive integer r_n with the following property: Let X be a smooth n-dimensional projective variety of non-negative Kodaira dimension. Then the rational map ϕ_{rK_X} is birationally equivalent to the Iitaka fibration for all sufficiently divisible integers $r \geq r_n$.*

The purpose of this paper is to prove Conjecture 1.1 under some extra hypotheses.

Theorem 1.2. *For any positive integers n, b, k, there exists an integer $m(n, b, k) > 0$ such that if $f : X \to Y$ is the Iitaka fibration with X and Y smooth projective varieties, $\dim X = n$, with generic fiber F of f of Kodaira dimension zero, such that*

1. *the variation of f is zero;*
2. *F has a good minimal model;*
3. *b is the smallest integer such that $h^0(F, bK_F) \neq 0$, and $\mathrm{Betti}_{\dim(E')}(E') \leq k$, where E' is a smooth model of the cover $E \to F$ of the generic fiber F associated to the unique element of $|bK_F|$;*

then the pluricanonical map

$$\phi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(mK_X))$$

is birationally equivalent to f, for any $m \in \mathbb{Z}_{>0}$ such that m is divisible by $m(n, b, k)$.

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Conjecture 1.1 has been extensively studied. In [FM00], Fujino and Mori prove that if $\kappa(X) = 1$, then (1.1) holds under the hypothesis (3) of Theorem 1.2. Veihweg and Zhang [VZ07] also obtain this uniformity result for $\kappa(X) = 2$ under the same hypothesis. A related result of [VZ07] for 3-folds has been obtained independently by Ringler [Rin07]. For arbitrary Kodaira dimension, Pacienza [Pac07b] recently has given an affirmative answer to (1.1) assuming that Y is not uniruled, the Iitaka fibration f has maximal variation and the hypotheses (2) and (3) of Theorem 1.2.

We now sketch the proof of Theorem 1.2. The main idea is to follow the approach of [HM06], [Tak06] and [Tsu06]. By the Canonical Bundle Formula (cf. Section 3), there are two $\mathbb{Q}$-divisors M_Y (the moduli part) and B_Y (the boundary part) on Y, such that for all $i > 0$, $H^0(X, \mathcal{O}_X(ibNK_X)) \cong H^0(Y, \mathcal{O}_Y(\lfloor ibN(K_Y + M_Y + B_Y)\rfloor))$, where N is a positive integer depending on the hypothesis (3) of Theorem 1.2 and M_Y is $\mathbb{Q}$-linearly trivial by the hypotheses (1) and (2) (Theorem 3.6). In order to prove Theorem 1.2, it remains to bound a multiple m of bN for which $\phi_{m(K_Y+M_Y+B_Y)}$ is birational. We first show that there exists such m of the form $\alpha(\mathrm{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta$ that $\phi_{m(K_Y+M_Y+B_Y)}$ is birational, where $n' = \dim Y$ and α, β are constants depending only on n, b and k. Then using techniques developed in [HMX10], we show that if M_Y is $\mathbb{Q}$-linearly trivial, $\mathrm{vol}(Y, K_Y + M_Y + B_Y)$ can be bounded from below. Hence m admits a uniform bound.

The main difficulty is that for a very general point $y \in Y$, we need to construct an effective $\mathbb{Q}$-divisor D_y which is $\mathbb{Q}$-linearly equivalent to $\lambda(K_Y + M_Y + B_Y)$, where λ depends on $\mathrm{vol}(Y, K_Y + M_Y + B_Y)$, such that y is an isolated non-klt center of (Y, D_y). There is a well established way for producing divisors with non-klt centers at y. The problem is that the smallest non-klt center V containing y may be of positive dimension. In order to produce an isolated non-klt center, we have to cut down the dimension of the non-klt centers. By [BCHM10], we can assume Y is the log canonical model, so $K_Y + M_Y + B_Y$ is ample. Then by Subadjunction (see Section 5) we prove that $\mathrm{vol}(V, (K_Y + M_Y + B_Y)|_V)$ is bounded by a number related to $\mathrm{vol}(Y, K_Y + M_Y + B_Y)$. Using techniques developed in [M^{c}K02] (see Section 4), we can produce a new divisor with a smaller dimensional non-klt center at y. Repeating this procedure at most $n' - 1$ times, we get the desired divisor D_y, see Section 6.

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2. Preliminaries

2.1. **Notation and conventions.** We work over the complex number field $\mathbb{C}$. Let X be a normal variety. We say that two $\mathbb{Q}$-divisor D_1, D_2 on X are $\mathbb{Q}$-linearly equivalent ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. If $D = \sum d_i D_i$ is a $\mathbb{Q}$-divisor, then the round down of D is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d \rfloor$ denotes the largest integer which is at most d, and the round up of D is $\lceil D \rceil = -\lfloor -D \rfloor$.

A **log pair** (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that (X, Δ) is **log smooth** if X is smooth and Δ is a $\mathbb{Q}$-divisor with simple normal crossings support. A projective morphism

$\mu : Y \longrightarrow X$ is a **log resolution** of the pair (X, Δ) if Y is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossings support. We write $K_Y = \mu^*(K_X + \Delta) + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors. We call a_i the **discrepancy** of the pair (X, Δ) at Γ_i. The pair (X, Δ) is **kawamata log terminal**, klt for short (resp. **log canonical**, lc for short), if there is a log resolution $\mu : Y \longrightarrow X$ as above such that the discrepancies of Γ are strictly greater than -1, i.e. $a_i > -1$ for all i (resp. $a_i \geq -1$). A subvariety V of X is called a **non-klt center** of (X, Δ) if it is the image of a divisor of discrepancy at most -1. The **non-klt locus** Non-klt(X, Δ) of the pair (X, Δ) is the union of the non-klt centers. A non-klt center V is called a **pure log canonical center** if (X, Δ) is log canonical at the generic point of V.

If D is a Weil divisor on a normal projective variety X, then ϕ_D denotes the rational map $X \dashrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(D))$ induced by global sections of $\mathcal{O}_X(D)$.

2.2. **Volumes and bounded pairs.**

Definition 2.1. Lex X be an irreducible projective variety of dimension n and D be a $\mathbb{Q}$-divisor. The **volume** of D is

$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

We say that D is **big** if $\mathrm{vol}(X, D) > 0$.

We refer the reader to [Laz1] for further details.

Lemma 2.2 ([HM06, Lemma 2.2])**.** *Let X be a projective variety, D a divisor such that ϕ_D is birational with image Z. Then the volume of D is at least the degree of Z and hence at least* 1.

Lemma 2.3 ([HMX10, Lemma 2.3.4])**.** *Let X be a normal projective variety of dimension n and let D be a big $\mathbb{Q}$-Cartier divisor on X. If ϕ_D is birational, then $\phi_{K_X+(2n+1)(D+M)}$ is birational for any numerically trivial Cartier divisor M.*

Definition 2.4 ([HMX10, Definiton 2.4.2])**.** A set $\mathfrak{D}$ of log pairs is **log birationally bounded** if there is a log pair (Z, B) and a projective morphism $Z \longrightarrow T$, where T is of finite type, such that for every $(X, \Delta) \in \mathfrak{D}$, there is a closed point $t \in T$ and a birational map $f : Z_t \dashrightarrow X$ such that the support of B_t contains the support of the strict transform of Δ and any f-exceptional divisor.

Theorem 2.5 ([HMX10, Theorem 3.1])**.** *Fix a positive integer n and two constants A and $\delta > 0$. Then the set of log pairs (X, Δ) satisfying*

1. *X is projective of dimension n,*
2. *(X, Δ) is log canonical,*
3. *the coefficients of Δ are at least δ,*
4. *there is a positive integer m such that $\mathrm{vol}(X, m(K_X + \Delta)) \leq A$,*
5. *$\phi_{K_X+m(K_X+\Delta)}$ is birational,*

is log birationally bounded.

Theorem 2.6 ([HMX10, Theorem 1.7])**.** *Fix a set $I \subset [0, 1]$ which satisfies the DCC. Let $\mathfrak{D}$ be a set of log smooth pairs (X, Δ), which is log birationally bounded, such that if $(X, \Delta) \in \mathfrak{D}$, then the coefficients of Δ belong to I. Then the set*

$$\{\mathrm{vol}(X, K_X + \Delta) | (X, \Delta) \in \mathfrak{D}\},$$

satisfies the DCC.

2.3. **Multiplier ideals and singularities of pairs.** Let X be a smooth variety. If D is an effective $\mathbb{Q}$-divisor on X, then the **multiplier ideal sheaf** associated to D is defined to be

$$\mathcal{J}(X, D) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor)$$

where $\mu : X' \to X$ is a log resolution of (X, D). It is known that a pair (X, D) is klt (resp. non-klt) at a point x, if and only if

$$\mathcal{J}(X, D)_x = \mathcal{O}_{X,x} \quad (\text{resp.} \quad \mathcal{J}(X, D)_x \neq \mathcal{O}_{X,x})$$

and a pair is klt if it is klt at each point $x \in X$. A pair (X, D) is lc at a point x, if and only if

$$\mathcal{J}(X, (1-\varepsilon)D)_x = \mathcal{O}_{X,x}$$

for all rational numbers $0 < \varepsilon < 1$ and a pair is lc if it is lc at each point $x \in X$. Note that we have the following relation for non-klt locus

$$\text{Non-klt}(X, D) = \text{Supp}(\mathcal{O}_X / \mathcal{J}(X, D))_{\text{red}}.$$

The following is a useful way to produce non-klt pairs.

Lemma 2.7 ([Laz2, Proposition 9.3.2])**.** *Assume that X is smooth of dimension n, and let D be an effective $\mathbb{Q}$-divisor on X. If $\text{mult}_x D \geq n$ at some point $x \in X$, then $\mathcal{J}(X, D)$ is non-trivial at x, i.e. $\mathcal{J}(X, D) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of x.*

We now recall Nadel's vanishing theorem.

Theorem 2.8 ([Laz2, Theorem 9.4.8])**.** *Let X be a smooth projective variety. Let D be an effective $\mathbb{Q}$-divisor on X, and L a divisor on X such that $L - D$ is nef and big. Then, for all $i > 0$, we have*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0.$$

2.4. **Iitaka fibration.** Here we recall some results regarding Iitaka fibrations.

Let L be a line bundle on an irreducible projective variety X. The semigroup $\mathbf{N}(L)$ of L is

$$\mathbf{N}(L) = \{m \in \mathbb{Z}_{>0} | H^0(X, mL) \neq 0\}.$$

Assuming $\mathbf{N}(L) \neq (0)$, all sufficiently large elements of $\mathbf{N}(L)$ are multiples of a largest single natural number $e = e(L) \geq 1$, which we call the **exponent** of L. If $\kappa(X, L) = \kappa \geq 0$, then $\dim(\phi_{mL}(X)) = \kappa$ for all sufficiently large $m \in \mathbf{N}(L)$.

Theorem 2.9 (Iitaka fibrations, see [Laz1, Theorem 2.1.33])**.** *Let X be a normal projective variety, and L a line bundle on X such that $\kappa(X, L) > 0$. Then for all sufficiently large $k \in \mathbf{N}(L)$, there exists a commutative diagram of rational maps and morphisms*

$$\begin{array}{ccc} X & \xleftarrow{u_\infty} & X_\infty \\ {\scriptstyle \phi_k} \downarrow & & \downarrow {\scriptstyle \phi_\infty} \\ Y_k & \xleftarrow[\nu_k]{} & Y_\infty \end{array}$$

where the horizontal maps are birational and u_∞ is a morphism. One has $\dim Y_\infty = \kappa(X, L)$. Moreover, if we set $L_\infty = u_\infty^* L$, and F is a very general fiber of ϕ_∞, we have $\kappa(F, L_\infty|_F) = 0$.

In this paper, we only deal with the case $L = \mathcal{O}_X(K_X)$ and simply write $\kappa(X) = \kappa(X, \mathcal{O}_X(K_X))$. The following results are important for our induction in the proof of the main theorem.

Lemma 2.10. *Let X and Y be smooth projective varieties and T an algebraic variety. Assume that $f : X \to Y$ is the Iitaka fibration of (X, K_X) and $\varphi : Y \to T$ is a surjective morphism. For a very general closed point $t \in T$, let $V = \varphi^{-1}(t)$ and $W = f^{-1}(\varphi^{-1}(t))$, then the restriction morphism $f_W : W \to V$ is the Iitaka fibration of (W, K_W).*

Proof. By assumption, we have the following diagram

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow{\scriptstyle f_W} & & \downarrow{\scriptstyle f} \\ V & \hookrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ t & \in & T \end{array}$$

Since t is very general, we may assume V and W are smooth and the very general fiber of f_W is just the very general fiber of f. Hence, in order to prove that f_W is the Iitaka fibration, we only need to show $\dim V \leq \kappa(W)$.

Fix an ample divisor H on Y, then there exists a positive integer m such that $mK_X \geq f^*(H)$. Since V is a smooth fiber, we have $K_X|_W = K_W$. It follows that $mK_W \geq f_W^*(H|_V)$, which implies

$$h^0(W, \mathcal{O}_W(imK_W)) \geq h^0(V, \mathcal{O}_V(iH|_V)) \qquad \forall\, i \in \mathbb{Z}_{>0}.$$

Since $H|_V$ is ample on V, then $\kappa(W) \geq \dim V$. Therefore, f_W is the Iitaka fibration. □

Theorem 2.11 ([Lai09, Theorem 4.4])**.** *Let X be a $\mathbb{Q}$-factorial normal projective variety with non-negative Kodaira dimension and at most terminal singularities. Suppose that the general fiber F of the Iitaka fibration has a good minimal model, then X has a good minimal model.*

3. Canonical bundle formula

In this section, we collect some of the results regarding the direct image of the relative dualizing sheaf.

Let X and Y be smooth projective varieties and $f : X \to Y$ an algebraic fiber space with generic fiber F of Kodaira dimension zero. Let b be the smallest integer such that the b-th plurigenus $h^0(F, bK_F)$ of F is non-zero. Then there exists a $\mathbb{Q}$-divisor $L_{X/Y}$ on Y such that

$$\mathcal{O}_Y(\lfloor iL_{X/Y} \rfloor) \cong (f_*\mathcal{O}_X(ibK_{X/Y}))^{**}$$

and

$$H^0(Y, \mathcal{O}_Y(\lfloor ibK_Y + iL_{X/Y} \rfloor)) \cong H^0(X, \mathcal{O}_X(ibK_X))$$

for all $i > 0$. We may write the divisor $L_{X/Y}$ as

$$L_{X/Y} = L^{ss}_{X/Y} + \Delta,$$

where $L^{ss}_{X/Y}$ is a $\mathbb{Q}$-Cariter divisor, called the **semistable part** or the **moduli part**, and Δ is an effective $\mathbb{Q}$-divisor, called the **boundary part**. Moreover, if f

satisfies the conditions as in [FM00, 4.4], then $L^{ss}_{X/Y}$ is nef and Δ has simple normal crossings support. Therefore, replacing Y by a smooth birational model, we may always assume that $L^{ss}_{X/Y}$ is nef and Δ is a simple normal crossings divisor.

In applications, it is important to bound the denominator of $L^{ss}_{X/Y}$.

Theorem 3.1 ([FM00, Theorem 3.1]). *Under the above notations and assumptions, let $E \to F$ be the cover associated to the b-th root of the unique element of $|bK_F|$. Let $\overline{E}$ be a nonsingular projective model of E and let B_m be its m-th Betti number. Then there is a natural number $N = N(B_m)$ depending only on B_m such that $NL^{ss}_{X/Y}$ is a divisor.*

Let $\Delta = \sum_P s_P P$. We have the following result about the coefficients s_P.

Proposition 3.2 ([FM00, Propostion 2.8]). *Under the notations and the assumptions as above, let $N \in \mathbb{Z}_{>0}$ be such that $NL^{ss}_{X/Y}$ is a Weil divisor. Then we have*

$$L_{X/Y} = L^{ss}_{X/Y} + \sum_P s_P P,$$

where $s_P \in \mathbb{Q}$ for every codimension one point P of Y is such that

1. *For each P, there exists $u_P, v_P \in \mathbb{Z}_{>0}$, such that $0 < v_P \leq bN$ and $s_P = (bNu_P - v_P)/(Nu_P)$.*
2. *$s_P = 0$ if $f^*(P)$ has only canonical singularities or if $X \to Y$ has a semistable resolution in a neighbourhood of P.*

Moreover, s_P depends only on $f|_{f^{-1}(U)}$ where U is an open set of Y containing P.

For convenience, we write $M_Y = L^{ss}_{X/Y}/b$ and $B_Y = \Delta/b$, then all non-zero coefficients of B_Y are contained in

$$A(b, N) := \{\frac{bNu - v}{bNu} | u, v \in \mathbb{Z}_{>0}; 0 < v \leq bN\} \backslash \{0\}.$$

Lemma 3.3 ([VZ07, Lemma 1.2]). *Under the notations as above, the following hold true.*

1. *The set $A(b, N)$ is a DCC set, and one has*
$$\frac{1}{bN} \leq \inf A(b, N).$$
2. *(Y, B_Y) is log smooth and has klt singularities.*
3. *The $\mathbb{Q}$-divisor $K_Y + M_Y + B_Y$ is big.*
4. *For every $s \in \mathbb{Z}_{>0}$, we have*
$$H^0(Y, \mathcal{O}_Y(\lfloor sb(K_Y + M_Y + B_Y)\rfloor)) \cong H^0(X, \mathcal{O}_X(sbK_X));$$
further the map ϕ_{sbK_X} is birational to the Iitaka fibration f if and only if $|sb(K_Y + M_Y + B_Y)|$ gives rise to a birational map.
5. *bNM_Y is an integral nef Cartier divisor.*
6. *If $m \in \mathbb{Z}_{>0}$ is divisible by bN, then $\lfloor mB_Y \rfloor \geq (m-1)B_Y$.*

Lemma 3.4. *Under the same notations and assumptions as in Lemma 3.3, $(Y, M_Y + B_Y)$ has a log terminal model and a log canonical model.*

Proof. Since $K_Y+M_Y+B_Y$ is big, we may write $K_Y+M_Y+B_Y\sim_{\mathbb{Q}} A+E$, where A is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. By (2) of Lemma 3.3, (Y,B_Y) is klt, so $(Y,B_Y+\epsilon E)$ is also klt for $0<\epsilon\ll 1$. By (5) of Lemma 3.3, M_Y is nef, so $M_Y+\epsilon A$ is ample. Thus there exist a sufficiently ample divisor A' and a rational number $0<\epsilon'\ll 1$ such that $M_Y+\epsilon A\sim_{\mathbb{Q}}\epsilon' A'$ and $(Y,B_Y+\epsilon E+\epsilon' A')$ is also klt. It follows that

$$\begin{aligned}(1+\epsilon)(K_Y+M_Y+B_Y) &\sim_{\mathbb{Q}} K_Y+M_Y+B_Y+\epsilon A+\epsilon E\\ &\sim_{\mathbb{Q}} K_Y+B_Y+\epsilon E+\epsilon' A'.\end{aligned}$$

By [BCHM10], $(Y,B_Y+\epsilon E+\epsilon' A')$ has a log terminal model Y^m and a log canonical model Y^c. It is easy to see that Y^m (resp. Y^c) is also a log terminal model (resp. log canonical model) of (Y,M_Y+B_Y). □

Lemma 3.5. *Under the notations and assumptions as in Lemma 2.10, the boundary part B_V of f_W is the restriction of B_Y to V and the moduli part M_V of f_W is $\mathbb{Q}$-linearly equivalent to the restriction of M_Y.*

Proof. Since (Y,B_Y) is log smooth and V is a very general fiber of $\varphi: Y\to T$, we may assume that $B_Y|_V$ has simple normal crossings support. Let $B_Y=\sum_P r_P P$ and $B_V=\sum_Q r'_Q Q$. Recall that $1-r_P$ is the log canonical threshold of f^*P with respect to $(X,-D_X/b)$ over the generic point of P and $1-r'_Q$ is the log canonical threshold of f_W^*Q with respect to $(W,-D_W/b)$ over the generic point of Q, where $D_X=bK_X-f^*(bK_Y+L_{X/Y})$ and $D_W=bK_W-f_W^*(bK_V+L_{W/V})$ (see [Fuj03, Defintion 3.4]). Since W is a very general fiber, we have $D_X|_W=D_W$. Hence $r'_Q=0$ when Q is not contained in the support of $B_Y|_V$ and $r'_Q=r_P$ when Q is the restriction of some component P of B_Y. Therefore $B_V=B_Y|_V$. On the other hand, we have $K_V+M_V+B_V\sim_{\mathbb{Q}}(K_Y+M_Y+B_Y)|_V$. Hence $M_V\sim_{\mathbb{Q}} M_Y|_V$. □

Variation. Let $f: X\to Y$ be an algebraic fiber space. Let $K\supset\mathbb{C}$ be an algebraically closed field contained in $\overline{\mathbb{C}(Y)}$ such that there is a finitely generated extension L of K such that $Q(L\otimes_K\overline{\mathbb{C}(Y)})\cong Q(\mathbb{C}(X)\otimes_{\mathbb{C}(Y)}\overline{\mathbb{C}(Y)})$ over $\overline{\mathbb{C}(Y)}$, where Q denotes the fraction field. The minimum of $\mathrm{tr.deg}_{\mathbb{C}}K$ for all such K is called the **variation** of f and denoted by $\mathrm{Var}(f)$.

Theorem 3.6. *Let $f: X\to Y$ be the Iitaka fibration as in* [FM00, 4.4]. *If the generic fiber F of f has a good minimal model, then the following are equivalent:*

1. *M_Y is numerically trivial.*
2. $M_Y\sim_{\mathbb{Q}}0$.
3. $\kappa(Y,M_Y)=0$.
4. $\mathrm{Var}(f)=0$.

Proof. (1)$\Longleftrightarrow$(2) is followed by [Amb05, Theorem 3.5]. The implication (2)$\Longrightarrow$(3) is trivial. Since F has a good minimal model, following [Kaw85, Theorem 1.1], we have (3)$\Longleftrightarrow$(4) (cf. [Fuj03, Remark 3.9]). Finally, Fujino [Fuj03, Theorem 3.11] proves the implication (4)$\Longrightarrow$(2). □

4. Birational covering families of pure log canonical centers

In this section, we construct a birational covering family of pure log canonical centers.

Recall that a subset P of a variety Y is called **countably dense** if it is not contained in the union of countably many closed subsets of Y.

Lemma 4.1. *Let (Y, Δ) be a log pair, where Y is projective and let D be a big $\mathbb{Q}$-Cartier divisor on Y. Suppose that for every point $y \in P$, where P is a countably dense subset of Y, we can find a pair (Δ_y, W_y) such that W_y is a pure log canonical center for $K_Y + \Delta + \Delta_y$ at y and $\Delta_y \sim_{\mathbb{Q}} D/w_y$ for some positive rational number w_y. Then there exists a diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ \big\downarrow{\scriptstyle \varphi} & & \\ T & & \end{array}$$

such that φ is a dominant morphism of normal projective varieties with connected fibers and for a general fiber V_t of φ there exists $y \in \varphi(V_t)$ so that $\varphi(V_t)$ is a pure log canonical center for $K_Y + \Delta + \Delta_t$ with $\Delta_t \sim_{\mathbb{Q}} D/w$ at y, for some weight w. Also π is a generically finite and dominant morphism of normal varieties.

Proof. See [M^{c}K02, Lemma 3.2] or [Tod08, Lemma 3.2]. □

Lemma 4.2 (M^{c}Kernan)**.** *Let (Y, Δ) be a log pair, where Y is a normal projective variety of dimension n'. Let D be a nef and big $\mathbb{Q}$-Cartier divisor. Let (Δ_t, V_t) be a covering family of weight less than w and dimension k.*

If (Δ_t, V_t) is not birational then we may find a covering family of (Γ_s, W_s) of weight $w/(n' - k)$ and dimension l, where either

(1) *$l > k$, or*
(2) *$l < k$ and (Γ_s, W_s) is a birational family.*

Remark 4.3. Lemma 4.2 still holds if we only assume that D is big instead of nef and big.

Proof. See [M^{c}K02, Lemma 4.2]. □

Corollary 4.4. *Let (Y, Δ) be a log pair, where Y is a normal projective variety of dimension n'. Let D be a big $\mathbb{Q}$-Cartier divisor. Let (Δ_t, V_t) be a covering family of weight w and dimension k. Then there exists a birational covering family of (Γ_s, W_s) of weight $w' \geq w/(n' - 1)!$.*

Proof. This is immediate from Lemma 4.2. □

By Lemma 3.3, $K_Y + M_Y + B_Y$ is a big $\mathbb{Q}$-divisor on Y, where Y is a smooth projective variety of dimension n', so for each point $y \in Y$, we can find a pair (D_y, V_y) such that

(1) $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$, for some rational number $\lambda > 0$,
(2) V_y is a pure log canonical center of (Y, D_y) at y.

Note that we can take the same λ for every point in a countably dense subset of Y with $\dim(V_y) = k$. Then by the previous corollary we obtain a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ \big\downarrow{\scriptstyle \varphi} & & \\ T & & \end{array}$$

such that

(1) π is birational and φ is dominant.

(2) Let $V_t = \pi(V'_t)$, where V'_t is a general fiber of φ. Then there exists a $\mathbb{Q}$-divisor $D_t \sim_{\mathbb{Q}} \lambda'(K_Y + M_Y + B_Y)$ on Y such that V_t is a pure log canonical center of (Y, D_t) and $\lambda' \leq \lambda(n'-1)!$.

Proposition 4.5. *Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2. Suppose that for any y in a countably dense subset of Y, there is an effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ such that $y \in \text{Non-klt}(Y, D_y)$. Then there exists a diagram*

$$\begin{CD} X' @>>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{\pi}>> Y \\ @V{\varphi}VV \\ T \end{CD}$$

such that

(1) *X' and Y' are smooth projective varieties.*
(2) *π is birational, φ is dominant with $\dim T \geq 0$ and f' satisfies the hypotheses of Theorem 1.2.*
(3) *For any very general fiber V'_t of φ, there exists an effective $\mathbb{Q}$-divisor $D'_t \sim_{\mathbb{Q}} \lambda'(K_{Y'} + M_{Y'} + B_{Y'})$ on Y' such that V'_t is a pure log canonical center of (Y', D'_t) and $\lambda' \leq \lambda(n'-1)!$, where $n' = \dim Y$.*

Proof. By our discussions above, there exists a covering family $Y' \overset{\varphi}{\to} T$ such that $Y' \overset{\pi}{\to} Y$ is birational. Now replace Y' by a smooth model and let X' be the resolution of the main component of $X \times_Y Y'$. It is easy to see that f' and f have the same generic fiber. Hence, (1) and (2) are satisfied. We only need to show (3).

Let $V_t = \pi(V'_t)$. By our assumptions and previous discussions, there is an effective $\mathbb{Q}$-divisor $D_t \sim_{\mathbb{Q}} \lambda'(K_Y + M_Y + B_Y)$ on Y such that V_t is a pure log canonical center of (Y, D_t) and $\lambda' \leq \lambda(n'-1)!$. Since π is birational, for all $m \in \mathbb{Z}_{>0}$ sufficiently divisible, we have

$$\begin{aligned} H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + M_{Y'} + B_{Y'}))) &\cong H^0(X', \mathcal{O}_{X'}(mK_{X'})) \\ &\cong H^0(X, \mathcal{O}_X(mK_X)) \\ &\cong H^0(Y, \mathcal{O}_Y(m(K_Y + M_Y + B_Y))). \end{aligned}$$

So there is an effective $\mathbb{Q}$-divisor $D'_t \sim_{\mathbb{Q}} \lambda'(K_{Y'} + M_{Y'} + B_{Y'})$ on Y' such that $\pi(D'_t) = D_t$. Since V'_t is a very general fiber of φ, (Y', D'_t, V'_t) and (Y, D_t, V_t) are isomorphic at the generic point of V'_t. Therefore, V'_t is a pure log canonical center of (Y', D'_t). □

Lemma 4.6 ([M^cK02, Lemma 5.3])**.** *Let (Y, Δ) be a log pair and let D be a $\mathbb{Q}$-divisor of the form $A + E$ where A is ample and E is effective. Let (Δ_t, V_t) be a covering family of weight greater than w and dimension k. Let A_t be the restriction of A to V_t. Suppose that for all very general points $t \in U$ we may find a covering family of $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight, with respect to A_t, greater than w'.*

Then we may find a covering family of (Γ_s, W_s) of dimension less than k and weight

$$w'' = \frac{ww'}{w + w'}.$$

Further if both (Δ_t, V_t) and $(\Gamma_{t,s}, W_{t,s})$ are birational families then so is (Γ_s, W_s).

5. Subadjunction

In his fundamental paper [Kaw98], Kawamata proves a remarkable subadjunction theorem. An immediate consequence of this theorem is that if (X, D) is a log canonical pair, V is a non-klt center of (X, D), then we have $(K_X + D)|_V \sim_{\mathbb{Q}} K_V + \Delta_V$, where Δ_V is a pseudoeffective divisor on V. Actually, one can prove a more precise result.

Proposition 5.1 (Subadjunction)**.** *Let X be a normal variety and D an effective $\mathbb{Q}$-divisor on X such that (X, D) is a log pair. If V is a pure log canonical center of (X, D) and $\nu : V^\nu \to V$ is the normalization, then we have*

$$(K_X + D)|_{V^\nu} \sim_{\mathbb{Q}} K_{V^\nu} + \Delta_{V^\nu},$$

where Δ_{V^ν} is an effective $\mathbb{Q}$-divisor.

Remark 5.2. Recently, Fujino and Gongyo [FG10] prove the much stronger result that if (X, D) is an lc pair and V is a minimal non-klt center of (X, D), then there exists an effective $\mathbb{Q}$-divisor Δ_V on V such that $(K_X + D)|_V \sim_{\mathbb{Q}} K_V + \Delta_V$ and (V, Δ_V) is klt.

This result depends on Ambro's results on the moduli (**b**-)divisor associated to an lc-trivial fibration .

Theorem 5.3 (Ambro)**.** *Let $f : (X, B) \to Y$ be an lc-trivial fibration such that the generic geometric fiber $X_{\bar{\eta}} = X \times_Y \operatorname{Spec}(\overline{k(Y)})$ is a projective variety and $B_{\bar{\eta}}$ is effective. Then there exists a diagram*

$$\begin{array}{ccccc} (X, B) & & & & (X^!, B^!) \\ \downarrow f & & & & \downarrow f^! \\ Y & \xleftarrow{\ \tau\ } & \bar{Y} & \xrightarrow{\ \varrho\ } & Y^! \end{array}$$

satisfying the following properties:

- *$f^! : (X^!, B^!) \to Y^!$ is an lc-trivial fibration.*
- *τ is generically finite and surjective and ϱ is surjective.*
- *There exists a nonempty open subset $U \subset \bar{Y}$ and an isomorphism*

$$\begin{array}{ccc} (X, B) \times_Y \bar{Y}|_U & \xrightarrow{\ \simeq\ } & (X^!, B^!) \times_{Y^!} \bar{Y}|_U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

- *Let $\mathbf{M}$ and $\mathbf{M}^!$ be the corresponding moduli $\mathbb{Q}$-b-divisors. Then $\mathbf{M}^!$ is b-nef and big and $\tau^*\mathbf{M} = \varrho^*(\mathbf{M}^!)$, which implies $\mathbf{M}$ is b-nef and good. In particular, $\mathbf{M}$ is $\mathbb{Q}$-linearly equivalent to an effective divisor.*

Proof. See [Amb05, Theorem 3.3]. □

Before giving the proof of 5.1, we need the following useful lemmas.

Lemma 5.4 (Hacon)**.** *Let X be a normal quasi-projective variety and B a boundary $\mathbb{R}$-divisor on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Then, there exists a projective birational morphism $f : Y \to X$ from a normal quasi-projective variety Y with the following properties.*

(1) *Y is $\mathbb{Q}$-factorial.*
(2) *$a(E, X, B) \leq -1$ for every f-exceptional divisor E on Y.*
(3) *We put*

$$B_Y = f_*^{-1}B + \sum_{E:\subset Ex(f)} E.$$

Then (Y, B_Y) is dlt and

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(E,X,B)<-1} (a(E, X, B) + 1)E.$$

In particular, if (X, B) is lc, then $K_Y + B_Y = f^*(K_X + B)$. Moreover, if (X, B) is dlt, then we can assume that f is small, that is, f is an isomorphism in codimension one.

Proof. See e.g. [Fuj09, Theorem 10.4]. □

Remark 5.5. Lemma 5.4 still holds if the coefficients of some components of B are greater than 1. But we need to replace (3) by

(3') Let

$$B_Y = f_*^{-1}B^{\leq 1} + \mathrm{Supp} f_*^{-1}B^{>1} + \sum_{E:\subset Ex(f)} E.$$

Then (Y, B_Y) is dlt and

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(F,X,B)<-1} (a(F, X, B) + 1)F.$$

Lemma 5.6 (Adjunction for dlt pairs)**.** *Let (X, D) be a dlt pair. We put $S = \lfloor D \rfloor$ and let $S = \sum_{i \in I} S_i$ be the irreducible decompostion of S. Then, W is a non-klt centre for the pair (X, D) with $\mathrm{codim}_X W = k$ if and only if W is an irreducible component of $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ for some $\{i_1, i_2, \cdots, i_k\} \subset I$. By adjunction, we obtain*

$$K_{S_{i_1}} + \mathrm{Diff}(D - S_{i_1}) = (K_X + D)|_{S_{i_1}},$$

and $(S_{i_1}, \mathrm{Diff}(D - S_{i_1}))$ is dlt. Note that S_{i_1} is normal, W is a non-klt center for the pair $(S_{i_1}, \mathrm{Diff}(D - S_{i_1}))$, $S_{i_j}|_{S_{i_1}}$ is a reduced component of $\mathrm{Diff}(D - S_{i_1})$ for $2 \leq j \leq k$, and W is an irreducible component of $(S_{i_2}|_{S_{i_1}}) \cap (S_{i_3}|_{S_{i_1}}) \cap \cdots \cap (S_{i_k}|_{S_{i_1}})$. By applying adjunction k times, we obtain a $\mathbb{Q}$-divisor $\Delta \geq 0$ on W such that

$$(K_X + D)|_W = K_W + \Delta$$

and (W, Δ) is dlt.

Proof. See [Cor07, Proposition 3.9.2]. □

Proof of Proposition 5.1. Applying Lemma 5.4 and Remark 5.5, we may get a morphism $f : Y \to X$ satisfying the properties of Lemma 5.4. Let $D_Y = f_*^{-1}D^{\leq 1} + \mathrm{Supp} f_*^{-1}D^{>1} + \sum_{E:\subset Ex(f)} E$. Then we have

$$f^*(K_X + D) = K_Y + D_Y - \sum_{a(F,X,D)<-1} (a(F, X, D) + 1)F,$$

and the pair (Y, D_Y) is dlt. Since V is a pure log canonical center of (X, D), F is vertical over V if $a(F, X, D) < -1$.

Let W be a minimal non-klt center of (Y, D_Y) over the generic point of V and $\nu : V^\nu \to V$ the normalization of V. We obtain the following diagram

$$\begin{array}{ccccccc} & & W & \hookrightarrow & & & Y \\ & {}^{s}\swarrow & \downarrow g & & & & \downarrow f \\ U & \xrightarrow{t} & V^\nu & \xrightarrow{\nu} & V & \hookrightarrow & X \end{array}$$

where $g : W \to V^\nu$ is the induced morphism and $W \xrightarrow{s} U \xrightarrow{t} V^\nu$ is the Stein factorization of g.

By Lemma 5.6, there exists a log pair (W, Δ_W), where $\Delta_W \geq 0$, such that

$$K_W + \Delta_W \sim_{\mathbb{Q}} (K_Y + D_Y - \sum_{a(F,X,D)<-1} (a(F, X, D) + 1)F)|_W \sim_{\mathbb{Q}} f^*(K_X + D)|_W,$$

and the non-klt centers of (W, Δ_W) are vertical over V^ν, so (W, Δ_W) has klt singularities over the generic point of V^ν. It follows that (W, Δ_W) is klt over the generic point of U. Moreover,

$$K_W + \Delta_W \sim_{\mathbb{Q}} g^*((K_X + D)|_{V^\nu}) \sim_{\mathbb{Q}} s^*((K_X + D)|_U).$$

Therefore, $s : (W, \Delta_W) \to U$ is an lc-trivial fibration as defined in [Amb04, Definition 2.1].

We may write $(K_X + D)|_U \sim_{\mathbb{Q}} K_U + M + B$, where M is the moduli part and B is the boundary part of this lc-trivial fibration. Since $\Delta_W \geq 0$, $B \geq 0$. By Theorem 5.3, we may assume that M is effective. Let $\Delta_U = M + B$, then,

$$(K_X + D)|_U \sim_{\mathbb{Q}} K_U + \Delta_U$$

and $\Delta_U \geq 0$. Since $t : U \to V^\nu$ is finite and $K_U + \Delta_U \sim_{\mathbb{Q}} t^*((K_X + D)|_{V^\nu})$, it is easy to see that there exists an effective $\mathbb{Q}$-divisor Δ_{V^ν} on V^ν such that

$$(K_X + D)|_{V^\nu} \sim_{\mathbb{Q}} K_{V^\nu} + \Delta_{V^\nu}.$$

□

6. Creating isolated non-klt centers

Proposition 6.1. *Assume that Theorem 1.2 holds for varieties of dimensions $< n$. Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2 with* $\dim X = n$ *and* $\dim Y = n'$. *Then there exist positive constants α and β depending on n, b and k, such that for any very general point $y \in Y$ there is an effective $\mathbb{Q}$-divisor D_y such that*

(1) $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$, *where* $\lambda < \dfrac{\alpha}{\mathrm{vol}(Y, K_Y + M_Y + B_Y)^{1/n'}} + \beta$;

(2) *y is an isolated point of* Non-klt(Y, D_y).

Proof. Take a very general point $y \in Y$. Since $K_Y + M_Y + B_Y$ is big, by the argument in the proof of [Pac07b, Theorem 6.2], we can pick an effective $\mathbb{Q}$-divisor $D_0 \sim_{\mathbb{Q}} \lambda_0(K_Y + M_Y + B_Y)$ which has multiplicity $> n_0$ at y, where $n_0 = n'$ and $\lambda_0 < n_0(\mathrm{vol}(Y, K_Y + M_Y + B_Y))^{-1/n_0} + \varepsilon_0$ with $1 \gg \varepsilon_0 > 0$. Hence there is a component V_0 of Non-klt(Y, D_0) passing through y. Multiplying D_0 by a positive rational number ≤ 1, we can assume that V_0 is a pure log canonical center of (Y, D_0).

By Proposition 4.5, we may replace Y with a higher smooth birational model such that there exists a morphism $\varphi : Y \to T$ satisfying the properties of 4.5. Therefore, the point y is contained in a very general fiber V_1 of φ and there is an effective $\mathbb{Q}$-divisor $D_1 \sim_{\mathbb{Q}} \lambda_1(K_Y + M_Y + B_Y)$ on Y with $\lambda_1 \leq \lambda_0(n_0-1)! < n_0!(\mathrm{vol}(Y, K_Y + M_Y + B_Y))^{-1/n_0} + \varepsilon_0(n_0-1)!$, such that V_1 is a pure log canonical center of (Y, D_1).

By Lemma 3.4, there is a log canonical model Y' of $(Y, M_Y + B_Y)$. Replacing Y with a higher smooth birational model, we may assume that there is a morphism $\phi : Y \to Y'$. Let $M_{Y'} = \phi_* M_Y$ and $B_{Y'} = \phi_* B_Y$. Then $K_{Y'} + M_{Y'} + B_{Y'}$ is $\mathbb{Q}$-Cartier and ample on Y'.

By our assumption, the generic fiber of f has a good minimal model. Applying Theorem 2.11, there exists a good minimal model X' of X. Replacing X with a higher smooth birational model, we may assume that there is a morphism $\psi : X \to X'$. Hence, we obtain a diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\phi} & Y' \\ {\scriptstyle \varphi}\downarrow & & \\ T & & \end{array}$$

where f' is the induced rational map.

Remark 6.2. The generic fiber of f may have changed after running the Minimal Model Program, so f may not satisfy the hypotheses of Theorem 1.2. But since our new X is a higher birational model of the original one, we do not change either M_Y or B_Y by the Canonical Bundle Formula.

Lemma 6.3. *We have the following:*

(1) *Y' is isomorphic to the weak canonical model $(X')^w$ of X' in the sense that*

$$(X')^w = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X', \mathcal{O}_{X'}(mK_{X'})).$$

(2) *f' is a morphism and $K_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + B_{Y'})$.*

Proof. X' is a good minimal model, so X' admits a morphism to its weak canonical model $(X')^w$. On the other hand, $K_{Y'} + M_{Y'} + B_{Y'}$ is ample on Y', so

$$Y' = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + M_{Y'} + B_{Y'}) \rfloor)).$$

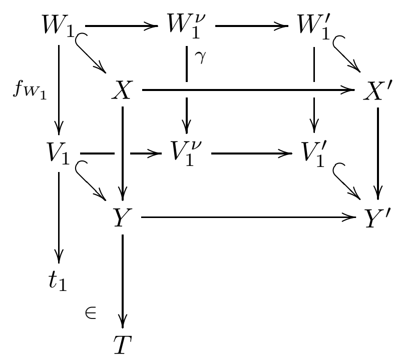
If $m \in \mathbb{Z}_{>0}$ is sufficiently divisible, by the Canonical Bundle Formula we have

$$\begin{aligned} H^0(X', \mathcal{O}_{X'}(mK_{X'})) &\cong H^0(X, \mathcal{O}_X(mK_X)) \\ &\cong H^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y) \rfloor)) \\ &\cong H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + M_{Y'} + B_{Y'}) \rfloor)). \end{aligned}$$

Hence Y' is the weak canonical model of X' and (2) follows from (1). □

Now let $y' = \phi(y)$, $V_1' = \phi(V_1)$, and $D_1' = \phi_*(D_1)$ and let $n_1 = \dim V_1 = \dim V_1'$. Since V_1 is a pure log canonical center of (Y, D_1) and y' is very general, it follows that

V_1' is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D_1')$ at y'. Let $W_1 = f^{-1}(V_1)$, $W_1' = f'^{-1}(V_1')$, V_1^ν the normalization of V_1', W_1^ν the normalization of W_1' and $\gamma : W_1^\nu \to V_1^\nu$ the induced morphism. We have the following diagram



By Lemma 2.10 and Lemma 3.5, the morphism $f_{W_1} : W_1 \to V_1$ is the Iitaka fibration of (W_1, K_{W_1}) and the moduli part M_{V_1} of f_{W_1} is $\mathbb{Q}$-linearly equivalent to the restriction of M_Y to V_1. Thus we can assume that f_{W_1} satisfies the hypotheses of Theorem 1.2.

Remark 6.4. As in Remark 6.2, the generic fiber of f_{W_1} may be different from the original one. However this does not affect the computation of M_{V_1} and B_{V_1}.

Lemma 6.5. *There exists a constant $\delta > 0$ depending on $n-1, b$ and k, such that* $\mathrm{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) \geq \delta$.

Proof. Since $\mathrm{dim} W_1 < n$, by our assumptions in Proposition 6.1, there exists a positive integer m_1 depending on $n-1, b$ and k, such that $\phi_{m_1(K_{V_1}+M_{V_1}+B_{V_1})}$ gives a birational map. Then $\mathrm{vol}(V_1, m_1(K_{V_1} + M_{V_1} + B_{V_1})) \geq 1$ by Lemma 2.2. Therefore,

$$\begin{aligned}
\mathrm{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) &= \frac{1}{m_1^{n_1}} \mathrm{vol}(V_1, m_1(K_{V_1} + M_{V_1} + B_{V_1})) \\
&\geq \frac{1}{m_1^{n_1}} \\
&\geq \frac{1}{m_1^{n-1}}.
\end{aligned}$$

Now let δ be $1/m_1^{n-1}$. □

We have the following fact.

Lemma 6.6. $\mathrm{vol}(V_1', (K_{Y'} + M_{Y'} + B_{Y'} + D_1')|_{V_1'}) \geq \delta$.

Proof. By Lemma 6.3, we have $K_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + B_{Y'})$. V_1' is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D_1')$ and y' is a very general point of Y', so W_1' is a pure log canonical center of $(X', f'^* D_1')$.

By Proposition 5.1, there exists an effective $\mathbb{Q}$-divisor $\Delta_{W_1^\nu}$ on W_1^ν, such that

$$(K_{X'} + f'^* D_1')|_{W_1^\nu} \sim_{\mathbb{Q}} K_{W_1^\nu} + \Delta_{W_1^\nu}.$$

On the other hand,

$$(K_{X'} + f'^* D_1')|_{W_1^\nu} \sim_{\mathbb{Q}} \gamma^*((K_{Y'} + M_{Y'} + B_{Y'} + D_1')|_{V_1^\nu}).$$

For all $m \in \mathbb{Z}_{>0}$ sufficiently divisible, by the Projection Formula we have

$$h^0(W_1^\nu, \mathcal{O}_{W_1^\nu}(m(K_{W_1^\nu}+\Delta_{W_1^\nu}))) = h^0(V_1^\nu, \mathcal{O}_{V_1^\nu}(m(K_{Y'}+M_{Y'}+B_{Y'}+D'_1)|_{V_1^\nu})). \quad (*)$$

By the Canonical Bundle Formula,

$$h^0(W_1, \mathcal{O}_{W_1}(mK_{W_1})) = h^0(V_1, \mathcal{O}_{V_1}(m(K_{V_1} + M_{V_1} + B_{V_1}))). \quad (**)$$

Since W_1 is smooth and $\Delta_{W_1^\nu} \geq 0$, it follows that

$$h^0(W_1^\nu, \mathcal{O}_{W_1^\nu}(m(K_{W_1^\nu} + \Delta_{W_1^\nu}))) \geq h^0(W_1, \mathcal{O}_{W_1}(mK_{W_1})).$$

Therefore, by equations $(*)$ and $(**)$,

$$h^0(V_1^\nu, \mathcal{O}_{V_1^\nu}(m(K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V_1^\nu})) \geq h^0(V_1, \mathcal{O}_{V_1}(m(K_{V_1} + M_{V_1} + B_{V_1}))),$$

which implies

$$\mathrm{vol}(V_1^\nu, (K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V_1^\nu}) \geq \mathrm{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}).$$

Note that the normalization $\nu : V_1^\nu \to V'_1$ is birational. Thus we have

$$\begin{aligned} \mathrm{vol}(V'_1, (K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V'_1}) &= \mathrm{vol}(V_1^\nu, (K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V_1^\nu}) \\ &\geq \mathrm{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) \\ &\geq \delta. \end{aligned}$$

□

Let $\phi_{V_1} : V_1 \to V'_1$ be the restriction of ϕ to V_1. We have

$$\phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_1} \sim_{\mathbb{Q}} \phi_{V_1}^*((K_{Y'} + M_{Y'} + B_{Y'})|_{V'_1}).$$

Recall that $D'_1 \sim_{\mathbb{Q}} \lambda_1(K_{Y'} + M_{Y'} + B_{Y'})$, so by Lemma 6.6 it follows that

$$\begin{aligned} \mathrm{vol}(V_1, \phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_1}) &= \mathrm{vol}(V'_1, (K_{Y'} + M_{Y'} + B_{Y'})|_{V'_1}) \\ &= \frac{\mathrm{vol}(V'_1, (K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V'_1})}{(1 + \lambda_1)^{n_1}} \\ &\geq \frac{\delta}{(1 + \lambda_1)^{n_1}}. \end{aligned}$$

Hence for any very general fiber V_t of φ, we always have

$$\mathrm{vol}(V_t, \phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_t}) \geq \delta(1 + \lambda_1)^{-n_1}.$$

Then for any point $p \in V_t$, there exists an effective $\mathbb{Q}$-divisor $E_{t,p} \sim_{\mathbb{Q}} \lambda_{t,p}(\phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_t})$ on V_t such that $\mathrm{mult}_p E_{t,p} > n_1$ and

$$\begin{aligned} \lambda_{t,p} &< \frac{n_1}{\mathrm{vol}(V_t, \phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_t})^{1/n_1}} + \varepsilon_1 \\ &< \frac{n_0! n_1}{\delta^{1/n_1} \mathrm{vol}(Y, K_Y + M_Y + B_Y)^{1/n_0}} + (1 + \varepsilon_0(n_0 - 1)!) \frac{n_1}{\delta^{1/n_1}} + \varepsilon_1 \end{aligned}$$

where $0 < \varepsilon_1 \ll 1$. This implies that there is a component of Non-klt$(V_t, E_{t,p})$ passing through p. Multiplying $E_{t,p}$ by a positive rational number ≤ 1, we can assume that p is contained in a pure log canonical center of $(V_t, E_{t,p})$.

Applying Lemma 4.1 and Corollary 4.4, there exists a birational covering family of $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight w' with respect to $\phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_t}$ such that

$\Gamma_{t,s} \sim_{\mathbb{Q}} (1/w')\phi^*(K_{Y'} + M_{Y'} + B_{Y'})|_{V_t}$ and the image of $W_{t,s}$ on V_t is a pure log canonical center of $(V_t, \Gamma_{t,s})$, where

$$\frac{1}{w'} < \frac{n_0!n_1!}{\delta^{1/n_1}\mathrm{vol}(Y, K_Y + M_Y + B_Y)^{1/n_0}} + (1 + \varepsilon_0(n_0 - 1)!)\frac{n_1!}{\delta^{1/n_1}} + \varepsilon_1(n_1 - 1)!.$$

By Lemma 4.6, we can find a new birational covering family of (D'_s, V''_s) on Y' of dimension less than n_1 and weight w'' such that

$$\begin{aligned} \frac{1}{w''} &= \lambda_1 + \frac{1}{w'} \\ &< \frac{n_0!n_1!\delta^{-1/n_1} + n_0!}{\mathrm{vol}(Y, K_Y + M_Y + B_Y)^{1/n_0}} \\ &\quad + (1 + \varepsilon_0(n_0 - 1)!)\frac{n_1!}{\delta^{1/n_1}} + \varepsilon_1(n_1 - 1)! + \varepsilon_0(n_0 - 1)!. \end{aligned}$$

Therefore, we obtain the following diagram

$$\begin{array}{ccc} Y'' & \xrightarrow{\phi''} & Y' \\ {\scriptstyle \varphi''}\downarrow & & \\ S & & \end{array}$$

where ϕ'' is birational and φ'' is surjective. For the very general point $y' \in Y'$, there are an effective $\mathbb{Q}$-divisor $D'_s \sim_{\mathbb{Q}} \lambda_2(K_{Y'} + M_{Y'} + B_{Y'})$ on Y' with $\lambda_2 = 1/w''$ and a very general fiber V''_s of φ'' such that $V'_2 = \phi''(V''_s)$ is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D'_s)$ at y' with $\mathrm{dim}V'_2 < \mathrm{dim}V'_1 = n_1$. Replacing Y'' with the common higher smooth model of Y, Y' and Y'', we can assume that Y'' is smooth and the dimension of any very general fiber of $\varphi'' : Y'' \to S$ is strictly less than that of $\varphi : Y \to T$. The moduli part $M_{Y''}$ on Y'' is still $\mathbb{Q}$-linearly trivial, since it is the pullback of M_Y.

Repeating above procedure at most $n'-1$ times, there exists an effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} \lambda(K_{Y'} + M_{Y'} + B_{Y'})$ on Y' with $\lambda < \alpha(\mathrm{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta$, where α and β depend only on n, k and b, such that y' is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D')$. By the standard tie-breaking technique, we can assume that y' is the unique non-klt center of $(Y', M_{Y'} + B_{Y'} + D')$ on a neighborhood of y', i.e. y' is an isolated point of Non-klt$(Y', M_{Y'} + B_{Y'} + D')$. Since Y' and Y are birational, there is a unique effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ on Y such that $\phi_*(D_y) = D'$. Then D_y satisfies the requirements in Proposition 6.1. This completes the proof. □

7. Proof of Theorem 1.2

Lemma 7.1. *Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2. Let m_0 be a positive integer and assume that for any very general point $y \in Y$, there exists an effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ where $\lambda \leq m_0 - 1$, such that y is an isolated point in* Non-klt(Y, D_y). *Then for all $m \geq m_0$ such that mM_Y is an integral divisor, i.e. m is divisible by bN, we have $h^0(X, \mathcal{O}_X(mK_X)) > 0$ and moreover, if $m \geq 2m_0$, then $h^0(X, \mathcal{O}_X(mK_X)) \geq 2$.*

Proof. Since $K_Y + M_Y + B_Y$ is big, there exist an ample $\mathbb{Q}$-divisor H and an effective $\mathbb{Q}$-divisor G on Y such that $K_Y + M_Y + B_Y \sim_{\mathbb{Q}} H + G$. Pick a very general point $y \in Y$ not contained in the support of $G + B_Y$. By Lemma 3.3, the divisor

$(\lfloor mB_Y\rfloor-(m-1)B_Y)$ is effective. Let $D'_y = D_y+(m-1-\lambda)G+\lfloor mB_Y\rfloor-(m-1)B_Y$. Then

$$\lfloor m(K_Y + M_Y + B_Y)\rfloor - K_Y - D'_y \sim_{\mathbb{Q}} (m-1-\lambda)H + M_Y$$

is ample so that $H^1(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y)\rfloor) \otimes \mathcal{J}(Y, D'_y)) = 0$.

Consider the short exact sequence of coherent sheaves on Y

$$0 \to \mathcal{O}_Y(\lfloor m(K_Y+M_Y+B_Y)\rfloor)\otimes \mathcal{J}(Y, D'_y) \to \mathcal{O}_Y(\lfloor m(K_Y+M_Y+B_Y)\rfloor) \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ denotes the corresponding quotient. By the discussion above, the map

$$H^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y)\rfloor)) \to H^0(Y, \mathcal{Q})$$

is surjective. Since y is an isolated point in Non-klt(Y, D'_y), $\mathbb{C}_y$ is a direct summand of $H^0(Y, \mathcal{Q})$. Thus, we have

$$h^0(X, \mathcal{O}_X(mK_X)) = h^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y)\rfloor)) > 0.$$

Pick a very general point $y_1 \in Y$. Then there is an effective $\mathbb{Q}$-divisor $D_{y_1} \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ such that y_1 is an isolated point in Non-klt(Y, D_{y_1}). Now we may pick a very general point $y_2 \in Y$ not contained in the support of D_{y_1}, and pick a very general divisor $D_{y_2} \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ such that y_2 is an isolated point in Non-klt(Y, D_{y_2}) and y_1 is not contained in the support of D_{y_2}. Hence y_1 and y_2 are isolated points in Non-klt$(Y, D_{y_1} + D_{y_2})$. Then $h^0(X, \mathcal{O}_X(mK_X)) \geq 2$ by an argument similar to the discussion above. □

Lemma 7.2. *Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2. Let m'_0 be a positive integer divisible by bN. Assume that $h^0(X, mK_X) \geq 2$ for all $m \geq m'_0$ such that m is divisible by bN. Let $X' \to Y' \to \mathbb{P}^1$ be any morphism induced by sections of $\mathcal{O}_X(m'_0K_X)$ on an appropriate birational model $f' : X' \to Y'$ of $f : X \to Y$. Let $p \in \mathbb{P}^1$ be a very general point. $f_W : W \to V$ denotes the restriction of f' to the fiber over p. If there is a positive integer s divisible by bN such that $|sK_W|$ induces the Iitaka fibration for any very general point p, then $|tK_X|$ induces the Iitaka fibration for all $t \geq m'_0(2s + 2) + s$ such that t is divisible by bN.*

$$\begin{array}{ccccc} W & \hookrightarrow & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f_W} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ V & \hookrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \\ p & \in & \mathbb{P}^1 & & \end{array}$$

Proof. Following [Kol86, Theorem 4.6] and its proof, $|(m'_0(2s + 1) + s)K_X|$ gives the Iitaka fibration. Since mK_X is effective for all $m \geq m'_0$ such that m is divisible by bN, the assertion follows. □

Proof of Theorem 1.2. Since the moduli part is $\mathbb{Q}$-linearly trivial by Theorem 3.6, we always have $\mathrm{vol}(Y, K_Y+M_Y+B_Y) = \mathrm{vol}(Y, K_Y+B_Y)$. The proof is by induction on the dimension of X. It is well known that the theorem holds for $n = 1$. Assume that the theorem holds when $\dim X \leq n - 1$. Let $f : X \to Y$ be the Iitaka firation satisfying the hypotheses of Theorem 1.2 with $\dim X = n$ and $\dim Y = n'$. By Proposition 6.1, for any very general point $y \in Y$, there exists an effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ with $\lambda < \alpha(\mathrm{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta$, where α and β are two positive constants depending only on n, b and k, such that y is an isolated point in Non-klt(Y, D_y).

If $\text{vol}(Y, K_Y + M_Y + B_Y) = \text{vol}(Y, K_Y + B_Y) \geq 1$, Proposition 6.1, Lemma 7.1 and Lemma 7.2 imply that there exists an positive integer m_n only depending on n, b and k such that mK_X gives the Iitaka fibration if $m \geq m_n$ and divisible by bN.

Now we prove the case when $\text{vol}(Y, K_Y + M_Y + B_Y) = \text{vol}(Y, K_Y + B_Y) < 1$. By induction, there exists a positive integer s such that $|sK_W|$ gives the Iitaka fibration for all W with $\dim W \leq n-1$ satisfying the hypotheses of Theorem 1.2. By Proposition 6.1, Lemma 7.1 and Lemma 7.2, $|mK_X|$ induces the Iitaka fibration, for

$$m = 8bNs\lceil \frac{\alpha}{\text{vol}(Y, K_Y + M_Y + B_Y)^{1/n'}} + \beta + 1 \rceil,$$

so $\phi_{m(K_Y+M_Y+B_Y)}$ gives a birational map. As mM_Y is a $\mathbb{Q}$-linearly trivial Cartier divisor, $\phi_{K_Y+(2n'+1)m(K_Y+B_Y)}$ is also birational by Lemma 2.3. We have

$$\begin{aligned} \text{vol}(Y, (2n'+1)m(K_Y + B_Y)) &= (2n'+1)^{n'} m^{n'} \text{vol}(Y, K_Y + B_Y) \\ &\leq (2n'+1)^{n'} (8bNs)^{n'} (\alpha + \beta + 2)^{n'} \\ &\leq (2n+1)^n (8bNs)^n (\alpha + \beta + 2)^n. \end{aligned}$$

It follows that there is a constant A such that $\text{vol}(Y, (2n'+1)m(K_Y + B_Y)) \leq A$. Then Lemma 3.3 and Theorem 2.5 imply that the set of such log pairs (Y, B_Y) is log birationally bounded.

By Theorem 2.6, there exists a constant $\delta_n > 0$ such that

$$\text{vol}(Y, K_Y + B_Y) \geq \delta_n.$$

So we are done by applying Proposition 6.1, Lemma 7.1 and Lemma 7.2 again. □

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