# Hemisystems of small flock generalized quadrangles

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**Abstract**

In this paper, we describe a complete computer classification of the hemisystems in the two known flock generalized quadrangles of order $(5^2, 5)$ and give numerous further examples of hemisystems in all the known flock generalized quadrangles of order $(s^2, s)$ for $s \leqslant 11$. By analysing the computational data, we identify two possible new infinite families of hemisystems in the classical generalized quadrangle $\mathsf{H}(3, s^2)$.

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## 1. Introduction

A *hemisystem of lines* of a generalized quadrangle of order $(s^2, s)$ is a set $\mathcal{H}$ of lines such that every point $P$ is incident with $(s+1)/2$ elements of $\mathcal{H}$; that is, exactly half of the lines incident with each point lie in $\mathcal{H}$. The complementary set of lines to a hemisystem is also a hemisystem that may or may not be equivalent under the automorphism group of the generalized quadrangle — if it is equivalent to its complement then we call it *self-complementary*. Hemisystems give rise to various other combinatorial objects, including partial quadrangles (Cameron [6]), strongly regular graphs with certain parameters, and 4-class imprimitive cometric $Q$-antipodal association schemes[1] that are not metric (see Martin, Muzychuk and van Dam [14]), all of which were thought to be somewhat rare.

The notion of a hemisystem was introduced in 1965 by Segre [20] in his work on *regular systems* of the Hermitian surface, and he proved that there is a unique hemisystem of lines (up to equivalence) of the classical generalized quadrangle $\mathsf{H}(3, 3^2)$. It was long thought that this was the only hemisystem in $\mathsf{H}(3, q^2)$ and indeed Thas [22] conjectured this as late as 1995. However, forty years after Segre's seminal paper, Cossidente and Penttila [8] constructed an infinite family of hemisystems of the classical quadrangles $\mathsf{H}(3, q^2)$ and other authors subsequently constructed sporadic examples in $\mathsf{H}(3, q^2)$ [4, 7] and a single example in the non-classical generalized quadrangle $\mathsf{FTWKB}(5)$ (see [2]). The first main result of this paper extends the complete classification of hemisystems to the (known) generalized quadrangles of order $(5^2, 5)$.

**Theorem 1.1.** *A hemisystem of the classical generalized quadrangle* $\mathsf{H}(3, 5^2)$ *is equivalent to one of the two self-complementary hemisystems described in Table 5 and a hemisystem of the Fisher-Thas-Walker-Kantor-Betten generalized quadrangle* $\mathsf{FTWKB}(5)$ *is equivalent to one of the three complementary pairs described in Table 6.*

All known generalized quadrangles of order $(s^2, s)$, $s$ odd, arise from flocks of the quadratic cone and hence are called *flock generalized quadrangles*. In [3] we gave a general construction for hemisystems that

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[1] In fact, these cometric association schemes have Krein array $\{(q^2+1)(q+1), (q^2-q+1)^2/q, (q^2-q+1)(q-1)/q, 1; 1, (q^2-q+1)(q-1)/q, (q^2-q+1)^2/q, (q^2+1)(q-1)\}$.

| $q$ | GQ | Type I hemisystems | Other hemisystems | Total |
|---|---|---|---|---|
| 3 | $\mathsf{H}(3,3^2)$ | 1 | 0 | 1 |
| 5 | $\mathsf{H}(3,5^2)$ | 2 | 0 | 2 |
| | $\mathsf{FTWKB}(5)$ | $1\times 2$ | $2\times 2$ | 6 |
| 7 | $\mathsf{H}(3,7^2)$ | 2 | 4 | 6 |
| | $\mathsf{K}_2(7)$ | $6\times 2+2$ | $6\times 2+2$ | 28 |
| 9 | $\mathsf{H}(3,9^2)$ | 3 | 4 | 7 |
| | $\mathsf{K}_1(9)$ | 3 | 2 | 5 |
| | $\mathsf{Fi}(9)$ | $6\times 2+9$ | $4\times 2+5$ | 34 |
| 11 | $\mathsf{H}(3,11^2)$ | 6 | $1\times 2+5$ | 13 |
| | $\mathsf{FTWKB}(11)$ | $10\times 2$ | | 20 |
| | $\mathsf{Fi}(11)$ | $42\times 2+6$ | $6\times 2$ | 102 |
| | $\mathsf{PM}(11)$ | $74\times 2+8$ | $18\times 2$ | 192 |

Table 1: Known hemisystems in the flock generalized quadrangles of order $(s^2,s)$ for $s\leqslant 11$

produces a hemisystem in every flock generalized quadrangle, known or unknown. In fact (as pointed out to us by Tim Penttila), our construction shows that the number of hemisystems in any infinite family of flock generalized quadrangles grows exponentially with the size of the generalized quadrangle. Therefore, far from being rare, hemisystems and their associated partial quadrangles, strongly regular graphs etc. actually exist in great profusion. Of course this is an asymptotic result only, and so in this companion paper to [3], we consider hemisystems in the small (known) flock generalized quadrangles, namely those of order $(s^2,s)$ for all (odd) $s\leqslant 11$. Using a mixture of computation and analysis driven by the computational data, we discover large numbers of new hemisystems that do not arise from our general construction.

Table 1 summarizes the results of our investigations, dividing the hemisystems into those of Type 1 arising from construction of [3] which we review in Section 3, and those that do not arise from this construction. In this table, notation of the form $6\times 2+2$ is used to indicate that, up to equivalence under the automorphism group of the generalized quadrangle, there are 6 complementary pairs of hemisystems and 2 self-complementary hemisystems, for a total of 14 hemisystems. In Theorem 3.3 we show that a hemisystem of Type 1 in a generalized quadrangle of order $(q^2,q)$ is invariant under an elementary abelian group of order $q^2$, so one way to verify that a hemisystem is not of Type I is to show that it is not invariant under such a group.

By analysing the computational data for the classical generalized quadrangles $\mathsf{H}(3,q^2)$, we identify patterns that suggest the existence of three possible new infinite families of hemisystems. For these candidate families, we extend the computations to higher values of $q$ and, based on these computations, conjecture that just two of the three candidate families continue indefinitely. These families are discussed in Section 4.

We end the paper in Section 8 by discussing a number of questions and directions for future research suggested by our results.

## 2. Some basic background theory

A *generalized quadrangle* is an incidence structure of points and lines such that if $P$ is a point and $\ell$ is a line not incident with $P$, then there is a unique line through $P$ which meets $\ell$ in a point. From this property, if there is a line containing at least three points or if there is a point on at least three lines, then there are constants $s$ and $t$ such that each line is incident with $s+1$ points, and each point is incident with $t+1$ lines. Such a generalized quadrangle is said to have *order* $(s,t)$, and its point-line dual is a generalized quadrangle of order $(t,s)$.

In this paper we are concerned with generalized quadrangles of order $(s^2,s)$, for $s$ odd. The classical example is the incidence structure of all points and lines of a non-singular Hermitian variety in $\mathsf{PG}(3,q^2)$, which forms the *classical* generalized quadrangle $\mathsf{H}(3,q^2)$ of order $(q^2,q)$ (see [16, 3.2.3]). Further examples can be constructed from BLT-sets using the Knarr model. We briefly outline this construction below.

*2.1. Flocks of quadratic cones and BLT-sets*

A *flock* of the quadratic cone $\mathcal{C}$ with vertex $v$ in $\mathsf{PG}(3,q)$ is a partition of the points of $\mathcal{C}\backslash\{v\}$ into conics. J. A. Thas [21] showed that a flock gives rise to an elation generalized quadrangle of order $(q^2,q)$, which we call a *flock quadrangle.* A *BLT-set of lines* of $\mathsf{W}(3,q)$ is a set $\mathcal{O}$ of $q+1$ lines of $\mathsf{W}(3,q)$ such that no line of $\mathsf{W}(3,q)$ is concurrent with more than two lines of $\mathcal{O}$. In [1], it was shown that, for $q$ odd, a flock of a quadratic cone in $\mathsf{PG}(3,q)$ gives rise to a BLT-set of lines of $\mathsf{W}(3,q)$. Conversely, a BLT-set gives rise to possibly many flocks, however we only obtain one flock quadrangle up to isomorphism (see [15]).

For $q$ odd, Knarr [11] gave a direct geometric construction of a flock quadrangle from a BLT-set of lines of $\mathsf{W}(3,q)$. Applying this construction to a *linear* BLT-set of lines (i.e., a *regulus* obtained from field reduction of a Baer subline) of $\mathsf{W}(3,q)$, yields a generalized quadrangle isomorphic to the classical object $\mathsf{H}(3,q^2)$.

The BLT-sets of lines of $\mathsf{W}(3,q)$ have been classified by Law and Penttila [13] for prime powers $q$ at most 29, and this has recently been extended by Betten [5] to $q \leqslant 67$. We outline the main infinite families in Section 5.

*2.2. The Knarr model*

The symplectic polar space $\mathsf{W}(5,q)$ of rank 3 is the geometry arising from taking the one-, two- and three-dimensional vector subspaces of $\mathsf{GF}(q)^6$ for which a given alternating bilinear form restricts to the zero form (i.e., the *totally isotropic* subspaces). For example, one can take this alternating bilinear form to be defined by

$$\beta(\boldsymbol{x},\boldsymbol{y}) = x_1y_6 - x_6y_1 + x_2y_5 - x_5y_2 + x_3y_4 - x_4y_3.$$

In particular $\beta(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}J'\boldsymbol{y}^T$ where

$$J' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This bilinear form also determines a null polarity $\perp$ of the ambient projective space $\mathsf{PG}(5,q)$, defined by $U \mapsto U^\perp := \{\boldsymbol{v} \in \mathsf{GF}(q)^6 : \beta(\boldsymbol{u},\boldsymbol{v}) = 0 \text{ for all } \boldsymbol{u} \in U\}$.

The ingredients of the Knarr construction are as follows:

- a null polarity $\perp$ of $\mathsf{PG}(5,q)$;
- a point $P$ of $\mathsf{PG}(5,q)$;
- a BLT-set of lines $\mathcal{O}$ of $\mathsf{W}(3,q)$.

Note that the totally isotropic lines and planes incident with $P$ yield the quotient polar space $P^\perp/P$ isomorphic to $\mathsf{W}(3,q)$. So we will abuse notation and identify $\mathcal{O}$ with a set of totally isotropic planes on $P$. Then we construct a generalized quadrangle $\mathcal{K}(\mathcal{O})$ as follows:

We now describe how the Knarr model leads to some obvious automorphisms of the resulting generalized quadrangle $\mathcal{K}(\mathcal{O})$. Let $G$ be the semisimilarity group of the form $\beta$, that is, the group of all semilinear transformations $g$ of $\mathsf{GF}(q)^6$ for which there exists $\lambda \in \mathsf{GF}(q)$ and $\sigma \in \mathsf{Aut}(\mathsf{GF}(q))$ such that $\beta(\boldsymbol{u}^g,\boldsymbol{v}^g) = \lambda\beta(\boldsymbol{u},\boldsymbol{v})^\sigma$ for all $\boldsymbol{u},\boldsymbol{v} \in \mathsf{GF}(q)^6$. Let $H$ be the group of similarities of $\beta$, that is, the group of all linear transformations that preserve $\beta$ up to a scalar. Then

$$H = \{A \in \mathsf{GL}(6,q) \mid AJ'A^T = \lambda J' \text{ for some } \lambda \in \mathsf{GF}(q)\} \cong \mathsf{GSp}(6,q).$$

| | Points | | Lines |
|---|---|---|---|
| (i) | points of $\mathsf{PG}(5,q)$ not in $P^\perp$ | (a) | totally isotropic planes not contained in $P^\perp$ and meeting some element of $\mathcal{O}$ in a line |
| (ii) | lines not incident with $P$ but contained in some element of $\mathcal{O}$ | (b) | elements of $\mathcal{O}$ |
| (iii) | the point $P$ | | |

Incidence is inherited from that of $\mathsf{PG}(5,q)$.

Let

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and take $P$ to be the span of $[1,0,0,0,0,0]$. Then $H_P = E \rtimes (Q \times R)$ where

$$\begin{aligned} E &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ J^T \boldsymbol{a}^T & I & 0 \\ z & \boldsymbol{a} & 1 \end{pmatrix} \;\middle|\; \boldsymbol{a} \in \mathsf{GF}(q)^4, z \in \mathsf{GF}(q) \right\} \\ Q &= \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \;\middle|\; \lambda \in \mathsf{GF}(q)\backslash\{0\} \right\} \cong C_{q-1} \\ R &= \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; A \in \mathsf{GL}(4,q), AJA^T = \lambda J \right\} \cong \mathsf{GSp}(4,q) \end{aligned}$$

and $(H_P)_\mathcal{O} = E \rtimes (Q \times R_\mathcal{O}) \cong E \rtimes (Q \times \mathsf{GSp}(4,q)_\mathcal{O})$. Moreover, $G_P = \langle H_P, \sigma\rangle$, where $\sigma$ is the standard Frobenius map. Note that $\langle R, \sigma\rangle \cong \Gamma\mathsf{Sp}(4,q)$ and acts on $E/Z(E)$ as in its natural action on a 4-dimensional vector-space over $\mathsf{GF}(q)$. Moreover, $(G_P)_\mathcal{O} = E \rtimes (Q \rtimes \langle R, \sigma\rangle_\mathcal{O}) \cong E \rtimes (Q \rtimes \Gamma\mathsf{Sp}(4,q)_\mathcal{O})$. The group $(G_P)_\mathcal{O}$ preserves the flock generalized quadrangle $\mathcal{K}(\mathcal{O})$ and contains the subgroup $Z$ of all scalar matrices. Hence $E \rtimes \Gamma\mathsf{Sp}(4,q)_\mathcal{O} \cong (G_P)_\mathcal{O}/Z \leqslant \mathsf{Aut}(\mathcal{K}(\mathcal{O}))$. In fact, if the flock quadrangle $\mathcal{K}(\mathcal{O})$ is not classical, then these are the only automorphisms that you get, that is, $\mathsf{Aut}(\mathcal{K}(\mathcal{O})) = E \rtimes \Gamma\mathsf{Sp}(4,q)_\mathcal{O}$ [17, IV.1 and IV.2]. (Note: In the paper [3] we incorrectly claimed that additional automorphisms could arise for the Kantor-Knuth generalized quadrangles.)

## 3. Hemisystems of Type I and their automorphisms

In this section we revise the construction given in [3] and discuss the stabiliser of the resulting hemisystems.

**Lemma 3.1 (Bamberg, Giudici and Royle [3]).** *Consider a set $\mathcal{O}$ of totally isotropic planes of $\mathsf{W}(5,q)$ each incident with a point $P$ such that $\{\pi/P : \pi \in \mathcal{O}\}$ is a BLT-set of lines of the quotient symplectic space $P^\perp/P \cong \mathsf{W}(3,q)$. Define a binary relation $\equiv_\ell$ on $\mathcal{O}$ by setting $\pi \equiv_\ell \pi'$ if and only if*

$$\pi = \pi' \quad or \quad \{\langle Y, Y^\perp \cap \pi\rangle \mid Y \in \ell\} \cap \{\langle Y, Y^\perp \cap \pi'\rangle \mid Y \in \ell\} = \varnothing.$$

*Then $\equiv_\ell$ is an equivalence relation yielding a partition of $\mathcal{O}$ into two parts of equal size.*

**Theorem 3.2 (Bamberg, Giudici and Royle [3]).** *Consider a set $\mathcal{O}$ of totally isotropic planes of $\mathsf{W}(5,q)$ each incident with a point $P$ such that $\{\pi/P : \pi \in \mathcal{O}\}$ is a BLT-set of lines of the quotient* *symplectic space $P^\perp/P \cong \mathsf{W}(3,q)$. Suppose that we have a line $\ell$ of $\mathsf{W}(5,q)$ not meeting any element of $\mathcal{O}$, and let $\equiv_\ell$ be the binary relation on $\mathcal{O}$ defined in Lemma 3.1 with equivalence classes $\mathcal{O}^+$ and $\mathcal{O}^-$. Let $\mathcal{S}$ be a subset of the totally isotropic planes on $\ell$ of size $(q-1)/2$, not containing $\langle P,\ell\rangle$, and let $\mathcal{S}^c$ be the complementary set of planes on $\ell$. Let*

*(i) $\mathcal{L}^+_{\mathcal{S}}$ be the totally isotropic planes that meet some element of $\mathcal{O}^+$ in a line, and which meet some element of $\mathcal{S}$ in a point; and*

*(ii) $\mathcal{L}^-_{\mathcal{S}^c}$ be the totally isotropic planes that meet some element of $\mathcal{O}^-$ in a line, and which meet some element of $\mathcal{S}^c$ in a point;*

*Then $\mathcal{O}^+ \cup \mathcal{L}^+_{\mathcal{S}} \cup \mathcal{L}^-_{\mathcal{S}^c}$ is a hemisystem of lines of $\mathcal{K}(\mathcal{O})$.*

Recall that Cossidente and Penttila showed that for each odd $q$, there exists a hemisystem $\mathcal{H}_q$ of $\mathsf{H}(3,q^2)$ admitting $\mathsf{P\Omega}^-(4,q)$. It was shown in [3] that these hemisystems could be constructed using Theorem 3.2. Moreover, the number of hemisystems produced by this construction grows exponentially with $q$. To see this, note that the number of choices of $(q+1)/2$ things from $(q+1)$ things is the binomial coefficient; asymptotically this has value $\frac{2^{q+1}\sqrt{2/\pi}}{\sqrt{q+2}}$, or basically, $\Theta(2^q/\sqrt{q})$. Whereas the automorphism group of the generalized quadrangle is polynomial in size and hence there are exponentially many inequivalent choices.

**Theorem 3.3.** *Let $\mathcal{H}$ be the hemisystem exhibited in Theorem 3.2 and let $G$ be the automorphism group of the generalized quadrangle $\mathcal{K}(\mathcal{O})$. Then $G_{\mathcal{H}}$ contains $T \rtimes \mathsf{Sp}(4,q)_{\mathcal{O}^+,\mathcal{O}^-,\ell'}$, where $T$ is an elementary abelian group of order $q^2$ and $\ell'$ is the line of $\mathsf{W}(3,q)$ obtained by projecting $\ell$ onto $P^\perp/P$. The group $T$ acts semiregularly on the set of lines of type (a) of $\mathcal{K}(\mathcal{O})$ and fixes each line of type (b).*

Proof. Consider the group

$$E = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ J^T\boldsymbol{a}^T & I & 0 \\ z & \boldsymbol{a} & 1 \end{pmatrix} \;\middle|\; \boldsymbol{a} \in \mathsf{GF}(q)^4, z \in \mathsf{GF}(q) \right\}$$

which acts on the generalized quadrangle $\mathcal{K}(\mathcal{O})$.

Let $\mathcal{O}$ be our BLT-set, considered as a set of lines in $\mathsf{W}(3,q)$. Each $\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle \in \mathcal{O}$ is identified with the 3-space $\langle P, [0,\boldsymbol{u}_1,0], [0,\boldsymbol{u}_2,0]\rangle$ in $V$. Note that

$$[0,\boldsymbol{u},0] \begin{pmatrix} 1 & 0 & 0 \\ J^T\boldsymbol{a}^T & I & 0 \\ z & \boldsymbol{a} & 1 \end{pmatrix} = [\boldsymbol{u}J^T\boldsymbol{a}^T, \boldsymbol{u}, 0]$$

Hence $E$ fixes each plane on $P$ and hence each element of $\mathcal{O}$. Moreover, given a line $\ell$ in $P^\perp$ that is disjoint from every element of $\mathcal{O}$, we have that $E$ fixes $\langle P,\ell\rangle$. Now $\langle P,\ell\rangle$ contains $q^2$ lines not on $P$. If we take $\ell' = \langle [0,\boldsymbol{w}_1,0], [0,\boldsymbol{w}_2,0]\rangle$ to be a line on $\langle \ell, P\rangle$ we see that

$$E_{\ell'} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ J^T\boldsymbol{a}^T & I & 0 \\ z & \boldsymbol{a} & 1 \end{pmatrix} \;\middle|\; z \in \mathsf{GF}(q), \boldsymbol{w}_1 J^T\boldsymbol{a}^T = \boldsymbol{w}_2 J^T\boldsymbol{a}^T = 0 \right\}$$

which has order $q^3$. Thus $E$ acts transitively on the set of lines of $\langle P,\ell\rangle$ not on $P$ and so we may choose $\ell = \langle [0,\boldsymbol{w}_1,0], [0,\boldsymbol{w}_2,0]\rangle$. We let $\ell' = \langle \mathbf{w}_1, \mathbf{w}_2\rangle$, a totally isotropic line in $\mathsf{W}(3,q)$.

Let $\mathcal{R}$ be the set of totally isotropic planes on $\ell$ other than $\langle P,\ell\rangle$. Note that $E_\ell$ fixes $\mathcal{R}$ setwise. These planes are of the form $\langle \ell, [x_1,0,0,0,0,1]\rangle$ with $x_1 \in \mathsf{GF}(q)$. Let $T$ be the elementary abelian subgroup of $E_\ell$ of order $q^2$ consisting of all elements with $z=0$. Then

$$[x_1,0,0,0,0,1] \begin{pmatrix} 1 & 0 & 0 \\ J^T\boldsymbol{a}^T & I & 0 \\ 0 & \boldsymbol{a} & 1 \end{pmatrix} = [x_1, \boldsymbol{a}, 1]$$

Since $\boldsymbol{a} \in \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle^\perp = \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle$, it follows that $[x_1, \boldsymbol{a}, 1] \in \langle \ell, [x_1, 0, 0, 0, 0, 1] \rangle$ and so $T$ fixes each element of $\mathcal{R}$.

Let $\mathcal{S}$ be a subset of size $(q-1)/2$ of $\mathcal{R}$ and $\mathcal{S}^c$ be the complementary set of totally isotropic planes of size $(q+1)/2$. Then $T$ fixes $\mathcal{S}$ and $\mathcal{S}^c$ elementwise. Hence $T$ fixes the hemisystem $\mathcal{H} = \mathcal{O}^+ \cup \mathcal{L}^+_{\mathcal{S}} \cup \mathcal{L}^-_{\mathcal{S}^c}$.

Let $B \in \mathsf{Sp}(4, q)_{\mathcal{O}}$ and consider the element

$$X = \begin{pmatrix} 1 & 0_{1\times 4} & 0 \\ 0_{4\times 1} & B & 0_{4\times 1} \\ 0 & 0_{1\times 4} & 1 \end{pmatrix}$$

which acts on the flock generalized quadrangle $\mathcal{K}(\mathcal{O})$. If $B \in \mathsf{Sp}(4, q)_{\mathcal{O}^+, \mathcal{O}^-, \ell'}$ then $X$ fixes each element of $\mathcal{R}$ setwise and hence stabilises the hemisystem $\mathcal{O}^+ \cup \mathcal{L}^+_{\mathcal{S}} \cup \mathcal{L}^-_{\mathcal{S}^c}$. Thus $T \rtimes \mathsf{Sp}(4, q)_{\mathcal{O}^+, \mathcal{O}^-, \ell'} \leqslant G_{\mathcal{H}}$.

The lines of $\mathcal{K}(\mathcal{O})$ are the elements of $\mathcal{O}$ and the totally isotropic planes not on $P$ and meeting some element of $\mathcal{O}$ in a line. We have seen already that $T$ fixes each of the elements of $\mathcal{O}$. Now let $U = \langle P, [0, \mathbf{u}_1, 0], [0, \mathbf{u}_2, 0] \rangle \in \mathcal{O}$ and recall that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \cap \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = \{0\}$. Then

$$T_{\langle [0,\mathbf{u}_1,0],[0,\mathbf{u}_2,0] \rangle} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ J^T \boldsymbol{a}^T & I & 0 \\ 0 & \boldsymbol{a} & 1 \end{pmatrix} \in T \,\middle|\, \boldsymbol{u}_1 J^T \boldsymbol{a}^T = \boldsymbol{u}_2 J^T \boldsymbol{a}^T = 0 \right\}$$

Since such elements lie in $T$ they also satisfy $\boldsymbol{w}_1 J^T \boldsymbol{a}^T = \boldsymbol{w}_2 J^T \boldsymbol{a}^T = 0$. If $\boldsymbol{a} \neq \boldsymbol{0}$, we have $\{\boldsymbol{x} \mid \boldsymbol{x} J^T \boldsymbol{a}^T = 0\}$ has dimension 3 but contains the complementary 2-spaces $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ and $\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle$. This is a contradiction and so $T_{\langle [0,\mathbf{u}_1,0],[0,\mathbf{u}_2,0] \rangle} = 1$. Thus $T$ acts regularly on the $q^2$ lines in $U$ not containing $P$, and hence acts semiregularly on the totally isotropic planes not on $P$ and meeting some element of $\mathcal{O}$ in a line. □

**Remark 3.4.** *The stabiliser $G_{\mathcal{H}}$ can be larger than the group given by Theorem 3.3. Sometimes extra automorphisms can arise from the structure of the Knarr model. For example, if $\mathcal{S}$ were chosen to be $\{\langle \ell, [x^2, 0, 0, 0, 0, 1] \rangle \mid x \in \mathsf{GF}(q)\}$ then the elements*

$$\begin{pmatrix} \lambda & 0_{1\times 4} & 0 \\ 0_{4\times 1} & I_{4\times 4} & 0_{4\times 1} \\ 0 & 0_{1\times 4} & \lambda^{-1} \end{pmatrix}$$

*will fix $\mathcal{H}$. Similarly, suitable choices of $\mathcal{S}$ may give rise to semisimilarities of $\beta$ that stabilise $\mathcal{H}$.*

*Alternatively, $\mathsf{H}(3, q^2)$ has more automorphisms than those arising from the Knarr model.. The Cossidente-Pentilla hemisystems in these generalized quadrangles admit at least $\mathsf{P\Sigma L}(2, q^2)$.*

## 4. Potential new infinite families of hemisystems of $\mathsf{H}(3, q^2)$

Examination of our computational data uncovered three promising candidates for new infinite families of hemisystems of $\mathsf{H}(3, q^2)$, and in this section we investigate these possible families in more detail.

### *4.1. Hemisystems that are invariant under a Singer type element*

In this section, we present a way of viewing hemisystems of $\mathsf{H}(3, q^2)$ that are invariant under a *Singer type element*, and we give some computational data which shows the existence of such hemisystems for all $q \leqslant 29$, except (curiously) $q \in \{13, 25\}$. For $q = 3$ we obtain the Segre hemisystem, for $q = 5$ we obtain the hemisystem invariant under $(3 \cdot A_7).2$ discovered by Cossidente and Penttila [8] and for $q = 7, 9$ the examples are given in [4]. In what follows, we will work in the dual generalized quadrangle; the points and lines of the elliptic quadric $\mathsf{Q}^-(5, q)$. A hemisystem of lines of $\mathsf{H}(3, q^2)$ transfers to a *hemisystem of points* of $\mathsf{Q}^-(5, q)$.

We begin with $\mathsf{GF}(q^6)$ and equip it with the following bilinear form over $\mathsf{GF}(q)$:

$$B(x, y) := \mathrm{Tr}_{q^6 \to q}(x y^{q^3}).$$

(Note that $\mathrm{Tr}_{q^6\to q}$ is the relative trace map $x \mapsto x + x^q + x^{q^2} + x^{q^3} + x^{q^4} + x^{q^5}$). This form is symmetric and defines an elliptic orthogonal space isomorphic to $\mathsf{Q}^-(5,q)$. Now let $\omega = \xi^{(q^3-1)(q+1)}$ where $\xi$ is a primitive element of $\mathsf{GF}(q^6)$. Let $K = \langle\omega\rangle$ and note that $K$ is independent of the choice of $\xi$ (it is just the set of elements $x$ such that $x^{q^2-q+1} = 1$). Then $K$ is irreducible and acts semiregularly on the totally isotropic points of $\mathsf{Q}^-(5,q)$, and is occasionally known as a *Singer type* isometry of $\mathsf{Q}^-(5,q)$. So the number of orbits of $K$ on totally isotropic points is $(q+1)^2$. It is not difficult to see that each point orbit is of the form

$$\{\langle u\rangle \mid u^{(q^2-q+1)(q-1)} = r\},$$

where $r$ is a singular element of $\mathsf{GF}(q^6)^*$ such that $r^{(q+1)} \in \mathsf{GF}(q^3)$. In what follows, we will use the underlying vectors instead of the projective points as the equations will be simpler. Note that the $K$-orbits on singular nonzero vectors are each of the form

$$\{u \in \mathsf{GF}(q^6)^* \mid u^{q^2-q+1} = r\}, \quad r \in R$$

where

$$R := \{r \in \mathsf{GF}(q^6) \mid r^{q+1} \in \mathsf{GF}(q^3), \mathrm{Tr}_{q^6\to q}(r^{q+1}) = 0\}.$$

The elements of $R$ lie on the mutually disjoint lines

$$\ell_a : X^{q^2} - aX = 0$$

where $a$ is an element of $\mathsf{GF}(q^3)$ such that $a^{q+1} + a + 1 = 0$.

So to construct a hemisystem, we need to construct a set of $\frac{1}{2}(q+1)^2$ elements of $R$. Of the hemisystems we found, all were invariant under the field automorphism $\tau : a \mapsto a^{q^2}$ fixing $\mathsf{GF}(q^2)$ elementwise, and it acts on the set of lines $\{\ell_a\}$. The orbits of $\langle\tau\rangle$ on $\{\ell_a\}$ are the zero sets of the $\mathsf{GF}(q^2)$-irreducible factors of the polynomial $X^{q+1} + X + 1$. Now every element $r \in R$ can be uniquely represented by the pair $(r^{q^2-1}, r^{q^3-1})$. The possible values of $r^{q^2-1}$ are the $q+1$ zeros of $X^{q+1} + X + 1$, and the possible values of $r^{q^3-1}$ are the $q+1$ solutions to $X^{q+1} = 1$; let this latter set be denoted by $N$. So a $\langle\tau\rangle$-orbit on $R$ is uniquely determined by a $\mathsf{GF}(q^2)$-irreducible factor $i(X)$ of $X^{q+1} + X + 1$ and an element $n \in N$:

$$\{r \in R : i(r^{q^2-1}) = 0, r^{q^3-1} = n\}.$$

The hemisystems we construct arise from unions of these orbits.

Below we list the hemisystems that we found for $3 \leqslant q \leqslant 9$. In each table we describe each solution by unions of $\langle\tau\rangle$-orbits on $R$. The constituents of these unions are described by which values of $N$ appear as right-hand values for each $i(X)$.

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**Example 4.1.** *For $q = 3$, $N = \{1, -1, z^2, z^6\}$, where $z$ is the primitive root of $\mathsf{GF}(q^2)$. The $\mathsf{GF}(q^2)$-irreducible factors of $X^{q+1} + X + 1$ are*

$$i_1(X) : X - 1 \quad \text{and} \quad i_2(X) : X^3 + X^2 + X - 1.$$

*Let $\Pi$ be the subset of the ordered pairs $\{i_1, i_2\} \times N$ described by specifying the right-hand coordinates per possible left-hand coordinate:*

| | |
|---|---|
| $X - 1$ | $1, z^6$ |
| $X^3 + X^2 + X - 1$ | $-1, z^2$ |

*Now let*

$$\mathcal{H}_\Pi^R := \{r \in R \mid i(r^{q^2-1}) = 0, (i(X), r^{q^3-1}) \in \Pi\}.$$

*Then our hemisystem of points of $\mathsf{Q}^-(5,q)$ is simply*

$$\mathcal{H}_\Pi := \left\{\langle u\rangle \mid u^{(q^2-q+1)(q-1)} \in \mathcal{H}_\Pi^R\right\}.$$

*Moreover, we know that $\mathcal{H}_\Pi$ is projectively equivalent to the Segre hemisystem.*

---

In each case below, $z$ denotes the primitive element of $\mathsf{GF}(q^2)$. For each $q$ below, we list one solution, and all the solutions can be obtained by taking the given solution and its orbit under the action of $\langle z\rangle$.

| $q$ | $i(X)$ | $N$ |
|---|---|---|
| 3 | $X-1$ | $1, z^6$ |
| | $X^3+X^2+X-1$ | $-1, z^2$ |
| 5 | $X^3+2X^2-X-1$ | $1, z^8, z^{16}$ |
| | $X^3+3X^2-1$ | $1, z^4, z^{20}$ |
| 7 | $X+3$ | $z^6, z^{12}, z^{30}, z^{36}$ |
| | $X+5$ | $1, -1, z^{18}, z^{42}$ |
| | $X^3+4X-1$ | $1, -1, z^{18}, z^{42}$ |
| | $X^3-X^2+3X-1$ | $z^6, z^{12}, z^{30}, z^{36}$ |
| 9 | $X-1$ | $1, z^8, z^{24}, z^{56}, z^{72}$ |
| | $X^3-X^2-X-1$ | $1, z^{16}, z^{32}, z^{48}, z^{64}$ |
| | $X^3+z^{50}X^2+z^{50}X-1$ | $1, z^8, z^{16}, z^{64}, z^{72}$ |
| | $X^3+z^{70}X^2+z^{70}X-1$ | $1, z^{24}, z^{32}, z^{48}, z^{56}$ |

Table 2: Sets $\Pi$ of ordered pairs $(i(X), r^{q^3-1})$.

We have found hemisystems for larger $q$ and we summarise them below.

| $q$ | $q^2-q+1$ | Stabiliser |
|---|---|---|
| 3 | 7 | $\mathrm{PSL}(3,4).2$ |
| 5 | 21 | $3\cdot A_7\cdot 2$ |
| 7 | 43 | $43:6$ |
| 9 | 73 | $73:6$ |
| 11 | 111 | $111:6$, $333:3$ |
| 17 | 273 | $273:3$ |
| 19 | 1715 | $1715:6$ |
| 23 | 507 | $507:6$ |
| 27 | 703 | at least $703:3$ |

**Problem 4.2.** *Does there exist a hemisystem invariant under a Singer type element for all odd prime powers $q\not\equiv 1 \pmod{12}$?*

*4.2. Hemisystems invariant under the stabiliser of a triangle: tyranny of the small?*

Another interesting sequence of hemisystems apparent in our data is that for $q=7,9$ and 11, the generalized quadrangle $\mathsf{H}(3,q^2)$ contains a hemisystem invariant under a group $K=C_{q+1}^2:S_3$. In fact, for $q=9$ and 11 there are several such hemisystems. Moreover, the stabiliser of the Segre hemisystem for $q=3$ contains such a subgroup, as does the group $(3\cdot A_7).2$ for $q=5$.

The group $K$ can be realised as follows. The stabiliser of a nondegenerate hyperplane of $\mathsf{H}(3,q^2)$ contains a group $H\cong C_{q+1}^3:(S_3\times C_{2f})$ where $q=p^f$ that fixes a set $T$ of mutually orthogonal nondegenerate points $\{\langle v_1\rangle,\langle v_2\rangle,\langle v_3\rangle\}$ of the underling projective space. In particular, taking $v_1,v_2,v_3$ as the first three elements of a basis of the underlying vector space, the pointwise stabiliser in $\mathsf{PGU}(4,q)$ of $T$ is the group $D$ of all diagonal matrices $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,1)$ such that $\lambda_1^{q+1}=\lambda_2^{q+1}=\lambda_3^{q+1}=1$. Letting $\sigma$ and $\tau$ be the permutation matrices such that $\sigma: v_1\mapsto v_2\mapsto v_3\mapsto v_1$ and $\tau: v_1\mapsto v_1, v_2\mapsto v_3\mapsto v_1$, we have $\langle\sigma,\tau\rangle\cong S_3$. Moreover, $H=D:(\langle\sigma,\tau\rangle\times\langle\phi\rangle)$ where $\phi$ is the field automorphism such that $\phi:\sum\lambda_i v_i\mapsto\sum\lambda_i^p v_i$. The group $H$ contains a normal subgroup $R$ isomorphic to $C_{q+1}^2$ given by

$$R:=\{\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,1)\mid\lambda_i^{q+1}=1,\lambda_1\lambda_2\lambda_3=1\}.$$

The group $K$ that leaves invariant a hemisystem for the values of $q$ examined is $R \rtimes \langle \sigma, \mathrm{diag}(\lambda, \lambda, \lambda, 1)\tau\phi^f \rangle$ where $\lambda$ is an element of order $q+1$.

So naturally we may ask if there exists a hemisystem of $\mathsf{H}(3, q^2)$ invariant under $K$ for all $q$? For $q = 13$ and $q = 17$, we constructed the group $K$ and, as anticipated, found hemisystems stabilised by $K$, but to our surprise the sequence appears to stop there and for $q = 19$, 23, 25 and 27 there are no hemisystems stabilised by $K$. (We were sufficiently surprised by this that we ran the linear program with a second integer programming package — GLPK — in addition to Gurobi.)

*4.3. Hemisystems invariant under* $2^4.A_5$

A further interesting sequence is that for $\mathsf{H}(3, 7^2)$ and $\mathsf{H}(3, 11^2)$ there is a hemisystem with stabiliser of shape $2^4.A_5$. The stabiliser of the Segre hemisystem for $q = 3$ also contains such a subgroup, and further calculations have verified the existence of a hemisystem invariant under $2^4.A_5$ when $q = 19$. The group $\mathsf{PGU}(3, q)$ contains a subgroup $H$ isomorphic to $2^4.A_6$ for all $q \equiv 3 \pmod 4$ (such a subgroup is usually referred to as a $\mathcal{C}_6$-group, or the normaliser of a symplectic type $r$-group, see for example [10, §4.6]). The group $H$ contains two groups of shape $2^4.A_5$, corresponding to the two classes of $A_5$ in $A_6$. The group which arises as a stabiliser of a hemisystem for $q = 3, 7, 11$ and 19 is the one for which the $A_5$ acts transitively on the nontrivial elements of the $2^4$.

**Problem 4.3.** *Is there a hemisystem of* $\mathsf{H}(3, q^2)$ *invariant under* $2^4.A_5$ *for all* $q \equiv 3 \pmod 4$*?*

These hemisystems are especially intriguing (and also potentially harder to search for) as the order of their stabiliser is constant.

## 5. *q*-clans and BLT-sets

A 2-by-2 matrix $M$ over $\mathsf{GF}(q)$ is *anisotropic* if $\boldsymbol{x}A\boldsymbol{x}^T = 0$ ($\boldsymbol{x} \in \mathsf{GF}(q)^2$) holds only when $\boldsymbol{x} = (0, 0)$. A *$q$-clan* is a set of 2-by-2 matrices over $\mathsf{GF}(q)$, of size $q$, the difference of any two being anisotropic. Payne introduced $q$-clans in [18], and used them to construct flock quadrangles of order $(q^2, q)$. In particular, a $q$-clan gives rise to a BLT-set of lines of $\mathsf{W}(3, q)$ which we describe as follows. Let $q$ be an odd prime power. Suppose we have a $q$-clan $\mathcal{C}$ written as symmetric matrices

$$\mathcal{C} := \left\{ \begin{pmatrix} t & f_t \\ f_t & g_t \end{pmatrix} : t \in \mathsf{GF}(q) \right\}.$$

We can find in Payne's 1988 paper [19, §2] how to obtain a BLT-set of lines of $\mathsf{W}(3, q)$ directly from a $q$-clan, but since our model is slightly different here, we provide the details behind the connection between these two objects.

**Lemma 5.1.** *Let $\ell_t, \ell_u, \ell_v$ be three 2-by-4 matrices over* $\mathsf{GF}(q)$ *representing three skew lines of* $\mathsf{PG}(3, q)$*. Suppose, without loss of generality, that $\ell_v = \ell_u + \ell_t$. Let*

$$\{m_x : x \in \mathsf{GF}(q) \cup \{\infty\}\}$$

*be the set of transversal lines to $\ell_t, \ell_u, \ell_v$, and let $M$ be the Gram matrix of a symplectic form for which $\ell_t, \ell_u, \ell_v$ are each totally isotropic. Then one of the $m_x$ is totally isotropic with respect to this symplectic form if and only if there exists a nonzero vector $(x, y) \in \mathsf{GF}(q)^2$ such that*

$$(x, y)\ell_u M \ell_v^T (x, y)^T = 0.$$

*That is, none of the $m_x$ are totally isotropic if and only if $\ell_u M \ell_v^T$ is anisotropic.*

Proof. First, it is a simple exercise to establish that the set of $q+1$ transversal lines to $\ell_t$, $\ell_u$, $\ell_v$ are given by

$$m_x = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & x \end{pmatrix} \begin{pmatrix} \ell_u \\ \ell_v \end{pmatrix}, \quad x \in \mathsf{GF}(q),$$

$$m_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ell_u \\ \ell_v \end{pmatrix}.$$

Suppose that $m_x$ is totally isotropic, for some $x \in \mathsf{GF}(q)$. Then

$$\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & x \end{pmatrix} \begin{pmatrix} \ell_u \\ \ell_v \end{pmatrix} M \begin{pmatrix} \ell_u^T & \ell_v^T \end{pmatrix} \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & x \end{pmatrix}^T = 0$$

which implies that

$$\left(\begin{array}{c|c} (1,x)\ell_u M \ell_u^T (1,x)^T & (1,x)\ell_u M \ell_v^T (1,x)^T \\ \hline (1,x)\ell_v M \ell_u^T (1,x)^T & (1,x)\ell_v M \ell_v^T (1,x)^T \end{array}\right) = \left(\begin{array}{c|c} 0 & (1,x)\ell_u M \ell_v^T (1,x)^T \\ \hline (1,x)\ell_v M \ell_u^T (1,x)^T & 0 \end{array}\right) = 0.$$

So $(1,x)\ell_u M \ell_v^T (1,x)^T = 0$ and the result follows. Similarly, if $m_\infty$ is totally isotropic, then $(0,1)\ell_u M \ell_v^T (0,1)^T = 0$. □

Now suppose we are in the 3-dimensional symplectic space $\mathsf{W}(3,q)$ defined by the form $\beta(x,y) = x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2$.

**Lemma 5.2.** *Let $\mathcal{C}$ be a set of matrices of the form*

$$A_t := \begin{pmatrix} t & f_t \\ f_t & g_t \end{pmatrix}, \quad t \in \mathsf{GF}(q)$$

*where $f$ and $g$ are maps on $\mathsf{GF}(q)$. Consider the following lines $\mathcal{L}$ of $\mathsf{W}(3,q)$:*

$$\ell_\infty := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \ell_t := \begin{pmatrix} 1 & 0 & f_t & t \\ 0 & 1 & g_t & f_t \end{pmatrix} \text{ for all } t \in \mathsf{GF}(q).$$

*Then $\mathcal{L}$ is a BLT-set of lines of $\mathsf{W}(3,q)$ if and only if $\mathcal{C}$ is a $q$-clan.*

Proof. If $M$ is the Gram matrix of the aforementioned form, it is not difficult to see that

$$\ell_u M \ell_v^T = A_v - A_u.$$

The result follows from Lemma 5.1. □

We now summarise the $q$-clans that generate the flock quadrangles used in this paper. Our information has been taken from [12]. Four of the families are outlined in Table 3.

| Flock quadrangle | Abbreviation | $f_t$ | $g_t$ | Conditions |
|---|---|---|---|---|
| Linear | $\mathsf{H}(3,q^2)$ | $0$ | $-nt$ | $n$ is a nonsquare in $\mathsf{GF}(q)$ |
| Fisher-Thas-Walker-Kantor-Betten | $\mathsf{FTWKB}(q)$ | $\frac{3}{2}t^2$ | $3t^3$ | $q \equiv 2 \pmod 3$ |
| Kantor Monomial | $\mathsf{K}_2(q)$ | $\frac{5}{2}t^3$ | $5t^5$ | $q \equiv \pm 2 \pmod 5$, 5 is a nonsquare in $\mathsf{GF}(q)$ |
| Kantor-Knuth | $\mathsf{K}_1(q)$ | $0$ | $-nt^\sigma$ | $n \in \mathsf{GF}(q)$ nonsquare, $q$ not prime, $1 \neq \sigma \in \mathsf{Aut}(\mathsf{GF}(q))$ |

Table 3: $q$-clans for some flock quadrangles. For each $q$-clan, the variable $t$ runs over $\mathsf{GF}(q)$.

For the remaining flock quadrangles considered in this paper, the $q$-clans are a little more difficult to write down but we outline them below.

*The Fisher q-clans:*

The following model was taken from Payne [19] (see also [12]). Let $q$ be an odd prime power, let $\zeta$ be a primitive element of $\mathsf{GF}(q^2)$, so that $\omega^{q+1}$ is a primitive element of $\mathsf{GF}(q)$. Let $i = \zeta^{(q+1)/2}$ and write $z = \zeta^{q-1} = a + bi$. Note that $a = (z + z^q)/2$ and $b = (z - z^q)/2$. For each $t \in \mathsf{GF}(q)$ such that $t^2 - 2/(1 + a)$ is a square, we set

$$A_t = \begin{pmatrix} t & 0 \\ 0 & -\omega t \end{pmatrix}.$$

For the remaining $(q + 1)/2$ values of $t$, we set

$$A_{t(j)} = \begin{pmatrix} -(z^{2j+1} + z^{-2j})/(z + 1) & -i(z^{2j+1} - z^{-2j})/(z + 1) \\ -i(z^{2j+1} - z^{-2j})/(z + 1) & -\omega(z^{2j+1} + z^{-2j})/(z + 1) \end{pmatrix}, \quad 0 \leqslant j \leqslant (q - 1)/2.$$

*The Penttila-Mondello q-clans:*

The authors are not aware of any nice representation of these $q$-clans, so we simply list here explicitly the $q$-clan that gives rise to $\mathsf{PM}(11)$, namely:

$$\begin{pmatrix} 0 & 8 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 2 & 7 \\ 7 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 8 \\ 8 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 1 \\ 1 & 10 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 5 & 9 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}.$$

## 6. Computational methods

The *point-line incidence matrix* of a generalized quadrangle is the matrix $A$ with rows indexed by points and columns by lines such that

$$A_{P,\ell} = \begin{cases} 1, & P \text{ is on } \ell; \\ 0, & \text{otherwise.} \end{cases}$$

In order to construct the point-line incidence matrix of a flock generalized quadrangle, we used the GAP package $\mathsf{FinInG}$[2]. This software can construct flock generalized quadrangles from a $q$-clan.

A hemisystem is a subset of the columns of $A$ that sum to $(s + 1)/2\, \boldsymbol{j}^T$ where $\boldsymbol{j}$ is the all-ones (row) vector or, equivalently, a $\{0, 1\}$-vector $\boldsymbol{h}$ such that

$$A\boldsymbol{h}^T = (s + 1)/2\, \boldsymbol{j}^T. \tag{1}$$

For all but the smallest generalized quadrangles, the matrix $A$ is so large that we cannot hope to solve the equations completely. To reduce the problem, we assume the existence of some group $G$ stabilizing the hemisystem. Suppose that $G$ has orbits $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ on points and $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n\}$ on lines. Then every point in a point-orbit $\mathcal{P}_i$ is incident with the same number of lines in the line-orbit $\mathcal{L}_j$. If we denote this number by $b_{ij}$ and define the $m \times n$ matrix $B = (b_{ij})$, then a $\{0, 1\}$-vector $\boldsymbol{h}$ such that

$$B\boldsymbol{h}^T = (s + 1)/2\, \boldsymbol{j}^T \tag{2}$$

determines a hemisystem that is stabilized by the group $G$.

There are a variety of approaches to solving equations such as (2). In particular, the system of equations can be viewed either as an *integer linear program* or as a *constraint satisfaction problem*. After experimenting with software for each type of problem, we determined that the commercial integer programming package Gurobi [9] (available with a free academic license) was the most effective for our purposes.

A linear program attempts to find values for variables $x_1, x_2, \ldots, x_n$ that maximize (or minimize) a linear objective function subject to linear constraints. An *integer* linear program, or just integer program, is a linear program with the additional restriction that the variables must take integral values. Solving

[2] This can be found at `http://cage.ugent.be/geometry/fining.php`. This package is currently in development.

(2) does not involve any maximizing or minimizing and so the objective function can be taken to be a constant, say 0, and then any feasible solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ to the following integer program yields a hemisystem:

$$\begin{array}{lrcl} \text{Maximize:} & 0 & & \\ \text{subject to:} & B\boldsymbol{x}^T & = & (s+1)/2\, \boldsymbol{j}^T \\ & x_i & \in & \{0,1\}. \end{array}$$

In order to find *all* the solutions to a given system of equations, the system is augmented as each solution is found with an additional constraint excluding that particular solution, and the system is then re-solved. When all the solutions have been found and excluded, the resulting system has no integer feasible solutions. In order to exclude a particular solution $\boldsymbol{h} = (h_1, h_2, \ldots, h_n)$ it suffices to add a constraint of the form

$$\sum_{\{i \mid h_i = 1\}} x_i < \sum_i h_i$$

which merely says that $\boldsymbol{x}$ cannot agree with $\boldsymbol{h}$ in every coordinate position, and so must differ in at least one place. In principle, a constraint of this form only eliminates vectors *identical* to $\boldsymbol{h}$ and still permits the solver to investigate vectors that have almost all of their entries identical to $\boldsymbol{h}$. However, if we know an upper bound, say $\alpha$, on the size of the intersection of two hemisystems, then we can strengthen this constraint to

$$\sum_{\{i \mid h_i = 1\}} x_i \leqslant \alpha \tag{3}$$

without missing any hemisystems. The exhaustive search for hemisystems in $\mathsf{H}(3, 5^2)$ was made feasible by using two basic techniques to shorten the search time:

- Use the automorphism group of $\mathsf{H}(3, 5^2)$ to determine the largest possible set of lines that can freely be assumed to be in a hemisystem.
- Use knowledge of the possible intersection sizes of a hemisystem with the two known hemisystems to add strong constraints of the same type as (3) during the search.

A more detailed description of the computation for $\mathsf{H}(3, 5^2)$ follows:

Proof of Theorem 1.1 for $\mathsf{H}(3, 5^2)$.

Let $G$ be the full automorphism group of $\mathsf{H}(3, 5^2)$ and let $\mathcal{H}$ be a hemisystem. As $G$ is transitive on the set of lines of $\mathsf{H}(3, 5^2)$ we can assume without loss of generality that $\ell_1 \in \mathcal{H}$. Then the stabiliser $G_{\ell_1}$ has two orbits on the remaining lines, those disjoint from $\ell_1$ and those that meet $\ell_1$. It is easy to see that any hemisystem containing $\ell_1$ must contain a line disjoint from $\ell_1$ and so we can arbitrarily pick a second line, say $\ell_2$ , and assume without loss of generality that $\ell_1, \ell_2 \in \mathcal{H}$. This process can be continued in a semi-automated fashion as follows: suppose that we have a set $\ell_1, \ldots, \ell_i$ of lines that we can already assume are contained in $\mathcal{H}$, and consider the orbits of the setwise stabiliser $G_{\{\ell_1,\ldots,\ell_i\}}$ on lines. An orbit $\mathcal{O}$ is denoted *essential* if a search for a hemisystem that contains $\ell_1, \ldots, \ell_i$ but does not contain *any* line from $\mathcal{O}$ is infeasible. If $\mathcal{O}$ is essential, then $\mathcal{H}$ contains at least one line from $\mathcal{O}$, and we can select $\ell_{i+1}$ arbitrarily from $\mathcal{O}$. This process can be continued until the set of lines is sufficiently large that its stabiliser is so small that it has no essential orbits. In this fashion, we found a particular set of 8 lines $\ell_1, \ldots, \ell_8$ that can be assumed to lie in $\mathcal{H}$.

The next important step was to determine that no hemisystem has a "large" intersection with either of the two known hemisystems. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be representatives of the two known hemisystems. First we found the maximum possible size in which *any* hemisystem (known or unknown) can intersect $\mathcal{H}_1$ by running the integer linear program where the objective function to be maximised is the sum of the variables corresponding to the lines in $\mathcal{H}_1$. This revealed that a hemisystem different from $\mathcal{H}_1$ can intersect $\mathcal{H}_1$ in at most 306 lines. By running the linear program again with the additional constraint that the intersection with $\mathcal{H}_1$ has size *exactly* 306, we determined all the hemisystems that intersect $\mathcal{H}_1$ in 306 lines and confirmed that no new hemisystems arose. We repeated this process with the "next largest" intersection, which proved to be size 300, then 282, then 270 and then 258, eventually confirming that any hemisystem that meets $\mathcal{H}_1$ in 258 or more lines is isomorphic to either $\mathcal{H}_1$ or $\mathcal{H}_2$. Similar results were obtained for $\mathcal{H}_2$ and similarly we determined that any hemisystem meeting $\mathcal{H}_2$ in 258 or more lines is isomorphic to $\mathcal{H}_1$ or $\mathcal{H}_2$.

Finally, the exhaustive search is run where the variables corresponding to $\ell_1, \ldots, \ell_8$ are initially set to 1 and every time a hemisystem is found, it is excluded by adding a constraint similar to (3) with $\alpha = 257$. Notice that this constraint is much stronger than simply excluding the hemisystem that has just been found and will exclude other hemisystems. However if the just-found hemisystem is one of the two known ones, then the "extra" hemisystems that are excluded by the constraint are necessarily isomorphic to the known ones, and hence not of interest. Therefore if unknown hemisystems do exist, then at least one of them will be discovered by the search. As this does not occur, we conclude that there are no other hemisystems of $\mathsf{H}(3, 5^2)$. $\square$

In this computation, there is a trade-off involved in choosing the value 258 used in the constraints to exclude solutions as they are found. Using a lower value would make the final exhaustive part of the search run faster, but it would take longer to establish that only known hemisystems intersect $\mathcal{H}_1$ or $\mathcal{H}_2$ in that many lines.

The computation for $\mathsf{FTWKB}(5)$ was done in an exactly analogous fashion.

## 7. A summary of the known hemisystems of flock quadrangles

In this section we catalogue all the known hemisystems of lines of flock quadrangles of order $(s^2, s)$ for $s \leqslant 11$. These include those which arise in the pre-existing literature, those obtained via Theorem 3.2, and numerous further examples constructed by computer. Each row of the table describes a complementary pair of hemisystems; the column SC (for "self-complementary") indicates whether the hemisystem is equivalent to its complement in which case it contributes just 1 to the total count of hemisystems.

The tables contain an exhaustive listing of all the hemisystems that arise by Theorem 3.2 and are complete for the known generalized quadrangles of order up to $(5^2, 5)$. However there may be many more hemisystems, though necessarily with small automorphism groups, that remain to be found.

**Proposition 7.1.** *Let $\mathcal{H}$ be a hemisystem of a flock quadrangle of order $(s^2, s)$ with $s \leqslant 9$ such that $\mathcal{H}$ arises from Theorem 3.2. Then $\mathcal{H}$ appears in one of the tables in this section.*

We also list all hemisystems arising from Theorem 3.2 for $\mathsf{H}(3, 11^2)$ in Table 12. Due to the large number of hemisystems of Type I for the remaining flock quadrangles of order $(11^2, 11)$, they are listed in Appendix A which is only included in the version of this paper on the arxiv.

The data given for the Type I hemisystems in our table is sufficient to reconstruct the actual hemisystem given some additional knowledge about the particular choices that have been made for the variables in the construction. The point $P$ is $(1, 0, 0, 0, 0, 0)$ and the BLT-sets are the ones given in Section 5. Each totally isotropic plane can be represented uniquely by a $3 \times 6$ matrix written in Hermite normal form, whose row space gives us the corresponding 3-dimensional vector subspace. The totally isotropic planes on $\ell$ are sorted into lexicographic order and indexed by $\{1, \ldots, q+1\}$, and the chosen subset $\mathcal{S}$ is given by a $(q-1)/2$ subset of this index set.

### 7.1. *Linear,* $\mathsf{H}(3, 3^2)$

Segre [20] established that there is just one example of a hemisystem (up to projectivity) in $\mathsf{H}(3, 3^2)$. The strongly regular graph (andpartial quadrangle) arising is the Gewirtz graph on 56 vertices.

| Group | Size | Self-complementary | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $\mathsf{PSL}(3,4).2$ | 40320 | true | Theorem 3.2, Segre [20], Sections 4.1, 4.2 and 4.3 | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&1&1&0 \end{smallmatrix}\right]$ | any |

Table 4: The hemisystem of $\mathsf{H}(3, 3^2)$.

*7.2. Linear,* $\mathsf{H}(3,5^2)$

The full automorphism group of this generalized quadrangle is $\mathsf{P\Gamma U}(4,5)$ which has order $2^9 \times 3^4 \times 5^6 \times 7 \times 13$. There were two previously known hemisystems in this generalized quadrangle and our computer searches have confirmed that there are no more.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $\mathsf{P\Sigma L}(2,25)$ | 15600 | true | Theorem 3.2, Cossidente–Penttila [8] | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \end{smallmatrix}\right]$ | any |
| $(3 \cdot A_7).2$ | 15120 | true | Cossidente–Penttila [8], Sections 4.1 and 4.2 | | |

Table 5: The hemisystems of $\mathsf{H}(3,5^2)$.

*7.3. Fisher-Thas/Walker/Kantor/Betten,* $\mathsf{FTWKB}(5)$

The full automorphism group of this generalized quadrangle is $5^{1+4} : (\mathsf{SL}(2,9) : C_4)$, which has order $2^6 \times 3^2 \times 5^6$. There was one previously known hemisystem of this generalized quadrangle in the literature and Theorem 3.2 yields a second example. Our computer searches uncovered a third example with group $S_3$, and confirmed that there are no more.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $C_5^2 : (C_4 \times S_3)$ | 600 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \end{smallmatrix}\right]$ | any |
| $\mathsf{AGL}(1,5) \times S_3$ | 120 | false | Bamberg–De Clerck–Durante [2] | | |
| $S_3$ | 6 | false | **New** | | |

Table 6: The hemisystems of $\mathsf{FTWKB}(5)$.

*7.4. Linear,* $\mathsf{H}(3,7^2)$

The full automorphism group of this generalized quadrangle is $\mathsf{P\Gamma U}(4,7)$, which has order $2^{13} \times 3^2 \times 5^2 \times 7^6 \times 43$. There were five previously known hemisystems in this quadrangle and our computer searches have uncovered a sixth.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $\mathsf{P\Sigma L}(2,49)$ | 117600 | true | Theorem 3.2, Cossidente–Penttila [8] | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&1&1&0 \end{smallmatrix}\right]$ | $\{1,3,4\}$ |
| $C_2 \times (C_7^2 : Q_{16})$ | 1568 | true | Penttila (personal communication), Theorem 3.2 | | $\{1,3,5\}$ |
| $2^4.A_5$ | 960 | true | Bamberg–Kelly–Law–Penttila [4] Section 4.3 | | |
| $C_2 \times (C_{43} : C_6)$ | 516 | true | Bamberg–Kelly–Law–Penttila [4] | | |
| $C_8^2 : S_3$ | 384 | true | **New**, Section 4.2 | | |
| $C_2 \times \mathsf{PSL}(2,7)$ | 336 | true | Cossidente–Penttila [7], Section 4.1 | | |

Table 7: Known hemisystems of $\mathsf{H}(3,7^2)$.

*7.5. Kantor Monomial,* $\mathsf{K}_2(7)$

The full automorphism group of this generalized quadrangle is $7^{1+4} : (C_3 \times (Q_8 : (\mathsf{SL}(2,3).2) : 2))$, which has order $2^8 \times 3^2 \times 7^5$. In addition to the 14 examples obtained by Theorem 3.2, we have found a further 15 hemisystems; all are listed in Table 8.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $C_7^2 : (C_3 \times \mathsf{SL}(2,3))$ | 3528 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \end{smallmatrix}\right]$ | $\{1,3,4\}$ |
| $C_7^2 : (\mathsf{SL}(2,3).2)$ | 2352 | false | Theorem 3.2 | | $\{1,3,5\}$ |
| $C_7^2 : (Q_{16} \times C_3)$ | 2352 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&4&0&0 \end{smallmatrix}\right]$ | $\{1,3,4\}$ |
| $(C_7^2 : Q_{16}) \times C_2$ | 1568 | false | Theorem 3.2 | | $\{1,3,5\}$ |
| $C_7^2 : (C_6 \times C_3)$ | 882 | true | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&3&0&0 \end{smallmatrix}\right]$ | $\{1,3,4\}$ |
| $C_7^2 : (C_3 : C_4)$ | 588 | true | Theorem 3.2 | | $\{1,3,5\}$ |
| $C_7^2 : C_{12}$ | 588 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \end{smallmatrix}\right]$ | $\{1,3,4\}$ |
| $C_7^2 : Q_8$ | 392 | false | Theorem 3.2 | | $\{1,3,5\}$ |
| $C_3 \times F_{42}$ | 126 | false | **New** | | |
| $C_3 \times F_{42}$ | 126 | true | **New** | | |
| $C_2 \times (C_7 : C_3)$ | 42 | false | **New** | | |
| $\mathsf{AGL}(1,7)$ | 42 | false | **New** | | |
| $(C_2 \times Q_8) : C_2$ | 32 | false | **New** | | |
| $(C_2 \times Q_8) : C_2$ | 32 | false | **New** | | |
| $C_7 : C_3$ | 21 | false | **New** | | |
| $C_7 : C_3$ | 21 | true | **New** | | |
| $C_3$ | 3 | true | **New** | | |

Table 8: Known hemisystems of $\mathsf{K}_2(7)$.

*7.6. Linear,* $\mathsf{H}(3,9^2)$

The full automorphism group of this generalized quadrangle is $\mathsf{P\Gamma U}(4,9)$, which has order $2^{12} \times 3^{12} \times 5^3 \times 41 \times 73$. In addition to the two previously known hemisystems, we found two more arising fromTheorem 3.2 and three others; all are listed in Table 9.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $\mathsf{P\Sigma L}(2,81)$ | 1062720 | true | Theorem 3.2, Cossidente–Penttila [8] | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \end{smallmatrix}\right]$ | $\{1,3,4,5\}$ |
| $C_3^4 : (C_{20} : C_4)$ | 6480 | true | Theorem 3.2 | | $\{1,3,5,6\}$ |
| $C_3^4 : (C_5 : C_8)$ | 3240 | true | Theorem 3.2 | | $\{1,3,5,9\}$ |
| $C_{73} : C_{12}$ | 876 | true | Bamberg–Kelly–Law–Penttila [4], Section 4.1 | | |
| $(C_{10}^2 : C_4) : C_3$ | 1200 | true | **New**, Section 4.2 | | |
| $C_{10}^2 : S_3$ | 600 | true | **New**, Section 4.2 | | |
| $(C_5 \times (C_5 : C_4)) : C_4$ | 400 | true | **New** | | |

Table 9: Known hemisystems of $\mathsf{H}(3,9^2)$.

*7.7. Kantor-Knuth,* $\mathsf{K}_1(9)$

The full automorphism group of this generalized quadrangle is $E_9 : (((\mathsf{SL}(2,9).C_4) : C_8) : C_2)$ where $E_9$ is the Heisenberg group of order $9^5$ with centre of order 9. The order of the automorphism group is $2^{10} \times 3^{12} \times 5$.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $C_3^4 : (C_4 \times C_8)$ | 2592 | true | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \end{smallmatrix}\right]$ | $\{1,3,5,9\}$ |
| $C_3^4 : (C_2 \times C_8)$ | 1296 | true | Theorem 3.2 | | $\{1,3,5,6\}$ |
| $C_3^4 : C_8$ | 648 | true | Theorem 3.2 | | $\{1,3,4,5\}$ |
| $C_4 \times \mathsf{AGL}(1,9)$ | 288 | true | **New** | | |
| $\mathsf{AGL}(1,9)$ | 72 | true | **New** | | |

Table 10: Known hemisystems of $\mathsf{K}_1(9)$.

### 7.8. *Fisher,* $\mathsf{Fi}(9)$

The full automorphism group of this generalized quadrangle is $E_9 : (C_5^2 : (D_{16}.Q_8))$ which has order $2^7 \times 3^{10} \times 5^2$. (Here $E_9$ is the Heisenberg group of order $9^5$ with centre of order 9.)

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $C_3^4 : C_5 : C_8$ | 3240 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{smallmatrix}\right]$ | $\{1, 3, 4, 5\}$ |
| $C_3^4 : C_{20} : C_4$ | 6480 | false | Theorem 3.2 | | $\{1, 3, 5, 6\}$ |
| $C_3^4 : C_5 : (C_4.(C_4 \times C_2))$ | 12960 | false | Theorem 3.2 | | $\{1, 3, 5, 9\}$ |
| $C_3^4 : C_2$ | 162 | false | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&z&0&0 \end{smallmatrix}\right]$ | $\{1, 3, 4, 5\}$ |
| $C_3^4 : C_4$ | 324 | false | Theorem 3.2 | | $\{1, 3, 5, 6\}$ |
| $C_3^4 : C_8$ | 648 | false | Theorem 3.2 | | $\{1, 3, 5, 9\}$ |
| $C_3^4 : C_4$ | 324 | true | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&1&1&0 \\ 0&0&1&2&1&0 \end{smallmatrix}\right]$ | $\{1, 3, 4, 5\}$ |
| $C_3^4 : C_4 \times C_2$ | 648 | true | Theorem 3.2 | | $\{1, 3, 5, 6\}$ |
| $C_3^4 : C_8 \times C_2$ | 1296 | true | Theorem 3.2 | | $\{1, 3, 5, 9\}$ |
| $C_3^4 : C_4$ | 324 | true | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&1&1&0 \\ 0&0&1&z^2&1&0 \end{smallmatrix}\right]$ | $\{1, 3, 4, 5\}$ |
| $C_3^4 : C_4 \times C_2$ | 648 | true | Theorem 3.2 | | $\{1, 3, 5, 6\}$ |
| $C_3^4 : C_8 \times C_2$ | 1296 | true | Theorem 3.2 | | $\{1, 3, 5, 9\}$ |
| $C_3^4 : C_4$ | 324 | true | Theorem 3.2 | $\left[\begin{smallmatrix} 0&1&0&1&z^2&0 \\ 0&0&1&z^3&1&0 \end{smallmatrix}\right]$ | $\{1, 3, 4, 5\}$ |
| $C_3^4 : C_4 \times C_2$ | 648 | true | Theorem 3.2 | | $\{1, 3, 5, 6\}$ |
| $C_3^4 : C_8 \times C_2$ | 1296 | true | Theorem 3.2 | | $\{1, 3, 5, 9\}$ |
| $C_2 \times \mathsf{AGL}(1,9)$ | 144 | true | **New** | | |
| $\mathsf{AGL}(1,9)$ | 72 | $4 \times 2 + 4$ | **New** | | |

Table 11: Known hemisystems of $\mathsf{Fi}(9)$.

### 7.9. *Linear,* $\mathsf{H}(3, 11^2)$

The full automorphism group of this generalized quadrangle is $\mathsf{P\Gamma U}(4, 11)$, which has order $2^{10} \times 3^4 \times 5^2 \times 11^6 \times 37 \times 61$.

| Group | Size | SC | Construction/Author(s) | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|---|
| $\mathsf{P\Sigma L}(2,121)$ | 1771440 | true | Theorem 3.2 | $\left[\begin{smallmatrix}0&1&0&1&0&0\\0&0&1&2&1&0\end{smallmatrix}\right]$ | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : C_2 \times (C_3 : Q_8)$ | 5808 | true | Theorem 3.2 | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_2 \times (C_3 : Q_8)$ | 5808 | true | Theorem 3.2 | | { 1, 3, 4, 5, 9 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | true | Theorem 3.2 | | { 1, 3, 4, 5, 10 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | true | Theorem 3.2 | | { 1, 3, 4, 5, 11 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | true | Theorem 3.2 | | { 1, 3, 4, 5, 6 } |
| $3.A_6.2$ | 2160 | true | **New** | | |
| $C_{333} : C_6$ | 1998 | true | **New**, Section 4.1 | | |
| $2^4.A_5$ | 960 | true | **New**, Section 4.3 | | |
| $C_{12}^2 : S_3$ | 864 | true | **New**, Section 4.2 | | |
| $C_{12}^2 : S_3$ | 864 | true | **New**, Section 4.2 | | |
| $C_{111} : C_6$ | 666 | false | **New**, Section 4.1 | | |

Table 12: Known hemisystems of $\mathsf{H}(3, 11^2)$.

### *7.10. Fisher-Thas-Walker-Kantor-Betten,* FTWKB(11)

The full automorphism group of this generalized quadrangle is $11^{1+4} \rtimes \mathrm{GL}(2, 11)$ which has order $2^4 \times 3 \times 5^2 \times 11^6$. There are 20 hemisystems of Type I, listed in Table A.15 of Appendix A[3], and we do not know any other hemisystems in this generalized quadrangle.

### *7.11. Fisher,* Fi(11)

The full automorphism group of this generalized quadrangle is $11^{1+4} : (C_5 \times (((C_3 \times (C_3 : C_4)) : Q_8) : C_2))$ which has order $2^6 \times 3^2 \times 5 \times 11^5$. There are 90 hemisystems of Type I, listed in Table A.16 of Appendix A[3], and we know 12 further hemisystems listed in Table 13.

| Group | Size | Number |
|---|---|---|
| $\mathsf{AGL}(1, 11)$ | 110 | $6 \times 2$ |

Table 13: Non Type I hemisystems of $\mathsf{Fi}(11)$

### *7.12. Penttila-Mondello, $q = 11$*

The full automorphism of this generalized quadrangle is $11^{1+4} \rtimes (C_5 \times (C_3 \times \mathrm{SL}(2, 3).2) : 2)$ which has order $2^5 \times 3^2 \times 5 \times 11^5$. There are 164 hemisystems of Type I, listed in Table A.17 of Appendix A[3], and we know 36 further hemisystems listed in Table 14.

| Group | Size | Number |
|---|---|---|
| $\mathsf{AGL}(1, 11)$ | 110 | $18 \times 2$ |

Table 14: Non Type I hemisystems of $\mathsf{PM}(11)$

[3] Due to its size, this Appendix is only included in the ARXIV version of this paper

## 8. Open Problems

We saw in Section 3 that in any infinite family of generalized quadrangles of order $(q^2, q)$ the number of hemisystems arising from Theorem 3.2 grows exponentially in $q$. Hemisystems that do not arise from Theorem 3.2 are then of particular interest, and so it is natural to ask the whether they always exist.

**Problem 8.1.** *Does every flock generalized quadrangle of order $(s^2, s)$ with $s \geq 7$ contain a hemisystem that does not arise from Theorem 3.2?*

We have found such hemisystems in all of the generalized quadrangles that we have examined with the exception of the small cases ($\mathsf{H}(3, 3^2)$ and $\mathsf{H}(3, 5^2)$) and $\mathsf{FTWKB}(11)$.

Although we have outlined two possibilities for infinite families of hemisystems in $\mathsf{H}(3, q^2)$ in Section 4, we do not have any proven general constructions for hemisystems other than Theorem 3.2.

**Problem 8.2.** *Find a natural construction for an infinite family of hemisystems (not of Type I) in $\mathsf{H}(3, q^2)$ or in one of the known families of non-classical generalized quadrangles.*

By Theorem 3.3, a hemisystem coming from Theorem 3.2 is invariant under a particular elementary abelian group of order $q^2$ denoted by $T$. Thus if a hemisystem is not invariant under such a group, then it does not arise from the construction. However, we do not know if the converse is true.

**Problem 8.3.** *Are there hemisystems invariant under the elementary abelian group $T$ of order $q^2$ described in Theorem 3.3 that do not arise from Theorem 3.2?*

At the other end of the symmetry spectrum, we currently do not know of any hemisystems with a trivial group. However this is not surprising, as almost all of our searches have assumed the existence of symmetries. Hence we expect a positive answer to the following question, although it may be challenging to find such a hemisystem.

**Problem 8.4.** *Is there a hemisystem with trivial group?*

Each hemisystem gives a strongly regular graph and the stabiliser of the hemisystem in the automorphism group of the generalized quadrangle gives a group of automorphisms of the strongly regular graph. In all cases investigated so far, the automorphism group of the strongly regular graph is induced by the stabiliser of the hemisystem in the automorphism group of the generalized quadrangle. It is not apparent why this should always be the case.

**Problem 8.5.** *Is the full automorphism group of the strongly regular graph obtained from a hemisystem always induced by the stabiliser of the hemisystem in the automorphism group of the generalized quadrangle?*

A related problem is whether isomorphic strongly regular graphs can arise from different generalized quadrangles.

**Problem 8.6.** *Are there hemisystems in different generalized quadrangles whose associated strongly regular graphs are isomorphic?*


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## Appendix A. Tables of Type I hemisystems

### *Appendix A.1. Type I hemisystems of* $\mathsf{FTWKB}(11)$

| Group | Size | SC | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&2&0&0 \end{smallmatrix}\right]$ | {1, 3, 4, 5, 6} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 5, 7} |
| $C_{11}^2 : C_{20}$ | 2420 | false | | {1, 3, 4, 5, 8} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 5, 9} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 5, 10} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 5, 11} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 5, 12} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 6, 10} |
| $C_{11}^2 : C_4$ | 484 | false | | {1, 3, 4, 7, 10} |
| $C_{11}^2 : C_{20}$ | 2420 | false | | {1, 3, 4, 7, 12} |

Table A.15: Type I hemisystems of $\mathsf{FTWKB}(11)$.

### *Appendix A.2. Type I hemisystems of* $\mathsf{Fi}(11)$

| Group | Size | SC | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|
| $C_{11}^2 : C_2$ | 242 | false | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \end{smallmatrix}\right]$ | { 1, 3, 4, 5, 6 } |
| $C_{11}^2 : C_4$ | 484 | false | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_{10}$ | 1210 | false | | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : C_4$ | 484 | false | | { 1, 3, 4, 5, 9 } |
| $C_{11}^2 : C_2$ | 242 | false | | { 1, 3, 4, 5, 10 } |
| $C_{11}^2 : C_2$ | 242 | false | | { 1, 3, 4, 5, 11 } |
| $C_{11}^2 : C_2$ | 242 | false | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&2&0&0 \end{smallmatrix}\right]$ | { 1, 3, 4, 5, 6 } |
| $C_{11}^2 : C_4$ | 484 | false | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_{10}$ | 1210 | false | | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : C_4$ | 484 | false | | { 1, 3, 4, 5, 9 } |
| $C_{11}^2 : C_2$ | 242 | false | | { 1, 3, 4, 5, 10 } |
| $C_{11}^2 : C_2$ | 242 | false | | { 1, 3, 4, 5, 11 } |
| $C_{11}^2 : C_2$ | 242 | true | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&10&0&0 \end{smallmatrix}\right]$ | { 1, 3, 4, 5, 6 } |
| $C_{11}^2 : C_4$ | 484 | true | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_{10}$ | 1210 | true | | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : C_4$ | 484 | true | | { 1, 3, 4, 5, 9 } |
| $C_{11}^2 : C_2$ | 242 | true | | { 1, 3, 4, 5, 10 } |
| $C_{11}^2 : C_2$ | 242 | true | | { 1, 3, 4, 5, 11 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | $\left[\begin{smallmatrix} 0&1&0&0&1&0 \\ 0&0&1&9&0&0 \end{smallmatrix}\right]$ | { 1, 3, 4, 5, 6 } |
| $C_{11}^2 : C_2 \times (C_3 : Q_8)$ | 5808 | false | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_5 \times (C_3 : Q_8)$ | 14520 | false | | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : C_2 \times (C_3 : Q_8)$ | 5808 | false | | { 1, 3, 4, 5, 9 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | { 1, 3, 4, 5, 10 } |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | { 1, 3, 4, 5, 11 } |
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&1&2&0 \\ 0&0&1&1&1&0 \end{smallmatrix}\right]$ | { 1, 3, 4, 5, 6 } |
| $C_{11}^2 : Q_8$ | 968 | false | | { 1, 3, 4, 5, 7 } |
| $C_{11}^2 : C_{20}$ | 2420 | false | | { 1, 3, 4, 5, 8 } |
| $C_{11}^2 : Q_8$ | 968 | false | | { 1, 3, 4, 5, 9 } |

| | | | | |
|---|---|---|---|---|
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&1&2&0 \\ 0&0&1&5&1&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&1&4&0 \\ 0&0&1&8&1&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&1&4&0 \\ 0&0&1&10&1&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |

Table A.16: Type I hemisystems of $\mathsf{Fi}(11)$.

*Appendix A.3. Type I hemisystems of* $\mathsf{PM}(11)$

| Group | Size | SC | $\ell$ | Subset $\mathcal{S}$ |
|---|---|---|---|---|
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&1&0&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&2&0&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_4$ | 484 | true | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&5&0&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : Q_8$ | 968 | true | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{20}$ | 2420 | true | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : Q_8$ | 968 | true | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_4$ | 484 | true | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_4$ | 484 | true | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\left[\begin{smallmatrix} 0&1&0&0&0&0 \\ 0&0&1&10&0&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 9 \}$ |

| | | | | |
|---|---|---|---|---|
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 12 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 6, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 7, 10 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 7, 12 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_2$ | 242 | true | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_2$ | 242 | true | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_2 \times Q_8$ | 1936 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_5 \times Q_8$ | 4840 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_2 \times Q_8$ | 1936 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : Q_8$ | 968 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_6$ | 726 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 8 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{30}$ | 3630 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_6$ | 726 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_6$ | 726 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 7 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 7 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_{10}$ | 1210 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_4$ | 484 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_2$ | 242 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | $\begin{bmatrix} 0 & 1 & 0 & 1 & 10 & 0 \\ 0 & 0 & 1 & 9 & 1 & 0 \end{bmatrix}$ | $\{ 1, 3, 4, 5, 6 \}$ |

| | | | | |
|---|---|---|---|---|
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_5 \times (C_3 : C_4)$ | 7260 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 11 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | $\left[\begin{smallmatrix} 0&1&0&2&2&0 \\ 0&0&1&0&2&0 \end{smallmatrix}\right]$ | $\{ 1, 3, 4, 5, 6 \}$ |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | $\{ 1, 3, 4, 5, 7 \}$ |
| $C_{11}^2 : C_5 \times (C_3 : C_4)$ | 7260 | false | | $\{ 1, 3, 4, 5, 8 \}$ |
| $C_{11}^2 : C_3 : Q_8$ | 2904 | false | | $\{ 1, 3, 4, 5, 9 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 10 \}$ |
| $C_{11}^2 : C_3 : C_4$ | 1452 | false | | $\{ 1, 3, 4, 5, 11 \}$ |

Table A.17: Type I hemisystems of $\mathsf{PM}(11)$