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Symmetric Invariant Subspaces of Complexifications of Linear Operators

1. Introduction.

Let $T : X \rightarrow X$ be a linear operator on a complex Banach space. Denote by $\sigma(T)$ and $R(\lambda, T)$ its spectrum and resolvent.

Suppose that $\sigma(T)$ is disconnected, $\sigma(T) = F_1 \cup F_2$, where F_1 and $F_2 = \sigma(T) \setminus F_1$ are some components of spectrum. Let γ be a contour surrounding F_1 . Consider the spectral projection $P = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, T) d\lambda$. Its image $[F_1]$ and kernel $[F_2]$ are both invariant subspaces and $\sigma(T|_{[F_i]}) = F_i$, $i = 1, 2$.

Suppose that the spectrum is connected. Cutting out the subset $F \subset \sigma(T)$ by a contour γ , we must multiply the resolvent to weight function g that is small in some neighborhood of the points of intersection $\gamma \cap F$: $f(T) = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, T) g(\lambda) d\lambda$. This method allows to obtain analogs of spectral subspaces with constraints on the growth of the resolvent near the spectrum (see [1, 2]). These constraints are fulfilled, for example, if the powers of $T^{\pm n}$ increase slowly. If the spectrum lies on the unit circle Λ then the condition of non-quasianalyticity $\frac{\sum_{n=-\infty}^{\infty} \|T^n\|}{1+n^2} < \infty$ is equivalent to the Levinson condition, an integral constraint on the growth of the resolvent near the spectrum; such operators also have a separated spectrum [3].

Suppose now that $T : X \rightarrow X$ and X is real. The spectral projection corresponding to the symmetric component of the spectrum of the complexification $T_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ gives us a symmetric invariant subspace $L_{\mathbb{C}}$. Its real part $L \subset X$ is T -invariant. See, for example, theorem 5.3 of [4].

Even in the case where the spectrum of $T_{\mathbb{C}}$ is connected, it is still easy to get a symmetric $T_{\mathbb{C}}$ -subspace by using the above method. We must integrate over a *symmetric* contour (with respect to the real axis) with a *symmetric* function g . The “real part” of $f(T_{\mathbb{C}})$ is a T -invariant subspace in X . In the second part of this note, we will expose a “realification” of one of such methods. As an application, we receive a theorem whose complex version was proved in [5]. The last statement of the “complex” theorem is used quite often and goes back to Theorem J of [6].

Theorem 1. *Suppose that $T : X \rightarrow X$, X is a real Banach space, T is an invertible linear operator such that $\|T^n\| = o(|n|^k)$ as $n \rightarrow \pm\infty$. If $\dim X > 2$ then T has an invariant subspace. In particular, a linear isometry $T : X \rightarrow X$ of a real space has an invariant subspace if $\dim X > 2$.*

Perhaps these results are already known, but the author has not found any direct reference. Let us observe, however, that theorems about invariant spaces of compact operators were generalized to the real case. See, for example, [7] and the references therein.

Of course, not all invariant subspaces can be obtained by spectral methods. There are operators of Volterra type such that, for each invariant subspace L , $\sigma(T|_L) = \sigma(T) = [0, 1]$ (see [8]). Generally speaking, if the operator $T_{\mathbb{C}}$ has invariant subspaces it is not known whether there are symmetric subspaces among them; the existence of such subspaces is equivalent to the validity of Conjecture 3 in [7].

2. Detailed Definitions and the Proof of the Theorem 1.

Below by an *invariant subspace* we mean a *nontrivial closed* invariant subspace taken to itself by an operator $T : X \rightarrow X$.

Suppose that $T : X \rightarrow X$, X is a complex Banach space, and $x \in X$. The map $\lambda \mapsto R(\lambda, T)x$ is an X -valued function holomorphic outside $\sigma(T)$. If this map has a single-valued analytic extension to some set $\rho(x)$ then the set $\sigma(x) := \mathbb{C} \setminus \rho(x) \subset \sigma(T)$ is called the local spectrum of x and the corresponding extension is called the local resolvent of x .

Now, suppose that X is real space. The complexification of X is the space $X_{\mathbb{C}}$ of elements of the form $(x + iy)$; it is natural to call the vectors $x, y \in X$ real ($\operatorname{Re}(z)$) and imaginary ($\operatorname{Im}(z)$) parts of z . The space $X_{\mathbb{C}}$ is endowed by the conjugation $J : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$, $J(x + iy) = (x - iy)$. Introduce a norm by $\|z\|^2 = \max\{\|\operatorname{Re}(\lambda z)\|^2 + \|\operatorname{Im}(\lambda z)\|^2 \mid \lambda \in \mathbb{C}, |\lambda| = 1\}$. This norm is equivalent to the norm of the direct sum $X \oplus X$.

The complexification $T_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ of an operator $T : X \rightarrow X$ is defined by $T_{\mathbb{C}}(x + iy) = (Tx + iTy)$.

We call a subset $F \subset \mathbb{C}$ *symmetric* if F is symmetric with respect to the real line, i.e. $F = \overline{F}$. Similarly, a subset $Z \subset X_{\mathbb{C}}$ is symmetric if $J(Z) = Z$. It is easy to see that if $Z \subset X_{\mathbb{C}}$ is a symmetric subspace then $Z = L_{\mathbb{C}}$, where $L = \operatorname{Re} Z = \operatorname{Im} Z \subset X$.

Lemma. *Suppose that an operator $T_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is the complexification of some real operator $T : X \rightarrow X$. Then the spectrum $\sigma(T_{\mathbb{C}}) \subset \mathbb{C}$ is symmetric. If $T_{\mathbb{C}}$ admits a local resolvent, then $\sigma(J(z)) = \overline{\sigma(z)}$ for each $z \in X_{\mathbb{C}}$.*

Proof. It is easy to verify the equality $R(\bar{\lambda}, T_{\mathbb{C}}) = J \circ R(\lambda, T_{\mathbb{C}}) \circ J$, and so the spectrum of $T_{\mathbb{C}}$ is symmetric (this is lemma 4.1 of [4]). If $z \in X_{\mathbb{C}}$ and a function f is an analytic extension of the resolvent $R(\lambda, T)z$ then the function $J \circ f \circ J : \lambda \rightarrow J \circ R(\lambda, T)(J(x))$ is an analytic extension of the resolvent $R(\bar{\lambda}, T_{\mathbb{C}})x$. Therefore, both maximal extensions coincide and $\sigma(J(z)) = \overline{\sigma(z)}$. The lemma is proved.

Now we prove Theorem 1. Let F be a *symmetric* arc containing a part of the spectrum of $T_{\mathbb{C}}$ and let $[F] \subset X_{\mathbb{C}}$ be the space of vectors whose local spectrum is included in F . The spectrum of $T_{\mathbb{C}}$ is separated [3] and hence the subspace $[F]$ is $T_{\mathbb{C}}$ -invariant. Lemma 1 implies that this subspace is symmetric. As is easy to verify, $\operatorname{Re}[F] \subset X$ is a T -invariant subspace.

The spectrum $T_{\mathbb{C}}$ may consist of at most two points $\eta, \bar{\eta} \in \Lambda$; therefore, no symmetric arc F contains a *part* of the spectrum. In this situation, the separatedness of the operator and the constraint on the growth of $\|T^{\pm n}\|$ allow us to use the Gelfand—Hille theorem involving which it is easy to conclude that $((T_{\mathbb{C}} - \eta)(T_{\mathbb{C}} - \bar{\eta}))^{k+1} = 0$. Hence, the closure of the range of the operator $(T_{\mathbb{C}} - \eta)(T_{\mathbb{C}} - \bar{\eta})$ is not equal to $X_{\mathbb{C}}$. The coefficients of the polynomial $T_{\mathbb{C}}^2 - aT_{\mathbb{C}} + bI = (T_{\mathbb{C}} - \eta)(T_{\mathbb{C}} - \bar{\eta})$ are real, and so the closure of the range $(T^2 - aT + bI)$ is not equal to X . Cf. the proof of Theorem 3 in [5]. Therefore, either it is an invariant subspace or $(T^2 - aT + bI)X = 0$. In the latter case, every vector $x \in X$ generates at most a two-dimensional invariant subspace.

Consider the case where T is an isometry. If T is bijective then $\|T^{\pm n}\| = 1$ for all $n \in \mathbb{N}$ and everything is proved. If $TX \neq X$ then TX is a desired invariant subspace. Theorem 1 is proved.

Remark 1. We use very general results of [3], but for proving the separatedness of $[F]$ in our theorem, it would have sufficed to refer to Leaf's work [2] and even Dunford's basic result (see Corollary 9 of Chapter XVI §5 [9]).

Remark 2. At the end of article [3], it is mentioned that Wermer has shown the existence of invariant subspaces, "of course, if the spectrum contains more than one point". This is not the case; in [5], the one-point case is considered separately in the proof of theorem 3 in [5]. That is the argument we have used in the proof of the Theorem 1 in the case of a two-point spectrum.

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