Uniform Estimates of the Prolate Spheroidal Wave Functions and Spectral Approximation in Sobolev Spaces.

Aline Bonami[a] and Abderrazek Karoui[b] [1]

[a] Fédération Denis-Poisson, MAPMO-UMR 6628, département de mathématiques, Université d'Orléans, 45067 Orléans cedex 2, France.
[b] Université du 7 Novembre à Carthage, Département de Mathématiques, Faculté des Sciences de Bizerte, Tunisia.

Abstract— For fixed c Prolate Spheroidal Wave Functions $\psi_{n,c}$ form a basis with remarkable properties for the space of band-limited functions with bandwith c and have been largely studied and used after the seminal work of Slepian. Recently, they have been used for the approximation of functions of the Sobolev space $H^s([-1,1])$. The choice of c is then a central issue, which we address. Such functions may be seen as the restriction to $[-1,1]$ of almost time-limited and band-limited functions, for which PSWFs expansions are still well adapted. To be able to give bounds for the speed of convergence one needs uniform estimates in n and c. To progress in this direction, we push forward the WKB method and find uniform approximation of $\psi_{n,c}$ in terms of the Bessel function J_0 while only pointwise asymptotic approximation was known up to now. Many uniform estimates can be deduced from this analysis. Finally, we provide the reader with numerical examples that illustrate in particular the problem of the choice of c.

2010 Mathematics Subject Classification. Primary 42C10, 65L70. Secondary 41A60, 65L15.
Key words and phrases. Prolate spheroidal wave functions, eigenvalues and eigenfunctions, asymptotic estimates, spectral approximation, Sobolev spaces.

1 Introduction

Traditionally, the prolate spheroidal wave functions (PSWFs) have been used for solving various problems from physics and signal processing, see [7, 14, 15, 27]. Perhaps, the PSWFs were specially known for their important contributions in solving many problems from antenna theory, see for example [14]. Note that for a given real number $c > 0$, called bandwidth, the PSWFs denoted by $(\psi_{n,c}(\cdot))_{n\geq 0}$, were known as the eigenfunctions of the Sturm-Liouville operator's L_c defined on $C^2([-1,1])$ by

$$L_c(\psi) = (1-x^2)\frac{d^2\psi}{d\,x^2} - 2x\frac{d\psi}{d\,x} - c^2x^2\psi. \tag{1}$$

The eigenvalues $(-\chi_n(c))_{n\geq 0}$ are fixed by the requirement that the eigenfunctions $\psi_{n,c}(x)$ are bounded as $|x| \to 1^-$. To the best of our knowledge, in [18], C. Niven was the first, in 1880, to give a remarkably detailed theoretical, as well as computational study of the eigenfunctions and the eigenvalues of the above Sturm-Liouville problem. In their pioneer work [12, 13, 22, 23, 24], D. Slepian, H. Landau and H. Pollak have shown various important properties of the PSWFs and their associated spectrum. Among these properties, they have proved that the PSWFs are also the

[1] This work was supported in part by the ANR grant "AHPI" ANR-07- BLAN-0247-01, the French-Tunisian CMCU 10G 1503 project and the DGRST research grant 05UR 15-02.
Part of this work was done while the second author was visiting the research laboratory MAPMO of the University of Orléans, France.

eigenfunctions of the compact integral operators F_c and Q_c, defined on $L^2([-1,1])$ by

$$F_c(\psi)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\,\psi(y)\,dy, \quad Q_c(f)(x) = \int_{-1}^{1} e^{i\,c\,x\,y} f(y)\,dy. \tag{2}$$

As a result, they have shown that the PSWFs exhibits the unique properties to form an orthogonal basis of $L^2([-1,1])$, an orthonormal system of $L^2(\mathbf{R})$ and an orthonormal basis of B_c, the Paley-Wiener space of $c-$band-limited functions defined by $B_c = \left\{ f \in L^2(\mathbf{R}),\ \text{Support}\ \ \widehat{f} \subset [-c,c] \right\}$. Here, $\widehat{f}$ denotes the Fourier transform of f. The PSWFs are normalized by using the following rule,

$$\int_{-1}^{1} |\psi_{n,c}(x)|^2\,dx = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2\,dx = \frac{1}{\lambda_n(c)}, \quad n \geq 0. \tag{3}$$

$(\lambda_n(c))_n$ is the infinite sequence of the eigenvalues of F_c, arranged in the decreasing order $1 > \lambda_0(c) > \lambda_1(c) > \cdots > \lambda_n(c) > \cdots$.

Recently, there has been a growing interest in the study of the quality of the spectral approximations by the PSWFs and the building of PSWFs based numerical schemes for solving various problems from numerical analysis, see [3, 4, 5, 6, 25]. In particular, in [4], the author has shown that a PSWFS approximation based method outperforms in terms of spatial resolution and stability of timestep, the classical approximation methods based on Legendre or Tchebyshev polynomials. The authors of [6] were among the first to study the quality of the approximation by the PSWFs in the Sobolev space $H^s([-1,1]), s > 0$. In particular, they have given an estimate of the decay of the PSWFs expansion coefficients of a function $f \in H^s([-1,1])$, see also [4]. Recently, in [25], the author studied the speed of convergence of the expansion of such a function in a basis of PSWFs. We should mention that the methods used in the previous three references are heavily based on the use of the properties of the PSWFs as eigenfunctions of the differential operator L_c, given by (1). They pose the problem of the best choice of the value of the band-width $c > 0$, for approximating well a given $f \in H^s([-1,1])$, but their answer is mainly experimental. It has been numerically checked in [4, 25] that the smaller the value of s, the larger the value of c should be.

When starting this work, our aim was to clarify the way c should be chosen. We also wanted to use the properties of the PSWFs related with the Fourier transform. These ones are not easy to use for numerical computation because of the rapid decrease of the eigenvalues of Q_c, but they are nevertheless fondamental. To choose both c and the index of the partial sum used in the approximation, the main difficulty is that this requires uniform bounds for the PSWFs for c variable. The first main contribution of this work is to derive a whole set of qualitative and quantitative properties of the PSWFs. Some of these results were already known in the literature from mainly numerical evidences and/or asymptotic and heuristic justifications.

More precisely, the asymptotic behaviors of eigenvalues $\lambda_n(c)$ and functions $\psi_{n,c}$ are well-known in different conditions, for c tending to 0 or ∞ for instance. These kinds of behaviors are confirmed by numerical evidences [20, 24]. We give rigorous proofs, which are valid for the whole range of values of n and c, or, for most of them, under the condition that $c^2 < \chi_n(c)$, which is asymptotically equivalent to the condition given in [4], that $\frac{c}{c_n^*} < 1$, where $c_n^* = \frac{\pi}{2}(n+\frac{1}{2})$. An emblematic estimate that is frequently used in the literature is the following one,

$$|\psi_{n,c}(1)| < \sqrt{n+1/2}. \tag{4}$$

This is stated in [20] without a rigorous proof, but justified by asymptotic expansions and numerical evidence. This is a key point to prove that the sequence $\lambda_n(c)$ decreases very rapidly. An analytic proof of estimate (4) for all n and c seems out of reach. Nevertheless, by pushing forward different methods, we obtain uniform estimates on the values of $\psi_{n,c}$ that are sufficient to obtain an exponential decay of $\lambda_n(c)$. We have a gain compared to these authors in the sense that we have a complete proof, while we loose on constants that appear in the exponential decay compared to theirs, which are comforted by numerical evidence. The two approaches are clearly complementary, and we do hope that having rigorous proofs valid in the whole range of c and n adds to the understanding of the expansions in PSWFs.

Our second main contribution is to use different properties of the PSWFs as eigenfunctions of the differential operator L_c and the finite Fourier integral operator Q_c and give a study of the quality of approximation by the PSWFs in the Sobolev spaces $H^s([-1,1])$. More importantly, our study tries to give a satisfactory answer to the previous important problem of the choice of the parameter c. To this end, we are led to study various inequalities and asymptotic results associated with the PSWFs.

This work is organized as follows. In Section 2, we prove various uniform estimates of the PSWFs. In particular, we use the WKB method and provide an explicit uniform approximation over $[0,1]$ of the PSWFs. Also, we derive several useful bounds of the values the PSWFs and their first derivatives. Moreover, we give two results on the approximation of the PSWFs by Legendre polynomials. In Section 3, we give some practical and useful estimates of the decay of the eigenvalues and the Legendre coefficients associated with the PSWFs. Our results improve the existence decay results given in [6, 26]. In Section 4, we first give the quality of approximation by the PSWFs in the set of almost time and band-limited functions. Then, we combine these results with those of Section 2 and give a new $L^2([-1,1])-$error bound of approximating a function $f\in H^s([-1,1])$ by its Nth terms truncated PSWFs series expansion. The proof of this bound is based on the use of the quality of approximation of almost bandlimited functions by the PSWFs. Also, by using the Fourier characterization of periodic Sobolev spaces, we give another approach for functions that extend into periodic functions with the same regularity. These new estimates provide us with a way for the choice of the appropriate bandwidth $c>0$ to be used by a PSWFs based method for the approximation in a given Sobolev space $H^s([-1,1])$. In Section 5, we first give an overview concerning the computational aspects of the PSWFs, then we provide the reader with two methods for the computation of the PSWFs approximate expansion coefficients of a function $f\in H^s([-1,1])$. Finally, we give some numerical examples that illustrate the different result of this work.

2 Uniform estimates for the PSWFs

In this section, we prove that the PSWFs $\psi_{n,c}$ are uniformly approximated in terms of the Bessel function J_0 when c is not too large, namely, when $c^2\leq\alpha\chi_n(c)$, where $\alpha<1$. The proof of this approximation result is based on the use of the well known WKB method for the study of asymptotic behavior of the solutions of the perturbed differential equations of the Sturm-Liouville type, see for instance [17]. We emphasize the fact that the asymptotic behavior that we shall describe is well-known for a single value of the variable and n tending to infinity (see for instance [4, 8] and the references there). What is new here is the possibility to push forward the methods in order to have uniformity. This allows us to have uniform bounds for the functions $\psi_{n,c}$ within the prescribed constraints on the parameters and recover partially the bounds given in [20] from numerical evidence.

2.1 Uniform approximation of the PSWFs by the WKB method

In the section we note $q_n(c)=\frac{c^2}{\chi_n(c)}$. Most of the results of this section are obtained under the relatively weak constraint that $q_n(c)<1$. Also we always assume that $\psi_{n,c}(1)\geq 0$. We prove here that $\psi_{n,c}$ can be approximated by a simple explicit function (once one knows $\chi_n(c)$), up to a multiplicative constant $A(n,c)$ that takes into account the normalization that we have chosen for $\psi_{n,c}$. This constant is not explicitly known, but tends to 1 when $q_n(c)$ tends to 0. On another side, we compute explicitly the dependance in q of constants and do not allow the notations O and o, except for functions of one variable and when there is no ambiguity.

To simplify notation, we now skip the parameters and note ψ, q and χ_n. We only leave the parameter n for the eigenvalue χ_n so that the asymptotic behavior of each term is clear. With these notations, ψ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\psi'(x)\right]+\chi_n(1-qx^2)\psi(x)=0, \quad x\in[-1,1]. \tag{5}$$

Because of the parity of the PSWFs, we can restrict to the interval $[0,1]$, which we will do from now on in this sub-section. We then note

$$S(x):=S_q(x)=\int_x^1\sqrt{\frac{1-qt^2}{1-t^2}}\,dt \tag{6}$$

which defines a homeomorphism on the whole interval and is smooth in $[0,+1)$. It is well-known as an elliptic integral of the second kind. We look for ψ under the form

$$\psi(x)=\varphi(x)U(S(x)), \quad \varphi(x)=(1-x^2)^{-1/4}(1-qx^2)^{-1/4}. \tag{7}$$

Replacing S' by its explicit value, Equation (5) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[-(1-x^2)^{1/2}(1-qx^2)^{1/2}\varphi U'+(1-x^2)\varphi' U\right]+\chi_n(1-qx^2)\varphi U=0. \tag{8}$$

One can easily check that terms in U' disappear with this choice of φ, and the equation satisfied by U on the interval $[0,+1)$ may be written as

$$U''+(\chi_n+h_1)\,U=0, \tag{9}$$

with

$$h_1(S(x)):=\varphi(x)^{-1}(1-qx^2)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\varphi'(x)\right].$$

Let us write

$$\mathcal{Q}(x):=(1-x^2)(1-qx^2), \tag{10}$$

so that

$$\varphi'/\varphi=-\frac{1}{4}\mathcal{Q}'/\mathcal{Q}.$$

It follows that $h_1\circ S$ is a rational function with poles in ±1 and $\pm\sqrt{\frac{1}{q}}$, which may be written

$$h_1\circ S=\frac{1}{16}(1-qx^2)^{-1}\left[(1-x^2)\left(\frac{\mathcal{Q}'}{\mathcal{Q}}\right)^2-4\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathcal{Q}'}{\mathcal{Q}}\right)\right].$$

Only the pole 1 is of interest (because of the assumption on q). One easily computes the residue at 1, which is $a=-\frac{1}{8}(1-q)^{-1}$. So $h_1(S(x))$ can be written as

$$h_1(S(x))=\frac{(1-q)^{-1}}{8(1-x)}+h_2(S(x)),$$

with $h_2\circ S$ a rational function without poles on $[0,1]$. Elementary explicit computation proves that this function is bounded by $2(1-q)^{-3}$.

Next we want to replace $(1-x)^{-1}$ by a function of $S(x)$. This possibility will be a consequence of the following lemma.

Lemma 1. *One has the following inequalities, valid on the interval* $[0, 1]$*.*

$$2(1-q)(1-x) \leq S_q^2(x) \leq 4(1-x). \tag{11}$$

At 0 *one has*

$$1 \leq S_q(0) \leq \frac{\pi}{2}.$$

Moreover $\frac{S_q(x)^2}{1-x}$ *extends into a holomorphic function in a neighborhood of* 1 *and takes the value* $2(1-q)$ *at* 1*. Finally*

$$0 \leq S_q(x) - \sqrt{2(1-q)(1-x)} \leq \frac{2}{3}\,\frac{(1-x)^{3/2}}{(1-q)^{1/2}}. \tag{12}$$

Proof. The first inequality comes from the facts that the function $\frac{1-qt^2}{1+t}$ is decreasing and $\int_x^1 \frac{dt}{\sqrt{1-t}} = 2\sqrt{1-x}$. Moreover the square root of $\frac{1-qt^2}{1+t}$ extends into a holomorphic function in a neighborhood of 1. Extending it into an entire series in $1-t$, multiplying it by $\sqrt{1-t}$ and taking the antiderivative, we find that $\frac{S(x)^2}{1-x}$ also extends into an entire series in $1-t$. For the last inequality, we use the elementary bound

$$0 \leq \sqrt{\frac{1-qt^2}{1+t}} - \sqrt{\frac{1-q}{2}} \leq \frac{1-t}{(1-q)^{1/2}}.$$

By dividing the previous inequalities by $\sqrt{1-t}$ and then integrating the different sides of the inequalities over $[x, 1)$ and using (6), the estimate (12) follows at once. $\square$

It follows from Lemma 1 that $\frac{1}{2(1-q)}\frac{1}{1-x} - \frac{1}{S(x)^2}$ extends into a holomorphic function in a neighborhood of 1. Moreover it is bounded by $\frac{4}{(1-q)^2}$. So we can write this difference as $h_3(S(x))$, with h_3 a bounded and continuous function on $[0, S(0)]$. If we define $F := -(h_2 + h_3/4)$, we have proved that F is continuous and satisfies

$$|F(s)| \leq \frac{3}{(1-q)^3}. \tag{13}$$

This is summarized in the following lemma.

Lemma 2. *For* $q < 1$ *there exists a function* $F := F_q$ *that is continuous on* $[0, S(0)]$ *and satisfies* (13) *such that* U *is a solution of the equation*

$$U''(s) + \left(\chi_n + \frac{1}{4s^2}\right)U(s) = F(s)U(s), \quad s \in [0, S(0)]. \tag{14}$$

Before using the properties of such an equation, let us translate for U the fact that ψ has norm 1.

Lemma 3. *We have the equality*

$$-2\int_0^1 (1-qx^2)^{-1}|U(S(x))|^2 dS(x) = 1. \tag{15}$$

In particular,

$$(1-q) \leq 2\int_0^{S(0)} |U(s)|^2 ds \leq 1. \tag{16}$$

Proof. Just use the expression of ψ in terms of U and the expression of S'. $\square$

Let us come back to Equation (14), which we will consider as an equation in U with second member FU. It is well known that the associated homogeneous equation has the two independent solutions

$$U_1(s) = \chi_n^{1/4}\sqrt{s}J_0(\sqrt{\chi_n}s), \quad U_2(s) = \chi_n^{1/4}\sqrt{s}Y_0(\sqrt{\chi_n}s),$$

where J_0 (resp. Y_0) denotes the Bessel function of the first (resp. second) type. The Wronskian of U_1 and U_2 is given by

$$\begin{aligned} W(U_1, U_2)(s) &= U_1(s)U_2'(s) - U_1'(s)U_2(s) \\ &= s\chi_n \left[J_0(\sqrt{\chi_n}s)Y_0'(\sqrt{\chi_n}s) - J_0'(\sqrt{\chi_n}s)Y_0(\sqrt{\chi_n}s)\right] \\ &= = \frac{2\sqrt{\chi_n}}{\pi} \end{aligned}$$

when using the well-known identity for the Wronskian of Bessel functions. Then the general solution of (14), according to the method of variations of constants, leads to the following identity.

$$\begin{aligned} U(s) &= AU_1(s) + BU_2(s) + \frac{\pi}{2\sqrt{\chi_n}} \\ &\times \int_0^s \sqrt{st\chi_n}\left[J_0(\sqrt{\chi_n}s)Y_0(\sqrt{\chi_n}t) - J_0(\sqrt{\chi_n}t)Y_0(\sqrt{\chi_n}s)\right]F(t)U(t)dt. \end{aligned} \tag{17}$$

Let us pose

$$K_n(s,t) := \sqrt{st\chi_n}\left[J_0(\sqrt{\chi_n}s)Y_0(\sqrt{\chi_n}t) - J_0(\sqrt{\chi_n}t)Y_0(\sqrt{\chi_n}s)\right].$$

Using the fact that

$$\sup_{s\geq 0}\sqrt{s}(|J_0(s)| + |Y_0(s)|) < \infty,$$

we find that the kernel K_n is uniformly bounded. Using Schwarz Inequality, (16) and the estimate on F, we find that

$$\left|\frac{\pi}{2}\int_0^s K_n(s,t)F(t)U(t)dt\right| \leq \frac{Cs^{1/2}}{(1-q)^3}. \tag{18}$$

Here C is a uniform constant, that does not depend on q.

In particular, this integral remains bounded when divided by $s^{1/2}$ for s tending to 0. Let us prove now that $B = 0$. Since ψ extends continuously to 1, the function

$$\frac{U(S(x))}{\sqrt{S(x)}} = \psi(x)\left(\frac{(1-x)}{S(x)^2}\right)^{1/4} \times ((1+x)(1-qx^2)^{1/4}$$

remains bounded for x tending to 1 and the same is valid for $U(s)/\sqrt{s}$ for s tending to 0. On another side, $J_0(s)$ remains bounded while $Y_0(s)$ is not bounded for s tending to 0. This forces B to be 0. So (17) can be rewritten as

$$U(s) = A(\sqrt{\chi_n}s)^{1/2}J_0(\sqrt{\chi_n}s) + \frac{\pi}{2\sqrt{\chi_n}}\int_0^s K_n(s,t)F(t)U(t)dt. \tag{19}$$

The following lemma is needed in the proof of the main result of this section, given by Theorem 1.

Lemma 4. *Let* $I = \frac{2}{\sqrt{\chi_n}}\int_0^{\sqrt{\chi_n}S_q(0)} t\left(J_0(t)\right)^2\, dt$. *Then there exists a constant C' independent of n and c, such that*

$$\left|I - \frac{2S_q(0)}{\pi}\right| \leq \frac{C'}{\sqrt{\chi_n}}. \tag{20}$$

Proof. We first recall that $1 \leq S_q(0) \leq \pi/2$. It is well known (see [2]) that

$$\int_0^x t(J_0(t))^2\,dt = \frac{x^2}{2}\left[(J_0(x))^2 + (J_1(x))^2\right], \quad x > 0.$$

Moreover, we recall the asymptotic behavior of Bessel functions, see [19]

$$\sup_{x\geq 0} |\sqrt{x} J_\nu(x)| \leq c_\nu, \quad \sup_{x\geq 0}\left| J_\nu(x) - \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right| \leq \frac{d_\nu}{x^{3/2}}, \quad \nu > -1. \tag{21}$$

Here, c_ν, d_ν are constants depending only on the order ν. We easily get

$$\left|\left[J_0(\sqrt{\chi_n} S_q(0))\right]^2 + \left[J_1(\sqrt{\chi_n} S_q(0))\right]^2 - \frac{2}{\pi\sqrt{\chi_n} S_q(0)}\right| \leq \frac{K}{\chi_n (S_q(0))^2}, \quad K = d_0c_0 + d_1c_1 + 2\sqrt{\frac{2}{\pi}}.$$

That is

$$\left| I - \frac{2S_q(0)}{\pi}\right| \leq \frac{C'}{\sqrt{\chi_n}}, \quad C' = 2(d_0c_0 + d_1c_1) + 4\sqrt{\frac{2}{\pi}}. \tag{22}$$

□

We are now able to state our main theorem. We mention explicitly the parameters. In the statement, we assume that $\psi_{n,c}$ is such that $\psi_{n,c}(1) \geq 0$.

Theorem 1. *There exist constants C, C' with the following properties. Assume that the parameters n, c are such that $q = c^2/\chi_n(c) < 1$. Then one can find a constant $A := A(n,c) \leq M$ such that, for $0 \leq x \leq 1$,*

$$\psi_{n,c}(x) = A\frac{\chi_n(c)^{1/4}\sqrt{S_q(x)}J_0(\sqrt{\chi_n(c)}S_q(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} + R_{n,c}(x) \tag{23}$$

with

$$\sup_{x\in[0,1]} |R_{n,c}(x)| \leq C_q \chi_n(c)^{-1/2}, \qquad C_q = \frac{C}{(1-q)^{13/4}}. \tag{24}$$

Moreover, for any integer n satisfying $\chi_n > \left(\frac{C'\pi}{2S_q(0)}\right)^2$, where C' is as given by Lemma 4, the constant A is bounded above and below as follows,

$$\frac{\pi}{2S_q(0)} \frac{\left(\sqrt{1-q} - \sqrt{2}\frac{C_q}{\sqrt{\chi_n}}\right)^2}{\left(1 + \frac{\pi}{2S_q(0)}\frac{C'}{\sqrt{\chi_n}}\right)} \leq A^2 \leq \frac{\pi}{2S_q(0)} \frac{\left(1 + \sqrt{2}\frac{C_q}{\sqrt{\chi_n}}\right)^2}{\left(1 - \frac{\pi}{2S_q(0)}\frac{C'}{\sqrt{\chi_n}}\right)}. \tag{25}$$

Proof. Using (19), we can write

$$\psi(x) = A\frac{\chi_n(c)^{1/4}\sqrt{S(x)}J_0(\sqrt{\chi_n(c)}S_q(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} + R_{n,c}(x),$$

with

$$R_{n,c}(x) = (1-x^2)^{-1/4}(1-qx^2)^{-1/4}\widetilde{R}(S_q(x)),$$

where

$$\widetilde{R}(s) = \frac{\pi}{2\sqrt{\chi_n(c)}} \int_0^s K_n(S(x),t)F(t)U(t)dt.$$

To bound $R_n(x)$, we proceed as follows. From (18), we have $\sup_{x\in[0,1]} \dfrac{\widetilde{R}(S(x))}{\sqrt{S(x)}} \leq \dfrac{C}{(1-q)^3\sqrt{\chi_n}}$. Also, from (11), we get $\sup_{x\in[0,1]} \dfrac{\sqrt{S(x)}}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} \leq \dfrac{\sqrt{2}}{(1-q)^{1/4}}$. Moreover, since

$$R_{n,c}(x) = \sqrt{S_q(x)}(1-x^2)^{-1/4}(1-qx^2)^{-1/4}\frac{\widetilde{R}(S_q(x))}{\sqrt{S_q(x)}},$$

then from the previous discussion, we conclude the bound of $|R_{n,c}(x)|$. To prove (25), we first pose $V(s) = (\sqrt{\chi_n}s)^{1/2}J_0(\sqrt{\chi_n}s)$ for $s \in [0, S_q(0)]$. Note that $U = AV + \widetilde{R}$. We deduce from the previous estimates that the L^2 norm of $\widetilde{R}$ is bounded by $\sqrt{2}C_q\chi_n(c)^{-1/2}$. Let us recall the inequalities (16). It follows that

$$\sqrt{1-q} - \sqrt{2}C_q\chi_n(c)^{-1/2} \leq |A|\|V\|_2 \leq 1 + \sqrt{2}C_q\chi_n(c)^{-1/2}. \tag{26}$$

To conclude it is sufficient to prove that the norm of V is conveniently bounded above and below. After a change of variable, we have to use the bounds above and below for the quantity I, given by Lemma 4. More precisely, by writing $A^2\|V\|_2^2 = A^2\left|I - \frac{2S_q(0)}{\pi}) + \frac{2S_q(0)}{\pi}\right|$ and by using (20) and (26), one gets (25). □

Theorem 1 may be used on the whole interval $[-1,+1]$, using the parity of the functions $\psi_{n,c}$.

It gives precise indications on the functions $\psi_{n,c}$, both quantitatively and qualitatively. We could as well have written the bounds for the rests in terms of n^{-1} instead of $\chi_n(c)^{-1/2}$. Indeed, from the inequality $n(n+1) \leq \chi_n(c) \leq n(n+1) + c^2$ one deduces that

$$n(n+1) \leq \chi_n(c) \leq \frac{n(n+1)}{1-q}. \tag{27}$$

The next corollary gives bounds for $\psi_{n,c}$. It is an immediate consequence of Theorem 1 and the fact that $|J_0(s)| + s^{1/2}|J_0(s)|$ is uniformly bounded for $s \geq 0$. We use the same notations as in Theorem 1.

Corollary 1. *There is a constant C such that, for $C_q = C(1-q)^{-4}$, the two following inequalities hold.*

$$\sup\nolimits_{|x|\leq 1} |\psi_{n,c}(x)| \leq C_q\chi_{n,c}^{1/4} \tag{28}$$

$$\sup\nolimits_{|x|\leq 1}(1-x^2)^{1/4}|\psi_{n,c}(x)| \leq C_q. \tag{29}$$

Remark that the first bound is sharp since

$$\psi_{n,c}(1) = A(n,c)\chi_n(c)^{1/4}. \tag{30}$$

Next we state as a lemma the fact that, for q close from 0, the constant A is close from 1.

Lemma 5. *Let $\alpha < 1$ and $0 < K < 1$. Let C' be as defined by Lemma 4. There are constants $H_1 = H_1(\alpha, K)$ and $H_2 = H_2(\alpha, K)$ such that, for $q \leq \alpha$ and n satisfying $\frac{C'}{\sqrt{\chi_n}} \leq K\sqrt{1-\alpha}$, the constant $A(n,c)$ in Theorem 1 satisfies the inequality*

$$|A^2(n,c) - 1| \leq H_1(\alpha,K)q + H_2(\alpha,K)\chi_n(c)^{-1/2}. \tag{31}$$

As a consequence, under the same assumptions on q,

$$\psi_{n,c}(1)^2 - n - \frac{1}{2} \leq H_3qn + H_2. \tag{32}$$

Proof. We first note that

$$\frac{\pi(1-q)^{1/2}}{2} \leq S_q(0) \leq \frac{\pi}{2}.$$

By using the previous inequalities as well as (25), we get

$$(1-q)\frac{1-2\sqrt{2}\frac{C_\alpha(1-q)^{-1/2}}{\sqrt{\chi_n}}}{1+\frac{C'(1-q)^{-1/2}}{\sqrt{\chi_n}}} \leq A^2 \leq (1-q)^{-1/2}\frac{\left(1+\sqrt{2}\frac{C_\alpha}{\sqrt{\chi_n}}\right)^2}{1-\frac{C'(1-\alpha)^{-1/2}}{\sqrt{\chi_n}}}. \qquad (33)$$

We find in particular that

$$1-A^2 \leq q + (C' + 2^{3/2}C_\alpha)\chi_n^{-1/2}.$$

Under the condition that $\frac{C'}{\sqrt{\chi_n}} \leq K\sqrt{1-\alpha}$, which implies that the denominator on the right hand side is larger than $1-K$, we also have

$$A^2 - 1 \leq \frac{q}{\sqrt{1-\alpha}} + (1-K)^{-1}(1-\alpha)^{-1/2}(2^{3/2}C_\alpha + 2C_\alpha^2)\chi_n^{-1/2}.$$

By combining the two estimates, one obtains (31) with $H_1 = (1-\alpha)^{-1/2}$ and $H_2 = (1-K)^{-1}(1-\alpha)^{-1/2}(2^{3/2}C_\alpha + 2C_\alpha^2)$.

Next, we use (30), (31) and the inequality

$$\chi_n(c)^{1/2} - n - \frac{1}{2} \leq \frac{c^2}{\sqrt{\chi_n}}$$

to conclude that

$$\begin{aligned}\left|\psi_{n,c}(1)^2 - (n+\frac{1}{2})\right| &\leq ||A^2-1|\sqrt{\chi_n} + |\sqrt{\chi_n} - (n+1/2)| \\ &\leq (H_1+1)q\sqrt{\chi_n} + H_2.\end{aligned}$$

□

Remark that (32) can be compared to the conjecture that the left hand side is negative. Even if much weaker, it will nevertheless be sufficient to prove the exponential decrease of the sequence $\lambda_n(c)$.

2.2 First approximation by Legendre polynomials

We now discuss the possibility to modify $\chi_n(c)$ in Formula (23). It is easily seen that Theorem 1 remains valid when $\chi_n(c)$ is replaced by $\chi_n(c)+\beta$, for some fixed constant. Indeed, one just has to change F into $F+\beta$. This is particularly interesting for $c=0$, where, instead of $\chi_n(0)$ one can take $(n+1/2)^2$. With this modification, the well-known fact that $\overline{P}_n(1) = \sqrt{n+\frac{1}{2}}$ leads to the fact that $A=1$. Also, $S(x) = \arccos(x)$. So, we have the following corollary of Theorem 1.

Theorem 2. *The normalized Legendre polynomials satisfy the following, uniformly for* $0 \leq x \leq 1$.

$$\overline{P}_n(x) = \sqrt{n+\frac{1}{2}}\left(\frac{\arccos(x)}{\sqrt{1-x^2}}\right)^{1/2} J_0\left((n+1/2)\arccos(x)\right) + O\left(\frac{1}{n}\right) \qquad (34)$$

The uniformity of the approximation seems new, even if the formula is not.

We can also replace $\chi_n(c)$ by $(n+1/2)^2$ in Formula (23) for all c, except that now the constant C has to be multiplied by $(1+c^2)$. If we want that the second term behaves like a rest, we have to assume that c^2 is sufficiently small compared to $\chi_n(c)^{1/2}$. This leads us to the condition:

- Condition $\mathfrak{C}(\alpha,\epsilon)$:

$$c^2 \le \alpha \chi_n(c)^{\frac{1}{4}-\epsilon}.$$

So let us assume this condition and replace $\chi_n(c)$ by $(n+1/2)^2$ in Formula (23). It is easy to see that the rest of the proof remains unchanged. We do not give too many details because we will use another method later on. So there there exists some constant $B = B(c,n)$ such that we can write

$$\psi_{n,c}(x) = B\frac{\sqrt{n+\frac{1}{2}}\sqrt{S_q(x)}J_0((n+\frac{1}{2})S_q(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} + R_{n,c}(x) \tag{35}$$

with $|R_{n,c}| \le C\chi_n(c)^{-\frac{1}{4}-\epsilon}$, with C a uniform constant depending only on α and ϵ. Moreover (we also proceed as in Lemma 5), the constant B is such that

$$|B(n,c) - 1| \le H\chi_n(c)^{-\frac{1}{4}-\epsilon}.$$

So we can replace it by 1, which generates an error of the order $\chi_n(c)^{-\frac{1}{4}-\epsilon}$. In order to go further, we want to replace $S_q(x))$ by $\arccos(x)$ without too much loss, in order to be able to prove that $\psi_{n,c}$ is close from the Legendre polynomial. Since $S_q(x) \ge (1-q)^{1/2}\arccos(x)$, the difference $\arccos(x) - S_q(x)$)is also bounded by $H\chi_n(c)^{-\frac{1}{2}-\epsilon}\arccos(x)$. This implies that $|J_0((n+\frac{1}{2})\arccos(x)) - J_0((n+\frac{1}{2})S_q(x))|$ is bounded by $C\chi_n(c)^{-\epsilon}$. The other factors are also easily shown to be close to the analogous terms for the Legendre polynomial. We have proved finally the following proposition.

Proposition 1. *Under the condition $\mathfrak{C}(\alpha,\epsilon)$, there exists C such that*

$$\sup_{|x|\le 1} |\psi_{n,c}(x) - \overline{P}_n(x)| \le Cn^{-2\epsilon}.$$

Different authors have proposed a second term as a candidate for the expansion of $\psi_{n,c}$ in terms of Legendre polynomials for q tending to zero, see for instance Boyd's paper [4]. Our proof cannot lead to this because our rest is not small compared to this second term.

The previous proposition does not answer a natural question, which is the approximation by $\overline{P}_n$ for c tending to 0. We will give another method for this.

2.3 Second approximation by the Legendre polynomials

Proposition 2. *Let $\alpha < 1$. There exists a constant M_α with the following property. For all n and $c \ge 0$ such that $q_n(c) \le \alpha$, then*

$$\sup_{x\in[-1,1]} \left|\psi_{n,c}(x) - \overline{P_n}(x)\right| \le M_\alpha \frac{c^2}{\sqrt{n+1/2}}, \tag{36}$$

Proof. The proof follows the same lines as the proof of Theorem 1. We now see the equation satisfied by $\psi := \psi_{n,c}$ as

$$(1-x^2)\psi'' - 2x\psi' + n(n+1)\psi_{n,c} = \left(n(n+1) - \chi_n + c^2x^2\right)\psi.$$

The homogeneous equation

$$(1-x^2)\psi'' - 2x\psi' + n(n+1)\psi_{n,c} = 0$$

has as solutions the Legendre polynomial $\overline{P}_n$ on one side, the Legendre function of the second kind Q_n on another side. Moreover (see [1]) the Wronskian is given by

$$W(\overline{P_n}, Q_n)(x) = \frac{n+1/2}{1-x^2}.$$

The function $G := n(n+1) - \chi_n + c^2x^2$ is uniformly bounded in modulus by c^2. By the method of variation of constants we can write

$$\psi(x) = A\overline{P_n}(x) + BQ_n(x) + \frac{1}{n+1/2}\int_x^1 (1-y^2)\left(\overline{P_n}(x)Q_n(y) - \overline{P_n}(y)Q_n(x)\right) G(y)\psi(y)\, dt.$$

We pose

$$L_n(x,y) = (1-y^2)\left(\overline{P_n}(x)Q_n(y) - \overline{P_n}(y)Q_n(x)\right).$$

We claim that L_n is uniformly bounded, independently of n, x, y. Let us take this for granted and go on for the proof. Then, for some uniform constant H, the error term satisfies

$$\frac{1}{n+1/2}\left|\int_x^1 L_n(x,y)G(y)\psi(y)dy\right| \leq H\frac{c^2}{n+1/2}.$$

The consideration of the behavior of each term when x tends to 1 implies that B is equal to 0 and we have, for again H a uniform constant,

$$|\psi - A\overline{P}_n| \leq H\frac{c^2}{n+1/2}.$$

Since both functions ψ and $\overline{P}_n$ have L^2 norm 1, we have the inequality

$$|1-A| \leq H\frac{c^2}{n+1/2}.$$

Consequently, we have

$$\begin{aligned}
\sup_{x\in[0,1]} |\psi(x) - \overline{P_n}(x)| &\leq \sup_{x\in[0,1]} |\psi_{n,c}(x) - A\overline{P_n}(x)| + |A-1| \sup_{x\in[0,1]} |\overline{P_n}(x)| \\
&\leq M_\alpha \frac{c^2}{\sqrt{n+1/2}}.
\end{aligned}$$

□

It remains to prove that L_n is uniformly bounded. It is a consequence of the fact that $\sqrt{1-x^2}|\overline{P}_n(x)|$ and $\sqrt{1-x^2}|Q_n(x)|$ are bounded independently of x and n. For $\overline{P}_n$, it is a consequence of (29). For Q_n it is given by the following lemma.

Lemma 6. *There exists a constant H such that, for all n, we have the inequality*

$$\sup_{x\in[-1,1]} \sqrt{1-x^2}|Q_n(x)| \leq H.$$

Proof. We recall that $Q_n(x) = \frac{1}{2}\overline{P_n}(x)\log\left(\frac{1+x}{1-x}\right) - V_{n-1}(x)$, where

$$V_{n-1}(x) = \frac{1}{2}\int_{-1}^1 \frac{\overline{P_n}(x) - \overline{P_n}(t)}{x-t}\, dt = \sqrt{n+1/2}\left[\sum_{m=1}^n \frac{1}{m}\frac{\overline{P_{m-1}}(t)}{\sqrt{m-1/2}}\frac{\overline{P_{n-m}}(t)}{\sqrt{n-m+1/2}}\right].$$

We can restrict to $x \geq 0$. We recall that, by (29), $(1-x^2)^{1/4}|P_n(x)|$ is uniformly bounded. So the required inequality follows at once for the first term. Next we consider V_{n-1}. Using again (29), it is sufficient to prove that

$$\sum_{m=1}^n \frac{\sqrt{n+1/2}}{m\sqrt{m-1/2}\sqrt{n-m+1/2}}$$

is uniformly bounded, which is elementary: just consider separately the cases $m < n/2$ and $m \geq n/2$. This finishes the proof of the proposition. □

2.4 Uniform bound estimates of the PSWFs and their derivatives.

In this paragraph, we give various estimates of the bounds of the $\psi_{n,c}$ and $\psi'_{n,c}$ over $[-1,1]$. Moreover, we give bounds for the successive derivatives of the PSWFs at $x=0$. The content of this paragraph will play later on a major role for the approximation of polynomials by their series in PSWFs. By using the results of Theorem 1 and its corollary, we prove the following proposition that provides us with a first set of uniform bounds.

Proposition 3. *There exists a constant C depending only on $\alpha<1$ such that, for $q=\dfrac{c^2}{\chi_n}<\alpha$, then*

$$\sup_{x\in[-1,1]}|\psi'(x)|\leq C{\chi_n}^{5/4}, \tag{37}$$

$$\sup_{x\in[-1,1]}(1-x^2)|\psi'(x)|\leq C\sqrt{\chi_n}. \tag{38}$$

Proof. To prove (37), it suffices to consider the case $x\in[0,1]$. Then by using the identity

$$(1-x^2)\psi'(x)=\chi_n\int_x^1(1-qt^2)\psi(t)\,dt, \tag{39}$$

one gets

$$(1-x^2)|\psi'(x)|\leq\chi_n(1-x^2)\int_x^1|\psi(t)|\,dt.$$

Finally, inequality (28) gives us (37).

To prove (38), we start again from (39). We let $s=S(t)$, $t\in[0,1]$. Here, $S(t)$ is given by (6). Let $L(s)$ be the function defined on $[0,S_q(0)]$ by

$$L(s)=L(S^{-1}t)=(1-qt^2)^{1/4}\left(\frac{(1-t^2)^{1/2}}{S(t)}\right)^{1/2}.$$

It is easy to see that $L(s)$ is of class $\mathcal{C}^1$ on $[0,S_q(0)]$, with a norm bounded by a constant that depends only on α. Moreover, by using (23), we get

$$\int_x^1(1-qt^2)\psi(t)\,dt=A\chi_n^{1/4}\int_0^{S(x)}L(s)sJ_0(\sqrt{\chi_n}s)\,ds+\int_x^1R_{n,c}(t)\,dt. \tag{40}$$

Since $(sJ_1(s))'=sJ_0(s)$, then an integration by parts gives us

$$\begin{aligned}\chi_n^{1/4}\int_0^{S(x)}L(s)sJ_0(\sqrt{\chi_n}s)\,ds &= \left[L(S(x))\sqrt{S(x)}\chi_n^{1/4}\sqrt{S(x)}J_0(\sqrt{\chi_n}S(x))\right]\frac{1}{\sqrt{\chi_n}}\\ &\quad-\frac{1}{\sqrt{\chi_n}}\int_0^{S(x)}L'(s)\chi_n^{1/4}sJ_1(\sqrt{\chi_n}s)\,ds.\end{aligned} \tag{41}$$

By combining (39), (40), (41) and using (21), (24), we easily conclude for (38). $\square$

Note that the constants in the previous proposition degenerate for q tending to 1. We will see below than one has nevertheless uniform bounds in a neighborhood of 0 for $|\psi_{n,c}|$ and $|\psi'_{n,c}|$. This is the subject of the following proposition.

Proposition 4. *There exists a constant C such that, for all $n\geq 0$ and $c\geq 0$,*

$$\sup_{x\in[-1,1]}(1-x^2)^{1/4}|\psi_{n,c}(x)| \leq (2\chi_n(c))^{1/4}. \tag{42}$$

$$\sup_{x\in[-1,1]}(1-x^2)|\psi'_{n,c}(x)| \leq C(c^2+\chi_n(c))^{3/4}. \tag{43}$$

Proof. We shall again use the differential equation (5) satisfied by $\psi_{n,c}$, which we note only ψ in this proof, and the identity

$$\int_{-1}^{+1}(1-x^2)|\psi'(x)|^2dx + c^2\int_{-1}^{+1}x^2|\psi(x)|^2dx = \chi_n(c). \tag{44}$$

The above identity is obtained by multiplying (5) from both sides by $\psi(x)$ and then integrating the result over $[-1,1]$. We can consider only $x \geq 0$ and replace $1-x^2$ by $1-x$. By a simple integration by parts applied to the quantity $2\int_x^1(1-t)\psi(t)\psi'(t)dt$, one obtains the following identity

$$(1-x)|\psi(x)|^2 = -\int_x^1|\psi(t)|^2dt + 2\int_x^1(1-t)\psi(t)\psi'(t)dt. \tag{45}$$

We prove that the left hand side of the previous identity is bounded by $(1-x)^{1/2}\sqrt{\chi_n(c)}$. To this end, it suffices to bound the second integral. This is done as follows.

$$\begin{aligned}2\left|\int_x^1(1-t)\psi(t)\psi'(t)dt\right| &\leq 2\sqrt{1-x}\left|\int_x^1\sqrt{1-t}\psi(t)\psi'(t)dt\right| \\ &\leq 2\sqrt{1-x}\left[\int_0^1(1-t^2)|\psi'(t)|^2\,dt\right]^{1/2}\left[\int_0^1|\psi(t)|^2\,dt\right]^{1/2} \\ &\leq 2\sqrt{1-x}\sqrt{\chi_n/2}\sqrt{1/2}\end{aligned} \tag{46}$$

The last inequality is a consequence of (44). By combining (45) and (46), we conclude for (42). Next, we write

$$(1-x)^2|\psi'(x)|^2 = -2\int_x^1(1-t)|\psi'(t)|^2dt + 2\int_x^1(1-t)^2\psi'(t)\psi''(t)dt. \tag{47}$$

Again, we use (5), multiply this later by $\sqrt{1-t^2}$ and get

$$(1-t^2)^{3/2}\psi''(t) = 2t\sqrt{1-t^2}\psi'(t) + (c^2t^2-\chi_n)\sqrt{1-t^2}\psi(t), \quad t\in[-1,1].$$

Since $|t|, \sqrt{1-t^2} \leq 1$ and by using Minkowski inequality, one gets

$$\|(1-t^2)^{3/2}\psi''\|_2 \leq 2\|\sqrt{1-t^2}\psi'\|_2 + (c^2+\chi_n)\|\psi\|_2. \tag{48}$$

That is

$$\left(\int_{-1}^{+1}(1-t^2)^3|\psi''(t)|^2dt\right)^{1/2} \leq 2\sqrt{\chi_n(c)} + \chi_n(c) + c^2.$$

Again, by using Schwarz inequality in the second integral of (47) and then combining (47) and (48), one gets (43). $\square$

The following proposition gives us interesting bounds for the successive derivatives of the PSWFs at $x=0$.

Proposition 5. *For any integers $n, k \geq 0$, satisfying $k(k+1) \leq \chi_n$, we have*

$$\left|\psi_{n,c}^{(k)}(0)\right| \leq \left(\sqrt{\chi_n}\right)^k\left|\psi_{n,c}(0)\right|, \tag{49}$$

for n even and k even, and

$$\left|\psi_{n,c}^{(k)}(0)\right| \leq \left(\sqrt{\chi_n}\right)^{k-1}\left|\psi'_{n,c}(0)\right|, \tag{50}$$

for n odd and k odd. In particular, under the assumption that $q = \dfrac{c^2}{\chi_n} < 1$, there exists a constant C, depending only on q and such that for any positive integer k satisfying $k(k+1) \leq \chi_n$, we have

$$\left|\psi_{n,c}^{(k)}(0)\right| \leq C(\sqrt{\chi_n})^k. \tag{51}$$

Proof. We first study the case where $n = 2m, k = 2i$ are even integers. We show that for a fixed n, $\psi_{n,c}^{(k)}(0)$ has alternating signs, that is $\psi_{n,c}^{(k)}(0) = (-1)^{k/2}\left|\psi_{n,c}^{(k)}(0)\right|$ or equivalently $\psi_{n,c}^{(k)}(0)\psi_{n,c}^{(k-2)}(0) < 0$. Note that by an iterative use of (5), one can easily check that the $\psi_{n,c}^{(k)}(0) = \psi^{(k)}(0)$ are given by the following recurrence relation,

$$\psi^{(k+2)}(0) = (k(k+1) - \chi_n)\psi^{(k)}(0) + k(k-1)c^2\psi^{(k-2)}(0), \quad k \geq 0, \tag{52}$$

with $\psi(0) > 0$, $\psi^{(2)}(0) = -\chi_n\psi(0)$. Note that $\psi^{(2)}(0)\psi(0) < 0$. By induction, assume that this is the case for the order $k-2$, that is $\psi^{(k)}(0)\psi^{(k-2)}(0) < 0$. Multiplying (52) from both sides by $\psi^{(k)}(0)$, using the assumption that $k(k+1) \leq \chi_n$ as well as the induction hypothesis, one concludes that the induction assumption holds for the order k. Consequently, we have,

$$\left|\psi^{(k+2)}(0)\right| = (\chi_n - k(k+1))\left|\psi^{(k)}(0)\right| + k(k-1)c^2\left|\psi^{(k-2)}(0)\right|, \quad k \geq 0. \tag{53}$$

Let $\gamma_k = \left|\psi^{(k)}(0)\right| = m_k(\sqrt{\chi_n})^k$. Then (53) is rewritten as follows,

$$m_{k+2} = \left(1 - \frac{k(k+1)}{\chi_n}\right)m_k + k(k-1)\frac{q}{\chi_n}m_{k-2}, \quad |\psi(0)| = m_0. \tag{54}$$

From $\psi^{(2)}(0) = -\chi_n\psi(0)$, we have $m_2 = m_0$. Moreover, from (54), we get

$$m_{k+2} \leq \left(1 - \frac{k(k+1)}{\chi_n}\right)m_k + k(k+1)\frac{q}{\chi_n}m_{k-2}.$$

A simple induction gives us

$$m_{k+2} \leq \left(1 - \frac{k(k+1)}{\chi_n}(1-q)\right)m_0 \leq m_0 = |\psi(0)|.$$

This concludes for (49). Similarly, one can easily prove (50). Finally, by combining (29), (38) and (49), cone concludes for (51). □

3 Decay estimates for the eigenvalues and the Legendre coefficients

3.1 Decay estimates for the eigenvalues

We first use the previous estimates to prove the exponential decay of the sequence $\lambda_n(c)$. More precisely we prove the following.

Theorem 3. *Let $\delta > 0$. There exists N and κ such that, for all $c \geq 0$ and $n \geq \max(N, \kappa c)$,*

$$\lambda_n(c) \leq e^{-\delta(n-\kappa c)}.$$

This can be compared with the numerical evidence that one has super-exponential decay from $\frac{2c}{\pi}$. It is a corollary of the following technical proposition.

Proposition 6. *Let $\alpha < 1$. Then there exists constants M_1, M_2 such that*

$$\lambda_n(c) \leq M_1 \frac{n}{2} (\frac{e}{4})^{2n} \left(\frac{c^2}{n^2} \exp(M_2 \frac{c^2}{n^2}) \right)^n . \tag{55}$$

Proof. We start from the well-known equality, see [20, 24], that for any positive integer n, we have $\lambda_n(c) = \lambda' \times \lambda''$, with

$$\lambda' : = \frac{c^{2n+1}(n!)^4}{2((2n)!)^2 (\Gamma(n+3/2))^2} \tag{56}$$

$$\lambda'' : = \exp\left(2 \int_0^c \frac{(\psi_{n,\tau}(1))^2 - (n+1/2)}{\tau} \, d\tau \right). \tag{57}$$

Let us first consider λ''. We will use (32) and (36)

$$\begin{aligned} |(\psi_{n,\tau}(1))^2 - (n+1/2)| &\leq M(1+\tau^2/n) \qquad && \tau \geq 1, \\ &\leq M\tau^2 && \tau \leq 1. \end{aligned}$$

It follows that

$$\lambda'' \leq e^{2M} \left(e^{2M \frac{c^2}{n^2}} \right)^n . \tag{58}$$

On the other hand, we have

$$\frac{c^{2n+1}(n!)^2}{2((2n)!)^2} \leq \left(\frac{c}{n} \right)^{2n} \frac{c}{2} \left[\prod_{j=1}^{n} \left(1 + \frac{j}{n} \right) \right]^{-2} . \tag{59}$$

Since $\displaystyle\frac{1}{n} \sum_{j=1}^{n} \log(1+j/n) \geq \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} \log(1+x)\, dx = 2\log 2 - 1$, then

$$\left[\prod_{j=1}^{n} \left(1 + \frac{j}{n} \right) \right]^{-2} \leq e^{-(4\log 2 - 2)n}, \quad \forall\, n \geq 1.$$

By using the previous inequality together with (58), one gets the required inequality (55). $\square$

Remark 1. *Numerical evidence, see [20], indicates that $(\psi_{n,\tau})^2 - (n+1/2) \leq 0, \forall\, t \geq 0$. If we accept this assertion, then we observe that the sequence $\lambda_n(c)$ decays faster than $\frac{c}{2}\left(\frac{ec}{4n}\right)^{2n}$ so that the exponential decay has started at $[ec/4]$.*

3.2 Decay estimate of the Legendre expansion coefficients

In this paragraph, we study some decay estimates of the Legendre coefficients β_k^n. Recall that $\psi_{n,c}(x) = \sum_{k\geq 0} \beta_k^n \overline{P_k}(x),\ \forall\, x \in [-1,1]$, with $\beta_k^n = \int_{-1}^{1} \overline{P_k}(x) \psi_{n,c}(x)\, dx$. It is well known, see [Theorem 3.4, [26]], that for a fixed positive integer n, we have

$$|\beta_k^n| \leq \frac{2}{|\mu_n(c)|} \frac{1}{2^k}, \quad \forall\, k \geq 2([ec]+1). \tag{60}$$

Here $\mu_n(c)$ is the eigenvalue of the operator Q_c defined in the introduction, for the eigenfunction $\psi_{n,c}$. It is related to $\lambda_n(c)$ through the identity

$$\lambda_n(c) = \frac{c}{2\pi} |\mu_n(c)|^2 .$$

The following lemma improves the decay estimate given by (60).

Lemma 7. *For any real $c>0$ and any integer $n\geq 0$, we have*

$$|\beta_k^n| = \left|\int_{-1}^1 \overline{P_k}(x)\psi_{n,c}(x)\,dx\right| \leq \left(\frac{c}{2}\right)^k \frac{\sqrt{\pi}}{\Gamma(k+3/2)}\frac{1}{|\mu_n(c)|},\quad \forall\, k\geq 0. \tag{61}$$

Proof. We start from the following identity relating Bessel functions of the first type and Legendre polynomials, see [2]

$$\int_{-1}^1 e^{ixy}P_n(y)\,dy = i^n\sqrt{\frac{2\pi}{x}}J_{n+\frac{1}{2}}(x),\quad \forall\; x\neq 0. \tag{62}$$

Since the $\psi_{n,c}$ are eigenfunctions of Q_c, that is

$$\int_{-1}^1 e^{i\,c\,x\,y}\psi_{n,c}(y)\,dy = \mu_n(c)\psi_{n,c}(x), \tag{63}$$

then, by combining (63) and (62) and by using Plancherel Theorem, one gets

$$\begin{aligned}
\beta_k^n &= \int_{-1}^1 \overline{P_k}(x)\psi_{n,c}(x)\,dx = \frac{1}{\mu_n(c)}\int_{-1}^1\int_{-1}^1\left(\int_{-1}^1 e^{icxy}\overline{P_k}(x)\,dx\right)\psi_{n,c}(y)\,dy\\
&= \frac{1}{\mu_n(c)}\int_{-1}^1 \sqrt{k+1/2}\sqrt{\frac{2\pi}{cy}}J_{k+1/2}(cy)\psi_{n,c}(y)\,dy.
\end{aligned}$$

By using the previous equality together with the well known bound of the Bessel function, see [2]

$$|J_\alpha(x)| \leq \frac{|x|^\alpha}{2^\alpha\Gamma(\alpha+1)},\quad \forall\,\alpha>-1/2,\quad \forall\, x\in\mathbb{R}, \tag{64}$$

one gets

$$\begin{aligned}
|\beta_k^n| &\leq \frac{\sqrt{\pi(2k+1)}c^k}{2^{k+1/2}\Gamma(k+3/2)|\mu_n(c)|}\int_{-1}^1 |y^k||\psi_{n,c}(y)|\,dy\\
&\leq \frac{\sqrt{\pi(2k+1)}c^k}{2^{k+1/2}\Gamma(k+3/2)|\mu_n(c)|}\sqrt{\frac{2}{2k+1}}\left(\int_{-1}^1|\psi_{n,c}(y)|^2\,dy\right)^{1/2} = \left(\frac{c}{2}\right)^k\frac{\sqrt{\pi}}{\Gamma(k+3/2)}\frac{1}{|\mu_n(c)|}.
\end{aligned}$$

□

Remark 2. *Our decay bound given by (61) outperforms in two directions, the one given by (60). Firstly, it is valid for any integer $k\geq 0$. Secondly, our decay bound decays much faster than the one given by (60). In fact, Since $\Gamma(s+1)\geq\sqrt{2\pi}s^{s+1/2}e^{-s}$ for all $s>0$, see [1], then it is easy to see that*

$$\left(\frac{c}{2}\right)^k\frac{\sqrt{\pi}}{\Gamma(k+3/2)}\frac{1}{|\mu_n(c)|}\leq\left(\frac{1}{2^{k+1}}\sqrt{\frac{e}{2k+1}}\right)\left(\frac{1}{2^{k-1}}\frac{1}{|\mu_n(c)|}\right),\forall\, k\geq 2([ec]+1).$$

4 Quality of the spectral approximations by the PSWFs

In this section, we first study the quality of approximation of almost band-limited functions by the classical PSWFs $\psi_{n,c}$ that are concentrated on $[-b,b]$, for some $b>0$. Then, we extend this study to the case of periodic and non periodic Sobolev space $H^s([-1,1]), s>0$.

4.1 Approximation of almost time and band-limited functions

In this paragraph, $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R})$. We show that the set $\{\psi_{n,c}(x),\ n\geq 0\}$ is well adapted for the representation of almost time-limited and almost band-limited functions, which are defined as follows.

Definition 1. *Let $T=[-a,+a]$ and $\Omega=[-b,+b]$ be two intervals. A function f, which we assume to be normalized in such a way that $\|f\|_2=1$, is said to be ϵ_T-concentrated in T and $\epsilon_\Omega-$band concentrated in Ω if*

$$\int_{T^c}|f(t)|^2\,dt\leq\epsilon_T^2,\qquad \frac{1}{2\pi}\int_{\Omega^c}|\widehat{f}(\omega)|^2\,d\omega\leq\epsilon_\Omega^2.$$

Up to a re-scaling of the function f, we can always assume that $T=[-1,1]$ and $\Omega=[-c,+c]$, with $c:=ab$. Indeed, for f that is ϵ_T-concentrated in $T=[-a,+a]$ and $\epsilon_\Omega-$band concentrated in $\Omega=[-b,+b]$, the normalized function $g(t)=\sqrt{a}f(at)$ is ϵ_T-concentrated in $[-1,+1]$ and $\epsilon_\Omega-$band concentrated in $[-ab,+ab]$.

Before stating the theorem, let us give some notations. For f an L^2 function on $\mathbb{R}$, we consider its expansion $f=\sum_{n\geq 0}a_n\psi_{n,c}$ in $L^2([-1,+1])$, with, due to the normalization of the functions $\psi_{n,c}$ given by (3), the following equality holds,

$$\int_{-1}^{+1}|f(t)|^2dt=\sum_{n\geq 0}|a_n|^2. \tag{65}$$

We call $S_{N,c}f$, the N-th partial sum, defined by

$$S_{N,c}f(t):=\sum_{n<N}a_n\psi_{n,c}(t). \tag{66}$$

We write more simply S_Nf when there is no ambiguity. In the next lemma, we prove that S_Nf tends to f rapidly when f belongs to the space of band-limited functions. This statement is both very simple and classical, see for instance [21, 22] or Theorem 3.1 in [25].

Lemma 8. *Let $f\in B_c$ be an L^2 normalized function. Then*

$$\int_{-1}^{+1}|f-S_Nf|^2dt\leq\lambda_N(c). \tag{67}$$

Proof. Since the set of functions $\psi_{n,c}$ is also an orthogonal basis of B_c, the function f may be written on $\mathbb{R}$ as $f=\sum_{n\geq 0}a_n\psi_{n,c}$, with

$$\int_{\mathbb{R}}|f(t)|^2dt=\sum_{n\geq 0}|\lambda_n(c)|^{-1}|a_n|^2. \tag{68}$$

The two expansions coincide on $[-1,+1]$, and, from (68) applied to $f-S_Nf$, it follows that

$$\int_{-1}^{+1}|f-S_Nf|^2dt\leq\sup_{n\geq N}|\lambda_n(c)|\sum_{n\geq N}|\lambda_n(c)|^{-1}|a_n|^2.$$

We use the fact that the sequence $|\lambda_n(c)|$ decreases and (68) to conclude. $\square$

Next we define the time-limiting operator P_T and the band-limiting operator Π_Ω by:

$$P_T(f)(x)=\chi_T(x)f(x),\qquad \Pi_\Omega(f)(x)=\frac{1}{2\pi}\int_\Omega e^{ix\omega}\widehat{f}(\omega)\,d\omega.$$

The following proposition provides us with the quality of approximation of almost time- and band-limited functions by the PSWFs.

Proposition 7. *If f is an L^2 normalized function that is ϵ_T-concentrated in $T=[-1,+1]$ and $\epsilon_\Omega-$band concentrated in $\Omega=[-c,+c]$, then for any positive integer N, we have*

$$\left(\int_{-1}^{+1}|f-S_Nf|^2dt\right)^{1/2}\leq\epsilon_\Omega+\sqrt{\lambda_N(c)} \tag{69}$$

and, as a consequence,

$$\|f-P_TS_Nf\|_2\leq\epsilon_T+\epsilon_\Omega+\sqrt{\lambda_N(c)}. \tag{70}$$

More generally, if f is an L^2 normalized function that is ϵ_T-concentrated in $T=[-a,+a]$ and $\epsilon_\Omega-$band concentrated in $\Omega=[-b,+b]$ then, for $c=ab$ and for any positive integer N, we have

$$\|f-P_TS_{N,c,a}f\|_2\leq\epsilon_T+\epsilon_\Omega+\sqrt{\lambda_N(c)} \tag{71}$$

where $S_{N,c,a}$ gives the N-th partial sum for the orthonormal basis $\frac{1}{\sqrt{a}}\psi_{n,c}(t/a)$ on $[-a,+a]$.

Proof. We first prove (69) by writing f as the sum of $\Pi_\Omega f$ and g. Remark first that $\int_{-1}^{+1}|g-S_Ng|^2dt\leq\|g\|_2\leq\epsilon_\Omega$. We then use Lemma 8 for the band limited function $\Pi_\Omega f$ to conclude. The rest of the proof follows at once. $\square$

Remark 3. *Let f be a normalized L^2 function that vanishes outside $[-1,+1]$ and is in $H^s(\mathbb{R})$. Then f gives an example of 0-concentrated in $[-1,+1]$ and ϵ_c-band concentrated in $[-c,+c]$, with $\epsilon_c\leq M_f/c^s$.*

Indeed, take

$$M_f^2=\frac{1}{2\pi}\int|\widehat{f}(\xi)|^2|\xi|^{2s}d\xi,$$

which is finite by assumption.

4.2 Approximation by the PSWFs in Sobolev spaces

In this paragraph, we study the quality of approximation by the PSWFs in the Sobolev space $H^s([-1,1])$. We provide an $L^2([-1,1])$–error bound of the approximation of a function $f\in H^s([-1,1])$ by the $N-$th partial sum of its expansion in the basis of PSWFs.

To simplify notation we will write $I:=[-1,1]$. In this paragraph, $\|\cdot\|_2$ denotes the norm in $L^2(I)$. We should mention that different spectral approximation by the PSWFs in $H^s=H^s(I)$ have been already given in [4, 6, 25]. More precisely, the following result has been proved in [6]. Here $a_k(f):=\int_{-1}^1 f(x)\psi_{k,c}(x)\,dx$.

Theorem 4. **(Theorem 3.1 in** [*6*]**).** *Let $f\in H^s(I)$, $s\geq 0$. Then*

$$|a_N(f)|\leq C\left(N^{-2/3s}\|f\|_{H^s}+\left(\sqrt{\frac{c^2}{\chi_N(c)}}\right)^{\delta N}\|f\|_{L^2(I)}\right),$$

where C,δ are independent of f,N and c.

In [25], the author has used a different approach for the study of the spectral approximation by the PSWFs. More precisely, by considering the weighted Sobolev space $\widetilde{H}^r(I)$, associated with the differential operator

$$\mathcal{D}_cu=-(1-x^2)u''+2xu'+c^2x^2u,$$

and given by

$$\widetilde{H}^r(I) = \left\{ f \in L^2(I),\, \|f\|^2_{\widetilde{H}^r} = \|\mathcal{D}_c^{r/2} f\|^2 = \sum_{k\geq 0} (\chi_k)^r |\widehat{f}_k|^2 < +\infty \right\},$$

the following result has been given in [25].

Theorem 5. (**Theorem 3.3 in** [*25*]). *For any $f \in \widetilde{H}^r(I)$, with $r \geq 0$, we have*

$$\|f - S_N f\|_2 \leq (\chi_N(c))^{-r/2} \|f\|_{\widetilde{H}^r} \leq N^{-r} \|f\|_{\widetilde{H}^r}.$$

It is important to mention that the error bounds of the spectral approximations given by the previous two theorems, do not indicate how to choose a convenient value of the bandwidth c to be used to approximate a given $f \in H^s(I)$. By a simultaneous use of the properties of the PSWFs as eigenfunctions of the differential operator L_c and the integral operator Q_c, we give a first answer to this question. This is the subject of the following theorem.

Theorem 6. *Let $c \geq 0$ be a positive real number. Assume that $f \in H^s(I)$, for some positive real number $s > 0$. Then for any integer $N \geq 1$, we have*

$$\|f - S_N f\|_2 \leq K(1+c^2)^{-s/2} \|f\|_{H^s} + K\sqrt{\lambda_N(c)} \|f\|_2. \tag{72}$$

Here, the constant K depends only on s. Moreover it can be taken equal to 1 when f belongs to the space $H^s_0(I)$.

Proof. To prove (72), we first use the fact that for any real number $s \geq 0$, there exists a linear and continuous extension operator $E : H^s(I) \to H^s(\mathbb{R})$. Moreover, if $f \in H^s(I)$ and $F = E(f) \in H^s(\mathbb{R})$, then there exists a constant $K > 0$ such that

$$\|F\|_{L^2(\mathbb{R})} \leq K \|f\|_2, \qquad \|F\|_{H^s(\mathbb{R})} \leq K \|f\|_{H^s}. \tag{73}$$

We recall that the Sobolev norm of a function F on $\mathbb{R}$ is given by

$$\|F\|^2_{H^s(\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi.$$

In particular, for F $c-$bandlimited, one has

$$\|F\|^2_{L^2(\mathbb{R})} \leq (1+c^2)^{-s} \|F\|^2_{H^s(\mathbb{R})}.$$

Next, if $\mathcal{F}$ denotes the Fourier transform operator and if

$$\mathcal{G} = \mathcal{F}^{-1}(\widehat{F} \cdot 1_{[-c,c]}), \quad \mathcal{H} = \mathcal{F}^{-1}(\widehat{F} \cdot (1 - 1_{[-c,c]})),$$

then $\mathcal{G}$ is $c-$bandlimited and $F = \mathcal{G} + \mathcal{H}$. Moreover, since $\|\widehat{\mathcal{G}}\|_{L^2(\mathbb{R})} \leq \|\widehat{F}\|_{L^2(\mathbb{R})}$ and $\|\mathcal{H}\|_{L^2(\mathbb{R})} \leq c^{-s} \|F\|_{H^s(\mathbb{R})}$, then by using (73), one gets

$$\|\mathcal{G}\|_{L^2(\mathbb{R})} \leq K \|f\|_2, \quad \|\mathcal{H}\|_2 \leq K(1+c^2)^{-s/2} \|f\|_{H^s}. \tag{74}$$

Finally, by using the previous inequalities and the fact that $\mathcal{G}$ is $c-$bandlimited, one concludes that

$$\begin{aligned} \|f - S_N f\|_2 &\leq \|\mathcal{G} - S_N \mathcal{G}\|_2 + \|\mathcal{H} - S_N \mathcal{H}\|_2 \\ &\leq \sqrt{\lambda_N(c)} \|\mathcal{G}\|_{L^2(\mathbb{R})} + \|\mathcal{H}\|_2 \\ &\leq \sqrt{\lambda_N(c)} K \|f\|_2 + K(1+c^2)^{-s} \|f\|_{H^s}. \end{aligned}$$

This concludes the proof for general f. When f is in the subspace $H^s_0(I)$, one can take as extension operator the extension by 0 outside $[-1, 1]$, so that the constant K can be replaced by 1. $\square$

Remark 4. *This should be compared with the results of [25], given by Theorem 5. This has the advantage to give an error term for all values of c, while the first term in* (72) *is only small for c large enough. On another side, Wang compares this specific Sobolev space with the classical one and finds that*

$$\|f\|_{\widetilde{H^s_c}} \leq C(1+c^2)^{s/2}\|f\|_{H^s}.$$

For large values of N we clearly have $\chi_N(1+c^2) \ll (1+c^2)^{-1}$, but it goes the other way around when χ_N and $1+c^2$ are comparable. So it may be useful to have both kinds of estimates in mind for numerical purpose and for the choice of the value of c.

Remark 5. *Remark that the second error term in (72) is negligible as soon as N is comparable to c, due to the super-exponential decay of the sequence $\lambda_N(c)$, and also to the fact that the L^2 norm may be very small compared to the norm in H^s. A convenient choice of the value of the truncation order N_c to be used is given by*

$$N_c = \min\{N \in \mathbb{N},\ \sqrt{\lambda_N(c)}\|f\|_2 \leq c^{-s}\|f\|_{H^s}\}.$$

If $N < N_c$, then the error bound in (72) is concentrated in the quantity $\sqrt{\lambda_N(c)}\|f\|_2$ which can be significantly reduced by considering larger value of N. On the other hand, if $N > N_c$, then the error bound is concentrated in the quantity $c^{-s}\|f\|_{H^s}$. Hence, larger values of N will not reduce in a significant manner this error bound.

It may be useful to consider in particular the subspace H^s_{per} of functions in $H^s(I)$ that extend into $2-$periodic functions of the same regularity. For such functions, one can also use the norm

$$\|f\|^2_{H^s_{per}} = \sum_{k\in\mathbb{Z}}(1+(k\pi)^2)^s|b_k(f))|^2.$$

Here,

$$b_k(f) = \frac{1}{\sqrt{2}}\int_{-1}^{+1} f(x)e^{-i\pi kx}\,dx = \frac{1}{\sqrt{2}}\widehat{f}(k\pi)$$

is the coefficient of the Fourier series expansion of f. A precise error analysis of the quality of approximation by the PSWFs in the space H^s_{per} is given by the following theorem. We should mention that the proof of this result is essentially based on combining the Fourier characterization of H^s_{per} with some properties of the PSWFs as the eigenfunctions of the finite Fourier transform operator Q_c.

Theorem 7. *Let $s > 0$, $c > 0$, be any positive real numbers and let $f \in H^s_{per}([-1,1])$. Then for any integer $N \geq 1$, we have*

$$\|f - S_N f\|_2 \leq \sqrt{(1/2+\frac{\pi}{4c})\sum_{n\geq N}\|\psi_{n,c}\|^2_\infty\lambda_n(c)}\|f\|_2 + c^{-s}\|f - f_{[c/\pi]}\|_{H^s_{per}}. \tag{75}$$

Here, $f_{[c/\pi]}$ is the truncated Fourier series expansion of f to the order $\left[\frac{c}{\pi}\right]$. In particular, for any positive integer N satisfying $q = c^2/\chi_N < 1$, we have

$$\|f - S_N f\|_2 \leq K_q\sqrt{\sum_{n\geq N}\sqrt{\chi_n}\lambda_n(c)}\|f\|_2 + c^{-s}\|f - f_{[c/\pi]}\|_{H^s_{per}}, \tag{76}$$

where $K_q = \sqrt{(1/2+\frac{\pi}{4c})}C_q$ and C_q is as given by Corollary 1.

Proof. Recall Plancherel Formula $\sum_{k\in\mathbb{Z}}|b_k(f)|^2 = \|f\|_2^2$. We consider two auxiliary functions $g, h \in H^s_{per}([-1,1])$, given by the following formulae,

$$g(x) = \sum_{|k|\leq\left[\frac{c}{\pi}\right]} b_k(f)\frac{e^{ik\pi x}}{\sqrt{2}}, \quad h(x) = \sum_{|k|\geq\left[\frac{c}{\pi}\right]+1} b_k(f)\frac{e^{ik\pi x}}{\sqrt{2}}, \tag{77}$$

so that $f = g + h$ on $[-1,1]$. Moreover, we have $\|g\|_2 \leq \|f\|_2$ and

$$\|h\|_2^2 = \sum_{|k|\geq\left[\frac{c}{\pi}\right]+1} |b_k(f)|^2 \leq \sum_{|k|\geq\left[\frac{c}{\pi}\right]+1} (1+(k\pi)^2)^{-s}(1+(k\pi)^2)^s|b_k(f)|^2 \leq c^{-2s}\|f - f_{[c/\pi]}\|^2_{H^s}. \tag{78}$$

We then proceed as in the proof of the previous theorem. The main point is the computation of the PSWFs series of g. We use the following computation of coefficients of the exponential $e^{i\lambda x}$.

$$a_n(e^{i\lambda\cdot}) = \int_{-1}^1 e^{i\lambda x}\psi_{n,c}(x)dx = \mu_n(c)\psi_{n,c}\left(\frac{\lambda}{c}\right). \tag{79}$$

It follows, by linearity, that

$$a_n(h) = \frac{1}{\sqrt{2}}\mu_n(c)\sum_{|k|\leq\left[\frac{c}{\pi}\right]} b_k(f)\psi_{n,c}\left(\frac{k\pi}{c}\right). \tag{80}$$

Then by using Schwarz Inequality as well as the fact that all quantities $k\pi/c$ ar contained in the interval $[-1,1]$, we get

$$\begin{aligned} |a_n(h)|^2 &\leq \frac{|\mu_n(c)|^2}{2}\|\psi_{n,c}\|^2_\infty\left(\sum_{|k|\leq\left[\frac{c}{\pi}\right]}|b_k(f)|^2\right)\left(\sum_{|k|\leq\left[\frac{c}{\pi}\right]}1\right) &(81)\\ &\leq \left(\frac{1}{2}+\frac{\pi}{4c}\right)\frac{2c}{\pi}|\mu_n(c)|^2\|\psi_{n,c}\|^2_\infty\|f\|_2^2 = \left(\frac{1}{2}+\frac{\pi}{4c}\right)\lambda_n(c)\|\psi_{n,c}\|^2_\infty\|f\|_2^2. &(82)\end{aligned}$$

One obtains

$$\begin{aligned}\|g - S_N g\|_2 &= \left(\sum_{n\geq N}|a_n(g)|^2\right)^{1/2}\\ &\leq \sqrt{(1/2+\frac{\pi}{4c})\sum_{n\geq N}\|\psi_{n,c}\|^2_\infty\lambda_n(c)}\|f\|_2,\end{aligned}$$

which allows to conclude for (75). Finally, by combining the previous inequality and the result of Corollary 1, we conclude for (76).

□

Remark 6. *The last theorem may also be used for trigonometric polynomials $\sum b_k e^{ik\pi x}$ as long as their degree is smaller than c/π, so that the second term in (75) vanishes.*

Remark 7. *We also have a bound of the error for ordinary polynomials. Indeed, if we consider the polynomial $f(x) := x^j$, then*

$$a_n(f) = \int_{-1}^1 y^j\psi_{n,c}(y)\,dy = (-i)^j c^{-j}\mu_n(c)\psi^{(j)}_{n,c}(0), \quad \text{with } i^2 = -1.$$

We can then use Proposition 5 to conclude that if $c^2/\chi_N < 1$, then

$$\|f - S_N f\|_2^2 \leq C^2 \sum_{k\geq N} \left(\frac{\chi_k(c)}{c^2}\right)^j |\mu_k(c)|^2. \tag{83}$$

This gives an alternative way to find and improve the results of Theorems 3 and 4, given in [6]. Indeed, the authors approach f by a polynomial, then consider coefficients of the PSWFs expansion for a polynomial.

Remark 8. *It is important to note that when the bandwidth c is comparable with the truncation order N, then the error bounds given by our previous two theorems are of the expected order $O(N^{-s})$, for the values of N in the decay region of the $\lambda_n(c)$.*

5 Numerical results

In the first part of this section, we give a brief description of the numerical methods of computation of the PSWFs as well as the PSWFs series expansion coefficients of a function from the Sobolev space $H^s([-1,1])$. Note that Flammer's method is among the first methods that have been developed for the computation of the PSWFs inside $[-1,1]$. This method is well described in [9]. The explicit analytic extension of the PSWFs to the whole real line is given by D. Slepian.

Recently, there is an extensive amount of work devoted to new highly accurate computational methods of the PSWFs, see [3, 10, 11, 26]. In particular, the methods given in [3, 26] are based on an efficient quadrature method on the unit circle that provides highly accurate values of the PSWFs inside $[-1,1]$, as well as accurate approximations of the different eigenvalues $\mu_n(c)$, $n \geq 0$. The methods developed in [10, 11] for computing the values of the $\psi_{n,c}(x)$ inside $[-1,1]$ and the eigenvalues $\mu_n(c)$ are either based on an appropriate matrix representation of the finite Fourier transform operator Q_c, given by (63) or a Gaussian type quadrature formula applied to Q_c.

In the sequel, we will adopt Flammer's method for computing the PSWFs. This choice is based on the facts that the values of the analytic extensions of the PSWFS over $\mathbb{R}$, are easily obtained by this method. This later is briefly described as follows. We write as before the Legendre expansion of the PSWFs,

$$\psi_{n,c}(x) = \sum_{k=\geq 0}^{\infty}{}' \beta_k^n \overline{P_k}(x). \tag{84}$$

where the sign $\sum_{k=0,1}^{\infty}{}'$ means that the sum is over even or odd integers depending on whether the order n is even or odd. It is well known that the different expansion coefficients $(\beta_k^n)_k$ as well as the corresponding eigenvalues $\chi_n(c)$ are obtained by solving the following eigensystem

$$\frac{(k+1)(k+2)}{(2k+3)\sqrt{(2k+5)(2k+1)}} c^2 \beta_{k+2}^n + \big(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)}\big) c^2 \beta_k^n$$
$$\frac{k(k-1)}{(2k-1)\sqrt{(2k+1)(2k-3)}} c^2 \beta_{k-2}^n = \chi_n(c)\beta_k^n, \quad k \geq 0.$$

In [22], the author has shown that the analytic extension of the PSWFs outside $[-1,1]$ is simply given by the following formula,

$$\psi_{n,c}(x) = \frac{\sqrt{2\pi}}{|\mu_n(c)|} \sum_{k\geq 0} (-1)^k \beta_k^n \sqrt{k+1/2} \frac{J_{k+1/2}(cx)}{\sqrt{cx}}, \quad \forall\, |x| > 1, \tag{85}$$

where

$$\mu_n(c) = \frac{2\pi}{c}\left[\frac{\sum'_{k\geq 0,1} i^k\sqrt{k+1/2}\;\beta_k^n\;J_{k+1/2}(c)}{\sum_{k\geq 0}\beta_k^n\sqrt{k+1/2}}\right], \tag{86}$$

is the exact value of the $n-$th eigenvalue of the finite Fourier transform operator Q_c.

Remark 9. *If $f\in H^s_{per}$, $s>0$, then its different PSWFs series expansion coefficients can be easily approximated by the use of formula (79). More precisely for a positive integer K, an approximation $a_n^K(f)$ to $a_n(f)$ is given by the following formula*

$$a_n^K(f) = \frac{\mu_n(c)}{\sqrt{2}}\sum_{k=-K}^{K} b_k(f)\psi_{n,c}\left(\frac{k\pi}{c}\right) = a_n(f)+\epsilon_K, \tag{87}$$

where the $b_k(f)$ are the Fourier coefficients of f and where $\epsilon_K = \frac{1}{\sqrt{2}}\sum_{|k|\geq K+1}\mu_n(c)b_k(f)\psi_{n,c}\left(\frac{k\pi}{c}\right)$. Moreover, from the well known asymptotic behavior of the $\psi_{n,c}(x)$, for large values of x, see for example [10], one can easily check that $\epsilon_K = o\left(\frac{1}{(K+1)\pi)^{1+s}}\right)$. This computational method of the $a_n(f)$ has the advantage to work for small as well as large values of the smoothness coefficient $s>0$.

Remark 10. *If $f\in H^s([-1,1])$, where $s>1/2+2m, m\geq 1$, is an integer, then $f\in C^{2m}([-1,1])$. Moreover since $\psi_{n,c}\in C^\infty(\mathbb{R})$, then the classical Gaussian quadrature method, see for example [2] gives us the following approximate value $\widetilde{a}_n(f)$ of the $(n+1)-th$ expansion coefficient $a_n(f)=<f,\psi_{n,c}>$,*

$$\widetilde{a}_n(f) = \sum_{l=1}^{m}\omega_l f(x_l)\psi_{n,c}(x_l) = a_n(f)+\epsilon_n, \tag{88}$$

with $|\epsilon_n|\leq \sup_{\eta\in[-1,1]}\frac{1}{b_m^2}\frac{(f\cdot\psi_{n,c})^{(2m)}(\eta)}{(2m)!}$. Here, b_m is the highest coefficient of $\overline{P_m}$, and the different weights ω_l and nodes x_l, are easily computed by the special method given in [2].

Next, to illustrate the quality of approximation by the PSWFs, as well as to explain the contribution of the bandwidth $c\geq 0$ in this quality of approximation, we give the following examples.

Example 1: In this example, we show that the PSWFs outperforms the Legendre polynomials in the approximation of a class of functions from the Sobolev space $H^s([-1,1])$, having significant large coefficients at some high frequency components. To fix the idea, let $\lambda>0$, be a relatively large positive real number and let $f_\lambda(x)=e^{i\lambda x}, x\in[-1,1]$. The Legendre series expansion coefficients of f_λ are given by

$$\alpha_n(0) = \int_{-1}^{1} e^{i\lambda x}\overline{P_n}(x)\,dx = (i)^n\sqrt{\frac{2\pi}{\lambda}}J_{n+1/2}(\lambda).$$

In this case, we have

$$\|f_\lambda - \sum_{n=0}^{N}\alpha_n(0)\overline{P_n}\|_2^2 = \frac{2\pi}{\lambda}\sum_{n\geq N+1}(J_{n+1/2}(\lambda))^2. \tag{89}$$

If $c>0$ is a positive real number, then the corresponding PSWFs series expansion coefficients of f_λ are simply given as follows,

$$\alpha_n(c) = \int_{-1}^{1} e^{i\lambda x}\psi_{n,c}(x)\,dx = \mu_n(c)\psi_{n,c}(\lambda/c).$$

In this case, the L^2-approximation error is given by

$$E_N(c) = \|f - \sum_{n=0}^{N} \alpha_n(c)\psi_{n,c}\|_2^2 = \sum_{n\geq N+1} |\mu_n(c)|^2 \left(\psi_{n,c}\left(\frac{\lambda}{c}\right)\right)^2. \tag{90}$$

In the special case where $c=\lambda$, the previous error bound becomes $E_N(\lambda) = \sum_{n\geq N+1} |\mu_n(c)|^2 \left(\psi_{n,c}(1)\right)^2$. Since $\psi_{n,c}(1) = O(\sqrt{n})$, and if we accept the numerical evidence that the exponential decay of the sequence $(|\mu_n(c)|^2)_{n\geq 0}$ starts at $[ec/4]$, then from (89) and (90), one concludes that the PSWFs are better adapted for the approximation of the f_λ by its $N-$th order truncated PSWFs series expansion with $c=\lambda$ and $N=[\lambda]$. More generally, if $0\leq c<\lambda$, then $\frac{\lambda}{c}>1$ and the blowup of the $\psi_{n,c}\left(\frac{\lambda}{c}\right)$ implies that $\alpha_n(c) = \mu_n(c)\psi_{n,c}(\lambda/c)$ has as lower decay that $\alpha_n(\lambda) = \mu_n(\lambda)\psi_{n,c}(1)$. Moreover, if $c>\lambda$, then the decay of the $|\mu_n(c)|^2$ and consequently, the fast decay of the $\alpha_n(c)$ is possible only if n lies beyond a neighborhood of $\frac{ec}{4} > \frac{e\lambda}{4}$. This means that $c=\lambda$ is the appropriate value of the bandwidth to be used to approximate the function $f_\lambda(x) = e^{i\lambda x}$ by its first $N-$th truncated PSWFs series expansion, with $N=[\lambda]$. This explains the numerical results given in [25] concerning the approximation of the test function $u(x) = \sin(20\pi x)$, where the author has checked numerically that $c = 20\pi$ is the appropriate value of the bandwidth for approximating $u(x)$ by the PSWFs $\psi_{n,c}$ with a given high precision and minimal number of the truncation order N. As another example, we consider the value of $\lambda = 50$, then we find that

$$\|f_\lambda - \sum_{n=0}^{50} \alpha_n(0)\overline{P_n}\|_2 \approx 3.087858E-01, \quad \|f_\lambda - \sum_{n=0}^{50} \alpha_n(50)\psi_{n,50}\|_2 \approx 1.356604E-08.$$

Example 2: In this example, we consider the Weirstrass function

$$W_s(x) = \sum_{k\geq 0} \frac{\cos(2^k x)}{2^{ks}}, \quad -1\leq x\leq 1. \tag{91}$$

Note that $W_s \in H^{s-\epsilon}([-1,1])$, $\forall \epsilon < s,\ s>0$. We have considered the value of $c=100$, and computed $W_{s,N}$, the $N-$th terms truncated PSWFs series expansion of W_s with different values of $\frac{3}{4}\leq s\leq 2$ and different values of $20\leq N\leq 100$. Also, for each pair (s,N), we have computed the corresponding approximate L^2- error bound $E_N(s) = \left[\frac{1}{50}\sum_{k=-50}^{50}\left(W_{s,N}(k/50) - W_s(k/50)\right)^2\right]^{1/2}$. Table 1 lists the obtained values of $E_N(s)$. Note that the numerical results given by Table 1, follow what has been predicted by the theoretical results of the previous section. In fact, the L^2-errors $\|W_s - \Pi_N W_s\|_2$ is of order $O(N^{-s})$, whenever $N\geq N_c \sim \frac{ec}{4} - 1$. In the case, where $c=100$, $N_c = 67$. The graphs of $W_{3/4}(x)$ and $W_{3/4,N}(x)$, $N=90$ are given by Figure 1.

Example 3: In this example, we consider the Weirstrass function

$$f(x) = \sum_{k\geq 0} \frac{\cos(2^k \pi x)}{2^{ks}}, \quad -1\leq x\leq 1, \quad s=1.4. \tag{92}$$

It is clear that $f\in H^1_{per}([-1,1])$, with $\|f\|_2^2 = \sum_{k\geq 0}\frac{1}{2^{2ks}}$. Also, we fix the value of the badnwidth c to 100. In this case, we have $\|f\|_2 \approx 1.0805838$, $\|f - f_{[c/\pi]}\|_{H^1} \approx 1.203854$. Next, we have computed

Table 1: Values of $E_n(s)$ for various values of n and s.

	$s = 0.75$	$s = 1$	$s = 1.25$	$s = 1.5$	$s = 1.75$	$s = 2.0$
n	$E_n(s)$	$E_n(s)$	$E_n(s)$	$E_n(s)$	$E_n(s)$	$E_n(s)$
20	4.57329E-01	4.66173E-01	4.85990E-01	5.05973E-01	5.23232E-01	5.37227E-01
30	3.15869E-01	3.11677E-01	3.28241E-01	3.48562E-01	3.67260E-01	3.82963E-01
40	1.06843E-01	1.52009E-01	1.91237E-01	2.20969E-01	2.43432E-01	2.60523E-01
50	4.09844E-02	6.88472E-02	1.01827E-01	1.26518E-01	1.44809E-01	1.58520E-01
60	3.30178E-02	2.09084E-02	3.25551E-02	4.28999E-02	5.06959E-02	5.65531E-02
70	3.15097E-02	8.82446E-03	2.51157E-03	7.35725E-04	2.33066E-04	1.04137E-04
80	3.01566E-02	8.55598E-03	2.40312E-03	6.87458E-04	1.98993E-04	5.80481E-05
90	2.67972E-02	7.64167E-03	2.14661E-03	6.15062E-04	1.78461E-04	5.22848E-05
100	2.39141E-02	6.72825E-03	1.82818E-03	5.10057E-04	1.45036E-04	4.19238E-05

Figure 1: (a) graph of $W_{3/4}(x)$, (b) graph of $W_{3/4,N}(x), N = 90$.

the different approximate L^2- error $E_N = \left[\frac{1}{50}\sum_{k=-50}^{50}(f(k/50) - S_N f(k/50))^2\right]^{1/2}$. In figure 2, we have plotted the graphs of the actual error E_N versus the error bound of Theorem 5. Note that once N becomes larger than the critical value for the decay of the $\lambda_n(c)$, which in our case, is given by $N_c = [ec/4] = 67$, the theoretical error bound, given by Theorem 5 becomes very close to the actual error.

Example 4: In this example, we let $s > 0$ be any positive real number and we consider the Brownian motion function $B_s(x)$ given by as follows.

$$B_s(x) = \sum_{k\geq 1}\frac{X_k}{k^s}\cos(k\pi x), \quad -1 \leq x \leq 1. \tag{93}$$

Here, X_k is a Gaussian random variable. It is well known that $B_s \in H^s([-1, 1])$. For the special case $s = 1$, we consider the band-width $c = 100$, a truncation order $N = 80$ and compute $B_{1,N}$ the approximation of B_1 by its $N-$th terms truncated PSWFs series expansion. The graphs of B_1 and $B_{1,N}$ are given by Figure 3.

Remark 11. *From the quality of approximation in the Sobolev spaces $H^s([-1, 1])$ given in this paper and in [4, 6, 25], one concludes that for any value of the bandwidth $c \geq 0$, the approximation error $\|f - S_N f\|_2$ has the asymptotic order $O(N^{-s})$. Nonetheless, for a given $f \in H^s([-1, 1]), s > 0$*

Figure 2: (a) Graphs of the theoretical and the actual error bound for example 2. (b) Graph of the decay of $\lambda_n(c)$.

Figure 3: (a) graph of $B_1(x)$, (b) graph of $B_{1,N}(x), N = 80$.

which we may assume to have a unit L^2-norm and for a given error tolerance ϵ, the appropriate value of the bandwidth $c \geq 0$, corresponding to the minimum truncation order N, ensuring that $\|f - S_N f\|_2 \leq \epsilon$, depends on whether or not, f has some significant Fourier expansion coefficients, corresponding to large frequency components. In other words, the faster decay to zero of the Fourier coefficients of f, the smaller the value of the bandwidth should be and vice versa.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions, Dover Publication, INC, New York.1972. pp.773-792.

[2] G. E. Andrews, R. Askey and R. Roy, Special Functions, Cambridge University Press, 2000.

[3] G. Beylkin and L. Monzon, On Generalized Gaussian Quadrature for Exponentiels and their Applications, *Appl. Comput. Harmon. Anal.* **12**, (2002), 332–373.

[4] J. P. Boyd, Prolate spheroidal wave functions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudo-spectral algorithms, *J. Comput. Phys.* **199**, (2004), 688–716.

[5] J. P. Boyd, Approximation of an analytic function on a finite real interval by a bandlimited function and conjectures on properties of prolate spheroidal functions, *Appl. Comput. Harmon. Anal.* **25**, No.2, (2003), 168–176.

[6] Q. Chen, D. Gottlieb and J. S. Hesthaven, Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, *SIAM J. Numer. Anal.,* **43**, No. 5, (2005), pp. 1912–1933.

[7] D. Donoho and P. B. Stark, Uncertainty Principles and Signal Recovery, *SIAM J. Appl. Math.,* **49,** No. 3, (1989), 906–931.

[8] Y. M. Dunster, Uniform asymptotic expansions for prolate spheroidal functions with large parameters, *SIAM J. Math. Anal.,* **17** No. 6, (1986), 14951524.

[9] C. Flammer, *Spheroidal Wave Functions,* Stanford Univ. Press, CA, 1957.

[10] A. Karoui and T. Moumni, New efficient methods of computing the prolate spheroidal wave functions and their corresponding eigenvalues, *Appl. Comput. Harmon. Anal.* **24**, No.3, (2008), 269–289.

[11] A. Karoui, Uncertainty Principles, Prolate Spheroidal Wave Functions and Applications, in *New Trends in Fractals and Related Fields,* Applied and Numerical Harmonic Analysis series, Birkhauser-Boston, (2010), 165–190.

[12] H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty-III. The dimension of space of essentially time-and band-limited signals, *Bell System Tech. J.* **41**, (1962), 1295–1336.

[13] H. J. Landau and H. Widom, Eigenvalue distribution of time and frequency limiting, *J. Math. Anal. Appl.,* **77,** (1980), 469–481.

[14] L. W. Li, X. K. Kang, M. S. Leong, Spheroidal wave functions in electromagnetic theory, Wiley-Interscience publication, 2001.

[15] W. Lin, N. Kovvali and L. Carin, Pseudospectral method based on prolate spheroidal wave functions for semiconductor nanodevice simulation, *Computer Physics Communications,* **175** (2006), pp. 78–85.

[16] I. C. Moore and M. Cada, Prolate spheroidal wave functions, an introduction to the Slepian series and its properties, *Appl. Comput. Harmon. Anal.* **16**, No.3, (2004), 208–230.

[17] A. N. Nikoforov and V. B. Uvarov, Special functions of mathematical physics, translated from the Russian edition, Birkhäser Verlag Basel, (1988).

[18] C. Niven, On the Conduction of Heat in Ellipsoids of Revolution, *Phil. Trans. R. Soc. Lond.,* **171,** (1880), 117-151.

[19] A. Ya, Olenko, Upper bound on $\sqrt{x}J_\nu(x)$ and its applications, *Integral Transf. Special functions,* **17**, No.6, (2006), 455–467.

[20] V. Rokhlin and H. Xiao, Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit, *Appl. Comput. Harmon. Anal.* **22**, (2007), 105–123.

[21] Y. Shkolnisky, M. Tygert and V. Rokhlin, Approximation of bandlimited functions, *Appl. Comput. Harm. Anal.,* **21**, (3), (2006), 413–420.

[22] D. Slepian and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertaintyI, *Bell System Tech. J.* **40** (1961), 43–64.

[23] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty–IV: Extensions to many dimensions; generalized prolate spheroidal functions, *Bell System Tech. J.* **43** (1964), 3009–3057.

[24] D. Slepian, Some Asymptotic Expansions for Prolate Spheroidal Wave Functions, *J. Math. Phys.,* **44**, No. 2, (1965), 99–140.

[25] L. L. Wang, Analysis of Spectral Approximations Using Prolate Spheroidal Wave Functions, *Math. Comput.,* (2009).

[26] H. Xiao, V. Rokhlin and N. Yarvin, Prolate spheroidal wave functions, quadrature and interpolation, *Inverse Problems,* **17**, (2001), 805–838.

[27] Q. X. Yang, M. A. Lindquist, L. Shepp, C. H. Zhang, J. Wang, and M. B. Smith, Two dimensional prolate spheroidal wave functions for MRI, *Journal of Magnetic Resonance,* **158**, (2002), 4351.