Incomplete Hypergeometric Systems Associated to 1-Simplex $\times$ $(n-1)$-Simplex

Kenta Nishiyama *

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Abstract

The $\mathcal{A}$-hypergeometric system was introduced by Gel'fand, Kapranov and Zelevinsky in the 1980's. Among several classes of $\mathcal{A}$-hypergeometric functions, those for 1-simplex $\times$ $(n-1)$-simplex are known to be a very nice class. We will study an incomplete analog of this class.

1 Introduction

The $\mathcal{A}$-hypergeometric systems was introduced by Gel'fand, Kapranov and Zelevinsky in the 1980's ([1]). It is a system of homogeneous differential equations with parameters associated to an integer matrix A and contains a broad class of hypergeometric functions as solutions. Recently, the incomplete $\mathcal{A}$-hypergeometric system was proposed toward applications to statistics and a detailed study was given in the case of $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ = 1-simplex $\times$ 1-simplex ([6]). The system includes the incomplete Gauss' hypergeometric integral $I_{(a,b)}(\alpha, \beta, \gamma; x) = \int_a^b t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{\alpha}dt$ and the incomplete elliptic integral of the first kind $F(z;k) = \int_0^z \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$ as solution. It is interesting to describe properties of these functions in a general framework. Among several classes of (complete) $\mathcal{A}$-hypergeometric functions, those for $\Delta_1 \times \Delta_{n-1}$ (1-simplex $\times$ $(n-1)$-simplex) are known to be a very nice class (see, e.g., [9, Section 1.5]).

In this paper, we study an incomplete analog of this class. In the section 2, we give a definition of an incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system and prove that the existence of a solution of the system. In the section 3, we give

*Department of Mathematics, Kobe University and JST CREST.

a particular solution of the system and describe general solutions by combining with a base of the solutions of (homogeneous) $\mathcal{A}$-hypergeometric system. In the last section 4, we give the complete list of contiguity relations for the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric function.

2 Incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system

We will work over the Weyl algebra in $2n$ variables $D = \mathbf{C}\left\langle \begin{array}{c} x_{11}, \cdots, x_{1n}, \partial_{11}, \cdots, \partial_{1n} \\ x_{21}, \cdots, x_{2n}, \partial_{21}, \cdots, \partial_{2n} \end{array} \right\rangle$.

Definition 1 We call the following system of differential equations the *incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system*:

$$\left\{ \begin{array}{lll} (\theta_{i1} + \theta_{i2} - \alpha_i) \bullet f & = 0, & (1 \le i \le n) \\ \left(\displaystyle\sum_{i=1}^{n} \theta_{2i} + \gamma + 1 \right) \bullet f & = [g(t,x)]_{t=a}^{t=b}, & \\ (\partial_{1i}\partial_{2j} - \partial_{1j}\partial_{2i}) \bullet f & = 0, & (1 \le i < j \le n) \end{array} \right. \tag{1}$$

where $g(t,x) = t^{\gamma+1} \prod_{k=1}^{n} (x_{1k} + x_{2k} t)^{\alpha_k}$ and $\alpha_i, \gamma \in \mathbf{C}$ are parameters. The operator $\theta_{ij} = x_{ij}\partial_{ij}$ is called the Euler operator.

If $g(t,x) = 0$ in (1), the system agrees with the $\mathcal{A}$-hypergeometric or GKZ hypergeometric system associated to $\Delta_1 \times \Delta_{n-1}$.

Remark 1 The incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system introduced in Definition 1 is a special but interesting case of the incomplete $\mathcal{A}$-hypergeometric system (see appendix, [6]). Let A be the following $(n+1) \times 2n$ matrix:

$$A = \begin{pmatrix} 1 & 1 & & & & & \\ & & 1 & 1 & & \text{\Large 0} & \\ & & & & \ddots & & \\ & \text{\Large 0} & & & & 1 & 1 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{pmatrix}.$$

We set $\beta = (\alpha_1, \ldots, \alpha_n, -\gamma - 1) \in \mathbf{C}^{n+1}$ and $g = (0, \ldots, 0, [g(t,x)]_{t=a}^{t=b})$. Then the incomplete $\mathcal{A}$-hypergeometric system associated to A, β, g is the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system.

We note that the ideal $\langle \partial_{1i}\partial_{2j} - \partial_{1j}\partial_{2i} \mid 1 \le i < j \le n \rangle$ generated by the third operators of (1) is called the affine toric ideal associated to the matrix A and it is denoted by I_A. Moreover, I_A is Cohen-Macaulay because A is normal ([2]).

We note that the inhomogeneous system (1) does not necessarily have a solution f, when the inhomogeneous part $[g(t,x)]_{t=a}^{t=b}$ is randomly given.

Proposition 1 *For any α_i, $\gamma \in \mathbf{C}$, there exists a classical solution of the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system.*

Proof. We may verify conditions (7) and (8) in Theorem 5 in the appendix with respect to $g = (0, \dots, 0, [g(t,x)]_{t=a}^{t=b})$. For $1 \le i \le n$, we have

$$
\begin{aligned}
&(\theta_{1i} + \theta_{2i} - \alpha_i) \bullet t^{\gamma+1} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \\
&= \alpha_i x_{1i} t^{\gamma+1} (x_{1i} + x_{2i}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \\
&\qquad + \alpha_i x_{2i} t^{\gamma+2} (x_{1i} + x_{2i}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} - \alpha_i t^{\gamma+1} \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \\
&= \{x_{1i} + x_{2i}t - (x_{1i} + x_{2i}t)\} \alpha_i t^{\gamma+1} (x_{1i} + x_{2k}t)^{\alpha_i - 1} \prod_{k \neq i}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \\
&= 0.
\end{aligned}
$$

Thus the condition (7) holds.

For $1 \le i < j \le n$, we have

$$
\begin{aligned}
\partial_{1i}\partial_{2j} \bullet g(t,x) &= \partial_{1i} \bullet \alpha_j t^{\gamma+2} (x_{1j} + x_{2j}t)^{\alpha_j - 1} \prod_{k \neq j}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \\
&= \alpha_i \alpha_j t^{\gamma+2} (x_{1i} + x_{2i}t)^{\alpha_i - 1} (x_{1j} + x_{2j}t)^{\alpha_j - 1} \prod_{k \neq i,j}^{n} (x_{1k} + x_{2k}t)^{\alpha_k}.
\end{aligned}
$$

Since this expression is symmetric in the indices i and j, we have $(\partial_{1i}\partial_{2j} - \partial_{1j}\partial_{2i}) \bullet g(t,x) = 0$. Thus the condition (8) holds. ∎

Our definition of the incomplete $\Delta_1 \times \Delta_{n-1}$ hypergeometric system is natural in terms of a definite integral with parameters.

Proposition 2 *If* $\operatorname{Re}\gamma$, $\operatorname{Re}\alpha_i > 0$*, then the integral*

$$
\Phi(\beta; x) = \int_a^b t^\gamma \prod_{k=1}^{n} (x_{1k} + x_{2k}t)^{\alpha_k} \, dt \tag{2}
$$

is a solution of the incomplete $\Delta_1 \times \Delta_{n-1}$ *hypergeometric system (Definition 1).*

Proof. From the general theory of $\mathcal{A}$-hypergeometric systems, $\Phi(\beta; x)$ is annihilated by the elements of I_A and $\theta_{i1} + \theta_{i2} - \alpha_i$ for $1 \le i \le n$ (see, e.g., [9, Section 5.4]). We will prove that

$$
\left(\sum_{i=1}^{n} \theta_{2i} + \gamma + 1 \right) \bullet \Phi(\beta; x) = [g(t,x)]_{t=a}^{t=b}.
$$

Applying $\sum_{i=1}^n \theta_{2i}$ to the integrand, we get

$$\begin{aligned}\left(\sum_{i=1}^n \theta_{2i}\right) \bullet t^\gamma \prod_{k=1}^n (x_{1k}+x_{2k}t)^{\alpha_k} &= \sum_{i=1}^n \alpha_i x_{2i}(x_{1i}+x_{2i}t)^{\alpha_i-1} t^{\gamma+1} \prod_{k\neq i}^n (x_{1k}+x_{2k}t)^{\alpha_k} \\ &= \sum_{i=1}^n t^{\gamma+1} \frac{\partial (x_{1i}+x_{2i}t)^{\alpha_i}}{\partial t} \prod_{k\neq i}^n (x_{1k}+x_{2k}t)^{\alpha_k} \\ &= t^{\gamma+1} \frac{\partial \left(\prod_{k=1}^n (x_{1k}+x_{2k}t)^{\alpha_k}\right)}{\partial t}.\end{aligned}$$

By Stokes' theorem, we obtain

$$\begin{aligned}\left(\sum_{i=1}^n \theta_{2i}\right) \bullet \Phi(\beta;x) &= \int_a^b \left(\sum_{i=1}^n \theta_{2i}\right) \bullet t^\gamma \prod_{k=1}^n (x_{1k}+x_{2k}t)^{\alpha_k} dt \\ &= \int_a^b t^{\gamma+1} \frac{\partial \left(\prod_{k=1}^n (x_{1k}+x_{2k}t)^{\alpha_k}\right)}{\partial t} dt \\ &= \left[t^{\gamma+1} \prod_{k=1}^n (x_{1k}+x_{2k}t)^{\alpha_k} \right]_{t=a}^{t=b} - (\gamma+1)\Phi(\beta;x).\end{aligned}$$

Thus the proposition is proved. ∎

Example 1 We consider the following system of differential equations:

$$\begin{cases} (\partial_{11}\partial_{22} - \partial_{12}\partial_{21}) \bullet f &= 0, \\ (\theta_{11}+\theta_{21}-\alpha_1) \bullet f &= 0, \\ (\theta_{12}+\theta_{22}-\alpha_2) \bullet f &= 0, \\ (\theta_{21}+\theta_{22}+\gamma+1) \bullet f &= [g(t,x)]_{t=a}^{t=b}. \end{cases}$$

Here, $g(t,x) = t^{\gamma+1}(x_{11}+x_{21}t)^{\alpha_1}(x_{12}+x_{22}t)^{\alpha_2}$.

This is the incomplete $\Delta_1 \times \Delta_1$ hypergeometric system for $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, $\beta = (\alpha_1, \alpha_2, -\gamma-1)$, and $g_1 = 0, g_2 = 0, g_3 = [g(t,x)]_{t=a}^{t=b}$.

A detailed study on the system is given in [6].

3 Series Solution

The Lauricella function F_D is defined by

$$\begin{aligned}& F_D(a, b_1, \ldots, b_n, c; z_1, \ldots, z_n) \\ = & \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}(1)_{m_1}\cdots(1)_{m_n}} z_1^{m_1}\cdots z_n^{m_n}.\end{aligned}$$

It is well-known that the Lauricella function F_D of $n-1$ variables gives a series solution of $\Delta_1 \times \Delta_{n-1}$-hypergeometric system. We can give series solutions of our incomplete system in terms of the Lauricella series when parameters are generic. We need F_D of n variables to give a solution.

Theorem 1 *If γ is not negative integer, the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system has a series solution which can be expressed in terms of the Lauricella function F_D as*

$$F(\beta;x) = \prod_{k=1}^{n} x_{1k}^{\alpha_k} \left(\frac{b^{\gamma+1}}{\gamma+1} F_D \left(\gamma+1; -\alpha_1, \dots, -\alpha_n; \gamma+2; \frac{-x_{21}b}{x_{11}}, \dots, \frac{-x_{2n}b}{x_{1n}} \right) \right.$$
$$\left. - \frac{a^{\gamma+1}}{\gamma+1} F_D \left(\gamma+1; -\alpha_1, \dots, -\alpha_n; \gamma+2; \frac{-x_{21}a}{x_{11}}, \dots, \frac{-x_{2n}a}{x_{1n}} \right) \right).$$

Proof. For simplicity, we introduce some multi-index notations. An n-dimensional multi-index is an n-tuple $m = (m_1, \dots, m_n)$ of non-negative integers. The norm of a multi-index is defined by $|m| = m_1 + \cdots + m_n$. For a vector $x_i = (x_{i1}, \dots, x_{in})$ $(i = 1, 2)$, define $x_i^m = x_{i1}^{m_1} \cdots x_{in}^{m_n}$ and for a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{C}^n$, define the Pochhammer symbol by $(\alpha)_m = (\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n}$. By using these notations, the series F can be written as

$$F = x_1^{\alpha} \sum_{m \geq 0} c_m \left(\frac{x_2}{x_1} \right)^m, \quad c_m = \frac{(-1)^{|m|}(-\alpha)_m}{(\gamma+|m|+1)(1)_m} (b^{\gamma+|m|+1} - a^{\gamma+|m|+1}).$$

We note that

$$\begin{aligned} \theta_{1k} \bullet F &= (\alpha_k - m_k)F, \\ \theta_{2k} \bullet F &= m_k F. \end{aligned}$$

We now prove that the series F satisfies the incomplete system (1). Firstly, $(\theta_{1i} + \theta_{2i} - \alpha_i) \bullet F = 0$ for $1 \leq i \leq n$ follows from above fact immediately.

Secondly, we will prove $\left(\sum_{i=1}^{n}\theta_{2i}+\gamma+1\right)\bullet F=[g(t,x)]_{t=a}^{t=b}$, which can be shown as

$$\begin{aligned}\left(\sum_{i=1}^{n}\theta_{2i}+\gamma+1\right)\bullet F &= (|m|+\gamma+1)F\\ &= x_1^{\alpha}\sum_{m\geq 0}\frac{(-1)^{|m|}(-\alpha)_m}{(1)_m}(b^{\gamma+|m|+1}-a^{\gamma+|m|+1})\left(\frac{x_2}{x_1}\right)^m\\ &= \left[t^{\gamma+1}x_1^{\alpha}\sum_{m\geq 0}\frac{(-\alpha)_m}{(1)_m}\left(-\frac{x_2 t}{x_1}\right)^m\right]_{t=a}^{t=b}\\ &= \left[t^{\gamma+1}x_1^{\alpha}\prod_{k=1}^{n}\left(1+\frac{x_{2k}t}{x_{1k}}\right)^{\alpha_k}\right]_{t=a}^{t=b}\\ &= \left[t^{\gamma+1}\prod_{k=1}^{n}(x_{1k}+x_{2k}t)^{\alpha_k}\right]_{t=a}^{t=b}.\end{aligned}$$

In the last two steps, we take a branch such that the equality holds.

Finally, we will prove $(\partial_{1i}\partial_{2j}-\partial_{1j}\partial_{2i})\bullet F=0$ for $1\leq i<j\leq n$. This follows from the following two calculations:

$$\begin{aligned}\left(\theta_{1i}\theta_{2j}-\frac{x_{2j}x_{1i}}{x_{1j}x_{2i}}\theta_{1j}\theta_{2i}\right)\bullet F &= x_1^{\alpha}\sum_{m\geq 0}(\alpha_i-m_i)m_j c_m\left(\frac{x_2}{x_1}\right)^m\\ &\quad - x_1^{\alpha}\sum_{m\geq 0}(\alpha_j-m_j)m_i c_m\left(\frac{x_2}{x_1}\right)^{m-e_i+e_j}\\ &= x_1^{\alpha}\sum_{m\geq 0}(\alpha_i-m_i)(m_j+1)c_{m+e_j}\left(\frac{x_2}{x_1}\right)^{m+e_j}\\ &\quad - x_1^{\alpha}\sum_{m\geq 0}(\alpha_j-m_j)(m_i+1)c_{m+e_i}\left(\frac{x_2}{x_1}\right)^{m+e_j}\end{aligned}$$

and

$$\begin{aligned}(\alpha_i-m_i)(m_j+1)c_{m+e_j} &= (\alpha_i-m_i)(m_j+1)\frac{(-1)^{|m+e_j|}(-\alpha)_{m+e_j}}{(\gamma+|m+e_j|+1)(1)_{m+e_j}}(b^{\gamma+|m+e_j|+1}-a^{\gamma+|m+e_j|+1})\\ &= \frac{(-1)^{|m|+1}(-\alpha)_{m+e_j+e_i}}{(\gamma+|m|+2)(1)_m}(b^{\gamma+|m|+2}-a^{\gamma+|m|+2})\\ &= (\alpha_j-m_j)(m_i+1)\frac{(-1)^{|m+e_i|}(-\alpha)_{m+e_i}}{(\gamma+|m+e_i|+1)(1)_{m+e_i}}(b^{\gamma+|m+e_i|+1}-a^{\gamma+|m_i|+1})\\ &= (\alpha_j-m_j)(m_i+1)c_{m+e_i}.\end{aligned}$$

Therefore, the theorem is proved. ∎

Gel'fand, Kapranov and Zelevinsky ([1]) gave a base of the solutions of the (complete) $\mathcal{A}$-hypergeometric system. We will give a base of solutions of our incomplete system by utilizing their result and Theorem 1.

For a parameter $\beta = (\alpha_1, \dots, \alpha_n, -\gamma - 1) \in \mathbf{C}^{n+1}$, we set a $2 \times n$ matrix

$$s^{(\ell)} = (s_{ij}^{(\ell)}) = \begin{pmatrix} \beta_1 & \cdots & \beta_{\ell-1} & \sum_{j=\ell}^{n+1} \beta_j & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\sum_{j=\ell+1}^{n+1} \beta_j & \beta_{\ell+1} & \cdots & \beta_n \end{pmatrix}$$

for $1 \leq \ell \leq n$. Let $M^{(\ell)}$ be a set of $2 \times n$ matrices

$$M^{(\ell)} = \sum_{k=1}^{\ell-1} \mathbf{N}_0 \cdot (e_{2k} + e_{1\ell} - e_{1k} - e_{2\ell}) + \sum_{k=\ell+1}^{n} \mathbf{N}_0 \cdot (e_{1k} + e_{2\ell} - e_{2k} - e_{1\ell}),$$

where e_{ij} is the $2 \times n$ matrix whose (i, j)-entry is 1 and the other entries are 0. We suppose that the condition of parameter β called "*T-nonresonant*", that is the sets $s^{(\ell)} \pm M^{(\ell)}$ $(1 \leq \ell \leq n)$ are pairwise disjoint ([1, Definition 3]). Define series $\Psi^{(\ell)}(x)$ as

$$\Psi^{(\ell)}(x) = \Gamma(s^{(\ell)} + 1) \sum_{k \in M^{(\ell)}} \frac{1}{\Gamma(s^{(\ell)} + k + 1)} x^{s^{(\ell)} + k},$$

where $\Gamma(s^{(\ell)}+k+1) = \prod_{i=1}^{2} \prod_{j=1}^{n} \Gamma(s_{ij}^{(\ell)}+k_{ij}+1)$ and $x^{s^{(\ell)}+k} = \prod_{i=1}^{2} \prod_{j=1}^{n} x_{ij}^{s_{ij}^{(\ell)}+k_{ij}}$. These series are linearly independent and have the open domain

$$\left|\frac{x_{21}}{x_{11}}\right| < \left|\frac{x_{22}}{x_{12}}\right| < \cdots < \left|\frac{x_{2n}}{x_{1n}}\right|$$

as a common domain of convergence. Moreover they span the solution space of (complete) $\Delta_1 \times \Delta_{n-1}$-hypergeometric system ([1, Theorem 3]).

By using this result, we obtain the following theorem.

Theorem 2 *Suppose the parameter β is T-nonresonant and γ is not negative integer.*

1. *The common domain of convergence of $F(\beta; x)$ and $\Psi^{(\ell)}(x)$ is*

$$U : \left|\frac{x_{21}}{x_{11}}\right| < \left|\frac{x_{22}}{x_{12}}\right| < \cdots < \left|\frac{x_{2n}}{x_{1n}}\right| < \frac{1}{\max(|a|, |b|)}.$$

2. *Any holomorphic solution of the incomplete system (1) on U can be written as*

$$F(\beta; x) + \sum_{\ell=1}^{n} c_i \Psi^{(\ell)}(x), \qquad c_i \in \mathbf{C}.$$

Proof. Since the domain of convergence of the series F is

$$U_0 : \left|\frac{x_{21}}{x_{11}}\right| < \frac{1}{\max(|a|, |b|)}, \left|\frac{x_{22}}{x_{12}}\right| < \frac{1}{\max(|a|, |b|)}, \cdots, \left|\frac{x_{2n}}{x_{1n}}\right| < \frac{1}{\max(|a|, |b|)},$$

we have the statement 1. The statement 2 is clear. ∎

Theorem 3 *For $\sigma \in \mathfrak{S}_n$, we suppose the parameter $\sigma(\beta)$ is T-nonresonant and γ is not negative integer.*

1. *The domain of convergence of the series $F(\beta; x)$ and $\sigma(\Psi^{(\ell)}(x))$ is*

$$\sigma(U) : \left|\frac{x_{2\sigma(1)}}{x_{1\sigma(1)}}\right| < \left|\frac{x_{2\sigma(2)}}{x_{1\sigma(2)}}\right| < \cdots < \left|\frac{x_{2\sigma(n)}}{x_{1\sigma(n)}}\right| < \frac{1}{\max(|a|, |b|)}.$$

2. *Any holomorphic solution of the incomplete system (1) on $\sigma(U)$ can be written as*

$$F(\beta; x) + \sum_{\ell=1}^{n} c_i \sigma(\Psi^{(\ell)}(x)), \qquad c_i \in \mathbf{C}.$$

Here, $\sigma(\Psi^{(\ell)}(x))$ is given by the permutations $x_{ij} \leftrightarrow x_{i\sigma(j)}$, $s_{ij}^{(\ell)} \leftrightarrow s_{i\sigma(j)}^{(\ell)}$ and $\beta_j \leftrightarrow \beta_{\sigma(j)}$.

Proof. The theorem follows immediately from the σ-invariance of F. ∎

Remark 2 The closure of the union of $\sigma(U)$ coincides with the closure of U_0. That is

$$\overline{U_0} = \bigcup_{\sigma \in \mathfrak{S}_n} \overline{\sigma(U)}.$$

4 Contiguity Relation

Contiguity relation is a relation among two functions of which parameters are different by integer. Miller ([5]) gave contiguity relations for Lauricella functions and Sasaki ([11]) gave contiguity relations for Aomoto-Gel'fand hypergeometric functions, which include the case of complete $\Delta_1 \times \Delta_{n-1}$-hypergeometric functions. In [8] and [9], they give algorithms to compute contiguity relations in the case of $\mathcal{A}$-hypergeometric functions. These results are for complete functions. In [6], an algorithm of computing contiguity relations under some conditions is given for incomplete $\mathcal{A}$-hypergeometric functions and the complete list of them for the incomplete $\Delta_1 \times \Delta_1$-hypergeometric function is derived.

We will give contiguity relations of our incomplete system in this section. We put $\delta = -\gamma - 1$ (i.e., $\beta = (\alpha_1, \ldots, \alpha_n, \delta)$) to make formulas of contiguity relations of the incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric function simpler forms. We put

$$\Phi(\beta; x) = \int_a^b t^{-\delta-1} \prod_{k=1}^{n} (x_{1k} + x_{2k} t)^{\alpha_k} dt,$$

and assume $\mathrm{Re}\,(-\delta - 1)$, $\mathrm{Re}\,\alpha_k > 0$. Then, we note that $\Phi(\beta; x)$ is a solution of the following incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric system:

$$\begin{cases} z_i \bullet f &= 0, & z_i := \theta_{1i} + \theta_{2i} - \alpha_i, \qquad (1 \le i \le n) \\ z \bullet f &= [g(t,x)]_{t=a}^{t=b}, & z := \sum_{i=1}^{n} \theta_{2i} - \delta, \\ I_A \bullet f &= 0, \end{cases}$$

where $g(t,x) = t^{-\delta} \prod_{k=1}^{n} (x_{1k} + x_{2k} t)^{\alpha_k}$. Let a_{1k} and a_{2k} be vectors corresponding to the $(2k-1)$-st and the $2k$-th columns of A respectively.

Theorem 4 *The incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric function $\Phi(\beta; x)$ satisfies the following contiguity relations.*

- *Shifts with respect to a_{1k}:*

$$S(\beta; -a_{1k})\Phi(\beta; x) = \alpha_k \Phi(\beta - a_{1k}; x), \tag{3}$$

$$S(\beta - a_{1k}; +a_{1k})\Phi(\beta - a_{1k}; x) = \left(\sum_{i=1}^{n} \alpha_i - \delta\right) \Phi(\beta; x) - [g(t,x)]_{t=a}^{t=b}, \tag{4}$$

where

$$S(\beta; -a_{1k}) = \partial_{1k},$$
$$S(\beta - a_{1k}; +a_{1k}) = \sum_{i=1, i\neq k}^{n} (x_{1i}x_{2k} - x_{1k}x_{2i})\partial_{2i} + \sum_{i=1}^{n+1} \alpha_i x_{1k}.$$

- *Shifts with respect to a_{2k}:*

$$S(\beta; -a_{2k})\Phi(\beta; x) = \alpha_k \Phi(\beta - a_{2k}; x), \tag{5}$$

$$S(\beta - a_{2k}; +a_{2k})\Phi(\beta - a_{2k}; x) = \delta \Phi(\beta; x) + [g(t,x)]_{t=a}^{t=b}, \tag{6}$$

where

$$S(\beta; -a_{2k}) = \partial_{2k},$$
$$S(\beta - a_{2k}; +a_{2k}) = \sum_{i=1, i\neq k}^{n} x_{1k}x_{2i}\partial_{1i} + \left(\sum_{i=1, i\neq k}^{n} \theta_{2i} + \alpha_k\right) x_{2k}.$$

Proof. The down-step relations (3) and (5) are easily verified. We will prove only the up-step relations (4) and (6).

Let L_1 be the operator

$$\left(\sum_{i=1, i\neq k}^{n} (x_{1i}x_{2k} - x_{1k}x_{2i})\partial_{2i} + \sum_{i=1}^{n} \alpha_i x_{1k}\right)\partial_{1k} - \alpha_k\left(\sum_{i=1}^{n} \alpha_i - \delta\right) + \alpha_k\left(\sum_{i=1}^{n} \theta_{2i} - \delta\right).$$

We now prove that $L_1 \bullet \Phi(\beta; x) = 0$ which together with (3) will prove the contiguity relation (4). The operator L_1 can be reduced by z_k $(1 \le k \le n)$ as follows:

$$\begin{aligned}
L_1 &= \sum_{i=1,i\neq k}^{n} x_{1i}x_{2k}\partial_{2i}\partial_{1k} - \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)\theta_{1k} + \theta_{2k}\theta_{1k} - \alpha_k\sum_{i=1}^{n}\alpha_i + \alpha_k\sum_{i=1}^{n}\theta_{2i} \\
&= \sum_{i=1,i\neq k}^{n} x_{1i}x_{2k}\partial_{2i}\partial_{1k} - \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)(\theta_{1k}+\theta_{2k}-\alpha_k) \\
&\qquad + \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)(\theta_{2k}-\alpha_k) + \theta_{2k}\theta_{1k} + \alpha_k\sum_{i=1}^{n}(\theta_{2i}-\alpha_i) \\
&= \sum_{i=1,i\neq k}^{n} x_{1i}x_{2k}\partial_{2i}\partial_{1k} - \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)z_k + \sum_{i=1}^{n}\theta_{2k}(\theta_{2i}-\alpha_i) + \theta_{2k}\theta_{1k} \\
&= \sum_{i=1,i\neq k}^{n} x_{1i}x_{2k}\partial_{2i}\partial_{1k} - \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)z_k + \sum_{i=1}^{n}\theta_{2k}(\theta_{1i}+\theta_{2i}-\alpha_i) - \sum_{i=1}^{n}\theta_{2k}\theta_{1i} + \theta_{2k}\theta_{1k} \\
&= \sum_{i=1,i\neq k}^{n} x_{1i}x_{2k}(\partial_{2i}\partial_{1k}-\partial_{2k}\partial_{1i}) - \sum_{i=1}^{n}(\theta_{2i}-\alpha_i)z_k + \sum_{i=1}^{n}\theta_{2k}z_i.
\end{aligned}$$

Since the $\partial_{2i}\partial_{1k} - \partial_{2k}\partial_{1i}$ are elements of the toric ideal I_A, we obtain $L_1 \bullet \Phi(\beta;x) = 0$.

Let L_2 be the operator

$$\left(\sum_{i=1,i\neq k}^{n} x_{1k}x_{2i}\partial_{1i} + \left(\sum_{i=1,i\neq k}^{n}\theta_{2i}+\alpha_k\right)x_{2k}\right)\partial_{2k} - \alpha_k\delta + \alpha_k\left(\sum_{i=1}^{n}\theta_{2i}-\delta\right).$$

Since L_2 can be written as

$$L_2 = \sum_{i=1,i\neq k}^{n} x_{1k}x_{2i}(\partial_{1i}\partial_{2k}-\partial_{2i}\partial_{1k}) + \sum_{i=1}^{n}\theta_{2i}z_k,$$

we obtain $L_2 \bullet \Phi(\beta;x) = 0$ in an analogous calculation with the case of L_1. ∎

Theorem 4 gives contiguity relations for $e_k = a_{1k} = (0,\ldots,0,\overset{k}{\check{1}},0,\ldots,0)$ $(1 \leq k \leq n)$, but it does not give those for $e_{n+1} = (0,\ldots,0,1)$. The set of vectors $\{e_1,\ldots,e_{n+1}\}$ is the standard basis of $\mathbf{Z}^{n+1}$. The contiguity relations for e_{n+1} can be obtained from Theorem 4 as follows.

Corollary 1 *The incomplete $\Delta_1 \times \Delta_{n-1}$-hypergeometric function $\Phi(\beta;x)$ satisfies the following contiguity relations.*

- *Shifts with respect to e_{n+1}:*

$$S(\beta+e_{n+2};-e_{n+2})\Phi(\beta+e_{n+2};x) = \alpha_k\left(\sum_{i=1}^{n}\alpha_i-\delta\right)\Phi(\beta;x) - \alpha_k[g(t,x)]_{t=a}^{t=b},$$

$$S(\beta-e_{n+2};+e_{n+2})\Phi(\beta-e_{n+2};x) = \alpha_k\delta\Phi(\beta;x) + \alpha_k[g(t,x)]_{t=a}^{t=b},$$

where

$$S(\beta + e_{n+2}; -e_{n+2}) = S(\beta - a_{1k}; +a_{1k})\partial_{2k},$$
$$S(\beta - e_{n+2}; +e_{n+2}) = S(\beta - a_{2k}; +a_{2k})\partial_{1k} \quad \textit{for } 1 \le k \le n+1.$$

Although we prove these contiguity relations for the integral representation of the incomplete $\Delta_1 \times \Delta_{n-1}$ function, they hold for functions which satisfy the system and the two conditions (3) and (5). By an easy calculation to check these conditions for the series solution $F(\beta; x)$, we obtain the following corollary.

Corollary 2 *The series solution $F(\beta; x)$ satisfies the same contiguity relations.*

We note that $\Phi(\beta; x)$ can be formally expanded in $F(\beta; x)$.

5 Appendix: A solvability of incomplete $\mathcal{A}$-hypergeometric systems

Let D be the Weyl algebra in n variables. We denote by $A = (a_{ij})$ a $d \times n$-matrix whose elements are integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$.

Definition 2 ([6]) We call the following system of differential equations $H_A(\beta, g)$ an *incomplete $\mathcal{A}$-hypergeometric system*:

$$\begin{aligned} (E_i - \beta_i) \bullet f &= g_i, \quad E_i - \beta_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i, \qquad (i = 1, \ldots, d) \\ \Box_{u,v} \bullet f &= 0, \quad \Box_{u,v} = \prod_{i=1}^{n} \partial_i^{u_i} - \prod_{j=1}^{n} \partial_j^{v_j} \end{aligned}$$

with $u, v \in \mathbf{N}_0^n$ running over all u, v such that $Au = Av$.

Here, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$ are parameters and $g = (g_1, \ldots, g_d)$ where g_i are given holonomic functions.

We denote by $E - \beta$ the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$ and I_A the affine toric ideal generated by $\Box_{u,v}$ ($Au = Av$) in $\mathbf{C}[\partial_1, \ldots, \partial_n]$.

Lemma 1 *If the first homology of the Euler-Koszul complex vanishes;*

$$H_1(K_\bullet(E - \beta; D/DI_A)) = 0,$$

then the syzygy module $\mathrm{syz}(E_1 - \beta_1, \ldots, E_d - \beta_d) \subset (D/DI_A)^d$ *is generated by* $(E_i - \beta_i)e_j - (E_j - \beta_j)e_i$ $(1 \le i < j \le d)$.

Proof. The Euler-Koszul complex of D/DI_A is the following complex

$$0 \longrightarrow D/DI_A \xrightarrow{d_d} \cdots \xrightarrow{d_3} (D/DI_A)^{\binom{d}{2}} \xrightarrow{d_2} (D/DI_A)^d \xrightarrow{d_1} D/DI_A \longrightarrow 0$$

and the differential is defined by

$$d_p(e_{i_1,\ldots,i_p}) = \sum_{k=1}^{p} (-1)^{k-1}(E_{i_k} - \beta_{i_k})e_{i_1,\ldots,\widehat{i_k},\ldots,e_p}.$$

Here, $e_{i_1,\ldots,i_p}$ are basis vectors of $(D/DI_A)^{\binom{d}{p}}$. The kernel of d_1 is $\mathrm{syz}(E_1 - \beta_1, \ldots, E_d - \beta_d)$ and the image of d_2 is generated by $(E_i - \beta_i)e_j - (E_j - \beta_j)e_i$ $(1 \le i < j \le d)$ over $(D/DI_A)^d$. Since the first homology is zero, the conclusion is obtained. ∎

Theorem 5 ([12]) *If the first homology $H_1(K_\bullet(E-\beta; D/DI_A))$ vanishes and the g_i are holonomic functions satisfying the following relations*

$$(E_i - \beta_i) \bullet g_j = (E_j - \beta_j) \bullet g_i, \qquad (i, j = 1, \ldots, d) \tag{7}$$

$$\Box_{u,v} \bullet g_i = 0, \qquad (i = 1, \ldots, d, Au = Av, u, v \in \mathbf{N}_0^n) \tag{8}$$

then the incomplete hypergeometric system has a (classical) solution.

Proof. By virtue of [3, Theorem 4.1], $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{D}/\mathcal{D}H_A(\beta), \mathcal{O})$ vanishes at generic points in $\mathbf{C}^n$. Therefore, it is sufficient to prove that $\ell_1 g_1 + \cdots + \ell_d g_d = 0$ for all $(\ell_1, \ldots, \ell_d, \ell_{d+1}, \cdots, \ell_{d+m}) \in \mathrm{syz}(E - \beta, \Box)$, where $\Box$ is a finite sequence $\Box_{u_1,v_1}, \ldots, \Box_{u_m,v_m}$ which are generators of I_A. Since for $(\ell_1, \ldots, \ell_d, \ell_{d+1}, \cdots, \ell_{d+m}) \in \mathrm{syz}(E - \beta, \Box)$, the relation $\sum_{i=1}^d \ell_i(E_i - \beta_i) + \sum_{i=1}^m \ell_{d+i}\Box_{u_i,v_i} = 0$ holds, we have $(\ell_1, \ldots, \ell_d) \in \mathrm{syz}(E - \beta)$ over $(D/DI_A)^d$. By Lemma 1,

$$\begin{aligned}
\ell_1 g_1 + \cdots + \ell_d g_d &= (\ell_1, \ldots, \ell_d) \cdot g \\
&= \sum_{1 \le i < j \le d} c_{ij}\{(E_i - \beta_i)e_j - (E_j - \beta_j)e_i\} \cdot g, \qquad c_{ij} \in \mathbf{C} \\
&= \sum_{1 \le i < j \le d} c_{ij}\{(E_i - \beta_i)g_j - (E_j - \beta_j)g_i\} \\
&= 0.
\end{aligned}$$

∎

Remark 3 Matusevich, Miller and Walther ([4, Theorem 6.3]) showed that if the toric ideal I_A is Cohen-Macaulay, the i-th homology of the Euler-Koszul complex vanishes for all positive integers i.

The following facts are known about Cohen-Macaulay property of toric ideals.

1. If the initial monomial ideal of I_A is square-free, then A is normal (see, e.g., [10, Proposition 13.15]).
2. If the matrix A is normal, then I_A is Cohen-Macaulay ([2]).

This is an easy tool for showing Cohen-Macaulayness of toric ideals. When A is $\Delta_1 \times \Delta_{n-1}$, we can easily verify the condition 1.

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