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CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS

ZHI-Hong Sun

School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, PR China
E-mail: szh6174@yahoo.com
Homepage: http://www.hytc.edu.cn/xsjl/szh

Abstract. Let p be an odd prime. In the paper, by using the properties of Legendre polynomials we prove some congruences for $\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2 m^{-k} \pmod{p^2}$. In particular, we confirm several conjectures of Z.W. Sun. We also pose 13 conjectures on supercongruences.

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1. Introduction.

Let p be an odd prime. In 2003, Rodriguez-Villegas [11] conjectured the following congruence:

$$(1.1)\qquad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}.$$

This was later confirmed by Mortenson [7] via the Gross-Koblitz formula. See also [9] and [10, p.204]. Recently my twin brother Zhi-Wei Sun [13] obtained the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2 m^{-k} \pmod p$ in the cases $m=8,-16,32$, and made several conjectures for $\sum_{k=0}^{p-1}\binom{2k}{k}^2 m^{-k} \pmod{p^2}$. For example, he conjectured

$$(1.2)\qquad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{32^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } 4\mid p-3,\\ 2a-\frac{p}{2a} \pmod{p^2} & \text{if } 4\mid p-1 \text{ and } p=a^2+b^2 \text{ with } 4\mid a-1.\end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n\quad (|t|<1).$$

It is well known that (see [6, pp. 228-232], [4, (3.132)-(3.133)])

$$(1.3)\qquad P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}=\frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

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and $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, where $[x]$ is the greatest integer not exceeding x.

In the paper, by using the expansions of Legendre polynomials we obtain some congruences for $P_{\frac{p-1}{2}}(x)$ modulo p^2, where p is an odd prime and x is a rational p-integer. For example, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv 0 \pmod{p^2}, \tag{1.4}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - \left(\frac{x}{p}\right)x^{-k}\right) \equiv 0 \pmod p \quad \text{for} \quad x \not\equiv 0 \pmod p, \tag{1.5}$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol. Taking $x=1$ in (1.4) we obtain (1.1) immediately, and taking $x=\frac{1}{2}$ in (1.4) we deduce (1.2) for $p \equiv 3 \pmod 4$. We also determine $\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} k(k-1)\cdots(k-r+1) \pmod{p^2}$ for $r \in \{1, 2, \dots, \frac{p-1}{2}\}$, $\sum_{k=0}^{[p/3]} \frac{(3k)!}{54^k \cdot k!^3} \pmod p$ and pose some conjectures on supercongruences concerning binary quadratic forms.

Throughout this paper we use $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_p$ to denote the sets of integers, positive integers and rational p-integers for a prime p, respectively.

2. Main results.

Lemma 2.1. *For $n \in \mathbb{N}$ we have*

$$P_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k.$$

Proof. From [4, (3.135)] we have the following result due to Murphy:

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k. \tag{2.1}$$

As $\binom{n+k}{2k}\binom{2k}{k} = \binom{n+k}{k}\binom{n}{k}$, we obtain the result.

Lemma 2.2. *Let p be an odd prime and $k \in \{1, 2, \dots, (p-1)/2\}$. Then*

$$\binom{\frac{p-1}{2}+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k}\left(1 - p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\right) \pmod{p^4}.$$

Proof. Clearly

$$\begin{aligned}\binom{\frac{p-1}{2}+k}{2k} &= \frac{(\frac{p-1}{2}+k)(\frac{p-1}{2}+k-1)\cdots(\frac{p-1}{2}-k+1)}{(2k)!} \\ &= \frac{(p+2k-1)(p+2k-3)\cdots(p-(2k-3))(p-(2k-1))}{2^{2k}\cdot(2k)!} \\ &= \frac{(p^2-1^2)(p^2-3^2)\cdots(p^2-(2k-1)^2)}{2^{2k}\cdot(2k)!} \\ &\equiv \frac{(-1)^k\cdot 1^2\cdot 3^2\cdots(2k-1)^2}{2^{2k}\cdot(2k)!}\Big(1-p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big) \pmod{p^4}.\end{aligned}$$

To see the result, we note that

$$\frac{1^2\cdot 3^2\cdots(2k-1)^2}{2^{2k}\cdot(2k)!} = \frac{(2k)!^2}{(2\cdot 4\cdots(2k))^2\cdot 2^{2k}\cdot(2k)!} = \frac{(2k)!}{2^{4k}\cdot k!^2} = \frac{\binom{2k}{k}}{16^k}.$$

Let p be an odd prime, and let $\{A(n)\}$ be the Apéry numbers given by

$$A(n)=\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2.$$

It is well known that (see [1],[10]) $A(\frac{p-1}{2})\equiv a(p) \pmod{p^2}$, where $a(n)$ is defined by

$$q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4=\sum_{n=1}^{\infty}a(n)q^n.$$

By the fact $\binom{n+k}{k}\binom{n}{k}=\binom{n+k}{2k}\binom{2k}{k}$ and Lemma 2.2 we have

$$A\Big(\frac{p-1}{2}\Big)=\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}+k}{2k}^2\binom{2k}{k}^2 \equiv \sum_{k=0}^{\frac{p-1}{2}}\Big(\frac{\binom{2k}{k}}{(-16)^k}\Big)^2\binom{2k}{k}^2 \pmod{p^2}.$$

Hence

$$a(p)\equiv A\Big(\frac{p-1}{2}\Big)\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^4}{4^{4k}} \pmod{p^2}. \tag{2.2}$$

Let $b(n)$ be given by $q\prod_{n=1}^{\infty}(1-q^{4n})^6=\sum_{n=1}^{\infty}b(n)q^n$. Then Mortenson [8] proved the following conjecture of Rodriguez-Villegas:

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{4^{3k}}\equiv b(p) \pmod{p^2}. \tag{2.3}$$

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(x^k-(-1)^{\frac{p-1}{2}}(1-x)^k\right)\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\left(x^k-(-1)^{\frac{p-1}{2}}(1-x)^k\right)\equiv 0 \pmod{p^2}.$$

Proof. For a variable t, by Lemmas 2.1 and 2.2 we have

$$P_{\frac{p-1}{2}}(t)=\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}+k}{2k}\binom{2k}{k}\left(\frac{t-1}{2}\right)^k\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-t}{2}\right)^k \pmod{p^2}. \tag{2.4}$$

It is known that (see [6]) $P_n(t)=(-1)^nP_n(-t)$. Thus, by (2.4),

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-t}{2}\right)^k\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\left(\frac{1+t}{2}\right)^k \pmod{p^2}.$$

Now taking $t=1-2x$ in the congruence we deduce $\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\left(x^k-(-1)^{\frac{p-1}{2}}(1-x)^k\right)\equiv 0 \pmod{p^2}$. To complete the proof, we note that for $k\in\{\frac{p+1}{2},\frac{p+3}{2},\dots,p-1\}$, $\binom{2k}{k}=2k(2k-1)\cdots(k+1)/k!\equiv 0 \pmod p$.

Theorem 2.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } 4\mid p-3,\\ 2a-\frac{p}{2a} \pmod{p^2} & \text{if } 4\mid p-1 \text{ and } p=a^2+b^2 \text{ with } 4\mid a-1.\end{cases}$$

Proof. When $p\equiv 3 \pmod 4$, taking $x=\frac12$ in Theorem 2.1 we obtain the result. Now suppose $p\equiv 1 \pmod 4$ and so $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1 \pmod 4$. It is well known that ([6])

$$P_{2n+1}(0)=0 \quad\text{and}\quad P_{2n}(0)=\frac{(-1)^n}{2^{2n}}\binom{2n}{n}. \tag{2.5}$$

Thus, by (2.4) and (2.5) we have

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k}\equiv P_{\frac{p-1}{2}}(0)=\frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod{p^2}.$$

According to the result due to Chowla, Dwork and Evans (see [2] or [3]), we have

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv\frac{2^{p-1}+1}{2}\left(2a-\frac{p}{2a}\right) \pmod{p^2}.$$

Set $q=(2^{\frac{p-1}{2}}-(-1)^{\frac{p-1}{4}})/p$. Then $2^{p-1}\equiv 1+2(-1)^{\frac{p-1}{4}}qp \pmod{p^2}$. Thus

$$\frac{2^{p-1}+1}{2\cdot 2^{\frac{p-1}{2}}}\equiv\frac{2+2(-1)^{\frac{p-1}{4}}qp}{2((-1)^{\frac{p-1}{4}}+qp)}=(-1)^{\frac{p-1}{4}} \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k}\equiv\frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv\frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}}\cdot\frac{2^{p-1}+1}{2}\Big(2a-\frac{p}{2a}\Big)\equiv 2a-\frac{p}{2a} \pmod{p^2}.$$

The proof is now complete.

Remark 2.1 Theorem 2.2 was conjectured by Zhi-Wei Sun ([13]), and the congruence for $\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k} \pmod p$ was also proved by Zhi-Wei Sun in [13].

Theorem 2.3. *Let p be an odd prime and $r\in\{1,2,\dots,(p-1)/2\}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k}k(k-1)\cdots(k-r+1)$$
$$\equiv\begin{cases}0 \pmod{p^2} & \text{if } 4\mid(p+1-2r),\\ (-1)^{\frac{p-1+2r}{4}}2^{-\frac{p-1}{2}}\frac{(\frac{p-1}{2}+r)!}{\frac{p-1-2r}{4}!\,\frac{p-1+2r}{4}!} \pmod{p^2} & \text{if } 4\mid(p-1-2r).\end{cases}$$

Proof. By (2.4) we have

$$\text{(2.6)}\qquad \begin{aligned}\frac{d^r\, P_{\frac{p-1}{2}}(t)}{dt^r}&\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-32)^k}\cdot\frac{d^r(t-1)^k}{dt^r}\\&=\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-32)^k}k(k-1)\cdots(k-r+1)(t-1)^{k-r} \pmod{p^2}.\end{aligned}$$

Hence

$$\frac{d^r\, P_{\frac{p-1}{2}}(t)}{dt^r}\Big|_{t=0}=(-1)^r\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{32^k}k(k-1)\cdots(k-r+1).$$

By (1.3) we have

$$\begin{aligned}\frac{d^r}{dt^r}P_{\frac{p-1}{2}}(t)&=\frac{1}{2^{(p-1)/2}}\cdot\frac{d^r}{dt^r}\sum_{m=0}^{[\frac{p-1}{4}]}\frac{(-1)^m(p-1-2m)!}{m!(\frac{p-1}{2}-m)!(\frac{p-1}{2}-2m)!}t^{\frac{p-1}{2}-2m}\\&=\frac{1}{2^{(p-1)/2}}\sum_{m=0}^{[\frac{p-1-2r}{4}]}\frac{(-1)^m(p-1-2m)!}{m!(\frac{p-1}{2}-m)!(\frac{p-1}{2}-2m)!}\\&\quad\times(\frac{p-1}{2}-2m)(\frac{p-1}{2}-2m-1)\cdots(\frac{p-1}{2}-2m-r+1)t^{\frac{p-1}{2}-2m-r}.\end{aligned}$$

Thus,

$$\frac{d^rP_{\frac{p-1}{2}}(t)}{dt^r}\Big|_{t=0}=\begin{cases}0 & \text{if } r\not\equiv\frac{p-1}{2} \pmod 2,\\ \frac{(-1)^m(p-1-2m)!}{2^{(p-1)/2}\cdot m!(\frac{p-1}{2}-m)!} & \text{if } r=\frac{p-1}{2}-2m.\end{cases}$$

Now combining all the above we obtain the result.

Corollary 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{k^2\binom{2k}{k}^2}{32^k}\equiv\begin{cases}(-1)^{\frac{p+3}{4}}2^{-\frac{p-1}{2}}\frac{\frac{p+3}{2}!}{\frac{p-5}{4}!\,\frac{p+3}{4}!}\pmod{p^2} & \text{if } p\equiv 1\pmod 4,\\ (-1)^{\frac{p+1}{4}}2^{-\frac{p-1}{2}}\frac{\frac{p+1}{2}!}{\frac{p-3}{4}!\,\frac{p+1}{4}!}\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. By Theorem 2.3 we have

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{k\binom{2k}{k}^2}{32^k}\equiv\begin{cases}0\pmod{p^2} & \text{if } p\equiv 1\pmod 4,\\ (-1)^{\frac{p+1}{4}}2^{-\frac{p-1}{2}}\frac{\frac{p+1}{2}!}{\frac{p-3}{4}!\,\frac{p+1}{4}!}\pmod{p^2} & \text{if } p\equiv 3\pmod 4\end{cases}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{k(k-1)\binom{2k}{k}^2}{32^k}\equiv\begin{cases}(-1)^{\frac{p+3}{4}}2^{-\frac{p-1}{2}}\frac{\frac{p+3}{2}!}{\frac{p-5}{4}!\,\frac{p+3}{4}!}\pmod{p^2} & \text{if } p\equiv 1\pmod 4,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Observe that $k^2=k(k-1)+k$. From the above we deduce the result.

Lemma 2.3. *Let p be a prime greater than 3 and let t be a variable. Then*

$$P_{[\frac p3]}(t)\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}\Big(\frac{1-t}{54}\Big)^k\pmod p.$$

Proof. Suppose $r=1$ or 2 according as $3\mid p-1$ or $3\mid p-2$. Then clearly

$$\begin{aligned}\binom{\frac{p-r}{3}+k}{2k}&=\frac{(\frac{p-r}{3}+k)(\frac{p-r}{3}+k-1)\cdots(\frac{p-r}{3}-k+1)}{(2k)!}\\&=\frac{(p+3k-r)(p+3k-r-3)\cdots(p-(3k+r-3))}{3^{2k}\cdot(2k)!}\\&\equiv(-1)^k\frac{(3k-r)(3k-r-3)\cdots(3-r)\cdot r(r+3)\cdots(3k+r-3)}{3^{2k}\cdot(2k)!}\\&=\frac{(-1)^k\cdot(3k)!}{3\cdot 6\cdots 3k\cdot 3^{2k}\cdot(2k)!}=\frac{(-1)^k\cdot(3k)!}{3^k\cdot k!\cdot 3^{2k}\cdot(2k)!}\pmod p.\end{aligned}$$

Hence, by Lemma 2.1 we have

$$\begin{aligned}P_{[\frac p3]}(t)&=\sum_{k=0}^{[p/3]}\binom{[\frac p3]+k}{2k}\binom{2k}{k}(\frac{t-1}{2})^k\equiv\sum_{k=0}^{[p/3]}\frac{(-1)^k\cdot(3k)!}{3^{3k}\cdot k!(2k)!}\cdot\frac{(2k)!}{k!^2}\Big(\frac{t-1}{2}\Big)^k\\&=\sum_{k=0}^{[p/3]}\frac{(3k)!}{27^k\cdot k!^3}\Big(\frac{1-t}{2}\Big)^k\pmod p.\end{aligned}$$

This proves the lemma.

Theorem 2.4. *Let p be a prime greater than 3 and let x be a variable. Then*

$$\sum_{k=0}^{[p/3]} \frac{(3k)!}{27^k \cdot k!^3}\left(x^k - (-1)^{[\frac{p}{3}]}(1-x)^k\right) \equiv 0 \pmod p.$$

Proof. As $P_n(t) = (-1)^n P_n(-t)$, using Lemma 2.3 we deduce

$$\sum_{k=0}^{[p/3]} \frac{(3k)!}{27^k \cdot k!^3}\left(\left(\frac{1-t}{2}\right)^k - (-1)^{[p/3]}\left(\frac{1+t}{2}\right)^k\right) \equiv 0 \pmod p.$$

Now putting $t = 1 - 2x$ in the congruence we obtain the result.

Corollary 2.2. *Let p be a prime greater than 3. Then*

$$\sum_{k=0}^{[p/3]} \frac{(3k)!}{27^k \cdot k!^3} \equiv \left(\frac{p}{3}\right) \pmod p.$$

Proof. Taking $x = 1$ in Theorem 2.4 and noting that $(-1)^{[p/3]} = (\frac{p}{3})$ we deduce the result.

Remark 2.2 By [8] or [10, p. 204] we have the following stronger supercongruence $\sum_{k=0}^{p-1} \frac{(3k)!}{27^k \cdot k!^3} \equiv (\frac{p}{3}) \pmod{p^2}$.

Lemma 2.4. *Let p be an odd prime and $k \in \{1, 2, \ldots, \frac{p-1}{2}\}$. Then*

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k}\left(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\right) \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned}\binom{\frac{p-1}{2}}{k} &= \frac{\frac{p-1}{2}(\frac{p-1}{2}-1)\cdots(\frac{p-1}{2}-k+1)}{k!} = \frac{(p-1)(p-3)\cdots(p-(2k-1))}{2^k \cdot k!} \\ &\equiv \frac{(-1)(-3)\cdots(-(2k-1))}{2^k \cdot k!}\left(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\right) \\ &= \frac{(-1)^k \cdot (2k)!}{(2^k \cdot k!)^2}\left(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\right) \pmod{p^2}.\end{aligned}$$

This yields the result.

Theorem 2.5. *Let p be a prime greater than 5. Then*

$$\sum_{k=0}^{[p/3]} \frac{(3k)!}{54^k \cdot k!^3} \equiv \begin{cases} 0 \pmod p & \text{if } 6 \mid p-5, \\ 2A \pmod p & \text{if } 6 \mid p-1 \text{ and } p = A^2 + 3B^2 \text{ with } 3 \mid A-1 \end{cases}$$

and

$$\sum_{k=0}^{[p/3]}\frac{k\cdot(3k)!}{54^k\cdot k!^3}\equiv\begin{cases}0\pmod p & \text{if } 6\mid p-1,\\ \frac13(-1)^{\frac{p+1}{6}}2^{-\frac{p+1}{3}}\binom{(p+1)/3}{(p+1)/6}\pmod p & \text{if } 6\mid p-5.\end{cases}$$

Proof. Taking $t=0$ in Lemma 2.3 and applying (2.5) and Lemma 2.4 we deduce that

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{54^k\cdot k!^3}\equiv\begin{cases}0\pmod p & \text{if } p\equiv 5\pmod 6,\\ \frac{(-1)^{(p-1)/6}}{2^{(p-1)/3}}\binom{(p-1)/3}{(p-1)/6}\equiv\binom{(p-1)/2}{(p-1)/6}\pmod p & \text{if } p\equiv 1\pmod 6.\end{cases}$$

Now suppose $p\equiv 1\pmod 6$ and so $p=A^2+3B^2$ with $A,B\in\mathbb{Z}$ and $A\equiv 1\pmod 3$. By [2, Theorem 9.4.4] we have $\binom{(p-1)/2}{(p-1)/6}\equiv 2A\pmod p$. Thus the first part follows.

By Lemma 2.3 we have

$$\frac{d}{dt}P_{[\frac p3]}(t)\equiv-\sum_{k=0}^{[p/3]}\frac{(3k)!}{54^k\cdot k!^3}\cdot k(1-t)^{k-1}\pmod p.$$

Thus, $\frac{d}{dt}P_{[\frac p3]}(t)\big|_{t=0}\equiv-\sum_{k=0}^{[p/3]}\frac{k\cdot(3k)!}{54^k\cdot k!^3}\pmod p$. From (1.3) we know that

$$\frac{d}{dt}P_{[\frac p3]}(t)\Big|_{t=0}=\begin{cases}0 & \text{if } p\equiv 1\pmod 6,\\ 2^{-\frac{p-2}{3}}\cdot(-1)^{\frac{p-5}{6}}\frac{\frac{p+1}{3}!}{\frac{p-5}{6}!\,\frac{p+1}{6}!} & \text{if } p\equiv 5\pmod 6.\end{cases}$$

Thus the second part is true.

Lemma 2.5. *Let p be an odd prime and $k\in\{1,2,\ldots,\frac{p-1}{2}\}$. Then*

$$\frac{(-1)^k\binom{(p-1)/2+k}{k}}{\binom{(p-1)/2}{k}}\equiv 1+2p\sum_{i=1}^k\frac{1}{2i-1}\equiv 3-2(-4)^k\frac{\binom{(p-1)/2}{k}}{\binom{2k}{k}}\pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned}\frac{(-1)^k\binom{(p-1)/2+k}{k}}{\binom{(p-1)/2}{k}}&=\frac{(\frac{p-1}{2}+k)(\frac{p-1}{2}+k-1)\cdots(\frac{p-1}{2}+1)}{(-1)^k\frac{p-1}{2}(\frac{p-1}{2}-1)\cdots(\frac{p-1}{2}-k+1)}\\&=\frac{(p+2k-1)(p+2k-3)\cdots(p+1)}{(-1)^k(p-1)(p-3)\cdots(p-(2k-1))}\\&\equiv\frac{1\cdot3\cdots(2k-1)(1+p\sum_{i=1}^k\frac{1}{2i-1})}{1\cdot3\cdots(2k-1)(1-p\sum_{i=1}^k\frac{1}{2i-1})}\\&\equiv\Big(1+p\sum_{i=1}^k\frac{1}{2i-1}\Big)^2\equiv 1+2p\sum_{i=1}^k\frac{1}{2i-1}\pmod{p^2}.\end{aligned}$$

This together with Lemma 2.4 yields the result.

Theorem 2.6. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv -1 \pmod p$. Then*

$$P_{\frac{p-1}{2}}(x) \equiv \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3-x}{1+x}\Big) \pmod p.$$

Proof. It is known that (see [4, (3.134)])

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \Big(\frac{x+1}{2}\Big)^{n-k} \Big(\frac{x-1}{2}\Big)^k.$$

Thus, using Lemma 2.5 and (2.1) we see that

$$\begin{aligned} P_{\frac{p-1}{2}}(x) &= \Big(\frac{x+1}{2}\Big)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \Big(\frac{x-1}{x+1}\Big)^k \\ &\equiv \Big(\frac{2(x+1)}{p}\Big) \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2}+k}{k} (-1)^k \Big(\frac{x-1}{x+1}\Big)^k \\ &= \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(1 + 2\cdot\frac{1-x}{1+x}\Big) = \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3-x}{1+x}\Big) \pmod p. \end{aligned}$$

This proves the theorem.

Corollary 2.3. *Let p be a prime of the form $4k+3$. Then $p \mid P_{\frac{p-1}{2}}(3)$.*

Proof. By Theorem 2.6 and (2.5) we have $P_{\frac{p-1}{2}}(3) \equiv (\frac{2}{p}) P_{\frac{p-1}{2}}(0) = 0 \pmod p$.

Theorem 2.7. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \Big(x^k - \Big(\frac{x}{p}\Big) x^{-k}\Big) \equiv 0 \pmod p.$$

Proof. Clearly the result is true for $x \equiv 1 \pmod p$. Now assume $x \not\equiv 1 \pmod p$. As $P_n(t) = (-1)^n P_n(-t)$ (see [6]), using Theorem 2.6 we see that for $t \in \mathbb{Z}_p$ with $t \not\equiv \pm 1 \pmod p$,

$$(\frac{2(t+1)}{p}) P_{\frac{p-1}{2}}\Big(\frac{3-t}{1+t}\Big) \equiv (-1)^{\frac{p-1}{2}} \Big(\frac{2(-t+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3+t}{1-t}\Big) \pmod p.$$

Thus,

$$P_{\frac{p-1}{2}}\Big(\frac{3-t}{1+t}\Big) \equiv \Big(\frac{t^2-1}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3+t}{1-t}\Big) \pmod p.$$

Now applying (2.4) we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \Big(\frac{1-\frac{3-t}{1+t}}{2}\Big)^k \equiv \Big(\frac{t^2-1}{p}\Big) \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k} \Big(\frac{1-\frac{3+t}{1-t}}{2}\Big)^k \pmod p$$

and so

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\Big(\Big(\frac{t-1}{t+1}\Big)^k-\Big(\frac{(t-1)/(t+1)}{p}\Big)\Big(\frac{t+1}{t-1}\Big)^k\Big)\equiv 0 \pmod p.$$

Set $t=(1+x)/(1-x)$. Then $t\not\equiv \pm 1 \pmod p$ and $x=(t-1)/(t+1)$. Hence the result follows.

Let p be an odd prime, and $x\in\mathbb{Z}_p$ with $x\not\equiv 0,1 \pmod p$. By Theorems 2.1 and 2.7 we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(16x)^k}\equiv\Big(\frac{x(x-1)}{p}\Big)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(16(1-x))^k} \pmod p. \tag{2.7}$$

Theorem 2.8. *Let p be an odd prime, $x\in\mathbb{Z}_p$ and $x\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\Big(\frac{x}{4}\Big)^{2k}\equiv\Big(\frac{-x}{p}\Big)P_{\frac{p-1}{2}}\Big(\frac{x+x^{-1}}{2}\Big) \pmod p.$$

Proof. From [4, (3.138)] we have the following result due to Kelisky:

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}x^{2k}=2^{2n}x^nP_n\Big(\frac{x+x^{-1}}{2}\Big). \tag{2.8}$$

Taking $n=(p-1)/2$ in (2.8) we have

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{p-1-2k}{\frac{p-1}{2}-k}\binom{2k}{k}x^{2k}=2^{p-1}x^{\frac{p-1}{2}}P_{\frac{p-1}{2}}\Big(\frac{x+x^{-1}}{2}\Big)\equiv\Big(\frac{x}{p}\Big)P_{\frac{p-1}{2}}\Big(\frac{x+x^{-1}}{2}\Big) \pmod p.$$

To see the result, using Lemma 2.2 we note that for $0\le k\le \frac{p-1}{2}$,

$$\begin{aligned}\binom{p-1-2k}{\frac{p-1}{2}-k}&=\frac{(p-1-2k)(p-2-2k)\cdots(p-(\frac{p-1}{2}+k))}{(\frac{p-1}{2}-k)!}\\&\equiv(-1)^{\frac{p-1}{2}-k}\frac{(2k+1)(2k+2)\cdots(\frac{p-1}{2}+k)}{(\frac{p-1}{2}-k)!}\\&=(-1)^{\frac{p-1}{2}-k}\binom{\frac{p-1}{2}+k}{2k}\equiv(-1)^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{16^k} \pmod p.\end{aligned} \tag{2.9}$$

Theorem 2.9. *Let p be a prime of the form $4k+1$ and $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv P_{\frac{p-1}{2}}(3)\equiv(-1)^{\frac{p-1}{4}}\Big(2a-\frac{p}{2a}\Big) \pmod{p^2}.$$

Proof. By Theorem 2.1 and (2.4) we have $\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2 8^{-k}\equiv\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2(-16)^{-k}\equiv P_{\frac{p-1}{2}}(3)\pmod{p^2}$. From Theorem 2.6, (2.5) and Gauss' congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2a\pmod p$ (see [2]) we have

$$P_{\frac{p-1}{2}}(3)\equiv\Big(\frac{2}{p}\Big)P_{\frac{p-1}{2}}(0)=\Big(\frac{2}{p}\Big)\frac{(-1)^{(p-1)/4}}{2^{(p-1)/2}}\binom{(p-1)/2}{(p-1)/4}\equiv(-1)^{\frac{p-1}{4}}\cdot 2a\pmod p.$$

Write $P_{\frac{p-1}{2}}(3)=(-1)^{\frac{p-1}{4}}\cdot 2a+qp$. Then $P_{\frac{p-1}{2}}(3)^2\equiv 4a^2+(-1)^{\frac{p-1}{4}}\cdot 4aqp\pmod{p^2}$. By a result due to Van Hamme [15], we have $(\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k})^2\equiv 4a^2-2p\pmod{p^2}$. This together with (2.1) yields $P_{\frac{p-1}{2}}(3)^2\equiv 4a^2-2p\pmod{p^2}$. Hence $(-1)^{\frac{p-1}{4}}\cdot 4aq\equiv -2\pmod p$ and so $P_{\frac{p-1}{2}}(3)\equiv(-1)^{\frac{p-1}{4}}\cdot 2a-\frac{p}{2(-1)^{(p-1)/4}a}\pmod{p^2}$. Now combining all the above we obtain the result.

Remark 2.3 For a prime $p=4k+1=a^2+b^2$ with $a\equiv 1\pmod 4$, the congruence $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2 8^{-k}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2(-16)^{-k}\equiv(-1)^{\frac{p-1}{4}}\cdot 2a\pmod p$ was proved by Zhi-Wei Sun in [13], and he also conjectured $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2 8^{-k}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2(-16)^{-k}\equiv(-1)^{\frac{p-1}{4}}(2a-\frac{p}{2a})\pmod{p^2}$.

Theorem 2.10. *Let p be a prime of the form $4k+1$ and so $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $a\equiv 1\pmod 4$. Then*

$$\sum_{k=0}^{\frac{p-1}{4}}\frac{\binom{4k}{2k}^2}{16^{2k}}\equiv\frac{1}{2}+(-1)^{\frac{p-1}{4}}a-(-1)^{\frac{p-1}{4}}\frac{p}{4a}\pmod{p^2}.$$

Proof. Since $\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2 16^{-k}+\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2(-16)^{-k}=2\sum_{k=0}^{\frac{p-1}{4}}\binom{4k}{2k}^2 16^{-2k}$, by (1.1) and Theorem 2.9 we deduce the result.

For a prime $p>3$ and $A,B,C\in\mathbb{Z}_p$ let $\#E_p(y^2=x^3+Ax^2+Bx+C)$ be the number of points on the curve $E_p:\ y^2=x^3+Ax^2+Bx+C$ over the field $\mathbb{F}_p$ of p elements.

Lemma 2.6 ([5]). *Let $p>3$ be a prime and $\lambda\in\mathbb{Z}_p$ with $\lambda\not\equiv 0,1\pmod p$. Then*

$$p+1-\#E_p(y^2=x(x-1)(x-\lambda))\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}(-\lambda)^k\pmod p.$$

Theorem 2.11. *Let $p>3$ be a prime and $t\in\mathbb{Z}_p$. Then*

$$P_{\frac{p-1}{2}}(t)\equiv\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\Big(\frac{1-t}{32}\Big)^k\equiv-\Big(\frac{-6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(t^2+3)x+2t(t^2-9)}{p}\Big)\pmod p.$$

Proof. For $t \equiv \pm 1 \pmod p$ we have

$$\begin{aligned}
&\sum_{x=0}^{p-1}\Big(\frac{x^3-3(t^2+3)x+2t(t^2-9)}{p}\Big)\\
&=\sum_{x=0}^{p-1}\Big(\frac{x^3-12x\mp 16}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(2x)^3-12(2x)\mp 16}{p}\Big)=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x\mp 2}{p}\Big)\\
&=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{(x\pm 1)^2(x\mp 2)}{p}\Big)=\Big(\frac{2}{p}\Big)\Big(\sum_{x=0}^{p-1}\Big(\frac{x\mp 2}{p}\Big)-\Big(\frac{\mp 3}{p}\Big)\Big)=-\Big(\frac{\mp 6}{p}\Big).
\end{aligned}$$

Thus applying (2.4) and the fact $P_n(\pm 1)=(\pm 1)^n$ (see [6]) we deduce the result.

Now assume $t \not\equiv \pm 1 \pmod p$. For $A, B, C \in \mathbb{Z}_p$, it is easily seen that (see for example [12, pp. 221-222])

$$\#E_p(y^2=x^3+Ax^2+Bx+C)=p+1+\sum_{x=0}^{p-1}\Big(\frac{x^3+Ax^2+Bx+C}{p}\Big).$$

Taking $\lambda=(1-t)/2$ in Lemma 2.6 and applying the above and (2.1) we see that

$$\begin{aligned}
P_{\frac{p-1}{2}}(t)&=\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\Big(\frac{t-1}{2}\Big)^k\\
&\equiv(-1)^{\frac{p-1}{2}}\big(p+1-\#E_p(y^2=x(x-1)(x-(1-t)/2))\big)\\
&=(-1)^{\frac{p+1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x(x-1)(x-(1-t)/2)}{p}\Big) \pmod p.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{x=0}^{p-1}\Big(\frac{x(x-1)(x-(1-t)/2)}{p}\Big)\\
&=\sum_{x=0}^{p-1}\Big(\frac{\frac{x}{2}(\frac{x}{2}-1)(\frac{x}{2}-\frac{1-t}{2})}{p}\Big)=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x(x-2)(x+t-1)}{p}\Big)\\
&=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+(t-3)x^2-2(t-1)x}{p}\Big)\\
&=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{(x-\frac{t-3}{3})^3+(t-3)(x-\frac{t-3}{3})^2-2(t-1)(x-\frac{t-3}{3})}{p}\Big)\\
&=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{t^2+3}{3}x+\frac{2t^3-18t}{27}}{p}\Big)=\Big(\frac{2}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{(\frac{x}{3})^3-\frac{t^2+3}{3}\cdot\frac{x}{3}+\frac{2t^3-18t}{27}}{p}\Big)\\
&=\Big(\frac{6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(t^2+3)x+2t(t^2-9)}{p}\Big),
\end{aligned}$$

by the above and (2.4) we obtain the result. The proof is now complete.

Theorem 2.12. *Let $p>3$ be a prime. Then*

$$P_{\frac{p-1}{2}}(-31)\equiv\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{256^k}\equiv-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-723x-7378}{p}\Big)\pmod p,$$

$$P_{\frac{p-1}{2}}(33)$$
$$\equiv\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{2k}{k}^2\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-256)^k}\equiv(-1)^{\frac{p+1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3-91x+330}{p}\Big)\pmod p,$$

$$P_{\frac{p-1}{2}}(-15)\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{2^k}\equiv\Big(\frac{2}{p}\Big)\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{128^k}\equiv(-1)^{\frac{p+1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3-19x-30}{p}\Big)\pmod p,$$

$$P_{\frac{p-1}{2}}(9)\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-4)^k}\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-64)^k}\equiv(-1)^{\frac{p+1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3-7x+6}{p}\Big)\pmod p,$$

$$P_{\frac{p-1}{2}}(5)\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-8)^k}\equiv\Big(\frac{-2}{p}\Big)\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-32)^k}\equiv-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-21x+20}{p}\Big)\pmod p.$$

Proof. Taking $t=-31$ in Theorem 2.11 and applying Theorem 2.7 we see that

$$\begin{aligned}P_{\frac{p-1}{2}}(-31)&\equiv\sum_{k=0}^{\frac{p-1}{2}}\binom{2k}{k}^2\equiv\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{256^k}\equiv-\Big(\frac{-6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(31^2+3)x-62(31^2-9)}{p}\Big)\\&=-\Big(\frac{-6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{(2x)^3-4\cdot723\cdot2x-8\cdot7378}{p}\Big)\\&=-\Big(\frac{-3}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-723x-7378}{p}\Big)\pmod p.\end{aligned}$$

Taking $t=33$ in Theorem 2.11 and applying Theorem 2.7 we see that

$$\begin{aligned}&P_{\frac{p-1}{2}}(33)\\&\equiv\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{2k}{k}^2\equiv\Big(\frac{-16}{p}\Big)\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{(-256)^k}\equiv-\Big(\frac{-6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3276x+71280}{p}\Big)\\&=-\Big(\frac{-6}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{(6x)^3-36\cdot91\cdot6x+216\cdot330}{p}\Big)\\&=-\Big(\frac{-1}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-91x+330}{p}\Big)\pmod p.\end{aligned}$$

The remaining congruences can be proved similarly.

For $a,b,n\in\mathbb{N}$, if $n=ax^2+by^2$ for some $x,y\in\mathbb{Z}$, we say that $n=ax^2+by^2$. In 2003, Rodriguez-Villegas[11] posed many conjectures on supercongruences. In particular, he conjectured that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{(4k)!}{256^k k!^4}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and so } p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } 3\mid p-2.\end{cases}$$

Recently the author's twin brother Zhi-Wei Sun ([13,14]) made a lot of conjectures on supercongruences. In particular, he conjectured that for a prime $p\neq 2,7$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\sum_{k=0}^{p-1}\frac{(4k)!}{81^k k!^4}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and so } p=x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

Inspired by their work, we pose the following conjectures.

Conjecture 2.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(4k)!}{648^k k!^4}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4 \text{ and } p=x^2+y^2 \text{ with } 2\nmid x,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Conjecture 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(4k)!}{(-144)^k k!^4}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Conjecture 2.3. *Let $p\neq 2,3,7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(4k)!}{(-3969)^k k!^4}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Conjecture 2.4. *Let $p\neq 2,3,11$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{66^{3k}(3k)!k!^3}\equiv\begin{cases}(\frac{p}{33})(4x^2-2p) \pmod{p^2} & \text{if } 4\mid p-1 \text{ and } p=x^2+y^2 \text{ with } 2\nmid x,\\ 0 \pmod{p^2} & \text{if } 4\mid p-3.\end{cases}$$

Conjecture 2.5. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{20^{3k}(3k)!k!^3}\equiv\begin{cases}(\frac{-5}{p})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and } p=x^2+2y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Conjecture 2.6. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{54000^k(3k)!k!^3}\equiv\begin{cases}(\frac{p}{5})(4x^2-2p)\pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } p=x^2+3y^2,\\ 0\pmod{p^2} & \text{if } 3\mid p-2.\end{cases}$$

Conjecture 2.7. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{(-12288000)^k(3k)!k!^3}$$
$$\equiv\begin{cases}(\frac{10}{p})(L^2-2p)\pmod{p^2} & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=L^2+27M^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 2.8. *Let $p > 7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{(-15)^{3k}(3k)!k!^3}\equiv\begin{cases}(\frac{p}{15})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and so } p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

Conjecture 2.9. *Let $p \neq 2,3,5,7,17$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(6k)!}{255^{3k}(3k)!k!^3}\equiv\begin{cases}(\frac{p}{255})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and so } p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

Conjecture 2.10. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{1458^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 3 \text{ and so } p=x^2+3y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 2.11. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15} \text{ and so } p=x^2+15y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15} \text{ and so } p=3x^2+5y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 7,11,13,14\pmod{15}.\end{cases}$$

Conjecture 2.12. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 3\mid p-1,\ p=x^2+3y^2 \text{ and } 5\mid xy,\\ p-2x^2\pm 6xy\pmod{p^2} & \text{if } 3\mid p-1,\ p=x^2+3y^2,\ 5\nmid xy\\ & \quad\text{and } x\equiv\pm y,\pm 2y\pmod 5,\\ 0\pmod{p^2} & \text{if } 3\mid p-2.\end{cases}$$

Conjecture 2.13. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(3k)!}{54^k\cdot k!^3}\equiv\begin{cases}(\frac{x}{3})(2x-\frac{p}{2x})\pmod{p^2} & \text{if } 3\mid p-1 \text{ and so } p=x^2+3y^2,\\ 0\pmod{p^2} & \text{if } 3\mid p-2.\end{cases}$$

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