DUALITY FOR COCHAIN DG ALGEBRAS

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Abstract. This paper develops a duality theory for connected cochain DG algebras, with particular emphasis on the non-commutative aspects.

One of the main items is a dualizing DG module which induces a duality between the derived categories of DG left-modules and DG right-modules with finitely generated cohomology.

As an application, it is proved that if the canonical module $A/A^{\geq 1}$ has a semi-free resolution where the cohomological degree of the generators is bounded above, then the same is true for each DG module with finitely generated cohomology.

0. Introduction

This paper develops a duality theory for connected cochain DG algebras. Some of the ingredients are dualizing DG modules, section and completion functors, and local duality.

Particular emphasis is given to the non-commutative aspects of the theory. For instance, Theorem B below says that the dualizing DG module defined in the paper induces a duality between the derived categories of DG left-modules and DG right-modules with finitely generated cohomology.

As an application, it is proved that if the canonical module $A/A^{\geq 1}$ has a semi-free resolution where the cohomological degree of the generators is bounded above, then the same is true for each DG module with finitely generated cohomology.

Setup 0.1. Throughout the paper, A is a connected cochain DG algebra over a field k; that is, $A = A^{\geq 0}$ and $A^0 = k$.

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See [4, chp. 10] for an introduction to DG homological algebra.

Since A is connected, it has a canonical DG bimodule $A/A^{\geq 1}$ which is also denoted by k. The canonical module can be viewed as a DG left-A-module ${}_Ak$ or a DG right-A-module k_A, and there are inclusions of localizing subcategories $\langle {}_Ak \rangle \hookrightarrow \mathsf{D}(A)$ and $\langle k_A \rangle \hookrightarrow \mathsf{D}(A^{\mathrm{op}})$ where $\mathsf{D}(A)$ and $\mathsf{D}(A^{\mathrm{op}})$ are the derived categories of DG left-A-modules, respectively DG right-A-modules. Under technical assumptions spelled out in Setup 3.1, the inclusions have right-adjoint functors Γ and Γ^{op} which behave like derived local section functors, and the following is our first main result, see Theorem 3.3.

Theorem A. *There is a single DG A-bimodule F such that $\Gamma(-) = F \overset{\mathrm{L}}{\underset{A}{\otimes}} -$ and $\Gamma^{\mathrm{op}}(-) = - \overset{\mathrm{L}}{\underset{A}{\otimes}} F$.*

The DG algebra A may be very far from commutative, but the DG bimodule F behaves like a two-sided Čech complex for A which links DG left- and right-modules. This is seen most clearly by passing to the k-linear dual $D = \mathrm{Hom}_k(F, k)$ which will be called a dualizing DG module of A. The following is our second main result, see Theorem 3.4; like Theorem C below it will be proved under the additional assumption that $\mathrm{H}(A)$ is noetherian with a balanced dualizing complex.

Theorem B. *Let $\mathsf{D}^{\mathrm{f}}(A)$ and $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$ be the derived categories of DG modules with finitely generated cohomology over $\mathrm{H}(A)$. Then there are quasi-inverse contravariant equivalences*

$$\mathsf{D}^{\mathrm{f}}(A) \underset{\mathrm{RHom}_{A^{\mathrm{op}}}(-,D)}{\overset{\mathrm{RHom}_A(-,D)}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}}).$$

As an application of the theory, we prove the following in Theorem 4.7.

Theorem C. *If ${}_Ak$ has a semi-free resolution where the cohomological degree of the generators is bounded above, then the same is true for each DG left-A-module with finitely generated cohomology.*

Note that despite the bound on the degree, there may be infinitely many generators in each semi-free resolution of ${}_Ak$. For a simple example, view $A = k[T]/(T^2)$ as a DG algebra with T in cohomological degree 1 and $\partial = 0$. Then the minimal semi-free resolution of ${}_Ak$ has all generators in degree 0, but there are infinitely many of them. Hence each semi-free resolution of ${}_Ak$ has infinitely many generators and ${}_Ak$ is not compact in the derived category.

The following describes three types of connected cochain DG algebras where the results apply, see Section 5.

(i) $\mathrm{H}(A)$ is noetherian AS regular.
(ii) A is commutative in the DG sense and $\mathrm{H}(A)$ is noetherian.
(iii) $\dim_k \mathrm{H}(A) < \infty$.

Between them, (i) and (ii) cover many DG algebras which arise in practice, for instance as cochain DG algebras of topological spaces. Note that in case (ii), Theorem A is trivial but Theorems B and C are not. In this case, the categories $\mathsf{D}^{\mathrm{f}}(A)$ and $\mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$ are the same, but Theorem B says that this category has the non-trivial property of being self dual.

Section 1 summarises part of Dwyer and Greenlees's theory of section and completion functors from [6]. Section 2 considers the Greenlees spectral sequence [9] in a version given by Shamir [13], evaluates it in the present situation, and gives a technical consequence. Section 3 proves Theorems A and B, Section 4 proves Theorem C, and Section 5 provides some examples, not least by showing that the theorems apply to the algebras described in (i)–(iii) above.

It should be mentioned that Mao and Wu [12] have provided some technical tools which will be important in the proofs, and that Ext regularity (with the opposite sign) was studied in their paper under the name width. There is previous work on duality for DG algebras in [8] and the more general $\mathbb{S}$-algebras in [7].

Notation 0.2. Opposite rings and DG algebras are denoted by the superscript "op". Right (DG) modules are identified with left (DG) modules over the opposite. Subscripts are sometimes used to indicate left or right (DG) module structures.

If M is a DG module and ℓ an integer, then $M^{\geq \ell}$ is the hard truncation with $(M^{\geq \ell})^n = 0$ for $n < \ell$.

Let D denote the derived category of an abelian category or of left DG modules over a DG algebra. In the case of the standing DG algebra A, let $\mathsf{D}^{\mathrm{f}}(A)$ be the full subcategory of objects $M \in \mathsf{D}(A)$ for which $\mathrm{H}(M)$ is finitely generated over $\mathrm{H}(A)$.

If M is an object of a derived category D, then $\langle M \rangle$ denotes the localizing subcategory generated by M.

If $\mathscr{N}$ is a subcategory, then

$$\mathscr{N}^{\perp} = \{ M \mid \operatorname{Hom}(N, M) = 0 \text{ for } N \in \mathscr{N} \},$$
$${}^{\perp}\mathscr{N} = \{ M \mid \operatorname{Hom}(M, N) = 0 \text{ for } N \in \mathscr{N} \}.$$

The notation $(-)^*$ stands for $\operatorname{Hom}_A(-, A)$ or $\operatorname{Hom}_{A^{\mathrm{op}}}(-, A)$ and we write $(-)^{\vee} = \operatorname{Hom}_k(-, k)$. These functors interchange DG left- and right-A-modules.

For the theory of (balanced) dualizing complexes over connected graded algebras see [15].

If M is a complex or a DG module, then we write

$$\inf M = \inf\{ \ell \mid \mathrm{H}^{\ell}(M) \neq 0 \}, \quad \sup M = \sup\{ \ell \mid \mathrm{H}^{\ell}(M) \neq 0 \}.$$

These are integers or $\pm\infty$. Note that inf and sup of the empty set are ∞ and $-\infty$, respectively, so $\inf(0) = \infty$ and $\sup(0) = -\infty$.

1. Dwyer-Greenlees theory

Setup 1.1. In this section and the next, we assume that K is a K-projective DG left-A-module which satisfies the following conditions as an object of $\mathsf{D}(A)$: It is compact and there is an equality of localizing subcategories $\langle K \rangle = \langle {}_A k \rangle$.

Remark 1.2 (Dwyer-Greenlees theory)**.** The DG module K can be used as an input for Dwyer and Greenlees's theory from [6]. Technically speaking, they only considered the case of K being a complex over a ring, but everything goes through for a DG module over a DG algebra. Let us give a brief recap of some of their results.

Consider

$$\mathscr{E} = \operatorname{Hom}_A(K, K)$$

which is a DG algebra with multiplication given by composition. Then K acquires the structure ${}_{A,\mathscr{E}}K$ while $K^* = \operatorname{RHom}_A(K, A)$ has the structure $K^*_{A,\mathscr{E}}$. Define functors

$$T(-) = - \overset{\mathrm{L}}{\underset{\mathscr{E}}{\otimes}} K,$$
$$E(-) = \operatorname{RHom}_A(K, -) \simeq K^* \overset{\mathrm{L}}{\underset{A}{\otimes}} -,$$
$$C(-) = \operatorname{RHom}_{\mathscr{E}^{\mathrm{op}}}(K^*, -)$$

which form adjoint pairs (T, E) and (E, C) between $\mathsf{D}(\mathscr{E}^{\mathrm{op}})$ and $\mathsf{D}(A)$.

Set $\mathscr{N} = \langle {}_A k \rangle^{\perp} = \langle {}_A K \rangle^{\perp}$ in $\mathsf{D}(A)$; in terms of these null modules we define the torsion and the complete DG modules by

$$\mathsf{D}^{\mathrm{tors}}(A) = {}^{\perp}\mathscr{N}, \quad \mathsf{D}^{\mathrm{comp}}(A) = \mathscr{N}^{\perp}.$$

Note that

$$\mathsf{D}^{\mathrm{tors}}(A) = \langle {}_A k \rangle = \langle {}_A K \rangle.$$

There are pairs of quasi-inverse equivalences of categories as follows.

$$\mathsf{D}^{\mathrm{comp}}(A) \underset{C}{\overset{E}{\rightleftarrows}} \mathsf{D}(\mathscr{E}^{\mathrm{op}}) \underset{E}{\overset{T}{\rightleftarrows}} \mathsf{D}^{\mathrm{tors}}(A)$$

In particular, EC and ET are equivalent to the identity functor on $\mathsf{D}(\mathscr{E}^{\mathrm{op}})$, so if we set

$$\Gamma = TE, \quad \Lambda = CE$$

then we get endofunctors of $\mathsf{D}(A)$ which form an adjoint pair (Γ, Λ) and satisfy

$$\Gamma^2 \simeq \Gamma, \quad \Lambda^2 \simeq \Lambda, \quad \Gamma\Lambda \simeq \Gamma, \quad \Lambda\Gamma \simeq \Lambda.$$

These functors are adjoints of inclusions as follows, where left-adjoints are displayed above right-adjoints.

$$\mathsf{D}^{\mathrm{comp}}(A) \underset{\mathrm{inc}}{\overset{\Lambda}{\leftrightarrows}} \mathsf{D}(A) \underset{\Gamma}{\overset{\mathrm{inc}}{\leftrightarrows}} \mathsf{D}^{\mathrm{tors}}(A)$$

Note that the counit and unit, $\Gamma(-) \xrightarrow{\epsilon} (-)$ and $(-) \xrightarrow{\eta} \Lambda(-)$, are K-equivalences, that is, they become isomorphisms when the functor $\mathrm{Hom}_{\mathsf{D}(A)}(\Sigma^{\ell} K, -)$ is applied. Equivalently, their mapping cones are in $\langle {}_A k \rangle^{\perp}$. Along with $\Gamma(-) \in \mathsf{D}^{\mathrm{tors}}(A)$ and $\Lambda(-) \in \mathsf{D}^{\mathrm{comp}}(A)$, this characterizes them up to unique isomorphism.

It is useful to remark that in particular, for $M \in \mathsf{D}^{\mathrm{tors}}(A) = \langle {}_A k \rangle$, the counit morphism $\Gamma(M) \xrightarrow{\epsilon_M} M$ is an isomorphism, and for $M \in \mathsf{D}^{\mathrm{comp}}(A)$, the unit morphism $M \xrightarrow{\eta_M} \Lambda(M)$ is an isomorphism. For $M \in \langle {}_A k \rangle^{\perp}$, we get $\Gamma(M) = 0$.

Definition 1.3. We will write

$$F = K^* \overset{\mathrm{L}}{\underset{\mathscr{E}}{\otimes}} K, \quad D = F^{\vee}$$

and refer to D as a dualizing DG module of A.

In a more laborious notation we have $F = K^*_{A,\mathscr{E}} \overset{\mathrm{L}}{\underset{\mathscr{E}}{\otimes}} {}_{A,\mathscr{E}}K$, so F has the structure ${}_A F_A$ and D the structure ${}_A D_A$. It is easy to check that

$$\Gamma(-) = F \overset{\mathrm{L}}{\underset{A}{\otimes}} -, \quad \Lambda(-) = \mathrm{RHom}_A(F, -)$$

and adjointness yields

$$\Gamma(-)^{\vee} = \mathrm{RHom}_A(-, D). \tag{1}$$

The DG module F plays the role of the Čech complex and Γ and Λ behave like derived local section and completion functors. Equation (1) is the local duality formula.

Remark 1.4. Since Γ and Λ are given as derived $\otimes$ and Hom with the DG A-bimodule F, they can be applied to DG A-bimodules and this will give new DG A-bimodules.

Specifically, $\Gamma({}_AM_A) = {}_AF_A \overset{\mathrm{L}}{\underset{A}{\otimes}} {}_AM_A$ has a left-structure which comes from the left-structure of F and a right-structure which comes from the right-structure of M. And $\Lambda({}_AM_A) = \mathrm{RHom}_A({}_AF_A, {}_AM_A)$ has a left-structure which comes from the right-structure of F and a right-structure which comes from the right-structure of M.

It is easy to check that when the functors are applied to DG A-bimodules, the counit and unit, $\Gamma(-) \xrightarrow{\epsilon} (-)$ and $(-) \xrightarrow{\eta} \Lambda(-)$, can be viewed as morphisms in $\mathsf{D}(A^e)$, the derived category of DG A-bimodules.

2. The Greenlees spectral sequence in a version given by Shamir

Remark 2.1. In this remark, assume that $\mathrm{H}(A)$ is noetherian.

The Greenlees spectral sequence was originally given for group cohomology in [9, thm. 2.1]. It was developed further by Benson, Dwyer, Greenlees, Iyengar, and Shamir in [3], [7], and [13]. The most general version is given by Shamir in [13]; we will apply it to the situation at hand.

The cohomology $\mathrm{H}(A)$ is a connected graded k-algebra with graded maximal ideal $\mathfrak{m} = \mathrm{H}^{\geq 1}(A)$. Let $\mathscr{T}$ denote the $\mathfrak{m}$-torsion graded left-$\mathrm{H}(A)$-modules, that is, the graded modules such that each element t has $\mathfrak{m}^{\ell}t = 0$ for $\ell \gg 0$. Then $\mathscr{T}$ is a hereditary torsion class in the abelian category $\mathsf{Gr}\,\mathrm{H}(A)$ of graded left-$\mathrm{H}(A)$-modules, in the sense of [13, def. 3.1]. For $X \in \mathsf{Gr}\,\mathrm{H}(A)$, view X as an object of the derived category $\mathsf{D}(\mathsf{Gr}\,\mathrm{H}(A))$ and, using the notation of [13, def. 2.1], consider a morphism $\mathrm{Cell}^{\mathrm{H}(A)}_{\mathscr{T}}(X) \xrightarrow{\eta} X$ in $\mathsf{D}(\mathsf{Gr}\,\mathrm{H}(A))$ characterized by the properties that $\mathrm{Cell}^{\mathrm{H}(A)}_{\mathscr{T}}(X) \in \langle \mathscr{T} \rangle$ and $\mathrm{Hom}_{\mathsf{D}(\mathsf{Gr}\,\mathrm{H}(A))}(\Sigma^{\ell}T, \eta)$ is an isomorphism for each integer ℓ and $T \in \mathscr{T}$. These properties determine

η up to unique isomorphism, and using that $\mathrm{H}(A)$ is noetherian, it is not hard to show that η can be obtained as the canonical morphism $\mathrm{R}\Gamma_{\mathfrak{m}}X \to X$ where $\mathrm{R}\Gamma_{\mathfrak{m}}$ is the functor on the derived category which underlies local cohomology; see [15, sec. 4].

Consider the class

$$\mathscr{C} = \{\, C \in \mathsf{D}(A) \mid \mathrm{H}(C) \in \mathscr{T} \,\}.$$

Shamir refers to objects of $\mathscr{C}$ as $\mathscr{T}$-cellular and objects of $\mathscr{C}^{\perp}$ as $\mathscr{T}$-null; see for instance [13, p. 1 and defs. 2.1 and 2.3]. For each DG left-A-module M, Shamir obtains in [13, lem. 5.4] a distinguished triangle $C \to M \to N$ in $\mathsf{D}(A)$ with $N \in \mathscr{C}^{\perp}$ and a spectral sequence

$$E^2_{p,q} = \mathrm{H}_{p,q}(\mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{H}M)) \Rightarrow \mathrm{H}_{p+q}(C).$$

On the left hand side, p comes from the numbering of the modules in an exact couple and q is an internal degree; see [13, proof of lem. 5.4]. A consequence is that p is homological degree along the complex $\mathrm{R}\Gamma_{\mathfrak{m}}$ and q is graded degree along the graded module $\mathrm{H}M$. The sequence can hence also be written

$$E^2_{p,q} = \mathrm{H}^{-p}_{\mathfrak{m}}(\mathrm{H}M)_q \Rightarrow \mathrm{H}_{p+q}(C) \tag{2}$$

where $\mathrm{H}^{\ell}_{\mathfrak{m}} = \mathrm{H}^{\ell} \circ \mathrm{R}\Gamma_{\mathfrak{m}}$ is local cohomology; see [15, sec. 4].

The spectral sequence is conditionally convergent to the colimit; compare [13, last part of proof of lem. 5.4] with [5, def. 5.10]. In fact, the p in [13] corresponds to the s in [5, eq. (0.1)], except that they have opposite signs. Now note that $E^2_{p,*} = 0$ for $p > 0$ by construction, so the spectral sequence is a half-plane spectral sequence in the sense of [5, sec. 7]. By [5, thm. 7.1], to get strong convergence, all we need is to check $\mathrm{R}E_{\infty} = 0$ in the notation of [5].

In particular, using [5, rmk. after thm. 7.1], the spectral sequence (2) is strongly convergent if

$$\dim_k E^2_{p,q} = \dim_k \mathrm{H}^{-p}_{\mathfrak{m}}(\mathrm{H}M)_q < \infty \tag{3}$$

for all p, q.

Lemma 2.2. *If $M \in \mathsf{D}(A)$ has $\mathrm{H}^{\ell}(M) = 0$ for $\ell \gg 0$ then $M \in \langle {}_A k \rangle$.*

Proof. Using [12, sec. 1.5] to truncate, we can suppose $M^{\ell} = 0$ for $\ell \gg 0$ and desuspending if necessary, we can suppose $M^{\ell} = 0$ for $\ell > 0$ so $M^{\geq 1} = 0$. There is a direct system

$$M^{\geq 0} \to M^{\geq -1} \to M^{\geq -2} \to \cdots$$

in $\mathsf{D}(A)$ with homotopy colimit M, so it is enough to see $M^{\geq n} \in \langle {}_A k \rangle$ for each n. However, there are distinguished triangles

$$M^{\geq n+1} \to M^{\geq n} \to N(n)$$

where $\mathrm{H}(N(n))$ is concentrated in cohomological degree n, and it is easy to check that hence $N(n) \cong \coprod \Sigma^{-n} k$ so $N(n) \in \langle {}_A k \rangle$. Induction starting with $M^{\geq 1} = 0$ gives $M^{\geq n} \in \langle {}_A k \rangle$ for each n as desired. □

The following proof uses the methods of [13, sec. 6].

Theorem 2.3. *Assume (in addition to Setup 1.1) that* $\mathrm{H}(A)$ *is noetherian with a balanced dualizing complex. For* $M \in \mathsf{D}^{\mathrm{f}}(A)$ *there is a spectral sequence which is strongly convergent in the sense of* [5, def. 5.2]*,*

$$E^2_{p,q} = \mathrm{H}^{-p}_{\mathfrak{m}}(\mathrm{H}M)_q \Rightarrow \mathrm{H}_{p+q}(\Gamma M).$$

Proof. The condition in Equation (3) holds because $\mathrm{H}(A)$ is noetherian with a balanced dualizing complex and $\mathrm{H}(M)$ is finitely generated; combine [14, thms. 5.1 and 6.3]. Hence the spectral sequence (2) is strongly convergent by the observation at the end of Remark 2.1.

To complete the proof, we must see $C \cong \Gamma M$ where C is the object in (2). There is a distinguished triangle $C \to M \to N$, so by [13, prop. 2.4] it is enough to see $C \in \langle {}_A k \rangle$ and $N \in \langle {}_A k \rangle^{\perp}$.

For the latter, it suffices to see $\mathrm{Hom}_A(\Sigma^{\ell} k, N) = 0$ for each ℓ, and this is clear because $N \in \mathscr{C}^{\perp}$ in the notation from Remark 2.1.

For the former, note that $\mathrm{H}(A)$ is noetherian with a balanced dualizing complex and $\mathrm{H}(M)$ is finitely generated. Hence $\mathrm{H}^{>n}_{\mathfrak{m}}(\mathrm{H}M) = 0$ for some n, and for each p the graded module $\mathrm{H}^{p}_{\mathfrak{m}}(\mathrm{H}M)$ is zero in sufficiently high degree; this is by [14, thms. 5.1 and 6.3] again. The degree in question stems from the cohomological grading of $\mathrm{H}(M)$ so we learn $\mathrm{H}^{p}_{\mathfrak{m}}(\mathrm{H}M)_q = 0$ for $q \ll 0$ since q figures as a subscript, hence with a sign change. So $E^2_{p,q}$ is concentrated in a vertical strip which is bounded below, and strong convergence to $\mathrm{H}_{p+q}(C)$ implies $\mathrm{H}_{\ell}(C) = 0$ for $\ell \ll 0$. But then $C \in \langle {}_A k \rangle$ by Lemma 2.2. □

Remark 2.4. Note that it is easy to show that $C \in \langle {}_A k \rangle$ implies $\mathrm{H}(C) \in \mathscr{T}$. If we also had the opposite implication, then we could conclude that $\Gamma(M)$ was $\mathrm{Cell}^{A}_{\mathscr{T}}(M)$ in the notation of [13], and obtain Theorem 2.3 as a special case of [13, thm. 1].

Corollary 2.5. *Assume that* $\mathrm{H}(A)$ *is noetherian with a balanced dualizing complex. Then* $M \in \mathsf{D}^{\mathrm{f}}(A)$ *implies* $(\Gamma M)^{\vee} \in \mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}})$.

Proof. We must show that if $\mathrm{H}(M)$ is finitely generated over $\mathrm{H}(A)$ then $\mathrm{H}(\Gamma M)^{\vee}$ is finitely generated over $\mathrm{H}(A)^{\mathrm{op}}$.

As indicated in Remark 2.1, in the spectral sequence, q is internal degree. The same hence applies to the spectral sequence of Theorem 2.3. But by [14, thms. 5.1 and 6.3], the graded modules $\mathrm{H}_{\mathfrak{m}}^{-p}(\mathrm{H}M)$ are the k-linear duals of finitely generated $\mathrm{H}(A)^{\mathrm{op}}$-modules, and only finitely many of them are non-zero.

So the terms $E^2_{p,*}$ are the k-linear duals of finitely many finitely generated $\mathrm{H}(A)^{\mathrm{op}}$-modules whence the same is true for the terms $E^{\infty}_{p,*}$. Hence $\mathrm{H}(\Gamma M)$ has a filtration where the quotients are the k-linear duals of finitely many finitely generated $\mathrm{H}(A)^{\mathrm{op}}$-modules. This proves the result. □

3. Properties of the Čech and dualizing DG modules

Setup 3.1. In this section and the next, we assume that ${}_AK$ and L_A are K-projective DG A-modules which satisfy the following conditions as objects of $\mathsf{D}(A)$ and $\mathsf{D}(A^{\mathrm{op}})$:

- ${}_AK$ is compact, $\langle {}_AK \rangle = \langle {}_Ak \rangle$, and $K^*_A \in \langle k_A \rangle$,
- L_A is compact, $\langle L_A \rangle = \langle k_A \rangle$, and ${}_AL^* \in \langle {}_Ak \rangle$.

Remark 3.2. The DG module K can be used as input for the theory of the previous sections. In particular, Section 1 used K to define various objects which will be important: $\mathscr{E}$, F, D, Γ, Λ.

Similarly, L can be used as input for the theory applied to A^{op}, that is, to DG right-A-modules. In this case we get the endomorphism DG algebra

$$\mathscr{F} = \mathrm{Hom}_{A^{\mathrm{op}}}(L, L),$$

the DG module L acquires the structure ${}_{\mathscr{F}}L_A$, and we can define the DG A-bimodules

$$G = L^* \overset{\mathrm{L}}{\underset{\mathscr{F}}{\otimes}} L, \quad E = G^{\vee}$$

along with the functors

$$\Gamma^{\mathrm{op}}(-) = - \overset{\mathrm{L}}{\underset{A}{\otimes}} G, \quad \Lambda^{\mathrm{op}}(-) = \mathrm{RHom}_{A^{\mathrm{op}}}(G, -).$$

Then $\mathscr{F}$, G, E, Γ^{op}, Λ^{op} are the right handed versions of $\mathscr{E}$, F, D, Γ, Λ.

The following is Theorems A and B of the introduction.

Theorem 3.3. *We have $F \cong G$ and $D \cong E$ in the derived category $\mathsf{D}(A^e)$ of DG A-bimodules.*

Proof. We know that ${}_{\mathscr{F}}L$ is built from ${}_{\mathscr{F}}\mathscr{F}$ using (de)suspensions, distinguished triangles, coproducts, and direct summands. The functor ${}_{A}L^{*}_{\mathscr{F}} \overset{\mathrm{L}}{\underset{\mathscr{F}}{\otimes}} -$ preserves these operations and $\langle {}_{A}k \rangle$ is closed under them, so ${}_{A}L^{*} \in \langle {}_{A}k \rangle$ implies ${}_{A}G = {}_{A}L^{*}_{\mathscr{F}} \overset{\mathrm{L}}{\underset{\mathscr{F}}{\otimes}} {}_{\mathscr{F}}L \in \langle {}_{A}k \rangle$. By symmetry, $F_A \in \langle k_A \rangle$. However, we have

$$ {}_{A}G_{A} \xleftarrow{\epsilon_G} \Gamma({}_{A}G_{A}) = {}_{A}F_{A} \overset{\mathrm{L}}{\underset{A}{\otimes}} {}_{A}G_{A} = \Gamma^{\mathrm{op}}({}_{A}F_{A}) \xrightarrow{\epsilon_F^{\mathrm{op}}} {}_{A}F_{A} $$

where the counit morphisms ϵ_G and ϵ_F^{op} are morphisms in $\mathsf{D}(A^e)$ as explained in Remark 1.4. Now, ${}_{A}G$ is in $\langle {}_{A}k \rangle$ so by the last paragraph of Remark 1.2, if we forget the right-A-structures, then ϵ_G is an isomorphism in $\mathsf{D}(A)$. This just means that its cohomology is bijective whence ϵ_G itself is an isomorphism in $\mathsf{D}(A^e)$. By symmetry, ϵ_F^{op} is an isomorphism in $\mathsf{D}(A^e)$ and the proposition follows. $\square$

Theorem 3.4. *Assume (in addition to Setup 3.1) that* $\mathrm{H}(A)$ *is noetherian with a balanced dualizing complex. Then there are quasi-inverse contravariant equivalences*

$$ \mathsf{D}^{\mathrm{f}}(A) \underset{\mathrm{RHom}_{A^{\mathrm{op}}}(-,D)}{\overset{\mathrm{RHom}_{A}(-,D)}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}(A^{\mathrm{op}}). $$

Proof. Definition 1.3, Remark 3.2, and Theorem 3.3 show that the two functors in the theorem are $\Gamma(-)^{\vee}$ and $\Gamma^{\mathrm{op}}(-)^{\vee}$. They take values in the correct categories by Corollary 2.5 and its analogue for Γ^{op}.

To see that the functors are quasi-inverse equivalences, first observe that by adjointness,

$$ \Gamma(-)^{\vee} = (F \overset{\mathrm{L}}{\underset{A}{\otimes}} -)^{\vee} \simeq \mathrm{RHom}_{A^{\mathrm{op}}}(F, (-)^{\vee}) = \Lambda^{\mathrm{op}}((-)^{\vee}). $$

This gives the first of the following natural isomorphisms for $M \in \mathsf{D}^{\mathrm{f}}(A)$.

$$ \Gamma^{\mathrm{op}}(\Gamma(M)^{\vee})^{\vee} \cong \Gamma^{\mathrm{op}}(\Lambda^{\mathrm{op}}(M^{\vee}))^{\vee} \overset{\text{(a)}}{\cong} \Gamma^{\mathrm{op}}(M^{\vee})^{\vee} \overset{\text{(b)}}{\cong} M^{\vee\vee} \cong M. $$

Here (a) is because $\Gamma^{\mathrm{op}}\Lambda^{\mathrm{op}} \simeq \Gamma^{\mathrm{op}}$ by Remark 1.2, and (b) is because when $\mathrm{H}(M)$ is finitely generated, it has $\mathrm{H}^{\ell}(M) = 0$ for $\ell \ll 0$ whence

$\mathrm{H}^{\ell}(M^{\vee}) = 0$ for $\ell \gg 0$; hence $M^{\vee} \in \langle k_A \rangle$ by the right-module version of Lemma 2.2 and so $\Gamma^{\mathrm{op}}(M^{\vee}) \cong M^{\vee}$.

The reverse composition of functors is handled by symmetry. □

4. An application to Ext regularity

Definition 4.1. For $M \in \mathsf{D}(A)$ we define the Ext and Castelnuovo-Mumford regularities by

$$\mathrm{Ext.reg}\, M = -\inf \mathrm{RHom}_A(M,k), \quad \mathrm{CMreg}\, M = \sup \Gamma M,$$

and similarly for $M \in \mathsf{D}(A^{\mathrm{op}})$.

Note that $\mathrm{Ext.reg}(0) = \mathrm{CMreg}(0) = -\infty$; see the last part of Notation 0.2.

Remark 4.2. If $M \in \mathsf{D}(A)$ has $\mathrm{H}^{\ell}(M) = 0$ for $\ell \ll 0$, then it follows from [12, prop. 2.4] that M has a minimal semi-free resolution P with generators between cohomological degrees $\inf M$ and $\mathrm{Ext.reg}\, M$. That is, if we write $i = \inf M$ and $r = \mathrm{Ext.reg}\, M$, then

$$P^{\natural} = \coprod_{-r \leq \ell \leq -i} \Sigma^{\ell} (A^{\natural})^{(\beta_\ell)} \tag{4}$$

where $\natural$ sends DG modules to graded modules by forgetting the differential and (β_ℓ) indicates a coproduct. If, additionally, $\mathrm{H}(M) \neq 0$, then $\inf M$ is a finite number and P has at least one generator. Hence

$$\mathrm{H}^{\ell}(M) = 0 \text{ for } \ell \ll 0 \text{ and } \mathrm{H}(M) \neq 0 \\ \Rightarrow\ -\infty < \inf M \leq \mathrm{Ext.reg}\, M. \tag{5}$$

If $\mathrm{H}(A)$ is noetherian with a balanced dualizing complex, then Equation (1) in Definition 1.3 along with Theorem 3.4 give

$$M \in \mathsf{D}^{\mathrm{f}}(A) \text{ and } \mathrm{H}(M) \neq 0 \ \Rightarrow\ -\infty < \mathrm{CMreg}\, M < \infty. \tag{6}$$

By considering $k_A \overset{\mathrm{L}}{\underset{A}{\otimes}} {}_A k$, one proves $\mathrm{Ext.reg}(k_A) = \mathrm{Ext.reg}({}_A k)$, and this common number will be denoted by $\mathrm{Ext.reg}\, k$. Equation (5) implies

$$0 \leq \mathrm{Ext.reg}\, k.$$

By using Theorem 3.3 we get $\Gamma^{\mathrm{op}}(A) = A \overset{\mathrm{L}}{\underset{A}{\otimes}} F \cong F_A$ and $\Gamma(A) = F \overset{\mathrm{L}}{\underset{A}{\otimes}} A \cong {}_A F$, so $\mathrm{CMreg}(A_A) = \mathrm{CMreg}({}_A A)$, and this common number will be denoted by $\mathrm{CMreg}\, A$.

Definition 4.3 (He and Wu [10, def. 2.1])**.** A DG A-module M is Koszul if it has a semi-free resolution P all of whose basis elements are in degree 0.

The DG algebra A is Koszul if ${}_Ak$ is a Koszul DG module.

Remark 4.4. If M is a DG A-module with $\mathrm{H}^\ell(M) = 0$ for $\ell \ll 0$, then it is immediate from Remark 4.2 that it is Koszul precisely if $\mathrm{H}(M) = 0$ or $\inf M = \operatorname{Ext.reg} M = 0$.

Consequently, the DG algebra A is Koszul precisely if $\operatorname{Ext.reg} k = 0$.

Lemma 4.5. *Suppose that $M \in \mathsf{D}(A)$ has $\mathrm{H}^\ell(M) = 0$ for $\ell \ll 0$ and $\dim_k \mathrm{H}^\ell(M) < \infty$ for each ℓ. Then $\Lambda(M) \cong M$.*

Proof. The assumptions on M imply $M \cong \mathrm{RHom}_k(M^\vee, k)$. This gives the first of the following isomorphisms, and the second one is by adjointness.

$$\begin{aligned}\Lambda(M) &= \mathrm{RHom}_A(F, M)\\ &\cong \mathrm{RHom}_A(F, \mathrm{RHom}_k(M^\vee, k))\\ &\cong \mathrm{RHom}_k(M^\vee \overset{\mathrm{L}}{\underset{A}{\otimes}} F, k)\\ &= (M^\vee \overset{\mathrm{L}}{\underset{A}{\otimes}} F)^\vee\\ &= \Gamma^{\mathrm{op}}(M^\vee)^\vee\\ &\overset{(a)}{\cong} (M^\vee)^\vee\\ &\cong M\end{aligned}$$

Here (a) follows from the right-module version of Lemma 2.2 and the final paragraph of Remark 1.2.

Note that the proof uses the two-sided theory of Section 3: It is necessary to know that Λ and Γ^{op} are given by formulae involving the *same* DG bimodule F. ☐

The following is a DG version of [11, thms. 2.5 and 2.6].

Proposition 4.6. *Let $M \in \mathsf{D}(A)$ have $\mathrm{H}^\ell(M) = 0$ for $\ell \ll 0$ and $\mathrm{H}(M) \neq 0$. Then*

(i) $\operatorname{CMreg} M \neq -\infty$.
(ii) $\operatorname{Ext.reg} M \leq \operatorname{CMreg} M + \operatorname{Ext.reg} k$.
(iii) $\operatorname{CMreg} M \leq \operatorname{Ext.reg} M + \operatorname{CMreg} A$.

Proof. (i) Observe that $\Lambda k \cong k$ by Lemma 4.5, so

$$\begin{aligned}\mathrm{RHom}_A(M,k) &\cong \mathrm{RHom}_A(M,\Lambda k)\\ &= \mathrm{RHom}_A(M,\mathrm{RHom}_A(F,k))\\ &\overset{(a)}{\cong} \mathrm{RHom}_A(F \overset{\mathrm{L}}{\underset{A}{\otimes}} M,k)\\ &= \mathrm{RHom}_A(\Gamma M,k) \qquad (7)\end{aligned}$$

where (a) is by adjointness. Hence

$$\mathrm{Ext.reg}\, M = -\inf \mathrm{RHom}_A(\Gamma M,k). \qquad (8)$$

Now, $\mathrm{CMreg}\, M = \sup \Gamma M = -\infty$ would mean $\Gamma M = 0$. By Equation (8) this would imply $\mathrm{Ext.reg}\, M = -\infty$, but this is false by Equation (5) in Remark 4.2.

(ii) By part (i) and Equation (5) we have $\mathrm{CMreg}\, M$ and $\mathrm{Ext.reg}\, k$ different from $-\infty$. Hence, despite the potential for either regularity to be ∞, the right hand side of the inequality in the proposition makes sense because it does not read $\infty - \infty$.

Set $X = \Gamma M$ and let P be a minimal semi-free resolution of k_A. Then

$$\mathrm{Ext.reg}\, M \overset{(b)}{=} -\inf \mathrm{RHom}_A(X,k) \overset{(c)}{=} -\inf \mathrm{Hom}_A(X,P^\vee) = (*)$$

where (b) is Equation (8) and (c) is because $P^\vee$ is a K-injective resolution of ${}_A k$.

We have $\sup X = \mathrm{CMreg}\, M$ so by truncation we can suppose $X^\ell = 0$ for $\ell > \mathrm{CMreg}\, M$, cf. [12, 1.6], and so

$$(X^\vee)^j = 0 \text{ for } j < -\mathrm{CMreg}\, M.$$

Write $i = \inf k = 0$ and $r = \mathrm{Ext.reg}\, k$. Then P satisfies Equation (4) in Remark 4.2 and a computation shows

$$\mathrm{Hom}_A(X,P^\vee)^\natural \cong \prod_{-r \leq \ell} \Sigma^{-\ell}((X^\vee)^\natural)^{\beta_\ell}$$

where the power β_ℓ indicates a product. The last two equations imply

$$(*) \leq \mathrm{CMreg}\, M + r = \mathrm{CMreg}\, M + \mathrm{Ext.reg}\, k$$

as desired.

(iii) Note that the right hand side of the inequality makes sense again, for the same reason as in part (ii).

Let P be a minimal semi-free resolution of M. Then

$$\operatorname{CMreg} M = \sup \Gamma M = \sup F \overset{\mathrm{L}}{\underset{A}{\otimes}} M = \sup F \underset{A}{\otimes} P = (**).$$

As noted in Remark 4.2 we have $F_A \cong \Gamma^{\mathrm{op}}(A)$ and since $\operatorname{CMreg} A = \sup \Gamma^{\mathrm{op}}(A)$ we can suppose by truncation that

$$F^j = 0 \text{ for } j > \operatorname{CMreg} A.$$

Write $i = \inf M$ and $r = \operatorname{Ext.reg} M$. Then P satisfies Equation (4) in Remark 4.2 and a computation shows

$$(F \underset{A}{\otimes} P)^\natural \cong \coprod_{-r \leq n} \Sigma^n (F^\natural)^{(\beta_n)}.$$

The last two equations imply

$$(**) \leq r + \operatorname{CMreg} A = \operatorname{Ext.reg} M + \operatorname{CMreg} A$$

as desired. □

Part (i) of the following establishes Theorem C of the introduction while (ii) is a DG version of [11, cor. 2.9].

Theorem 4.7. *Assume (in addition to Setup 3.1) that* $\mathrm{H}(A)$ *is noetherian with a balanced dualizing complex. Let* $M \in \mathsf{D}^{\mathrm{f}}(A)$ *have* $\mathrm{H}(M) \neq 0$.

- (i) *If* $\operatorname{Ext.reg} k < \infty$ *then* $\operatorname{Ext.reg} M < \infty$.
- (ii) *If A is a Koszul DG algebra and* $\operatorname{CMreg} M \leq t$ *for an integer t, then* $\Sigma^t(M^{\geq t})$ *is a Koszul DG module.*

Proof. (i) follows by combining Equation (6) in Remark 4.2 with Proposition 4.6(ii).

As for (ii), it holds trivially if $\mathrm{H}(\Sigma^t(M^{\geq t})) = 0$, so suppose that we have $\mathrm{H}(\Sigma^t(M^{\geq t})) \neq 0$.

There is a short exact sequence of DG modules $0 \to M^{\geq t} \to M \to M/M^{\geq t} \to 0$ which induces a distinguished triangle $\Sigma^{-1}(M/M^{\geq t}) \to M^{\geq t} \to M$ in $\mathsf{D}(A)$, and hence a distinguished triangle

$$\Gamma(\Sigma^{-1}(M/M^{\geq t})) \to \Gamma(M^{\geq t}) \to \Gamma M$$

in $\mathsf{D}(A)$.

Lemma 2.2 and the last paragraph of Remark 1.2 imply the isomorphism $\Gamma(\Sigma^{-1}(M/M^{\geq t})) \cong \Sigma^{-1}(M/M^{\geq t})$, so $\sup \Gamma(\Sigma^{-1}(M/M^{\geq t})) \leq t$.

By assumption, $\sup \Gamma M = \operatorname{CMreg} M \leq t$. So the distinguished triangle implies $\sup \Gamma(M^{\geq t}) \leq t$. Hence $\sup \Gamma(\Sigma^t(M^{\geq t})) \leq 0$, that is $\operatorname{CMreg} \Sigma^t(M^{\geq t}) \leq 0$. But then $\operatorname{Ext.reg} \Sigma^t(M^{\geq t}) \leq 0$ by Proposition 4.6(ii).

On the other hand, it is clear that $\inf \Sigma^t(M^{\geq t}) \geq 0$, and Equation (5) in Remark 4.2 now implies $\inf \Sigma^t(M^{\geq t}) = \operatorname{Ext.reg} \Sigma^t(M^{\geq t}) = 0$, so $\Sigma^t(M^{\geq t})$ is a Koszul DG module. □

5. Examples

Recall from Setup 0.1 that A is a connected cochain DG algebra over a field k.

Example 5.1. If $\mathrm{H}(A)$ is noetherian AS regular [1], then all results in the paper apply to A.

To see so, we must find K and L as in Setup 3.1 and show that $\mathrm{H}(A)$ has a balanced dualizing complex. The latter is true by [15, cor. 4.14].

We know $\dim_k \operatorname{Tor}_*^{\mathrm{H}(A)}(k,k) < \infty$, and using the Eilenberg-Moore spectral sequence shows

$$\dim_k \mathrm{H}(k \overset{\mathrm{L}}{\underset{A}{\otimes}} k) < \infty$$

whence k is compact from either side. Also, $\dim_k \operatorname{Tor}_*^{\mathrm{H}(A)}(\mathrm{H}(A)^\vee, k) = \dim_k \operatorname{Ext}^*_{\mathrm{H}(A)}(k, \mathrm{H}(A)) < \infty$, and using the Eilenberg-Moore spectral sequence shows

$$\dim_k \mathrm{H}(\mathrm{RHom}_A(k,A)) = \dim_k \mathrm{H}(A^\vee \overset{\mathrm{L}}{\underset{A}{\otimes}} k) < \infty$$

whence $\sup \mathrm{RHom}_A(k,A) < \infty$ so $({}_A k)^* \in \langle k_A \rangle$ by Lemma 2.2. Hence the K-projective resolution of ${}_A k$ can be used for ${}_A K$. Similarly, the k-projective resolution of k_A can be used for L_A.

Example 5.2. If A is commutative in the DG sense and $\mathrm{H}(A)$ is noetherian, then all results in the paper apply to A.

To see so, we must again find K and L as in Setup 3.1 and show that $\mathrm{H}(A)$ has a balanced dualizing complex. The former can be done by using a DG version of the construction of the Koszul complex.

Since $\mathrm{H}(A)$ is graded commutative noetherian, it is a quotient of a tensor product $B \underset{k}{\otimes} C$ where B is a polynomial algebra with finitely many generators in even degrees and C is an exterior algebra with finitely many generators in odd degrees. It follows from [14, thm. 6.3] that $B \underset{k}{\otimes} C$ has a balanced dualizing complex. Now combine [14, thm. 6.3], [2, prop. 7.2(2)], and [2, thm. 8.3(2+3)] to see that so does any quotient of $B \underset{k}{\otimes} C$.

Example 5.3. If $\dim_k \mathrm{H}(A) < \infty$ then all results in the paper apply to A. Namely, $\langle {}_A k \rangle = \langle {}_A A \rangle = \mathsf{D}(A)$ and $\langle k_A \rangle = \langle A_A \rangle = \mathsf{D}(A^{\mathrm{op}})$ so we can use ${}_A K = {}_A A$ and $L_A = A_A$. Moreover, $\mathrm{H}(A)$ has the balanced dualizing complex $\mathrm{H}(A)^{\vee}$.

In this case, we can easily find the dualizing DG module. Since we have ${}_A A \in \langle {}_A k \rangle$, the counit morphism $\Gamma(A) \xrightarrow{\epsilon_A} A$ is an isomorphism. Hence $F \overset{\mathrm{L}}{\underset{A}{\otimes}} A \cong A$, that is, $F \cong A$, and this is an isomorphism in $\mathsf{D}(A^e)$; see the last paragraph of Remark 1.2 and Remark 1.4. So the dualizing DG module of A is

$$D \cong A^{\vee}$$

and the functors in Theorem 3.4 are just $(-)^{\vee}$.

Example 5.4. Let $A = k[T]$ have T in cohomological degree $d \geq 1$ and differential $\partial = 0$. All results in the paper apply to A by Example 5.1. Let us compute the dualizing DG module.

While A is commutative as a ring, it is not necessarily commutative in the DG sense because this means $xy = (-1)^{|x||y|} yx$ for graded elements x, y. This fails if d is odd and k has characteristic different from 2.

However, it remains the case that $1 \mapsto T$ extends to a unique homomorphism of DG A-bimodules $\Sigma^{-d} A \xrightarrow{\varphi} A$. The homomorphism is injective with cokernel ${}_A k_A$, so there is a distinguished triangle

$$\Sigma^{-d} A \xrightarrow{\varphi} A \longrightarrow k \tag{9}$$

in $\mathsf{D}(A^e)$.

We can consider $N = k[T, T^{-1}]$ as a DG A-bimodule with T in cohomological degree d and differential $\partial = 0$. Then there is a short exact sequence of DG A-bimodules $0 \to A \to N \to C \to 0$ which induces a distinguished triangle

$$\Sigma^{-1} C \to A \to N \tag{10}$$

in $\mathsf{D}(A^e)$.

It is easy to check that applying $\mathrm{RHom}_A(-, N)$ to (9) sends φ to an isomorphism, so $\mathrm{RHom}_A(k, N) = 0$ whence ${}_A N \in \langle {}_A k \rangle^{\perp}$ and $\Gamma(N) = 0$; see the last paragraph of Remark 1.2. We have ${}_A(\Sigma^{-1} C) \in \langle {}_A k \rangle$ by Lemma 2.2, so $\Gamma(\Sigma^{-1} C) \cong \Sigma^{-1} C$. Note that this is an isomorphism in

$\mathsf{D}(A^e)$, see Remark 1.4, so it follows that applying Γ to (10) produces an isomorphism $\Sigma^{-1}C \cong \Gamma(A)$ in $\mathsf{D}(A^e)$. However, $\Gamma(A) = F \overset{\mathrm{L}}{\underset{A}{\otimes}} A \cong F$ in $\mathsf{D}(A^e)$, so we get

$$\Sigma^{-1}C \cong F$$

in $\mathsf{D}(A^e)$. Hence the dualizing DG module of A is

$$D = F^{\vee} = (\Sigma^{-1}C)^{\vee}.$$

More explicitly, C is the DG quotient module $k[T, T^{-1}]/k[T]$, and based on this, a concrete computation of $D = (\Sigma^{-1}C)^{\vee}$ yields the following: As a graded vector space, D has a generator e_ℓ in cohomological degree $d\ell + d - 1$ for each $\ell \geq 0$. It has differential $\partial = 0$, and the left and right actions of A on D are given by

$$T^j e_\ell = e_{j+\ell}, \quad e_\ell T^j = (-1)^{jd} e_{j+\ell}.$$

As a DG left-A-module, D is just $\Sigma^{-(d-1)}A$. The right action of A is twisted by the DG algebra automorphism $\alpha : T^j \mapsto (-1)^{jd}T^j$. Denoting the twist by a superscript, we finally have

$$D \cong (\Sigma^{-(d-1)}A)^{\alpha}$$

in $\mathsf{D}(A^e)$.

There is no way to get rid of the twist: If D were isomorphic to $\Sigma^{-(d-1)}A$ in $\mathsf{D}(A^e)$, then the cohomologies of the two DG modules would be isomorphic as graded $\mathrm{H}(A)$-bimodules. However, $\mathrm{H}(\Sigma^{-(d-1)}A)$ is a symmetric $\mathrm{H}(A)$-bimodule, but if d is odd and k has characteristic different from 2, then $\mathrm{H}(D)$ is not.

The presence of twists in the theory of two sided duality is not surprising since it occurs already for rings, see for instance [15, thm. 7.18 and the remark preceding it].

Remark 5.5. In Definition 1.3, the dualizing DG module D depends on the choice of the object K made in Setup 1.1. However, in the two previous examples, the computations show that any choice of K produces the same D. It would be interesting to know if D is unique in general.

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