GENERALIZED COMPOSITIONS WITH A FIXED NUMBER OF PARTS

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ABSTRACT. We investigate compositions of a positive integer with a fixed number of parts, when there are several types of each natural number. These compositions produce new relationships among binomial coefficients, Catalan numbers, and numbers of the Catalan triangle.

1. Introduction

A k-tuple $(i_1, i_2, \ldots, i_k)$ of positive integers, such that $i_1 + i_2 + \cdots + i_k = n$, is called a composition of n with k parts. In [MJ], the following generalization of compositions is considered: Let $\mathbf{b} = (b_1, b_2, \ldots,)$ be a sequence of nonnegative integers, and let n be a positive integer. The composition of n is a k-tuple $(i_1, i_2, \ldots, i_k)$ such that $i_1+i_2+\cdots+i_k = n$, assuming that there are b_1 different types of 1, b_2 different types of 2, and so on. We call such a composition the generalized composition of n with k parts.

The generalized compositions extend several types of compositions which are investigated in some earlier papers. First of all this is the case with usual compositions, which are obtained when $b_i = 1$ for each i. In [DS], the author considers the compositions in which there are two different types of 1, and one type of each other natural number. Next, in [AG], the case $b_i = i$, $(i = 1, 2, \ldots)$ is investigated.

The generalized compositions may be described as the colored compositions, in which the part i is colored by one of b_i colors. Different kinds of compositions have already been called colored compositions. For example, the m-colored compositions, as they are defined in [DK], are, freely speaking, the generalized compositions in which $b_i \in \{\omega, \omega^2, \ldots \omega^{m-1}\}$, where ω is a primitive mth root of 1. As well, the composition in which $b_i = i$, for any i, considered in [AG], is also called an m-colored compositions. The above-mentioned compositions, as well as many other interesting results on compositions can be found in a recently-published book [HU].

In [MJ], several recursions and some closed formulas for the number of all generalized compositions are obtained.

In this paper, we investigate the generalized compositions with a fixed number of parts. The paper is organized as follows. In Section 2 we outline some basic properties of the generalized compositions with a fixed number of parts. We also show that they extend the notion of the matrix composition, considered in [MU]. Then we derive several recurrence equations and closed formulas, by choosing for b_i different functions of i. In particular, we obtain the formula for the number of n-colored compositions, given in [DK], as well as the formula for the number of n-colored compositions, given in [AG]. Section 3 deals with the case when b_i is

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a binomial coefficient. Several closed formulas will be derived. Also, if b_i is of the form $b_i = \binom{i+p-1}{q}$, we prove that the numbers of all generalized compositions satisfy a homogenous recurrence equation with constant coefficients, of order $q+1$. In particular, the m-matrix compositions satisfy such a recurrence equation. For the case $p=1$, we derive a closed formula for both the number of the generalized compositions with a fixed number of parts and for the number of all generalized compositions. In Section 4, we investigate relationships of the generalized compositions with the Catalan numbers. Finally, a result which connects the Catalan numbers, the numbers of the Catalan triangle, and the binomial coefficients is derived.

2. Some preliminary results

Let $\mathbf{b} = (b_0, b_1, \ldots)$ be a sequence of nonnegative integers, and n, k be positive integers. We let $C^{(\mathbf{b})}(n,k)$ denote the number of the generalized compositions of n with k parts. We also define $C^{(\mathbf{b})}(0,0) = 1,\ C^{(\mathbf{b})}(i,0) = 0,\ (i > 0)$.

In [MJ], the number of all generalized compositions of n is denoted by $C^{(\mathbf{b})}(n)$. Obviously,

$$C^{(\mathbf{b})}(n) = \sum_{k=1}^{n} C^{(\mathbf{b})}(n,k). \tag{1}$$

In the following two propositions we state some basic properties of the generalized compositions.

Proposition 2.1. *The following equations are true:*

$$C^{(\mathbf{b})}(i,1) = b_i,\ (i = 1, 2, \ldots),\ C^{(\mathbf{b})}(n,n) = b_1^n,\ C^{(\mathbf{b})}(n,k) = 0,\ (k > n).$$

Proof. All equations are easy to verify. □

Proposition 2.2. *The following recursions are true:*

$$C^{(\mathbf{b})}(n,k) = \sum_{i=1}^{n-k+1} b_i C^{(\mathbf{b})}(n-i, k-1).\ (k \le n). \tag{2}$$

$$C^{(\mathbf{b})}(n) = \sum_{i=1}^{n} b_i C^{(\mathbf{b})}(n-i), \tag{3}$$

providing that $C^{(\mathbf{b})}(0) = 1$.

Proof. Equation (2) is true since there are $b_i C^{(\mathbf{b})}(n-i, k-1)$ generalized compositions ending with one of the i's, for $i = 1, \ldots, n-k+1$. A similar argument proves equation (3). □

We next prove that the matrix compositions, considered in [MU], are a particular case of the generalized compositions. A k- matrix composition of n is a matrix with k rows, which entries are nonnegative integers, no column consists of zeroes only, and the sum of all entries equals n. We let $MC(n)$ denote its number.

Proposition 2.3. *If* $b_i = \binom{i+k-1}{i},\ (1 = 1, 2, \ldots)$, *then*

$$MC(n) = C^{(\mathbf{b})}(n).$$

Proof. It is a well-known that, for a given positive integer k, the equation $x_1+x_2+\cdots+x_k=i$ has $\binom{i+k-1}{i}$ nonnegative solutions. This means that $\binom{i+k-1}{k}MC(n-i)$ is the number of k-compositions of n, ending with a column in which the sum of all elements equals i. Taking $MC(0)=1$ we obtain

$$MC(n)=\sum_{i=1}^{n}\binom{i+k-1}{i}MC(n-i),$$

Comparing this equation with (3) we easily conclude that

$$MC(n)=C^{(\mathbf{b})}(n),$$

and the proposition is proved. □

In the rest of this section we shall choose for b_i different functions of i and obtain several closed formulas. We first consider the case when $\mathbf{b}$ is a constant sequence.

Proposition 2.4. *Let n,p be positive integers, and let $b_i=p,\ (i=1,2,\ldots)$. Then*

$$C^{(\mathbf{b})}(n,k)=p^k\binom{n-1}{k-1}.$$

Proof. In this case, the connection between compositions and generalized compositions is simple. From a composition of n with k parts we obtain p^k different generalized compositions with k parts, since each part my take p different values. In this way we obtain all generalized compositions of n with k parts. Moreover, there are $\binom{n-1}{k-1}$ compositions of n with k parts, and the proposition follows. □

Corollary 2.5. *In the conditions of Proposition 2.4 we have*

$$C^{(\mathbf{b})}(n)=p(1+p)^{n-1}.$$

Proof. The formula (1) now takes the form:

$$C^{(\mathbf{b})}(n)=\sum_{k=1}^{n}\binom{n-1}{k-1}p^k,$$

and the assertion follows from the binomial formula. □

Remark 2.6. The number $C^{(\mathbf{b})}(n)$ from the preceding corollary equals the number of the p-colored compositions, as they are defined in [DK].

Next, we investigate the case when $\mathbf{b}$ is a constant sequence with several leading zeroes.

Proposition 2.7. *Let p,m,n be positive integers, and let $b_i=0,\ (i=1,2,\ldots,m-1),\ b_i=p,\ (i\geq m)$. Then*

$$C^{(\mathbf{b})}(n,k)=p^k\binom{n-(m-1)k-1}{k-1}.$$

Proof. In this case, we consider the set X of the generalized compositions of n with k parts, all of which are $\geq m$. There is a bijection between the set X and the set Y of the generalized compositions of $n-(m-1)k$ with k parts, which are considered in Proposition 2.4. Namely, subtracting $m-1$ from each term of an element of X, we obtain an element of Y. Conversely, adding $m-1$ to each term of an arbitrary element of Y, we obtain an element of X. The proposition now follows from Proposition 2.4. □

As an immediate consequence of (1) we state

Corollary 2.8. *In the conditions of Proposition 2.7 we have*

$$C^{(\mathbf{b})}(n)=\sum_{k=1}^{n}\binom{n-(m-1)k-1}{k-1}p^k.$$

We shall now consider the case when b_i is an exponential function of i.

Proposition 2.9. *Let p,n,k be positive integers, and let $b_i=p^{i-1},\ (i=1,2,\ldots)$. Then*

$$C^{(\mathbf{b})}(n,k)=p^{n-k}\binom{n-1}{k-1}.$$

Proof. Equation (2) has the form:

$$C^{(\mathbf{b})}(n,k)=\sum_{i=1}^{n-k+1}p^{i-1}C^{(\mathbf{b})}(n-i,k-1).\ (k\le n).$$

We prove the formula by induction on k. It is obviously true for $k=1$. Suppose it is also true for $k-1$. Then the preceding equation takes the form:

$$C^{(\mathbf{b})}(n,k)=p^{n-k}\sum_{i=1}^{n-k+1}\binom{n-i-1}{k-2}.$$

On the other hand, by a well-known horizontal recursion for the binomial coefficients we have

$$\binom{n-1}{k-1}=\sum_{i=1}^{n-k+1}\binom{n-i-1}{k-2},$$

and the formula is true. □

Using the binomial formula, for the number of all generalized compositions, we obtain

$$C^{(\mathbf{b})}(n)=(1+p)^{n-1}.$$

This is the formula (i), Corollary 13, in [MJ].

In the next two results we consider the case when b_i is a linear function of i.

Proposition 2.10. *Let p,m,n be positive integers, and let $b_i=m(i-1),\ (i=1,2,\ldots)$. Then*

$$C^{(\mathbf{b})}(n,k)=m^k\cdot\binom{n-1}{2k-1}.$$

Proof. The proposition is obviously true for $k=1$. Assume that it is true for $k-1$. Equation (2) has the form:

$$C^{(\mathbf{b})}(n,k)=m\sum_{i=1}^{n-k+1}(i-1)C^{(\mathbf{b})}(n-i,k-1),\ (k\le n).$$

Using the induction hypothesis yields

$$C^{(\mathbf{b})}(n,k)=m^k\sum_{i=1}^{n-k+1}(i-1)\binom{n-i-1}{2k-3},\ (k\le n).$$

It follows that

$$C^{(\mathbf{b})}(n,k) = m^k \sum_{i=1}^{n-k} i\binom{n-i-2}{2k-3},\ (k \le n).$$

Denote $S = \sum_{i=1}^{n-k} i\binom{n-i-2}{2k-3}$. Then,

$$S = \sum_{i=0}^{n-k} \binom{n-i-2}{2k-3} + \sum_{i=2}^{n-k} \binom{n-i-2}{2k-3} + \cdots + \sum_{i=n-k}^{n-k} \binom{n-i-2}{2k-3}.$$

Using the horizontal recursion for the binomial coefficients we obtain

$$S = \binom{n-2}{2k-2} + \binom{n-3}{2k-2} + \cdots + \binom{2k-2}{2k-2}.$$

Using the same recursion once more we obtain

$$S = \binom{n-1}{2k-1}.$$

□

For the number of all generalized composition we get

Corollary 2.11. *In the conditions of Proposition 2.10 we have*

$$C^{(\mathbf{b})}(n) = \sum_{k=1}^{n} \binom{n-1}{2k-1} m^k.$$

In a similar way we may prove the following:

Proposition 2.12. *If* $b_i = mi,\ (i = 1, 2, \ldots),$ *then*

$$C^{(\mathbf{b})}(n,k) = m^k \cdot \binom{n+k-1}{2k-1}.$$

Also,

$$C^{(\mathbf{b})}(n) = \sum_{k=1}^{n} \binom{n+k-1}{2k-1} m^k.$$

Remark 2.13. Taking in particular $m = 1$ in the preceding equation, we obtain Theorem 3.23, in [HU], about the so called n-colored compositions, defined in [AG].

3. Binomial coefficients

In this section we investigate the generalized compositions, when the b's are some binomial coefficients. We first derive two closed formulas.

Proposition 3.1. *Let* k, p, n *be positive integers, and let* $b_i = \binom{p}{i-1},\ (i = 1, 2, \ldots).$ *Then,*

$$C^{(\mathbf{b})}(n,k) = \binom{pk}{n-k}.$$

Also,

$$C^{(\mathbf{b})}(n) = \sum_{k=1}^{n} \binom{pk}{n-k}.$$

Proof. We go by induction on k. For $k=1$ the proposition is obviously true. Using the induction hypothesis we see that the first assertion is equivalent to the following identity:

$$\binom{pk}{n-k}=\sum_{i=1}^{n-k+1}\binom{p}{i-1}\binom{pk-p}{n-i-k+1},$$

which is merely the Vandermonde convolution. □

The next result concerns the figured numbers.

Proposition 3.2. *Let p,k,n be positive integers, and let*

$$b_i=\binom{p+i-1}{p},\ (i=1,2,\ldots).$$

Then,

$$C^{(\mathbf{b})}(n,k)=\binom{n+pk-1}{pk+k-1}.$$

Proof. We use induction on k. For $k=1$ the proposition is obviously true. Using the induction hypothesis we see that the the assertion is equivalent to the following identity:

$$\binom{n+pk-1}{pk+k-1}=\sum_{i=1}^{n-k+1}\binom{p+i-1}{p}\binom{n-i+pk-p-1}{pk-p+k-2}.$$

To prove this identity, we shall count $pk+k-1$-subsets of the set $X=\{1,2,\ldots,n+pk-1\}$ according to the place of its $(p+1)$the element in such a subset. Suppose that this element is the $(p+i)$th element of X. Such a subset may be chosen in $\binom{p+i-1}{p}\cdot\binom{n-i+pk-p-1}{pk-p+k-2}$ ways. We also conclude that i ranges from 1 to $n-k+1$, which proves the proposition.

□

The following two results concern the number of all generalized compositions. We first prove that, in the case $b_i=\binom{i+p-1}{q},\ (i=1,2,\ldots)$, where p,q are positive integers, the numbers $C^{(\mathbf{b})}(n)$ satisfy a homogenous linear recurrence equation of the $(q+1)$th order, with constant coefficients.

Proposition 3.3. *Let p,q,n be positive integers, and let $b_i=\binom{i+p-1}{q},\ (i=1,2,\ldots)$. Then there exist integers $m_i(p,q),(i=0,1,\ldots,q)$, not depending on n, such that*

$$C^{(\mathbf{b})}(n+q+1)=\sum_{i=0}^{q}m_i(p,q)C^{(\mathbf{b})}(n+i),\ (n\geq 2). \tag{4}$$

Proof. We define the function $F(n,j)$ in the following way:

$$F(n,j)=\sum_{i=1}^{n-1}\binom{n-i+p}{q-j}C^{(\mathbf{b})}(i-1), \tag{5}$$

where $0\leq j\leq q,\ 2\leq n$. We want to prove that the following equation holds

$$F(n,j)=\sum_{i=0}^{j+1}c(i,j)C^{(\mathbf{b})}(n+i-1), \tag{6}$$

where $c(i,j)$ are integers, depending only on p and q.

The proof goes by induction on j. Taking $n=1$ in (1) we get $C^{(\mathbf{b})}(1)=\binom{p}{q}$. For $n>1$ we get

$$C^{(\mathbf{b})}(n)=\binom{p}{q}C^{(\mathbf{b})}(n-1)+\sum_{i=1}^{n-1}\binom{n-i+p}{q}C^{(\mathbf{b})}(i-1). \tag{7}$$

It follows that

$$F(n,0)=C^{(\mathbf{b})}(n)-\binom{p}{q}C^{(\mathbf{b})}(n-1). \tag{8}$$

Hence, taking

$$c(0,0)=-\binom{p}{q},\ c(1,0)=1,$$

we see that (6) holds for $j=0$ and $n\geq 2$.

Suppose that (6) holds for some $j\geq 0$. Replacing n by $n+1$ in (5) yields

$$F(n+1,j)=\sum_{i=1}^{n}\binom{n+1-i+p}{q-j}C^{(\mathbf{b})}(i-1).$$

Using the standard recursion for the binomial coefficients one obtains

$$F(n,j+1)=F(n+1,j)-F(n,j)-\binom{p+1}{q-j}C^{(\mathbf{b})}(n-1).$$

Using the induction hypothesis yields

$$F(n,j+1)=$$

$$=\sum_{i=0}^{j+1}c(i,j)C^{(\mathbf{b})}(n+i)-\sum_{i=0}^{j+1}c(i,j)C^{(\mathbf{b})}(n+i-1)-\binom{p+1}{q-j}C^{(\mathbf{b})}(n-1).$$

Denoting

$$c(0,j+1)=-c(0,j)-\binom{p+1}{q-j},\ c(j+2,j+1)=c(j+1,j),$$

$$c(i,j+1)=c(i-1,j)-c(i,j),\ (1\leq i\leq j+1),$$

implies

$$F(n,j+1)=\sum_{i=0}^{j+2}c(i,j+1)C^{(\mathbf{b})}(n+i-1),\ (n\geq 2),$$

and (6) is true.

Since $F(n,q)=\sum_{i=1}^{n-1}C^{(\mathbf{b})}(i-1)$, we have

$$\sum_{i=0}^{q+1}c(i,q)C^{(\mathbf{b})}(n+i-1)=\sum_{i=1}^{n-1}C^{(\mathbf{b})}(i-1). \tag{9}$$

Replacing n by $n+1$ in (9) yields

$$\sum_{i=0}^{q+1}c(i,q)C^{(\mathbf{b})}(n+i)=\sum_{i=1}^{n}C^{(\mathbf{b})}(i-1). \tag{10}$$

Subtracting (10) from (9) we obtain

$$\sum_{i=0}^{q+1} c(i,q)\left[C^{(\mathbf{b})}(n+i-1)-C^{(\mathbf{b})}(n+i)\right]+C^{(\mathbf{b})}(n-1)=0. \tag{11}$$

Further, we obviously have $c(q+1,q)=1$. Also, we may easily obtain the values for $c(0,q+1)$. First, we have

$$c(0,1)=-c(0,0)-\binom{p+1}{q}=\binom{p}{q}-\binom{p+1}{q}=-\binom{p}{q-1}.$$

Using induction easily implies that

$$c(0,j)=-\binom{p}{q-j},\ (j=0,1,\ldots,q). \tag{12}$$

In particular, $c(0,q)=-1$, which means that $C^{(\mathbf{b})}(n-1)$ vanishes in equation (11). Hence, equation (11) becomes (4), if we take

$$m_i(p,q)=-c(i+1,q+1),\ (i=0,1,\ldots,q).$$

□

Remark 3.4. We have seen, in Proposition 2.3, that in the case $p-1=q$, the number $C^{(\mathbf{b})}(n)$ is the number of q-matrix compositions, as they are defined in [MU]. Thus the numbers of q-matrix compositions satisfy a $(q+1)$th order homogenous linear recurrence equation with constant coefficients.

Remark 3.5. The coefficients $c(i,j),\ (j=0,1,\ldots;\ i=0,1,\ldots,j+1)$ form a kind of a Pascal-like triangle.

We shall now consider the particular case $p=1,\ q>1$, and show that then the coefficients $m_i(1,q)$ can be obtained explicitly.

Proposition 3.6. *Let q be a positive integer, and let $b_i=\binom{i}{q},\ (i=1,2,\ldots)$. Then,*

$$C^{(\mathbf{b})}(n+q+1)=\sum_{i=0}^{q}(-1)^{i+q}\binom{q+1}{i}C^{(\mathbf{b})}(n+i)+C^{(\mathbf{b})}(n+1),\ (n\geq 2).$$

Proof. Firstly, we have

$$c(0,0)=0,\ c(1,0)=1.$$

For $j\geq 1$, by (12), we have

$$c(0,j)=-\binom{1}{q-j}.$$

It follows that

$$c(0,q-1)=c(0,q)=-1,\ \text{and}\ c(0,j)=0\ \text{otherwise}\ .$$

Furthermore, for $j<q$ we have

$$c(1,j)=c(0,j-1)-c(1,j-1)=-c(1,j-1)=c(1,j-2)=\ldots=(-1)^j,$$

and

$$c(1,q)=c(0,q-1)-c(1,q-1)=-1-c(1,q-1)=\ldots=-1+(-1)^q.$$

Also,

$$c(2,j)=(-1)^{j-1}j,\ (j\leq q).$$

We next prove that for j, satisfying the condition $2 \leq j \leq q$, we have

$$c(i,j) = (-1)^{j-i+1}\binom{j}{i-1},\ (i = 2,\ldots,j).$$

The equation is true for $i = 2$, by the preceding equation. Suppose that it is true for some $i-1 \geq 2$. From the equation

$$c(i,j) = c(i-1,j-1) - c(i,j-1),$$

using the induction hypothesis we obtain

$$c(i,j) = (-1)^{j-i+1}\binom{j-1}{i-2} - c(i,j-1).$$

From this we easily conclude that

$$c(i,j) = (-1)^{j-i+1}\left[\binom{j-1}{i-2} + \binom{j-2}{i-2} + \cdots + \binom{i-2}{i-2}\right].$$

The assertion is true, by the horizontal recursion for the binomial coefficients. In particular, we have

$$(-1)^{i+q}[c(i+1,q) - c(i,q)] = \binom{q}{i} + \binom{q}{i-1} = \binom{q+1}{i}.$$

□

Now, we shall derive the closed formula for the recursion from the preceding proposition.

Proposition 3.7. *Let q be a positive integer, and let $b_i = \binom{i}{q}$, $(i = 1,2,\ldots)$. Then,*

$$C^{(\mathbf{b})}(n,k) = \binom{n+k-1}{qk+k-1}.$$

Proof. We first conclude that each term of any generalized composition is $\geq q$. It follows that $C^{(\mathbf{b})}(n,k) = 0$, if $n < qk$. This means that the assertion holds for $n < qk$. Assume that $n \geq qk$.

Using induction we easily conclude that the assertion is equivalent to the following binomial identity:

$$\binom{n+k-1}{qk+k-1} = \sum_{i=1}^{n-k+1}\binom{i}{q}\binom{n+k-2-i}{qk-q+k-2},\ (qk \leq n).$$

Adjusting the lower and the upper bounds in the sum on the right-hand side, we obtain the following identity:

$$\binom{n+k-1}{qk+k-1} = \sum_{i=q}^{n-qk+q}\binom{i}{q}\binom{n+k-2-i}{qk-q+k-2},\ (qk \leq n).$$

To prove this identity we shall count $(qk+k-1)$- subsets of the set $X = \{1,2,\ldots,n+k-1\}$ in the following way: Suppose that x is the $(q+1)$th element of a $(qk+k-1)$-subset of X, and suppose that we have i elements of X in the subset, which are less than x. It follows that there are

$$\binom{i}{q}\binom{n+k-2-i}{qk-q+k-2}$$

subsets with this property. The assertion is true, since i ranges from q to $n-qk+q$. □

As an immediate consequence we have

Corollary 3.8. *If* $b_i=\binom{i}{q},(i=1,2,\ldots)$, *then*

$$C^{(\mathbf{b})}(n)=\sum_{k=1}^{n}\binom{n+k-1}{qk+k-1}.$$

Remark 3.9. The preceding equation is the closed formula for the recurrence equation from Proposition 3.6.

4. Catalan numbers

In this section we consider the case when the b's are Catalan numbers. In the first result we shall prove that the numbers of generalized compositions with a fixed number of parts, may be expressed in terms of the numbers of the so called Catalan triangle, introduced by Chapiro, [SH]. We let $\mathbf{c}_i$ denote the ith Catalan number. Also, $B(n,k)$ denotes a number of Catalan triangle. Thus,

$$B(n,k)=\frac{k}{n}\binom{2n}{n+k},\ (k\le n).$$

Proposition 4.1. *Let* n,k *be positive integers, and let* $b_i=\mathbf{c}_i,(i=1,2,\ldots)$. *Then,*

$$C^{(\mathbf{b})}(n,k)=B(n,k).$$

Further,

$$C^{(\mathbf{b})}(n)=\binom{2n-1}{n}.$$

Proof. Equation (2), in this case, has the form:

$$C^{(\mathbf{b})}(n,k)=\sum_{i=1}^{n-k+1}\mathbf{c}_iC^{(\mathbf{b})}(n-i,k-1),\ (k\le n).$$

The assertion follows by induction, using Theorem 14.3, [KS]. The second assertion follows from Theorem 14.2, [KS].

Remark 4.2. Note that, in the preceding proposition, we have an example when the number of all generalized compositions is a binomial coefficient.

□

We now slightly change the conditions of the preceding corollary to obtain a relationship among Catalan numbers, binomial coefficients, and the numbers of Catalan triangle.

Proposition 4.3. *Let* n,k *be positive integers, and let* $b_i=\mathbf{c}_i,\ (i=0,1,\ldots)$. *Then, for* $n\ge k$, *we have*

$$C^{(\mathbf{b})}(n,k)=\sum_{i=0}^{k-1}\binom{k}{i}B(n-k,k-i). \tag{13}$$

Proof. We shall first prove that, for $1 \le k \le n$, the following equation

$$C^{(\mathbf{b})}(n,k) = \sum_{i_1+i_2+\cdots+i_k=n-k} \mathbf{c}_{i_1} \cdot \mathbf{c}_{i_2} \cdots \mathbf{c}_{i_k}, \tag{14}$$

holds. The sum is taken over $i_1 \ge 0, i_2 \ge 0, \ldots, i_k \ge 0$. We use induction on k. For $k = 1$, by (2), we have $C^{(\mathbf{b})}(n,1) = \mathbf{c}_{n-1}$. On the other hand, (14) has the form:

$$C^{(\mathbf{b})}(n,1) = \sum_{i_1=n-1} c_{i_1} = \mathbf{c}_{n-1},$$

and the proposition is true. Suppose that the proposition is true for $k \ge 1$. Then,

$$C^{(\mathbf{b})}(n,k+1) = \sum_{i=1}^{n-k} \mathbf{c}_{i-1} C^{(\mathbf{b})}(n-i,k).$$

Using the induction hypothesis yields

$$C^{(\mathbf{b})}(n,k+1) = \sum_{i=1}^{n-k} \mathbf{c}_{i-1} \sum_{i_1+i_2+\cdots+i_k=n-i-k} \mathbf{c}_{i_1} \cdot \mathbf{c}_{i_2} \cdots \mathbf{c}_{i_k}.$$

Denote $i - 1 = i_{k+1}$ to obtain

$$C^{(\mathbf{b})}(n,k+1) = \sum_{i_1+i_2+\cdots+i_k+i_{k+1}=n-k-1} \mathbf{c}_{i_1} \cdot \mathbf{c}_{i_2} \cdots \mathbf{c}_{i_k} \cdot \mathbf{c}_{i_{k+1}},$$

and (14) is true.

Collecting terms with a fixed number of zeroes in (14) we obtain

$$C^{(\mathbf{b})}(n,k) = \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i_1+i_2+\cdots+i_{k-j}=n-k} \mathbf{c}_{i_1} \cdot \mathbf{c}_{i_2} \cdots \mathbf{c}_{i_{k-j}},$$

where all sums on the right-hand side are taken over $i_t \ge 1$. According to Theorem 14. 4, [KS], we have

$$B(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} \mathbf{c}_{i_1} \cdot \mathbf{c}_{i_2} \cdots \mathbf{c}_{i_k},$$

where $i_1 \ge 1, \ldots, i_k \ge 1$, and the proposition is true. □

In [MJ] it is proved that the sum on the right-hand side of equation (14) equals the number of the weak compositions of $n-k$ in which exactly k parts equal 0. We thus have

Corollary 4.4. *Let n, k be positive integers, and let $b_i = \mathbf{c}_i$, $(i = 0, 1, \ldots)$. Then $C^{(\mathbf{b})}(n,k)$ is the number of the weak generalized compositions of $n-k$ in which there are exactly k zeroes.*

It is proved in Proposition 3, [MJ], that in this case $\mathbf{c}_n$ is the number of all generalized compositions. We thus obtain a formula which shows that Catalan numbers are some kind of convolution of the numbers of Pascal and Catalan triangles.

Corollary 4.5. *Let n be a positive integer. Then*

$$\mathbf{c}_n = 1 + \sum_{k=1}^{n-1} \sum_{i=1}^{k-1} \binom{k}{i} B(n-k, k-i).$$

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