

ON THE HEIGHT OF CYCLOTOMIC POLYNOMIALS

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ABSTRACT. Let A_n denote the height of cyclotomic polynomial Φ_n , where n is a product of k distinct odd primes. We prove that $A_n \leq \varepsilon_k \varphi(n)^{k-1} 2^{k-1-1}$ with $-\log \varepsilon_k \sim c 2^k$, $c > 0$. The same statement is true for the height C_n of the inverse cyclotomic polynomial Ψ_n .

Additionally, we improve on a bound of Kaplan for the maximal height of divisors of $x^n - 1$, denoted by B_n . We show that $B_n < \eta_k n^{(3^k-1)/(2^k)-1}$, with $-\log \eta_k \sim c 3^k$ and the same c .

1. INTRODUCTION

The polynomial

$$\Phi_n(x) = \sum_{0 \leq m \leq \varphi(n)} a_n(m) x^m = \prod_{k \leq n, (k,n)=1} (x - \zeta_n^k)$$

where $\zeta_n = e^{2i\pi/n}$, is called the n -th cyclotomic polynomial. We are interested in estimating its coefficients, so we define

$$A_n = \max_m |a_n(m)| \quad \text{and} \quad S_n = \sum_{m=0}^{\varphi(n)} |a_n(m)|.$$

We define also

$$\Psi_n(x) = \frac{1}{\Phi_n(x)} = \sum_{m \geq 0} c_n(m) x^m, \quad C_n = \max_m |c_n(m)|.$$

The polynomial $(1 - x^n)\Psi_n(x)$ is called the n -th inverse cyclotomic polynomial (see [10] for details). We remark that $c_n(m)$ is equal to the m' -th coefficient of the n -th inverse cyclotomic polynomial, where $0 \leq m' < n$ and $m' \equiv m \pmod{n}$.

We consider the numbers n which are odd and square free only, since it is known that $A_{\ker(n)} = A_n = A_{2n}$, where $\ker(n)$ is the product of all distinct prime factors of n (see [13] for details). The same fact is true for inverse cyclotomic polynomials.

The order of Φ_n is the number $\omega(n)$ of primes dividing n . For $\omega(n) \leq 4$ the following bounds are known:

$$(1) \quad A_p = 1, \quad A_{pq} = 1, \quad A_{pqr} \leq \epsilon_3 p, \quad A_{pqrs} \leq \epsilon_4 p^3 q.$$

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The first of them is obvious. The second one is due to A. Migotti [9].

The third one with $\epsilon_3 = 1$ is due to A. S. Bang [2]. It has been improved by some authors. Presently it is known that one can take $\epsilon_3 = 3/4$ (see [1, 4, 6]) and that one cannot replace ϵ_3 by a constant smaller than $2/3$ (see [7]). It is strongly believed that the estimate holds with $\epsilon_3 = 2/3$ (J. Zhao and X. Zhang [14], preprint). This conjecture is known as the Corrected Beiter Conjecture (see [7]).

The fourth inequality with $\epsilon_4 = 1$ was established by Bloom [5]. We use a simple argument from [3] to show that the inequality is true with $\epsilon_4 = \epsilon_3$.

For inverse cyclotomic polynomials we know the following bounds

$$C_p = 1, \quad C_{pq} = 1, \quad C_{pqr} \leq p - 1.$$

The first and the second of them are easy to obtain. The third was proved by P. Moree [10] who in the same paper proved that $p - 1$ cannot be replaced by a smaller number.

In the general case, we know the following result by P. T. Bateman, C. Pomerance and R. C. Vaughan [3] for standard cyclotomic polynomials.

$$(2) \quad A_{p_1 \dots p_k} \leq M_k \leq n^{k-1} 2^{k-1-1},$$

where $M_k = \prod_{i=1}^{k-2} p_i^{2^{k-i}-1}$ (this notation we use throughout the paper). The same authors came up with the following conjecture (cf. [3], p. 175).

Conjecture 1. *In (2) one can replace n by $\varphi(n)$.*

We prove this conjecture and moreover, we improve it by multiplying the right hand side by a constant depending on k only and decreasing quickly when k grows. We prove also a similar result for the inverse cyclotomic polynomials and give the bound for the maximal magnitude of the coefficient of any divisor of $x^n - 1$, improving on an earlier result of N. Kaplan [8]. The idea of estimating the maximal magnitude of coefficient of any divisor of $x^n - 1$ comes from C. Pomerance and N. C. Ryan [11].

By ϵ_k we denote the smallest positive real number for which the inequality $A_{p_1 \dots p_k} \leq \epsilon_k M_k$ holds with any distinct primes p_1, \dots, p_k . In the same way we define ϵ_k^{inv} for the inverse cyclotomic polynomial and E_k . Let

$$(3) \quad d = \max_{p,q,r} \frac{S_{pqr}}{p^2 qr}, \quad \rho = \prod_{i=0}^{\infty} \left(\frac{2i+5}{2i+6} \right)^{1/2^i}, \quad C = \left(\frac{3}{4} \epsilon_3^{3/2} d \rho^{1/4} \right)^{1/32}.$$

Note that $C < 1$. Our main results are the four following theorems.

Theorem 1. *We have $\log \epsilon_k \leq 2^k \log(C + o(1))$*

Theorem 2. *We have $\log \epsilon_k^{inv} \leq 2^k \log(C + o(1))$*

Theorem 3. *If $B_n = \eta_k n^{(3^k-1)/(2k)-1}$, then $\log \eta_k < 3^k \log(C + o(1))$ for every n free of squares.*

Theorem 4. *Conjecture 1 holds true, that is we have $M_k \leq \varphi(n)^{k-12^{k-1}-1}$ with $n = p_1 \dots p_k$.*

In the proof of Theorem 1 we also establish the following bounds

$$(4) \quad A_{pqrs} \leq \frac{3}{4}p^3q, \quad A_{pqrst} \leq \frac{135}{512}p^7q^3r, \quad A_{pqrst} \leq \frac{18225}{262144}p^{15}q^7r^3s,$$

where we assumed $\epsilon_3 = 3/4$. For $\epsilon_3 = 2/3$ we establish constants $\frac{2}{3}$, $\frac{2}{9}$, $\frac{32}{729}$, respectively.

Also for the inverse cyclotomic polynomial

$$(5) \quad C_{pqrs} \leq \frac{3}{4}p^3q, \quad C_{pqrst} \leq \frac{9}{16}p^7q^3r, \quad C_{pqrst} \leq \frac{10935}{131072}p^{15}q^7r^3s$$

for $\epsilon_3 = 3/4$. If $\epsilon_3 = 2/3$, then we obtain constants $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{81}$, respectively

Let us remark that Theorem 1, but with larger constant, can be obtained by the original method of P. T. Bateman, C. Pomerance and R. C. Vaughan. Our method is a bit different. It is based on a different recursive formula given in Lemma 1. We use also some basic combinatorics.

2. PRELIMINARIES

Our primary tool is the following lemma.

Lemma 1. *Let p_1, \dots, p_k be distinct primes. Then*

$$(6) \quad \Phi_{p_1 \dots p_k}(x) = f(x) \cdot \prod_{j=1}^{k-2} P_j(x),$$

where

$$(7) \quad f(x) = (1 - x^{p_1 \dots p_k}) \cdot \frac{\prod_{i=2}^k (1 - x^{p_2 \dots p_k / p_i})}{\prod_{i=1}^k (1 - x^{p_1 \dots p_k / p_i})}$$

and $P_j = \prod_{i=j+2}^k \Phi_{p_1 \dots p_j}(x^{p_{j+2} \dots p_k / p_i})$.

As $\deg(\Phi_n) = \varphi(n) < n$, we may replace f by $f^* \equiv f \pmod{x^{p_1 \dots p_k}}$ in (6), where $\deg(f^*) < p_1 \dots p_k$. Then we have congruence modulo $x^{p_1 \dots p_k}$ in (6) instead of equality, which does not matter for our purposes. In addition in the next section we prove the following lemma.

Lemma 2. *We have $H(f^*) \leq b_{k-2} = \binom{k-2}{\lfloor (k-2)/2 \rfloor}$.*

Lemmas 1 and 2 allow us to give the following recursive bound on ϵ_k .

Lemma 3. *We have $\epsilon_k \leq E_k = \frac{b_{k-2} d^{k-4}}{2^{k-3}} \prod_{j=1}^{k-2} \epsilon_j^{k-j-1}$.*

To start the induction we need also the following estimates.

Lemma 4. *We have $\epsilon_4 \leq \epsilon_3$.*

Proof. It is known that $S_1 = 2$ and $S_{pq} \leq pq/2$ (see [5] for a proof of the second equality). By Lemma 4 on pages 182–183 in [3],

$$A_{pqrs} \leq A_{pqr} S_{pq} S_p S_1 \leq \epsilon_3 \cdot p^3 q,$$

so the estimate holds. \square

Lemma 5. *For d defined in (3) we have $d \leq \epsilon_3(2 - \epsilon_3)/2$.*

Proof. Bloom [5] proved that

$$|a_{pqr}(m)| = |a_{pqr}(\varphi(pqr) - m)| \leq 2(\lfloor m/qr \rfloor + 1).$$

Thus

$$\begin{aligned} S_{pqr} &\leq 2 \sum_{k=0}^{\varphi(pqr)/2} \min\{\epsilon_3 p, 2(\lfloor m/qr \rfloor + 1)\} \\ &\leq \epsilon_3 p(\varphi(pqr) + 2 - 2\lfloor \epsilon_3 p/2 \rfloor qr) + 2qr \sum_{a=0}^{\lfloor \epsilon_3 p/2 \rfloor - 1} (2a + 2) \\ &= \epsilon_3 p(p-1)(q-1)(r-1) + 2\epsilon_3 p - 2\lfloor \epsilon_3 p/2 \rfloor \epsilon_3 pqr \\ &\quad + 2\lfloor \epsilon_3 p/2 \rfloor (2\lfloor \epsilon_3 p/2 \rfloor + 1)qr \\ &< \epsilon_3(2 - \epsilon_3)p^2 qr/2, \end{aligned}$$

which completes the proof. \square

3. PROOF OF LEMMA 1, 2 AND 3

Proof of Lemma 1. We prove this lemma by induction on k . By (see [5]) it holds for $k < 5$. Let us define

$$\tilde{f}(x) = (1 - x^{p_2 \cdots p_k}) \cdot \frac{\prod_{i=3}^k (1 - x^{p_3 \cdots p_k / p_i})}{\prod_{i=2}^k (1 - x^{p_2 \cdots p_k / p_i})}$$

and $\tilde{P}_j(x) = \prod_{i=j+2}^k \Phi_{p_2 \cdots p_j}(x^{p_{j+2} \cdots p_k / p_i})$. By the inductive assumption,

$$(8) \quad \Phi_{p_2 \cdots p_k} = \tilde{f}(x) \cdot \prod_{j=2}^{k-2} \tilde{P}_j(x).$$

It is known that $\Phi_{np}(x) = \Phi_n(x^p)/\Phi_n(x)$ for a prime p not dividing n (see [13]). Then also

$$\Phi_{p_1 \cdots p_k}(x) = \frac{\Phi_{p_2 \cdots p_k}(x^{p_1})}{\Phi_{p_2 \cdots p_k}(x)} \quad \text{and} \quad P_j(x) = \frac{\tilde{P}_j(x^{p_1})}{\tilde{P}_j(x)}.$$

By this and (8)

$$\Phi_{p_1 \cdots p_k}(x) = \frac{\tilde{f}_k(x^{p_1}) \cdot \prod_{j=2}^{k-2} \tilde{P}_j(x^{p_1})}{\tilde{f}_k(x) \cdot \prod_{j=2}^{k-2} \tilde{P}_j(x)} = \frac{\tilde{f}(x^{p_1})}{\tilde{f}(x) P_1(x)} \cdot \prod_{j=1}^{k-2} P_j(x).$$

Finally,

$$\frac{\widetilde{f}(x^{p_1})}{\widetilde{f}(x)} = P_1(x)(1 - x^{p_1 \cdots p_k}) \cdot \frac{\prod_{i=2}^k (1 - x^{p_2 \cdots p_k / p_i})}{\prod_{i=1}^k (1 - x^{p_1 \cdots p_k / p_i})} = P_1(x)f(x),$$

which completes the proof. \square

Proof of Lemma 2. Let $n = p_1 \cdots p_k$ and $f^*(x) = \sum_{m=0}^{n-1} d_m x^m$. By (7) we have

$$(9) \quad f^*(x) \equiv \prod_{i=2}^k (1 - x^{p_2 \cdots p_k / p_i}) \sum_{\alpha_1, \dots, \alpha_k \geq 0} x^{\alpha_1 n / p_1 + \dots + \alpha_k n / p_k} \pmod{x^n}.$$

Let

$$\Lambda = \{\lambda = (\lambda_2, \dots, \lambda_k) : \lambda_i \in \{0, 1\} \text{ for } i = 2, \dots, k\}, \quad s(\lambda) = (-1)^{\lambda_2 + \dots + \lambda_k}.$$

By (9)

$$(10) \quad d_m = \sum_{\lambda \in \Lambda} s(\lambda) \chi(m - \langle \lambda, v / p_1 \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^{k-1} , $v = (n/p_2, \dots, n/p_k)$ and

$$\chi(m) = \begin{cases} 1 & \text{if } m \text{ is of the form } \alpha_1 n / p_1 + \dots + \alpha_k n / p_k, \\ 0 & \text{otherwise.} \end{cases}$$

We define a number $\beta(\lambda)$ and a vector $\alpha(\lambda) = (\alpha_2(\lambda_2), \dots, \alpha_k(\lambda_k))$ by the congruence

$$(11) \quad m - \langle \lambda, v / p_1 \rangle \equiv \beta(\lambda) n / p_1 + \langle \alpha(\lambda), v \rangle \pmod{n}.$$

The numbers $\alpha_i(0)$ and $\alpha_i(1)$ depend only on the residue class of m modulo p_i , so (11) holds for every $\lambda \in \Lambda$. We have the following equivalences

$$\begin{aligned} & \chi(m - \langle \lambda, v / p_1 \rangle) = 1 \\ \iff & \langle \lambda, v / p_1 \rangle + \langle \alpha(\lambda), v \rangle \leq m \\ \iff & \langle \lambda, v / p_1 \rangle + \langle \alpha(\lambda) - \alpha(\theta_{k-1}), v \rangle \leq m - \langle \alpha(\theta_{k-1}), v \rangle, \end{aligned}$$

where $\theta_{k-1} = (0, \dots, 0)$. We have

$$\langle \alpha(\lambda) - \alpha(\theta_{k-1}), v \rangle = \sum_{i=2}^k (\alpha_i(\lambda_i) - \alpha_i(0)) v_i = \sum_{i=2}^k (\alpha_i(1) - \alpha_i(0)) v_i \lambda_i = \langle \lambda, w \rangle,$$

where $w = ((\alpha_i(1) - \alpha_i(0)) v_i)_{i=2}^k$. Therefore

$$\chi(m - \langle \lambda, v / p_1 \rangle) = 1 \iff \langle \lambda, u \rangle \leq D,$$

where $u = v / p_1 + w$ and $D = m - \langle \alpha(\theta_{k-1}), v \rangle$. By (10)

$$(12) \quad d_m = \sum_{\lambda \in \Lambda, \langle \lambda, u \rangle \leq D} s(\lambda).$$

Without loss of generality we may assume that $0 \leq u_k \leq u_2, \dots, u_{k-1}$.

There is a natural bijection between Λ and the family of subsets of $\{2, 3, \dots, k\}$, defined by

$$S_\lambda = \{i \in \{2, \dots, k\} : \lambda_i = 1\} \quad \text{for } \lambda \in \Lambda.$$

We say that $\lambda = (\lambda_2, \dots, \lambda_{k-1}, 0)$ is maximal if $\langle \lambda, u \rangle \leq D$ and for every $\lambda' = (\lambda'_2, \dots, \lambda'_{k-1}, 0)$ such that $S_\lambda \subset S_{\lambda'}$ we have $\langle \lambda', u \rangle > D$. Note that for

$$\lambda^0 = (\lambda_2, \dots, \lambda_{k-1}, 0) \quad \text{and} \quad \lambda^1 = (\lambda_2, \dots, \lambda_{k-1}, 1)$$

the following statements are true.

- If λ^0 is not maximal and $\langle \lambda^0, u \rangle \leq D$ then $\langle \lambda^1, u \rangle \leq D$.
- If $\langle \lambda^1, u \rangle \leq D$ then $\langle \lambda^0, u \rangle \leq D$.
- $s(\lambda^0) + s(\lambda^1) = 0$.

By this observation and (12) we conclude that

$$(13) \quad |d_m| \leq \#\{\lambda \in \Lambda : \lambda \text{ is maximal}\}.$$

Let $\lambda^1, \dots, \lambda^t \in \Lambda$ be maximal. By the definition of maximal λ , we have $S_{\lambda^i} \subset \{2, \dots, k-1\}$ and $S_{\lambda^i} \not\subset S_{\lambda^j}$ for every $i \neq j$.

Theorem 5 (E. Sperner, 1928). *Let $A_1, \dots, A_t \subset A$, where $\#A \leq \infty$. If $A_i \not\subset A_j$ for every $i \neq j$, then $t \leq \binom{\#A}{\lfloor \#A/2 \rfloor}$.* \square

For the proof see [12].

By Theorem 5 and (13), $|d_m| \leq t \leq \binom{k-2}{\lfloor (k-2)/2 \rfloor}$. \square

Proof of Lemma 3. For a formal power series $f(x) = \sum_{m \geq 0} a_m x^m \in \mathbb{Z}[[x]]$ we define $H, S \in [0, \infty]$

$$H(f) = \max_{m \geq 0} |a_m|, \quad S(f) = \sum_{m \geq 0} |a_m|.$$

We call $H(f)$ the height of f . Note that

$$(14) \quad H\left(f(x) \prod_{i=1}^k Q_i(x)\right) \leq H(f) \prod_{i=1}^k S(Q_i),$$

$$(15) \quad S\left(\prod_{i=1}^k Q_i(x)\right) \leq \prod_{i=1}^k S(Q_i)$$

for polynomials $Q_1, Q_2, \dots, Q_k \in \mathbb{Z}[x]$ and a formal power series f . By (15) we have for $j < k$

$$S_{p_1 \dots p_j} \leq (\deg(\Phi_{p_1 \dots p_j}) + 1) A_{p_1 \dots p_j} \leq \epsilon_j \cdot p_j \cdot p_1^{2^{j-2}} p_2^{2^{j-3}} \dots p_{j-2}^2 p_{j-1},$$

as $\deg(\Phi_n) = \varphi(n) < n$ for $n > 1$. Then again by (15)

$$(16) \quad S(P_j) \leq \epsilon_j^{k-j-1} \left(p_j \cdot p_1^{2^{j-2}} p_2^{2^{j-3}} \dots p_{j-2}^2 p_{j-1}\right)^{k-j-1},$$

where P_j is defined in Lemma 1. Additionally,

$$(17) \quad S_{p_1 p_2} < p_1 p_2 / 2, \quad S_{p_1 p_2 p_3} \leq d \cdot p_1^2 p_2 p_3.$$

Applying (14), (16), (17) and Lemma 2 to Lemma 1 we receive

$$\begin{aligned} A_{p_1 \dots p_k} &\leq \frac{b_{k-2} d^{k-4}}{2^{k-3}} \cdot \prod_{j=1}^{k-2} \epsilon_j^{k-j-1} \cdot \prod_{j=1}^{k-2} \left(p_j \cdot p_1^{2^{j-2}} p_2^{2^{j-3}} \dots p_{j-2}^{2^{j-1}} \right)^{k-j-1} \\ &= E_k M_k, \end{aligned}$$

which completes the proof. \square

4. PROOF OF THEOREM 1, 2, 3 AND 4

Proof of Theorem 1. Consider a sequence (e) given by the following conditions:

$$\begin{aligned} e_1 = e_2 = 1, \quad e_3 = e_4 = \epsilon_3, \\ e_k = \frac{b_{k-2} d^{k-4}}{2^{k-3}} \prod_{j=1}^{k-2} e_j^{k-j-1} \quad \text{for } k \geq 5. \end{aligned}$$

By Lemmas 3 and 4 we have $\epsilon_k \leq e_k$. We can easily compute that

$$(18) \quad e_5 = \frac{3}{4} \epsilon_3 d, \quad e_6 = \frac{9}{16} \epsilon_3^3 d^2, \quad \dots$$

For $k \geq 7$

$$\frac{e_k/e_{k-1}}{e_{k-1}/e_{k-2}} = e_{k-2} \cdot \frac{b_{k-2} b_{k-4}}{b_{k-3}^2},$$

then

$$e_k = e_{k-1}^2 \cdot \frac{b_{k-2} b_{k-4}}{b_{k-3}^2},$$

therefore

$$e_k = e_6^{2^{k-6}} \cdot \prod_{i=7}^k \left(\frac{b_{i-2} b_{i-4}}{b_{i-3}^2} \right)^{2^{k-i}}.$$

Note that

$$\frac{b_{i-2} b_{i-4}}{b_{i-3}^2} = \begin{cases} \frac{i-2}{i-1}, & \text{for odd } i \\ \frac{i-2}{i-3}, & \text{for even } i. \end{cases}$$

Then

$$\begin{aligned} e_k &= e_6^{2^{k-6}} \cdot \left(\frac{5}{6} \right)^{2^{k-7}} \cdot \left(\frac{6}{5} \right)^{2^{k-8}} \cdot \left(\frac{7}{8} \right)^{2^{k-9}} \cdot \left(\frac{8}{7} \right)^{2^{k-10}} \dots \\ &= \left(\frac{9}{16} \epsilon_3^3 d^2 \right)^{2^{k-6}} \cdot (1 + o(1)) \prod_{i=4}^{\lfloor k/2 \rfloor} \left(\frac{2i-3}{2i-2} \right)^{2^{k-2i}} \\ &= \left(\frac{3}{4} \epsilon_3^{3/2} d \rho^{1/4} + o(1) \right)^{2^{k-5}}, \end{aligned}$$

which completes the proof of the Theorem 1. \square

Note that (18) implies the bounds from (4).

Proof of Theorem 2. By the well known formula $\Psi_{np}(x) = \Psi_n(x^p)\Phi_n(x)$ we have

$$c_{np}(m) = \prod_{i=1}^{\lfloor m/p \rfloor} c_n(k) a_n(m - kp).$$

We note that $a_n(t) = 0$ for $t \notin \{0, \dots, \varphi(n)\}$, and therefore

$$C_{p_1 \dots p_k} \leq \left(\left\lfloor \frac{\varphi(p_1 \dots p_{k-1})}{p_k} \right\rfloor + 1 \right) A_{p_1 \dots p_{k-1}} C_{p_1 \dots p_{k-1}} \leq p_1 \dots p_{k-2} \cdot A_n C_n$$

for $k \geq 2$. Thus

$$C_{p_1 \dots p_k} \leq C_{p_1 p_2} \prod_{j=2}^{k-1} (p_1 \dots p_{j-1} \cdot A_{p_1 \dots p_j}) \leq \epsilon_2 \dots \epsilon_{k-1} M_k.$$

Therefore

$$\epsilon_k^{inv} \leq \epsilon_2 \dots \epsilon_{k-1} \leq e_2 \dots e_{k-1} = \frac{b_{k-2}}{b_{k-3}} e_k$$

for $k \geq 6$. It completes the proof. \square

We can also prove that

$$\epsilon_4^{inv} \leq \epsilon_3, \quad \epsilon_5^{inv} \leq \epsilon_3^2, \quad \epsilon_6^{inv} \leq \frac{3}{4} \epsilon_3^3 d$$

to justify (5).

Proof of Theorem 3. We recall that every divisor of $x^n - 1$ is of the form $\prod_{d \in D} \Phi_d(x)$, where D is a subset of the set of divisors of n . By (14) and Theorem 1

$$\begin{aligned} B_n &\leq A_n \prod_{d|n, d < n} S_d \leq \frac{2}{n} \prod_{d|n} d A_d \\ &\leq \frac{2}{n} \left(\prod_{d|n} d \right) \left(\prod_{d|n} \epsilon_{\omega(d)} \right) \left(\prod_{d|n} M_k(d) \right), \end{aligned}$$

where $M_k(d) = \prod_{i=1}^{\kappa-2} p_{\delta_i}^{2^{\kappa-i-1}-1}$ for $d = p_{\delta_1} \dots p_{\delta_\kappa}$, $p_{\delta_1} < \dots < p_{\delta_\kappa}$. We have

$$\begin{aligned} \frac{1}{n} \prod_{d|n} d &= n^{2^{k-1}-1}, \\ \prod_{d|n} M_k(d) &\leq \prod_{\omega=1}^k \left(\left((\sqrt[\omega]{n})^\omega \right)^{2^{\omega-1}/\omega-1} \right)^{\binom{k}{\omega}} = n^{(3^k-1)/(2k)-2^{k-1}}. \end{aligned}$$

and by Theorem 1

$$\begin{aligned} \log \left(2 \prod_{d|n} \epsilon_{\omega(d)} \right) &\leq \log 2 + \sum_{d|n} 2^{\omega(d)} \log(C + \xi'_{\omega(d)}) \\ &\sim 3^k \log C + \sum_{\omega=0}^k \binom{k}{\omega} 2^{\omega} \xi_{\omega}, \end{aligned}$$

where $\xi'_{\omega}, \xi_{\omega} \rightarrow 0$ with $\omega \rightarrow \infty$. It remains to prove that the sum equals $o(3^k)$. Indeed,

$$\begin{aligned} \sum_{\omega=0}^k \binom{k}{\omega} 2^{\omega} \xi_{\omega} &\leq \xi_0 \sum_{\omega=0}^{\lfloor \log k \rfloor} \binom{k}{\omega} 2^{\omega} + \xi_{\lfloor \log k \rfloor} \sum_{\omega=0}^k \binom{k}{\omega} 2^{\omega} \\ &= O(2^{\log k} e^{\log^2 k} \log k) + o(3^k) = o(3^k), \end{aligned}$$

and the proof is done. \square

In case $n = p_1 \dots p_k$ and $p_i \not\gg p_{i-1}$ for $i = 2, \dots, k$ Theorem 3 improves the result of N. Kaplan [8] showing that

$$B_n < \prod_{j=1}^{k-1} p_j^{4 \cdot 3^{k-2} - 1} \leq n^{(4 \cdot 3^{k-2} - 1)(k-1)/k}.$$

Proof of Theorem 4. We have $M_1 = M_2 = 1$, so theorem holds for $k = 1, 2$. We prove it by induction on k . We assume that $p_1 < \dots < p_k$. Then for $k \geq 3$

$$\begin{aligned} M_k &\leq p_1^{2^{k-2}-1} \cdot \varphi(p_2 \dots p_k)^{2^{k-2}/(k-1)-1} \\ &= \left(\frac{p_1}{p_1 - 1} \right)^{\frac{2^{k-1}}{k}-1} \cdot \left(\frac{p_1^{k-1}}{\varphi(p_2 \dots p_k)} \right)^{\frac{2^{k-2}}{k-1} - \frac{2^{k-1}}{k(k-1)}} \cdot (\varphi(p_1 \dots p_k))^{\frac{2^{k-1}}{k}-1} \\ &\leq \left(\frac{p_1 + 1}{p_1} \right)^{\frac{2^{k-1}}{k}-1} \cdot \left(\frac{p_1}{p_1 + 1} \right)^{2^{k-2} - \frac{2^{k-1}}{k}} \cdot (\varphi(p_1 \dots p_k))^{\frac{2^{k-1}}{k}-1} \\ &\leq (\varphi(p_1 \dots p_k))^{2^{k-1}/k-1}, \end{aligned}$$

which completes the proof of Theorem 4. \square

5. CONCLUDING REMARKS

We analyze the value of the constant C . It is proved that $\epsilon_3 \in [2/3, 3/4]$, however we do not know the exact value of ϵ_3 . Similarly, we can only estimate the value of d . In the case $\epsilon_3 = 3/4$ and if we have the equality in Lemmas 4 and 5, then $C \approx 0.953$. If the Corrected Beiter Conjecture holds, then $C \approx 0.946$.

Let us remark, that there exist a constant $\epsilon > 0$ such that for $C < \epsilon$ the bound from Theorem 1 is false. Indeed, if p_j is the j -th odd prime number for $j \geq 1$, then

$$1 \leq A_{p_1 \dots p_k} \leq (C + o(1))^{2^k} M_k$$

and therefore

$$C + o(1) \geq M_k^{-2^k} = \prod_{j=1}^{\infty} p_j^{-2^{3-j}} + o(1).$$

Using the prime number theorem we easily obtain that the product is convergent to a positive constant.

Recall the following conjecture of P. T. Bateman, C. Pomerance and R. C. Vaughan [3].

Conjecture 2. *For every k there exist a constant ϵ'_k such that*

$$A_n \geq \epsilon'_k n^{2^{k-1}/k-1}$$

for infinitely many cyclotomic polynomials Φ_n of order k .

If the conjecture is true, one of the most interesting questions is whether the maximal ϵ'_k is of the form $(C' + o(1))^{2^k}$ for some constant $0 < C' < 1$.

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