REALIZABILITY OF POLYTOPES AS A LOW RANK MATRIX COMPLETION PROBLEM

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Abstract. Here we show that the problem of realizing a polytope with specified combinatorics is equivalent to a low rank matrix completion problem. This is comparable to known results reducing realizability to solving systems of multinomial equations and inequalities, but the conditions we give here are more simply stated. We see how this matrix is related to a matrix appearing in a similar result by Díaz.

1. Definitions and Statement of Theorem

We being by briefly reviewing some definitions used in this article. The reader is advised to skim over the words in bold. A **polytope** P is the convex hull of finitely many points $\{v_i\}$ in $\mathbb{R}^d$, $\{\sum_i \lambda_i v_i : \sum_i \lambda_i = 1\}$. The **faces** of P are subsets of P where a linear inequality that is satisfied at every point in P, (a,b) s.t. $\langle a, x\rangle \leq b \ \forall x \in P$, is an equality, $F = \{x \in P : \langle a, x\rangle = b\}$. We call The 0 dimensional faces **vertices** and the $d-1$ dimensional faces **facets**.

A **lattice** is a partially ordered set where every pair of comparables $c\,d$ has a unique minimal upper bound, the **join** denoted $c \vee d$, and a unique maximal lower bound, the **meet** denoted $c \wedge d$. The **face lattice** of a polytope is the partially ordered set consisting of its faces ordered by containment. As the name suggests this is a lattice. A **flag** of a poset is a maximal totally ordered subset. It is not hard to see that in a lattice L with finite flags any subset of comparables C has a unique minimal upper bound $\bigvee C$ and as such there is a unique maximal comparable $\top = \bigvee L$, and likewise a unique maximal lower bound $\bigwedge C$ and minimal comparable $\bot = \bigwedge L$. We say a poset is **graded** when all flags have the same length. The **flag graph** of a poset is a graph consisting of a node for each flag $\mathcal{F}$, and edges connecting $\mathcal{F}$ to other flags containing all but one of $\mathcal{F}$'s comparables $N(\mathcal{F}) = \{\mathcal{F}' : |\mathcal{F} \setminus \mathcal{F}'| = 1\}$. An **abstract polytope lattice** is a lattice satisfying the following conditions:

1 It is a graded lattice.
2 Every interval of length 2 consists of the bounds along with two other elements between them.
3 The flag graph of every interval is connected.

Note that every face lattice is an abstract polytope lattice, but the reverse may not hold. When it does we say the abstract polytope lattice $\mathcal{P}$ is **realizable**. That is when there is some polytope P with face lattice isomorphic $\mathcal{P}$. In this case we call P a **realization** of $\mathcal{P}$. Abstract polytope lattices inherit all of the terminology of faces, vertices, facets, dimension, and containment by associating each comparable f with everything below it $[\bot, f]$ and defining dimension d so the length of flags are $d+1$. If one wants to consider purely combinatorial objects resembling a polytope there are more faithful things one may consider, but abstract polytopes suffice here.

For an abstract polytope lattice $\mathcal{P}$, we call a $|\mathrm{vert}(\mathcal{P})| \times |\mathrm{facet}(\mathcal{P})|$ matrix M with $M_{i,j} = 1$ when vertex i is contained in facet j, $v_i \in F_j$, and $M_{i,j} < 1$ for all other entries, a **filled incidence matrix** of $\mathcal{P}$. We are now ready to state the theorem of this paper.

Theorem 1. *An abstract polytope lattice $\mathcal{P}$ is realizable iff it has a rank d filled incidence matrix.*

2. Background and Remarks

Having an algebraic statement for determining when posets are realizable allows us to at least use Tarski's theorem on decidability of quantified algebraic statements to decide whether a given poset is realizable. The most obvious way to get this is to go directly to the definition, as Grünbaum does in [2].

Theorem 2. *A poset $\mathcal{P}$, given by a collection of subsets of $I := \{1, \cdots, n\}$ ordered by containment that includes $\{i\}$ for each $i \in I$ but not I, is realizable iff there are vectors $v_i \in \mathbb{R}^d$ such that for any subset $f \subsetneq I$ there is a vector $a \in \mathbb{R}^d$ with $\langle a, v_i \rangle = 1$ for $v_i \in f$ and $\langle a, v_i \rangle < 1$ for $v_i \notin f$ iff $f \in \mathcal{P}$.*

We first note that the combinatorics of $\mathcal{P}$ are much more relaxed here than in theorem 1, but this is compensated for by more stringent algebraic conditions. If we put the required vectors v_i in a matrix V and restrict the conditions to just the maximal comparables of $\mathcal{P}$, which would be facets, and put the required vectors a for each in a matrix A, then V^*A would give us a rank d filled incidence matrix of $\mathcal{P}$. The algebraic part of theorem 2 requires us to additionally find such vectors a for all faces, and show that no such vectors exist for all other subsets of I.

There are, however, other simpler statements like this that tell us more about the geometry of what can be realized. Díaz in [1] provides such alternative conditions. To state her theorem we need some additional definitions. Here she considers polytopes in the d-sphere $\mathbf{S}^d = \{x \in \mathbb{R}^d : \|x\| = 1\}$. For us it is enough to define these as the intersection of positive linear combinations of points in $\mathbb{R}^d \times 1$ with the unit sphere in $\mathbb{R}^{d+1}$. Projecting through the origin provides a bijection between polytopes defined this way in the sphere and in the euclidean plane that preserves combinatorics. We may also allow rotations of these. We call a sequence of facets $F_{j_1}, \cdots, F_{j_s}$ of a polytope a **truncated oriented cycle** when $\bigcap_{k=1}^{s} F_{j_k}$ is a face of the polytope, and when $s = d+1$ we call this a **maximal oriented cycle**. We say two maximal oriented cycles have the **same orientation** when the induced flags $\varnothing = \bigcap_{k=1}^{d+1} F_{j_k}, \bigcap_{k=1}^{d} F_{j_k}, \cdots, F_{j_1}, \mathcal{P}$ are an even distance apart in the flag graph. Here we denote the minor of a matrix G with rows $j_1 \cdots j_n$ and columns $k_1 \cdots k_n$ by $G\left[\begin{smallmatrix} j_1 \cdots j_m \\ k_1 \cdots k_n \end{smallmatrix}\right]$.

Theorem 3 (Díaz)**.** *Let G be a $|\mathrm{facet}(\mathcal{P})| \times |\mathrm{facet}(\mathcal{P})|$ matrix with diagonals all 1. There is a polytope $P \subset \mathbf{S}^d$ realizing $\mathcal{P}$ iff G satisfies the following:*

1 For every vertex of $\mathcal{P}$ and all facets $F_{j_1} \cdots F_{j_n}$ incident to it, the submatrix $G\left[\begin{smallmatrix} j_1 \cdots j_n \\ j_1 \cdots j_n \end{smallmatrix}\right]$ has rank d.
2 For every face of rank $d-s$ with $2 \le s \le d+1$ and truncated oriented cycle $F_{j_1}, \cdots, F_{j_s}$ incident to it, $\det(G\left[\begin{smallmatrix} j_1 \cdots j_s \\ j_1 \cdots j_s \end{smallmatrix}\right]) > 0$.
3 For every pair of maximal cycles $F_{j_1}, \cdots, F_{j_{d+1}}$ and $F_{k_1}, \cdots, F_{k_{d+1}}$ with the same orientation, $\det(G\left[\begin{smallmatrix} j_1 \cdots j_{d+1} \\ k_1 \cdots k_{d+1} \end{smallmatrix}\right]) > 0$.

Notice all variables here are existentially quantified. The matrix G is the grammian the corresponding facets' outward normal vectors in $\mathbb{R}^{d+1}$. That is its entries are the inner products of these vectors. This is nice because it tells us what the dyhedral angles of a spherical polytope can be. Díaz also provides a similar theorem for finite volume hyperbolic polytopes.

We construct the polytope P from the filled incidence matrix M in theorem 1 by first computing its compact singular value decomposition $U\Sigma V^* = M$. The entries of M are the inner products of corresponding vertices and covertices of the polytope. The **covertices** are the vertices of P's **polar polytope** $P^* := \{a : \forall x \in P, \langle a, x \rangle \le 1\}$. We will also refer to the facets of an abstract polytope lattice as covertices. P's vertices are given by the rows of $U\sqrt{\Sigma}$, and covertices by the rows of $V\sqrt{\Sigma}$. Each covertex corresponds to a facet of P and scaling it by the distance from the origin to the affine span of this facet gives the outward normal. We can construct the grammian of the vertices and covertices together as follows.

$$\tilde{G} = \begin{bmatrix} \sqrt{MM^*} & M \\ M^* & \sqrt{M^*M} \end{bmatrix} = \begin{bmatrix} U\Sigma U^* & U\Sigma V^* \\ V\Sigma U^* & V\Sigma V^* \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \begin{bmatrix} U^* & V^* \end{bmatrix}$$

Symmetrically rescaling the diagonal blocks of $\tilde{G}$ so the diagonal entries are all 1 gives a Euclidean analog of Díaz's matrix G for polytopes P and P^*. We also note that neither the span of U nor of V can contain the vector $\mathbf{1}$ with all entries equal to 1, in the spaces of appropriate dimension. If we complete these to the full singular value decomposition so that they include a normalized copy of $\mathbf{1}$, then the remaining columns give the gale transform of these polytopes.

Theorem 1 as stated requires us to provide an abstract polytope lattice along with the filled incidence matrix, but the only combinatorial data appearing in the matrix are the incidences between vertices and facets. These are in fact equivalent, and we can generate the lattice with these incidences as we shall see. We could just as well have stated the theorem as only asking for the matrix and requiring the resulting lattice to be an abstract polytope lattice. We get this lattice in a very intuitive way; we just recognize that each face can be identified by the vertices and facets that are incident to it.

3. Reasoning

To see how we generate the lattice $\mathcal{P}$ from the the incidences between vertices and facets we need some definitions. A **bipartite graph** is a graph with nodes in two disjoint parts and edges only between nodes in different parts. A **biclique** is a bipartite graph where every pair with one node from each part is connected by an edge. A **maxbiclique** of a bipartite graph is a maximal set of nodes and the edges between them such that this is a biclique. The **maxbiclique lattice** of a bipartite graph with one part specified is the poset consisting of it's maxbicliques ordered by containment of nodes in the specified part. A **join irreducible** is a comparable of a lattice that cannot be expressed as the join of other comparables, and a **meet irreducible** is similarly defined with order reversed.

Lemma 1. *Every lattice where all flags are finite is isomorphic to the maxbiclique lattice of its meet and join irreducibles. For abstract polytope lattices these are the vertices and covertices respectively.*

The proof will make use of the following fact about lattices. The argument appearing here comes from lemma 2.8 of [3], which proves a slightly more general result.

Lemma 2. *In a lattice with finite flags, every comparable c can be identified uniquely as the join of all join irreducibles below it J or the meet of all meet irreducibles above it M.*

Proof. Suppose this is not the case, then there is some minimal comparable c that is not the join of join irreducibles below it. c cannot itself be a join irreducible since $c = \bigvee\{c\}$, so it can be expressed as $c = \bigvee D$ where each comparable $d \in D$ is below c and as such can be expressed as the join of join irreducibles $d = \bigvee J_d$, so we have $c = \bigvee \bigcup_{d \in D} J_d$. Introducing more join irreducibles can only increase their join, so $c = \bigvee J$, and therefor, there can be no such c. Likewise the dual statement holds for meet irreducibles by symmetry. □

Proof of Lemma 1. The pair (J, M) we get for a comparable c from lemma 2 induces a biclique in the incidence graph of the meet and join irreducibles of the lattice, since for any pair $(j, m) \in (J, M)$, $j \le c \le m$. From the definition for comparables $c_i = \bigvee J_i$, we have that $c_1 \le c_2$ iff $J_1 \subseteq J_2$. Now we only have to see that these bicliques are maximal. Suppose they are not, then by symmetry we can assume there is a join irreducible $j \notin J$ that is not less than c, but is less than all meet irreducibles M of c, $j \nleq c = \bigwedge M$. We see this is impossible, since by the construction of M we have $j \in \bigcap_{m \in M} \{\cdot \le m\} \le c$, so c must correspond to a maxbiclique.

For abstract polytope lattices if a r-face f is not a vertex, $r \ne 0$, then there is some $(r-2)$-face c contained in f, and the interval $[c, f]$ contains exactly two other faces $\{d, e\}$ of dimension $r-1$. This gives $d \vee e = f$, so f is not join irreducible. If f is a vertex then there is only one face below it $\perp$, so it must be join irreducible. By symmetry the facets are the meet irreducibles. □

The proof of theorem 1 works by constructing a polytope and showing that there is an order preserving injection from the abstract polytope lattice given to its face lattice. The following lemma shows that this is sufficient for it to be a realization.

Lemma 3. *An order monomorphism between abstract polytope lattices of the same dimension is an isomorphism.*

Proof. Suppose not, then there are abstract polytope lattices $\mathcal{P}$ and $\mathcal{Q}$ of dimension d with a monomorphism sending $\mathcal{P}$ into $\mathcal{Q}$ that misses some face $f \in \mathcal{Q}$. Without loss of generality we can take $\mathcal{P}$ to be a subset of $\mathcal{Q}$ and the monomorphism to be the identity, otherwise just replace $\mathcal{P}$ with its image. Consider now the flag graphs G of $\mathcal{P}$ and H of $\mathcal{Q}$. Every flag of $\mathcal{P}$ is a flag of $\mathcal{Q}$ and two flags differ by one face in $\mathcal{P}$ iff they do so in $\mathcal{Q}$, so G is an induced subgraph of H. There is some flag in G, and f must belong to some flag of $\mathcal{Q}$, so $\varnothing \ne G \subsetneq H$. By property 2 of abstract polytope lattices, their flag graphs are d-regular. That is every node is connected to d other nodes. And, by property 3 their flag graphs are connected. This means G is a d-regular proper induced subgraph of a connected d-regular graph, namely H, which is impossible.

To see this consider a path from a node that is in G to one that is not. Let n be the last node along this path that is in G. This node must have d neighbors in G and a neighbor that is not in G, the next node in the path, so n must have at least $d+1$ neighbors in H contradicting the fact that H is also d-regular. □

Proof of Theorem 1. We have the "only if" direction immediately since this is just the matrix with entries equal to the inner products of vertices and covertices. For the "if" direction we will construct a realization of the polytope from a realization of its filled incidence matrix. Let $\mathcal{P}$ be a d dimensional abstract polytope lattice, and $M = U\Sigma V^*$ be the compact singular value decomposition of a rank d filled incidence matrix of $\mathcal{P}$. Now, let $\{w_i\}$ be the rows of $W = U\sqrt{\Sigma}$, and $\{h_j\}$ be the rows of $H = V\sqrt{\Sigma}$, and $P = \mathrm{conv}(\{w_i\})$. We will show that P is a realization of $\mathcal{P}$.

Let $\mathcal{M} \subset (I \times J)$ be the pairs of indices for entries of M that are 1. This is the incidence relation between vertices and covertices of $\mathcal{P}$. By lemma 1 each face a of $\mathcal{P}$ can be identified with each maxbiclique of vertices and covertices (I_a, J_a) of $\mathcal{P}$, and moreover w_i is in the hyperplane $h_j^{=1} := \{\langle h_j, \cdot\rangle = 1\}$ for $(i,j) \in \mathcal{M}$, but is in the open half space $h_j^{<1} := \{\langle h_j, \cdot\rangle < 1\}$ for $(i,j) \notin \mathcal{M}$. Also, since M has rank d so does W, and P has dimension at least d.

We will now construct a map from $\mathcal{P}$ to the face lattice of P and show that it is an isomorphism. Let $f_a = P \bigcap_{j\in J_a} h_j^{=1}$ be the face of P we get by intersecting it with the hyperplanes corresponding to covertices of a. We know this is a face of P since these are all supporting hyperplanes of P.

First we see that $a \mapsto f_a$ preserves order. In this context we require the very strong condition that $f_a \subseteq f_b$ iff $a \leq b$. Suppose $a \leq b$, then $J_a \subseteq J_b$ and $\bigcap_{j\in J_a} h_j^{=1} \subseteq \bigcap_{j\in J_b} h_j^{=1}$ so $f_a \subseteq f_b$. For the other direction suppose $a \nleq b$, then there is some $i \in I_a$ but $i \notin I_b$, so $w_i \in f_a$ but $w_i \notin f_b$ and $f_a \nsubseteq f_b$. Thus order is maintained.

We also have that $a \mapsto f_a$ is an injection. To see this consider a pair of faces $a\,b$ of $\mathcal{P}$ that map to the same face $f_a = f_b = f$ of P. With this $w_i \in f \subset h_j^{=1}$ for any $i \in I_a \cup I_b$ and $j \in J_a \cup J_b$, so $m_{ij} = 1$ and $v_i \leq F_j$ where v_i and F_j are the corresponding vertices and facets of $\mathcal{P}$ respectively. Since (I_a, J_a) and (I_b, J_b) are maxbicliques that are subsets of the same biclique $(I_a \cup I_b, J_a \cup J_b)$ we must have that $a = b$, and f is a monomorphism.

This induces an injection from a flag of $\mathcal{P}$ to a totally ordered set of P's faces, which must be of the same size or less. A larger set cannot be injected into a smaller one, so they must be the same size, and P must be of dimension d. Now this is a monomorphism between abstract polytope lattices of the same dimension, and by lemma 3 is therefore an isomorphism. Thus, P is a realization of $\mathcal{P}$. $\square$

References

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