Large B_d-free and union-free subfamilies

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Abstract

For a property Γ and a family of sets $\mathcal{F}$, let $f(\mathcal{F},\Gamma)$ be the size of the largest subfamily of $\mathcal{F}$ having property Γ. For a positive integer m, let $f(m,\Gamma)$ be the minimum of $f(\mathcal{F},\Gamma)$ over all families of size m. A family $\mathcal{F}$ is said to be B_d-free if it has no subfamily $\mathcal{F}' = \{F_I : I \subseteq [d]\}$ of 2^d distinct sets such that for every $I, J \subseteq [d]$, both $F_I \cup F_J = F_{I\cup J}$ and $F_I \cap F_J = F_{I\cap J}$ hold. A family $\mathcal{F}$ is a-union free if $F_1 \cup \ldots F_a \neq F_{a+1}$ whenever $F_1, \ldots, F_{a+1}$ are distinct sets in $\mathcal{F}$. We verify a conjecture of Erdős and Shelah that $f(m, B_2\text{-free}) = \Theta(m^{2/3})$. We also obtain lower and upper bounds for $f(m, B_d\text{-free})$ and $f(m, a\text{-union free})$.

1 Introduction, results

Moser proposed the following problem: Let $A_1, A_2 \ldots, A_m$ be a collection of m sets. A subfamily $A_{i_1}, A_{i_2} \ldots, A_{i_r}$ is *union-free* if $A_{i_{j_1}} \cup A_{i_{j_2}} \neq A_{i_{j_3}}$ for every triple of distinct

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sets $A_{j_1}, A_{j_2}, A_{j_3}$ with $1 \le j_1 \le r$, $1 \le j_2 \le r$, and $1 \le j_3 \le r$. Erdős and Komlós [1] considered the following problem of Moser: what is the size of the largest union-free subfamily $A_{i_1}, \dots, A_{i_r}$?

Put $f(m) = \min r$, where the minimum is taken over all families of m distinct sets. As mentioned in [1], Riddel pointed out that $f(m) > c\sqrt{m}$. Erdős and Komlós [1] showed $\sqrt{m} \le f(m) \le 2\sqrt{2}\sqrt{m}$. Kleitman proved $\sqrt{2m} - 1 < f(m)$; Erdős and Shelah [2] obtained

$$f(m) < 2\sqrt{m} + 1. \tag{1}$$

The latter two conjectured $f(m) = (2 + o(1))\sqrt{m}$.

We define $f(\mathcal{F}, \Gamma)$ as the size of the largest subfamily of $\mathcal{F}$ having property Γ,

$$f(\mathcal{F}, \Gamma) := \max\{|\mathcal{F}'| : \mathcal{F}' \subseteq \mathcal{F},\ \ \mathcal{F}' \text{ has property } \Gamma\}.$$

In this context, $f(E(K_r^n), \mathcal{H}\text{-free})$ is the Turán number $\mathrm{ex}_r(n, \mathcal{H})$. Let $f(m, \Gamma) = \min\{f(\mathcal{F}, \Gamma) : |\mathcal{F}| = m\}$. Generalizing the union-free property, a family $\mathcal{F}$ is *a-union free* if there are no distinct sets $F_1, F_2 \dots, F_{a+1}$ satisfying $F_1 \cup F_2 \cup \dots \cup F_a = F_{a+1}$.

Erdős and Shelah [2] also considered Γ to be the property that no four distinct sets satisfy $F_1 \cup F_2 = F_3$ and $F_1 \cap F_2 = F_4$. Such families are called *B_2-free*. Erdős and Shelah [2] gave an example showing $f(m, B_2\text{-free}) \le (3/2)m^{2/3}$ and they also conjectured $f(m, B_2\text{-free}) > c_2 m^{2/3}$.

A family $\mathcal{B}$ of 2^d distinct sets is forming a Boolean algebra of dimension d if the sets can be indexed with the subsets of $[d] = \{1, 2, \dots, d\}$ so that $B_I \cap B_J = B_{I \cap J}$ and $B_I \cup B_J = B_{I \cup J}$ hold for any $I, J \subseteq [d]$. If $\mathcal{F}$ does not contain any subfamily forming a Boolean algebra of dimension d, then it is called *B_d-free*, or we say that $\mathcal{F}$ *avoids* any Boolean algebra of dimension d. A result by Gunderson, Rödl, and Sidorenko [4] states that $f(2^{[n]}, B_d\text{-free}) = \Theta(2^n/n^{2^{-d}})$. In Section 2, we prove the aforementioned conjecture by Erdős and Shelah in the following more general form.

Theorem 1.1. *For any integer d, $d \ge 2$, there exist constants $c_d, c'_d > 0$, and exponents*

$$e_d := \frac{2^d - \lceil \log_2(d+2) \rceil}{2^d - 1}, \quad e'_d := \frac{2^d - 2}{2^d - 1}$$

such that

$$c_d m^{e_d} \le f(m, B_d\text{-free}) \le c'_d m^{e'_d}.$$

In particular,

$$(3 \cdot 2^{-7/3} + o(1))m^{2/3} \le f(m, B_2\text{-free}) \le \frac{3}{2} m^{2/3}. \tag{2}$$

In Section 4 we consider a-union free families. We generalize the construction giving (1) and prove the following

Theorem 1.2. *For any integer a, $a \geq 2$,*

$$\sqrt{2m} - \frac{1}{2} \leq f(m, a\text{-union free}) \leq 4a + 4a^{1/4}\sqrt{m}. \tag{3}$$

Since the first version of this manuscript, Fox, Lee, and Sudakov [3] verified the present authors' conjecture (see later as Problem 5) and proved a matching lower bound showing that $f(m, a\text{-union free}) \geq \max\{a, \frac{1}{3}\sqrt[4]{a}\sqrt{m})\}$.

2 Subfamilies avoiding Boolean algebras of dimension d

In this section we prove the lower bounds in Theorem 1.1 by a probabilistic argument applying the first moment method.

Suppose that $\mathcal{B} = \{B_I : I \subseteq [d]\}$ is forming a Boolean algebra of dimension d. Thus we have nonempty, pairwise disjoint sets, $A_0, A_1, \ldots, A_d$, called *atoms*, such that $B_I = A_0 \cup \{A_i : i \in I\}$. A subfamily $\mathcal{C} \subseteq \mathcal{B}$ *determines* the Boolean algebra $\mathcal{B}$ if every member of $\mathcal{B}$ can be obtained as a Boolean expression (using unions, intersections, differences, but not complements) of some sets of $\mathcal{C}$. Obviously, the d sets of the form $\{A_0 \cup A_i : i \in [d]\}$ determine $\mathcal{B}$. Much more is true.

Lemma 2.1. *Suppose that the sets of $\mathcal{B}$ are forming a Boolean algebra of dimension d. Then there exists a subfamily $\mathcal{C} \subseteq \mathcal{B}$ of size $\lceil \log_2(d+2) \rceil$ and determining $\mathcal{B}$. Moreover, there is no subfamily of smaller size with the same property.*

Proof. Let $k := \lceil \log_2(d+2) \rceil$. We define an appropriate $\mathcal{C}$ of size k by considering a standard construction used for non-adaptive binary search. Namely, write each integer $i \in [d]$ in base 2, $i = \sum_{1 \leq j \leq k} \varepsilon_{i,j} 2^{j-1}$ and define $C_j = A_0 \cup \{A_i : \varepsilon_{i,j} = 1\}$, $j = 1, 2, \ldots, k$.

On the other hand, any $\mathcal{C}$ determines at most $2^{|\mathcal{C}|} - 1$ nonempty atoms, we obtain $2^{|\mathcal{C}|} - 1 \geq d + 1$. $\square$

Corollary 2.2. *Given any family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of m sets, $\mathcal{F}$ contains at most $\binom{m}{\lceil \log(d+2) \rceil}$ subfamilies forming a Boolean algebra of dimension d.*

Lemma 2.1 gives the right order of magnitude on the number of possible subfamilies forming a Boolean algebra of dimension d contained in a family of m sets, as shown by the family $\mathcal{F} = 2^{[n]}$, where $m = 2^n$.

Proof of the lower bound in **Theorem 1.1**. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be any family of m sets. Let us consider a random subfamily $\mathcal{F}'$, that is, we select every set in $\mathcal{F}$

independently with probability p. Let X be the random variable denoting the number of sets in $\mathcal{F}'$, and let Y be the random variable denoting the number of subfamilies in $\mathcal{F}'$ forming a Boolean algebra of dimension d. By Corollary 2.2,

$$\mathbb{E}(X-Y) \geq mp - p^{2^d}\binom{m}{\lceil \log_2(d+2)\rceil}.$$

If we remove a set from each subfamily in $\mathcal{F}'$ forming a Boolean algebra of dimension d, then we obtain a B_d-free subfamily $\mathcal{F}''$ of size at least $X-Y$. Substituting $p = m^{e_d}$ where $e_d = \frac{\lceil \log(d+2)\rceil - 1}{2^d-1}$ yields the lower bound. To get a better constant in the case $d=2$, put $p = 2^{-1/3}m^{-1/3}$. ∎

One might try to improve the constant of the lower bound by improving Lemma 2.1 for families without large chains and antichains. However, the construction of Erdős and Shelah shows, one cannot hope for anything better than $(\frac{1}{2}+o(1))\binom{m}{2}$, which would improve the constant of the lower bound in (2) only to 3/4.

3 Upper bound using Turán theory

In this section we prove the upper bounds in Theorem 1.1 by generalizing the ideas of Erdős and Shelah [2].

Let $\mathcal{K}(a_1,\ldots,a_d)$ denote the complete, d-partite hypergraph with parts of sizes $a_1,\ldots,a_d$, i.e., $V(\mathcal{K}) := X_1 \cup \cdots \cup X_d$ where $X_1,\ldots,X_d$ are pairwise disjoint sets with $|X_i| = a_i$, and $E(\mathcal{K}) := \{E : |E| = d,\ |X_i \cap E| = 1 \text{ for all } i \in [d]\}$. For short we use $\mathcal{K}_d^{(k)}$ for $\mathcal{K}(k, k^2, \ldots, k^{2^{d-1}})$ and K_{d*2} for $\mathcal{K}(2,\ldots,2)$. The (generalized) *Turán number* of the d-uniform hypergraph $\mathcal{H}$ with respect to the other hypergraph $\mathcal{G}$, denoted by $\mathrm{ex}(\mathcal{G},\mathcal{H})$, is the size of the largest $\mathcal{H}$-free subhypergraph of $\mathcal{G}$.

Theorem 3.1. *For $k, d \geq 2$, $\mathrm{ex}(\mathcal{K}_d^{(k)}, K_{d*2}) < \left(2 - \frac{1}{2^{d-1}}\right)k^{2^d-2}$.*

Proof. We proceed by induction on d. Let $d=2$, and let H be a $K_{2,2}$-free subgraph of K_{k,k^2}. Let $v_1, v_2, \ldots, v_{k^2}$ be the vertices of the larger part of K_{k,k^2}, and $d_i := \deg_H(v_i)$. Each pair of vertices in the smaller part of K_{k,k^2} has at most one common neighbor in H. Therefore, $\sum\binom{d_i}{2} \leq \binom{k}{2}$. It yields

$$|E(H)| = \sum_{i=1}^{k^2} d_i \leq \sum_{i=1}^{k^2}\left(\binom{d_i}{2}+1\right) \leq \binom{k}{2} + k^2.$$

Fix d, $d>2$, and a K_{d*2}-free subhypergraph $\mathcal{H}$ of $\mathcal{K}_d^{(k)}$. Let v_i $1 \leq i \leq k^{2^{d-1}}$ be the vertices of the largest part of $\mathcal{K}_d^{(k)}$, and $d_i := \deg_{\mathcal{H}}(v_i)$. Let $\mathcal{H}_i$ be the $(d-1)$-uniform

$(d-1)$-partite hypergraph, which we get by taking the set of edges of $\mathcal{H}$ containing v_i and deleting v_i from all of them. We have $|\mathcal{H}_i| = d_i$. The hypergraph $\mathcal{H}_i$ contains at least $d_i - \mathrm{ex}(\mathcal{K}_{d-1}^{(k)}, K_{(d-1)*2})$ copies of $K_{(d-1)*2}$. Since $\mathcal{H}$ is K_{d*2}-free, each copy of $K_{(d-1)*2}$ belongs to no more than one of the hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{k^{2^{d-1}}}$. This implies

$$\sum_{i=1}^{k^{2^{d-1}}} \left[d_i - \left(2 - \frac{1}{2^{d-2}}\right) k^{2^{d-1}-2} \right] \leq \binom{k}{2}\binom{k^2}{2}\cdots\binom{k^{2^{d-2}}}{2} < \frac{k^{2(2^{d-1}-1)}}{2^{d-1}},$$

and the claim follows by rearranging the inequality. □

Proof of the upper bound in **Theorem 1.1**. For $m = k^{2^d-1}$ we define a family $\mathcal{F}$ of size m such that every subfamily $\mathcal{F}'$ avoiding B_d has size at most $2k^{2^d-2}$. Then $f(m, B_d\text{-free}) \leq O(m^{e'_d})$ follows for all m by the monotonicity of f.

Let $\mathcal{F}$ be a product of d chains, the ith of which has size $k^{2^{i-1}}$, i.e., for $1 \leq i \leq d, 1 \leq j \leq k^{2^{i-1}}$, let S_j^i be sets satisfying
- $|S_j^i| = j$, $S_{j_1}^i \subset S_{j_2}^i$ if $j_1 \leq j_2$,
- $S_{k^{2^{i-1}}}^i \cap S_{k^{2^{j-1}}}^j = \emptyset$ if $i \neq j$, and
- $\mathcal{F} := \{S_{j_1}^1 \cup S_{j_2}^2 \cup \cdots \cup S_{j_d}^d : 1 \leq i \leq d, 1 \leq j_i \leq k^{2^{i-1}}\}$.

Each set in $\mathcal{F}$ corresponds to a hyperedge in $\mathcal{K}_d^{(k)}$, and each copy of B_d in $\mathcal{F}$ corresponds to a copy of $\mathcal{K}_{d*2}$ in $\mathcal{K}_d^{(k)}$. The B_d-free subfamilies of $\mathcal{F}$ correspond to $\mathcal{K}_{d*2}$-free subhypergraphs of $\mathcal{K}_d^{(k)}$. The bound in Theorem 3.1 on the size of a $\mathcal{K}_{d*2}$-free subfamily completes the proof. □

4 Union-free subfamilies

Proof of **Theorem 1.2**. The proof of our lower bound is based on Kleitman [6], the proof by Erdős and Shelah [2] does not work in the general a-union free setting.

Let $\mathcal{F}$ be an arbitrary family of size m and let ℓ be the size of a longest chain in it. Split $\mathcal{F}$ according the rank of the sets, $\mathcal{F} = \cup_{1 \leq k \leq \ell} \mathcal{F}_k$. Each $\mathcal{F}_k$ together with a chain of size k with a top member from $\mathcal{F}_k$ form an a-union free subfamily implying $f(\mathcal{F}, a\text{-union free}) \geq |\mathcal{F}_k| + k - 1$ for all k. Adding up we have $\ell \times f \geq m + \binom{\ell}{2}$ implying $f(\mathcal{F}, a\text{-union free}) \geq |\mathcal{F}|/\ell + (\ell-1)/2$. Since the lower bound by Fox, Lee, and Sudakov [3] supersedes ours, we omit the details.

For the proof of the upper bound (3), first we consider the family $\mathcal{F}_{ES}(k)$ of size k^2, what Erdős and Shelah [2] used to obtain the upper bound (1) on $f(k^2, 2\text{-union free})$. The family $\mathcal{F}_{ES}$ is a product of two vertex disjoint chains of lengths k, that is, given the chains $\emptyset \neq A_1 \subset A_2 \subset \cdots \subset A_k$ and $\emptyset \neq B_1 \subset B_2 \subset \cdots \subset B_k$ with $A_k \cap B_k = \emptyset$ we define $\mathcal{F}_{ES}(k) := \{A_i \cup B_j : 1 \leq i, j \leq k\}$. We have $|\mathcal{F}_{ES}| = k^2$.

Lemma 4.1. *If $\mathcal{G}$ is an a-union free subfamily of $\mathcal{F}_{ES}(k)$, then*

$$|\mathcal{G}| \leq 2(\lceil\sqrt{a+1}\rceil - 1)k.$$

Proof. Associate a point set P of the 2-dimensional grid to the family $\mathcal{G}$ as $P := \{(i,j) : \text{when } A_i \cup B_j \in \mathcal{G}\}$. The rectangle $R(i,j)$ is defined as $R(i,j) := \{(x,y) : 1 \leq x \leq i \text{ and } 1 \leq y \leq j\}$. The set $A_i \cup B_j$ is a union of a distinct members of $\mathcal{G}$ if and only if the rectangle $R = R(i,j)$ contains at least a distinct points apart from (i,j) and at least one of these lies on the top boundary of R, i.e., on the segment $[(1,j),(i,j)]$ and at least one on the rightmost column $[(i,1),(i,j)]$.

Construct $P' \subseteq P$ by deleting the bottom $\lceil\sqrt{a+1}\rceil - 1$ elements of P in each column of the grid. Suppose that P' has a row with at least $\lceil\sqrt{a+1}\rceil$ elements, and let (i,j) be the rightmost point. Then P has at least $\lceil\sqrt{a+1}\rceil^2 \geq a+1$ points in the rectangle $R(i,j)$, also points on the top and the right most sides, a contradiction. Therefore, P has at most $2(\lceil\sqrt{a+1}\rceil - 1)k$ elements. □

Now we are ready to define a family $\mathcal{F}$ of size qk^2, such that

$$f(\mathcal{F}, a\text{-union free}) < a - 2 + 2k(\lceil\sqrt{a+1}\rceil - 1) + (2k-1)(q-1). \tag{4}$$

The family $\mathcal{F}$ consists of q *levels*, each of them isomorphic to $\mathcal{F}_{ES}(k)$. For all $1 \leq \ell \leq q$, let $\emptyset \neq A_1^\ell \subset A_2^\ell \subset \cdots \subset A_k^\ell$ and $\emptyset \neq B_1^\ell \subset B_2^\ell \subset \cdots \subset B_k^\ell$ be chains of length k such that the $2q$ top sets A_k^ℓ and $B_k^{\ell'}$ are pairwise disjoint. Let us define

$$\mathcal{F}_\ell = \left\{ \bigcup_{s=1}^{\ell-1} (A_k^s \cup B_k^s) \cup A_i^\ell \cup B_j^\ell : 1 \leq i,j \leq k \right\} \text{ and } \mathcal{F} := \bigcup_{\ell=1}^{q} \mathcal{F}_\ell.$$

Observe that $|\mathcal{F}| = m = qk^2$ and indeed each $\mathcal{F}_\ell$ is isomorphic to $\mathcal{F}_{ES}$. Note that if $\ell < \ell'$ and $F \in \mathcal{F}_\ell, F' \in F_{\ell'}$ then $F \subset F'$. Let $\mathcal{G}$ be an a-union free subfamily of $\mathcal{F}$ and let us write $\mathcal{G}_\ell = \mathcal{G} \cap \mathcal{F}_\ell$. Let t be the smallest integer with $\sum_{\ell=1}^{t} |\mathcal{G}_\ell| \geq a-2$. If there exists no such t, then $|\mathcal{G}| < a-2$, and we are done. We have:
- $\sum_{\ell=1}^{t-1} |\mathcal{G}_\ell| < a-2$, by the definition of t,
- $|\mathcal{G}_t| \leq 2(\lceil\sqrt{a+1}\rceil - 1)k$ by Lemma 4.1 since $\mathcal{F}_t$ is isomorphic to $\mathcal{F}_{ES}$,
- the family $\mathcal{G}_\ell$ is 2-union free for each ℓ with $t < \ell \leq k$.

To see the latest statement, suppose, on the contrary, that $G' \cup G'' = G$ for some $G, G', G'' \in \mathcal{G}_\ell$. Pick any $a-2$ sets $G_1, G_2, \ldots, G_{a-2}$ from $\cup_{s=1}^{t} \mathcal{G}_s$, and we have $G = G' \cup G'' \cup G_1 \cup \cdots \cup G_{a-2}$, contradicting $\mathcal{G}$ being a-union free. Therefore $|\mathcal{G}_\ell| \leq 2k-1$ by a slight strenghtening of the result of Erdős and Shelah (see [3]). Putting these observations together, using $|\mathcal{G}| = \sum |\mathcal{G}_\ell|$ and $t \geq 1$, we obtain (4). Finally, substituting $q = \lceil\sqrt{a+1}\rceil$ and $k = \lceil\sqrt{m/q}\rceil$ into (4) we have $f(m, a\text{-union free}) \leq a + (4k-1)(2q-1)$. A little calculation yields (3). □

5 Problems, concluding remarks

Conjecture 5.1. *If $m = 2^n$, then the family consisting of m sets that contains the highest number of subfamilies forming a Boolean algebra of dimension d is $2^{[n]}$.*

In Theorem 3.1 we have considered d-partite hypergraphs with very uneven part sizes. There is a number of results of this type, see, e.g., Győri [5]. Also the sizes grow exponentially, one can easily generalize it for other sequences.

Concerning a-union free families we had the modest conjecture

$$\lim_{a\to\infty}\left(\liminf_{m\to\infty}\frac{f(m, a\text{-union free})}{\sqrt{m}}\right)\to\infty \tag{5}$$

Knowing the results of Fox, Lee, and Sudakov [3] it is natural to ask

Problem 5.2. Given a, what is the limit

$$\lim_{m\to\infty}\frac{f(m, a\text{-union free})}{a^{1/4}\sqrt{m}}?$$

If it exists, it is between 1/3 and 4.

One can improve the coefficient 4 of the factor $a^{1/4}$ in Theorem 1.2 if in Section 4 we use different sizes. Namely we construct $\mathcal{F}$ by using $\mathcal{F}_\ell = \mathcal{F}_{ES}(k_\ell)$ where $k_\ell = k\left(\frac{b-1}{b-2}\right)^{2(\ell-1)}$ with $b = \lceil\sqrt{a+1}\rceil$. If q/b tends to infinity, we obtain

$$f(m, a\text{-union free}) \le \sqrt{8}a^{1/4}\sqrt{m} + O(a).$$

A family $\mathcal{F}$ is (a,b)-*union free* if there are no distinct sets $F_1, F_2 \dots, F_{a+b}$ satisfying $F_1\cup F_2\cup\dots\cup F_a = F_{a+1}\cup\dots\cup F_{a+b}$. This is another frequently investigated property. However $f(m,(a,b)\text{-free}) = a+b-1$ if $a, b \ge 2$, as it is shown by the family consisting of all $(m-1)$-subsets of an m-set.

Many more problems remained open.

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