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NON-COMPACT VERSIONS OF EDWARDS' THEOREM

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Abstract. Edwards' Theorem establishes duality between a convex cone in the space of continuous functions on a compact space and the set of representing or Jensen measures for this cone. In this paper we prove non-compact versions of this theorem.

1. Introduction

Let X be a Hausdorff topological space X and let $C(X)$ be the set of all continuous functions on X with the topology of uniform convergence on compacta. With each convex cone $\mathcal{S} \subset C(X)$ containing constants and a point $x \in X$ we associate the set $J_x^{\mathcal{S}}$ of $\mathcal{S}$-Jensen measures which are probability measures μ with compact support such that $\mu(\phi) \geq \phi(x)$ for all $\phi \in \mathcal{S}$. If ϕ is a function on X then its $\mathcal{S}$-envelope is

$$\mathcal{S}(\phi) = \sup\{\psi : \psi \in \mathcal{S}, \psi \leq \phi\}.$$

In 1965 Edwards proved ([E]) the following duality theorem:

Theorem 1.1 (Edwards' Theorem)**.** *Let X be compact and let ϕ be a lower semicontinuous function on X. Then*

$$\mathcal{S}(\phi)(x) = \inf\{\mu(\phi) : \mu \in J_x^{\mathcal{S}}(X)\}.$$

Moreover, the infimum is attained.

This theorem found many applications in uniform algebras and pluri-potential theory (see [Ga, G, W]). The theorem does not hold when X is not compact (see an example in Section 3). However, the third author proved in [P] the following

Theorem 1.2. *Let X be a domain in $\mathbb{C}^n$ and let $\mathcal{S}$ be the cone of continuous plurisubharmonic functions on X. If ϕ is an upper semicontinuous function on X, then*

$$\widetilde{\mathcal{S}}(\phi)(x) = \inf\{\mu(\phi) : \mu \in J_x^{\mathcal{S}}(X)\},$$

where

$$\widetilde{\mathcal{S}}(\phi) = \sup\{u : u \text{ is plurisubharmonic on } X, u \leq \phi\}.$$

Moreover, the function $\widetilde{\mathcal{S}}(\phi)$ is plurisubharmonic.

While both theorems are almost identically shaped there are crucial differences. The space in the second theorem is not compact and lower semicontinuity is replaced by upper semicontinuity. Moreover, the replacement is natural because plurisubharmonic functions are upper semicontinuous by definition.

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So the question appears: what is the natural version of Edwards' Theorem on non-compact spaces? In this paper we provide an answer to this question. Since the original proof in [E] is based on the description of superlinear positive functionals on $C(X)$ in Section 2 we give the description of such functionals on $C(X)$ when X is a locally compact Hausdorff space countable at infinity. It allows us in Section 3 to prove the first non-compact version of Edwards' Theorem.

Theorem 1.3. *Let X be a locally compact Hausdorff space countable at infinity. If $\phi \in C(X)$ then either*

$$\mathcal{S}\phi(x) = \inf\{\mu(\phi) : \mu \in J_x^{\mathcal{S}}\phi\}$$

or $\mathcal{S}(\phi) \equiv -\infty$.

As an example at the same section shows the dichotomy in this theorem cannot be resolved. To get rid of it we introduce the notion of lower semicontinuous multifunctions. Let $\mathcal{P}(X)$ be the set of regular Borel probability measures on X with compact support. A multifunction (or a set) $J \subset X \times \mathcal{P}(X)$ is *lower semicontinuous* if for any $x \in X$ and $\mu \in J_x = \{\nu : (x,\nu) \in J\}$ and every neighborhood V of μ in $C^*(X)$ there is a neighborhood W of x in X such that the natural projection of $(X \times V) \cap J$ onto X contains W (see [G]).

As it happens the multifunction $J^{\mathcal{S}} = \{(x,\mu) : x \in X, \mu \in J_x^{\mathcal{S}}\}$ is lower semicontinuous when X is a domain in $\mathbb{R}^n$ or $\mathbb{C}^n$ and $\mathcal{S}$ is the cone of continuous subharmonic functions or continuous plurisubharmonic functions. It is easy to see this because a small translation $\mu_y(\phi) = \mu(\phi(x+y))$ of an $\mathcal{S}$-Jensen measure is also $\mathcal{S}$-Jensen.

Under this assumption we obtain another non-compact version of Edwards' Theorem. Given a convex cone $\mathcal{S} \subset C(X)$ we denote by $\widetilde{\mathcal{S}}$ a convex cone of upper semicontinuous functions ϕ on X such that $\mu(\phi) \geq u(x)$ for every $x \in X$ and every $\mu \in J_x^{\mathcal{S}}$.

Theorem 1.4. *Let X be a locally compact space countable at infinity and let $\mathcal{S} \in C(X)$ be a convex cone containing constants. If the multifunction $J^{\mathcal{S}}$ is lower semicontinuous then*

$$\widetilde{\mathcal{S}}(\phi)(x) = \inf\{\mu(\phi) : \mu \in J_x^{\mathcal{S}}\}$$

whenever ϕ is an upper semicontinuous function on X. Moreover, $\widetilde{\mathcal{S}}(\phi) \in \widetilde{\mathcal{S}}$.

2. Superlinear operators

A functional F mapping $C(X)$ into $[-\infty, \infty)$ is called *superlinear* if:

(1) $F(c\phi) = cF(\phi), \quad c \geq 0$;
(2) $F(\phi_1 + \phi_2) \geq F(\phi_1) + F(\phi_2)$,

and *positive* if $F(\phi) \geq 0$ when $\phi \geq 0$.

If F is superlinear then $F(-\phi) \leq -F(\phi)$ because

$$F(-\phi) + F(\phi) \leq F(-\phi + \phi) = 0.$$

If, additionally, F is positive then $F(\phi_1) \leq F(\phi_2)$ if $\phi_1 \leq \phi_2$, because

$$F(\phi_2) = F(\phi_2 - \phi_1 + \phi_1) \geq F(\phi_2 - \phi_1) + F(\phi_1).$$

In further, we will consider spaces $C(X)$ of continuous functions on a locally compact Hausdorff space X countable at infinity. This means that every point of X has a neighborhood with the compact closure and X is the union of countably many compact sets. The space $C(X)$ will be endowed with the topology of uniform convergence on compacta.

We will need two facts about spaces above.

Lemma 2.1. *Let X be a locally compact Hausdorff space countable at infinity. Then:*

(1) *X is the union of compact sets X_j, $j = 1, 2, \dots$, such that each X_j lies in the interior X^o_{j+1} of X_{j+1};*
(2) *X is normal.*

Proof. (1) By the definition X is the union of an increasing sequence of compact sets K_j. Since X is locally compact we can cover K_1 by finitely many open sets V_m, $1 \leq m \leq n$, with compact closures and let $X_1 = K_1 \cup (\cup_{m=1}^n \overline{V}_j)$. Note that X_1 is compact. Let j_1 be the first number such that $K_{j_1} \setminus X_1 \neq \emptyset$. We cover $K_{j_1} \cup X_1$ by finitely many open sets W_m, $1 \leq m \leq n$, with compact closures and let $X_2 = K_{j_1} \cup X_1 \cup (\cup_{m=1}^n \overline{W}_j)$. Note that X_1 lies in the interior of X_2. Continuing this procedure we cover X by countably many compact sets X_j such that each X_j lies in the interior of X_{j+1}.

(2) Let F and G be closed disjoint sets in X. We take compact sets X_j from (1) and let $F_j = F \cap X_j$ and $G_j = G \cap X_j$. Since any compact Hausdorff space is normal there are open sets U_j and V_j in X such that $F_j \subset U_j$, $G_j \subset V_j$ and $U_j \cap V_j \cap X_j = \emptyset$. Define $U'_j = U_j \cap X^o_j$ and $V'_j = V_j \cap X^o_j$. Now we let $\tilde{U}_1 = U'_1$ and $\tilde{V}_1 = V'_1$ and define by induction

$$\tilde{U}_{j+1} = \tilde{U}_j \cup (U'_{j+1} \setminus X_j) \cup (U_j \cap U'_{j+1} \cap \partial X_j)$$

and

$$\tilde{V}_{j+1} = \tilde{V}_j \cup (V'_{j+1} \setminus X_j) \cup (V_j \cap V'_{j+1} \cap \partial X_j).$$

Clearly, $\tilde{U}_j \subset \tilde{U}_{j+1}$, $\tilde{V}_j \subset \tilde{V}_{j+1}$ and $\tilde{U}_j \cap \tilde{V}_j = \emptyset$. Let us show by induction that the sets $\tilde{U}_{j+1}$ and $\tilde{V}_{j+1}$ are open. Set $X_0 = \emptyset$. Note that $\tilde{U}_1$ is open and $U'_1 \setminus X_0 \subset \tilde{U}_1$. Suppose that $\tilde{U}_j$ is open and contains $U'_j \setminus X_{j-1}$. Since the sets $\tilde{U}_j$ and $U'_{j+1} \setminus X_j$ are open, to show that the set $\tilde{U}_{j+1}$ is open it suffices to show that any point $x \in U_j \cap U'_{j+1} \cap \partial X_j$ has a neighborhood W lying in $\tilde{U}_{j+1}$. For this we take $W \subset (U_j \cap U'_{j+1}) \setminus X_{j-1}$. Now $W \setminus X_j \subset U'_{j+1} \setminus X_j \subset \tilde{U}_{j+1}$ and

$$W \cap X^o_j \subset (U_j \setminus X_{j-1}) \cap X^o_j = U'_j \setminus X_{j-1} \subset \tilde{U}_j.$$

Hence $W \subset \tilde{U}_{j+1}$. Thus the sets $\tilde{U}_j$ are open. The same reasoning shows that the sets $\tilde{V}_j$ are also open.

Evidently, $\tilde{U}_j$ and $\tilde{V}_j$ form increasing sequences of open sets, $F_j \subset \tilde{U}_{j+1}$ and $G_j \subset \tilde{V}_{j+1}$. Moreover, the sets $\tilde{U}_j$ and $\tilde{V}_j$ are disjoint. Hence, the sets $U = \cup_j \tilde{U}_j$ and $V = \cup_j \tilde{V}_j$ are disjoint and $F \subset U$ and $G \subset V$. □

The main advantage of a locally compact Hausdorff space X countable at infinity is the following lemma claiming that any positive linear functional on $C(X)$ has a compact support.

Lemma 2.2. *Let X be a locally compact Hausdorff space countable at infinity. If F is a positive linear functional on $C(X)$, then there is a compact set $K \subset X$ such that $F\phi = 0$ whenever $\phi|_K = 0$.*

Proof. If X is compact then there is nothing to prove. Suppose that X is not compact. Then by Lemma 2.1 X is the union of compact sets X_j, $j = 1, 2, \ldots,$ such that each X_j lies in the interior X^o_{j+1} of X_{j+1}. Let us show that there is j_0 such that $F(\phi) = 0$ whenever $\operatorname{supp}\phi \subset X \setminus X_{j_0}$ and $\phi \geq 0$. If not then for every j there is $\phi_j \in C(X)$ such that $\operatorname{supp}\phi_j \subset X \setminus X_j$, $\phi_j \geq 0$ and $F(\phi_j) \neq 0$. Multiplying ϕ_j by an appropriate positive constant we also may assume that $F(\phi_j) = 1$.

The function $\phi = \sum \phi_j$ is defined and is continuous because every point has a neighborhood where only finitely many functions $\phi_j \not\equiv 0$. Since $\phi \geq \sum_{j=1}^n \phi_j$ we see that $F(\phi) \geq n$. Thus F is not defined on ϕ and this contradiction proves the statement.

If $\phi \in C(X)$ is arbitrary then $\phi = \phi^+ + \phi^-$, where $\phi^+ = \max\{\phi, 0\}$ and $\phi^- = -\max\{-\phi, 0\}$. If $\operatorname{supp}\phi \subset X \setminus X_{j_0}$ then $\operatorname{supp}\phi^*$ and $\operatorname{supp}\phi^-$ lie in $X \setminus X_{j_0}$. Hence $F(\phi^+) = F(\phi^-) = 0$. Thus $F(\phi) = 0$. □

The following proposition describes positive linear functional on $C(X)$.

Proposition 2.3. *Let X be a locally compact space Hausdorff countable at infinity. If μ is a positive linear functional on $C(X)$, then $\mu \in C^*(X)$. Moreover, there is a regular Borel measure with compact support which we will denote also by μ such that*

$$\mu(\phi) = \int \phi \, d\mu.$$

Proof. By Lemma 2.2 and linearity of F there is a compact set $K \subset X$ such that $F\phi = 0$ whenever $\phi|_K = 0$. Let us define a functional μ' on $C(K)$ in the following way: if $\phi \in C(K)$ then we take its continuous extension ϕ' to X and let $\mu'(\phi) = \mu(\phi')$. If $\phi_1, \phi_2 \in C(X)$ and $\phi_1 = \phi_2$ on K then $\mu(\phi_1) = \mu(\phi_2)$. Therefore, the functional μ' is well-defined. Moreover, if $\phi \geq 0$ on K, then replacing ϕ' with $|\phi'|$, we see that $\mu'(\phi) \geq 0$. By the Riesz' Representation Theorem there is a regular measure μ on K such that

$$\mu'(\phi) = \int_K \phi \, d\mu.$$

If $\phi \in C(X)$ and ϕ' is its restriction to K, then $\mu(\phi) = \mu'(\phi')$ and the lemma is proved. □

The following theorem describes positive superlinear functionals on $C(X)$. To state it we will need the following constructions: if C is a convex cone in $C(X)$ then G_C is a functional on $C(X)$ equal to 0 on C and $-\infty$ otherwise. Note that G_C is superlinear. Also for a given superlinear functional F we denote by F^* the set of measures μ on X with compact support such that $\mu(\phi) \geq F(\phi)$ for every $\phi \in C(X)$ and let

$$F'(\phi) = \inf\{\mu(\phi);\, \mu \in F^*\}$$

.

Theorem 2.4. *Let X be a locally compact Hausdorff space countable at infinity. A functional F on $C(X)$ is positive and superlinear if and only if there is a convex cone $C \subset C(X)$ containing all non-negative functions such that*

$$F(\phi) = F'(\phi) + G_C(\phi).$$

Proof. To prove the necessity we let $C = \{\phi \in C(X) : F(\phi) > -\infty\}$. Clearly C is a convex cone containing all non-negative functions. Let $\phi \in C$. If $t \geq 0$ then $F(t\phi) = tF(\phi)$. If $t > 0$ then $F(-t\phi) \leq -tF(\phi)$. Hence, on the line $\mathcal{M} = \{t\phi, t \in \mathbb{R}\}$ the functional $f(t\phi) = tF(\phi) \geq F(t\phi)$. It is easy to see that the Hahn–Banach theorem still holds when superlinear functionals can take $-\infty$ as their values. Hence there is a linear functional G_ϕ on $C(X)$ such that $G_\phi \geq F$ on $C(X)$ and $G_\phi = f$ on $\mathcal{M}$.

If $\psi \geq 0$ then $F(\psi) \geq 0$ and, consequently, $G_\phi(\psi) \geq 0$. By Lemma 2.3 there is a compactly supported measure $\mu_\phi \in F^*$ such that $G_\phi(\psi) = \mu_\phi(\psi)$.

If $F(\phi) \neq -\infty$ then $F(\phi) = \mu_\phi(\phi)$. On the other hand, $\mu(\phi) \geq F(\phi)$ for any $\mu \in F^*$. Therefore

$$F\phi = \inf\{\mu(\phi);\, \mu \in F^*\} + G_C(\phi).$$

If $\phi \notin C$ then $F(\phi) = -\infty$ and $G_C(\phi) = -\infty$ and again

$$F\phi = F'(\phi) + G_C(\phi).$$

To prove the converse we, firstly, note that the functional $F'(\phi)$ is positive and superlinear. If C is a convex cone in $C(X)$ containing all non-negative functions, then $F = F' + G_C$ is positive because $G_C(\phi) = 0$ when $\phi \geq 0$. Secondly, $F(c\phi) = cF(\phi)$, $c \geq 0$, because both F' and G_C have this property. And, thirdly, F is superlinear because both F' and G_C have this property. □

In general, $F' \neq F$. For example, let $X = (0,1)$ and let $F(\phi) = -\infty$ if $\liminf_{x\to 1^-} \phi(x) = -\infty$ and $F(\phi) = 0$ otherwise. Clearly, F is superlinear and positive but $F^* = \{0\}$ and $F' \equiv 0$.

This examples misses an important property: there is a decreasing sequence $\phi_j \in C(X)$ converging to $\phi \in C(X)$ but $\lim F(\phi_j) \neq F(\phi)$. However, this property is not sufficient for F to be equal to F'. Indeed, let $F(\phi) = -\infty$ if $\inf_{x\in(0,1)} \phi(x) < 0$ and $F(\phi) = 0$ otherwise. In this case, also $F^* = \{0\}$ and $F' \equiv 0$ but if a decreasing sequence $\phi_j \in C(X)$ converges to $\phi \in C(X)$ then $\lim F(\phi_j) = F(\phi)$. Note that $F(-1) = -\infty$.

In the assumption that $F(-1) > -\infty$ the following theorem gives the necessary and sufficient condition for $F' = F$.

Theorem 2.5. *Let X be a locally compact Hausdorff space countable at infinity and let F be a positive and superlinear functional on $C(X)$ such that $F(-1) > -\infty$. Then $F = F'$ if and only if $\lim F(\phi_j) = F(\phi)$ for every decreasing sequence $\phi_j \in C(X)$ converging to $\phi \in C(X)$.*

Proof. If $F = F'$ and a decreasing sequence $\phi_j \in C(X)$ converges to $\phi \in C(X)$, then $F(\phi) \leq \lim F(\phi_j)$. On the other hand, if $\mu \in F^*$ then

$$\lim F(\phi_j) \leq \lim \mu(\phi_j) = \mu(\phi).$$

Hence,

$$\lim F(\phi_j) \leq \inf\{\mu(\phi);\, \mu \in F^*\} = F'(\phi) = F(\phi).$$

Thus $\lim F(\phi_j) = F(\phi)$.

For the converse, we, firstly, note that if ϕ is bounded below by a constant c, then $F(\phi) \geq F(c) > -\infty$. Hence by Theorem 2.4 $F(\phi) = F'(\phi)$.

If $F(\phi) > -\infty$, then again Theorem 2.4 confirms that $F(\phi) = F'(\phi)$. If $F(\phi) = -\infty$ then ϕ is unbounded below and the decreasing sequence of $\phi_j = \max\{\phi, -j\}$

converges to ϕ. Consequently, $\lim F(\phi_j) = -\infty$. Since $F(\phi_j) = F'(\phi_j)$ we can find measures $\mu_j \in F^*$ such that $\lim \mu_j(\phi_j) = -\infty$. Hence

$$\lim \mu_j(\phi) \leq \lim \mu_j(\phi_j) = -\infty$$

and we see that $F(\phi) = F'(\phi)$. □

An operator E defined on $C(X)$ and whose values are functions on X taking values in $[-\infty, \infty)$ is called *superlinear* if:

(1) $E(c\phi) = cE(\phi), \quad c \geq 0;$
(2) $E(\phi_1 + \phi_2) \geq E\phi_1 + E\phi_2,$

and *positive* if $E(\phi_1) \leq E(\phi_2)$ when $\phi_1 \leq \phi_2$.

Such an operator generates for every $x \in X$ a set E_x^* of measures μ on X with compact support such that $\mu(\phi) \geq E(\phi)(x)$ for every $\phi \in C(X)$. Let

$$E'(\phi)(x) = \inf\{\mu(\phi);\ \mu \in E_x^*\}.$$

As an immediate consequence of the previous results we can get the following description of positive superlinear operators on $C(X)$.

Corollary 2.6. *Let X be a locally compact Hausdorff space countable at infinity. An operator E on $C(X)$ is positive and superlinear if and only if for every $x \in X$ there is a convex cone $C_x \subset C(X)$ containing all non-negative functions such that*

$$E(\phi)(x) = E'(\phi)(x) + G_{C_x}(\phi).$$

Moreover, if $E(-1) > -\infty$, then

$$E(\phi)(x) = E'\phi(x)$$

if and only if $\lim E(\phi_j) = E(\phi)$ for every decreasing sequence $\phi_j \in C(X)$ converging to $\phi \in C(X)$.

3. Envelopes

Let us give two important examples of positive superlinear operators. Let $\mathcal{S}$ be a convex cone in $C(X)$ containing constants. This cone generates an operator defined on the space of all functions on X by the formula

$$\mathcal{S}(g)(x) = \sup\{u(x) :\ u \in \mathcal{S}, u \leq g\}$$

if the set $\{u \in \mathcal{S}, u \leq g\}$ is non-empty and we let $\mathcal{S}g = -\infty$ otherwise. The lower-semicontinuous function $\mathcal{S}(g)$ is called the $\mathcal{S}$-envelope of a function g on X. Clearly, $\mathcal{S}$ is a positive superlinear operator such that $\mathcal{S}(c) = c$, c is a constant function, and $\mathcal{S}\phi \leq \phi$.

The cone $\mathcal{S}$ also generates a multifunction $J^{\mathcal{S}} \subset X \times \mathcal{P}(X)$ whose fiber $J_x^{\mathcal{S}}$ at x is the set of all compactly supported measures μ such that $\mu(\phi) \geq \phi(x)$ for all $\phi \in \mathcal{S}$. Since $\mathcal{S}$ contains constants any $\mu \in J^{\mathcal{S}}$ is a probability measure. Clearly, $\delta_x \in J_x^{\mathcal{S}}$ and $J_x^{\mathcal{S}}$ is convex and weak-$*$ closed. Moreover, the set $J^{\mathcal{S}}$ is closed with respect to the product topology on $X \times C^*(X)$, where $C^*(X)$ is equipped with the weak-$*$ topology.

At its turn the operator $\mathcal{S}$ generates a multifunction $\mathcal{S}^* \subset X \times \mathcal{P}(X)$ whose fiber at x is the set of all compactly supported measures μ such that $\mu(\phi) \geq \mathcal{S}(\phi)(x)$ for all $\phi \in C(X)$.

Now we can prove the first non-compact version of Edwards' theorem.

Proof of Theorem 1.3. By Corollary 2.6 for every $x \in X$ there is a convex cone $C_x \subset C(X)$ containing all non-negative functions such that

$$S(\phi)(x) = S'(\phi)(x) + G_{C_x}(\phi).$$

Suppose that $\mathcal{S}(\phi) \not\equiv -\infty$. Then there is a function ψ in $\mathcal{S}$ such that $\psi \leq \phi$. Hence $\mathcal{S}(\phi)(x) \neq -\infty$ for all $x \in X$ and, therefore, all cones G_{C_x} are empty and $S(\phi)(x) = S'(\phi)(x)$.

Let us show that $\mathcal{S}^* = J^{\mathcal{S}}$. Indeed, $\mathcal{S}^*_x \subset J^{\mathcal{S}}_x$ since $\mathcal{S}\phi = \phi$ whenever $\phi \in \mathcal{S}$. On the other hand, if $\mu \in J^{\mathcal{S}}_x$ then $\mu(\phi) \geq \phi(x)$ for every $\phi \in \mathcal{S}$ and this means that $\mu(\psi) \geq \mathcal{S}\psi(x)$ for every $\psi \in C(X)$. Hence, $\mu \in \mathcal{S}^*_x$. Thus

$$\mathcal{S}'(\phi) = \inf\{\mu(\phi) : \mu \in J^{\mathcal{S}}_x \phi\}.$$

□

The dichotomy in Theorem 1.3 is unavoidable in the classical settings even if we allow functions from $\mathcal{S}$ to take $-\infty$ as their values. Indeed, let $X = \mathbb{D}^2$ be the unit polydisk in $\mathbb{C}^2$ and let $\mathcal{S}$ be the cone of all continuous plurisubharmonic functions on X taking values at $[-\infty, \infty)$. Take a negative subharmonic function v on $\mathbb{D}$ such that $v(1/n) = -\infty$, $n = 2, 3, \ldots$ and $v(0) = -1$. Let $\phi(z_1, z_2) = \max\{v(z_2), -1/(1 - |z_1|^2)\}$. Then any continuous plurisubharmonic function $u \leq \phi$ on X is equal to $-\infty$ when $z_2 = 1/n$ and, consequently, is equal to $-\infty$ when $z_2 = 0$. Thus $\mathcal{S}(\phi)(z_1, 0) = -\infty$. On the other hand, $v(z_2) \leq \phi(z_1, z_2)$. Hence

$$\inf\{\mu(\phi) : \mu \in J^{\mathcal{S}}_{(0,0)}\} \geq \inf\{\mu(v) : \mu \in J^{\mathcal{S}}_{(0,0)}\}.$$

But by Theorem 1.2 the right side is equal -1 and we see that Theorem 1.3 fails in this case.

4. The second version of Edwards' Theorem

Lemma 4.1. *Let X be a locally compact Hausdorff space countable at infinity. Let ϕ be a locally bounded above function on X, let U be an open set in X and let $\psi \geq \phi$ be a continuous function on U. Then for every compact set K in X there is a function $f \in C(X)$ such that $f \geq \phi$ on X and $f = \psi$ on K.*

Proof. By Lemma 2.1 X is the union of compact sets X_j, $j = 1, 2, \ldots$, such that each X_j lies in the interior X^o_{j+1} of X_{j+1}. We may assume that $K \subset X^o_1$ and there are numbers $a_j \geq 0$ such that $\phi \leq a_j$ on X_j. Since X is normal for $j \geq 2$ there are continuous functions $\phi_j \geq 0$ on X such that $\phi_j = a_j$ on $X_j \setminus X^0_{j-1}$ and $\phi_j = 0$ on $X_{j-2} \cup K$ (we let $X_0 = \emptyset$). To define ϕ_1 we take an open neighborhood W of K such that $\bar{W} \subset X^o_1 \cap U$ and let ϕ_1 to be a non-negative continuous function on X equal to a_1 on $X \setminus W$ and to 0 on K. We set ϕ_0 as a non-negative continuous function on X equal to ψ on $\bar{W}$.

It is easy to check that the function $f = \sum_{j=0}^{\infty} \phi_j$ has all needed properties. □

This lemma has some important corollaries. To state them we introduce the *upper regularizations* of a function ϕ on X. We define ϕ^*_1 as the infimum of all functions $\psi \in C(X)$ such that $\psi \geq \phi$ and let

$$\phi^*_2(x) = \inf\{\sup_{y \in U} \phi(y) : U \text{ is open and } x \in U\}.$$

Clearly, ϕ^*_1 is upper semicontinuous and $\phi^*_1 \geq \phi^*_2$.

Corollary 4.2. *If ϕ is a locally bounded above function on a locally compact Hausdorff space X countable at infinity, then $\phi_1^* = \phi_2^* = \phi^*$.*

Proof. Let $\varepsilon > 0$, $x \in X$ and let U be a neighborhood of x such that $c = \sup_{y\in U} \phi(y) \le \phi_2^*(x) + \varepsilon$. We take another neighborhood W of x such that $\bar{W}$ is compact and lies in U. By Lemma 4.1 there is a continuous function ψ on X equal to c on $\bar{W}$ and greater or equal to ϕ on X. Hence, $\phi_1^*(x) \le c \le \phi_2^*(x) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary we see that $\phi_1^* = \phi_2^*$. □

It is known that on metric spaces every upper semicontinuous function is the limit of a decreasing sequence of continuous functions. Since our spaces are not supposed to be metric the following corollary has some value.

Corollary 4.3. *Let ϕ be an upper semicontinuous function on a locally compact Hausdorff space X countable at infinity and let μ be a regular Borel measure on X. Then for every $\varepsilon > 0$ there is a function $\psi \in C(X)$ such that $\psi \ge \phi$ and $\mu(\psi) < \mu(\phi) + \varepsilon$ if $\mu(\phi) > -\infty$ and $\mu(\psi) < -1/\varepsilon$ if $\mu(\phi) = -\infty$.*

Proof. We will prove this statement when $\mu(\phi) > -\infty$. The case when $\mu(\phi) = -\infty$ can be done in the same way. Let $K = \operatorname{supp}\mu$. There is $c > 0$ such that $\phi < c$ on K and $\mu(\{\phi \le -c\}) < \varepsilon$. We divide the interval $(-c, c]$ into consecutive intervals $(c_j, c_{j+1}]$, $j = 0, \dots, n$, of length less than ε. Let $K_j = \phi^{-1}((c_j, c_{j+1}])$. Since μ is a regular Borel measure we can find compact sets $K_{jm} \subset K_j$ such that $\mu(K_{jm}) > \mu(K_j) - 1/m$. By Lemma 4.1 there are continuous functions $\psi'_m \ge \phi$ on X such that $\psi'_m = c_{j+1}$ on all K_{jm}. Let $\psi_m = \min\{\psi'_1, \dots, \psi'_m\}$. The sequence ψ_m is decreasing to a function ψ and $\psi = c_{j+1}$ on K_j except of a set of measure 0. Hence $\mu(\psi) \le \mu(\phi) + \varepsilon\mu(K) + \varepsilon$. Since the sequence ψ_m is decreasing there is m such that $\mu(\psi_m) \le \mu(\phi) + \varepsilon\mu(K) + 2\varepsilon$. □

The upper regularizations is frequently used when it preserves the subaveraging inequality. This means that if a function ϕ is absolutely measurable, i. e., measurable with respect to any regular Borel measure, and satisfies the inequality $\mu(\phi) \ge \phi(x)$ for every $x \in X$ and $\mu \in J_x^{\mathcal{S}}$, then ϕ^* also has this property. As the following theorem shows the lower semicontinuity of the multifunction $J^{\mathcal{S}}$ suffice for the upper regularizations to preserve the subaveraging inequality.

Theorem 4.4. *Let X be a locally compact Hausdorff space countable at infinity and let $J \subset X \times \mathcal{P}(X)$ be a lower semicontinuous multifunction. If a function ϕ is absolutely measurable and $\mu(\phi) \ge \phi(x)$ for any $x \in X$ and $\mu \in J_x$, then ϕ^* also has the latter property.*

Proof. Let $\mu \in J_x$. By Corollary 4.3 for any $\varepsilon > 0$ there is a function $\psi \in C(X)$ such that $\psi \ge \phi$ and $\mu(\psi) < \mu(\phi) + \varepsilon$. Let $V = \{\nu \in C^*(X) :\ \nu(\psi) < \mu(\psi) + \varepsilon\}$ and let W be a neighborhood of x lying in the natural projection of $(V \times X) \cap J$ onto X. For any $y \in W$ we select $\mu_y \in J_y \cap V$. Then

$$\phi(y) \le \mu_y(\phi) \le \mu_y(\psi) \le \mu(\psi) + \varepsilon \le \mu(\phi) + 2\varepsilon \le \mu(\phi^*) + 2\varepsilon.$$

Hence $\phi^*(x) \le \mu(\phi^*) + 2\varepsilon$ for any $\varepsilon > 0$ and this means that $\phi^*(x) \le \mu(\phi^*)$. □

Let

$$\widetilde{\mathcal{S}}(g)(x) = \sup\{u(x) :\ u \in \widetilde{\mathcal{S}}, u \le g\}$$

if the set $\{u \in \widetilde{\mathcal{S}}, u \le g\}$ is non-empty and we let $\mathcal{S}g = -\infty$ otherwise.

Now we can prove the second non-compact version of Edwards' Theorem.

Proof of Theorem 1.4. Firstly, we assume that ϕ is continuous and bounded below. Let us show that the function

$$E_\phi(x) = \inf\{\mu(\phi) : \mu \in J_x^{\mathcal{S}}\}$$

is upper semicontinuous. For this we take some $\varepsilon > 0$, $x \in X$ and $\mu \in J_x^{\mathcal{S}}$ such that $E_\phi(x) \geq \mu(\phi) - \varepsilon$. Let $V = \{\nu \in C^*(X) : \nu(\phi) \leq \mu(\phi) + \varepsilon\}$. Since $J^{\mathcal{S}}$ is lower semicontinuous there is a neighborhood W of x in X such that for each point $y \in W$ we can find a measure $\nu \in J_y^{\mathcal{S}} \cap V$. Hence $E_\phi(y) \leq \mu(\phi) + \varepsilon \leq E_\phi(x) + 2\varepsilon$ and this proves the upper semicontinuity of $E\phi$.

Since $\mathcal{S}(\phi) \not\equiv -\infty$ by Theorem 1.3 $\mathcal{S}(\phi) = E_\phi$. But the function $\mathcal{S}(\phi)$ is lower semicontinuous. Therefore, E_ϕ is continuous and, consequently, belongs to $\mathcal{S}$. Since $E_\phi \leq \phi$ this shows that $\widetilde{\mathcal{S}}(\phi) \geq E_\phi$. On the other hand, if $u \in \widetilde{\mathcal{S}}$ and $u \leq \phi$, then $u(x) \leq E_\phi(x)$. Hence $\widetilde{\mathcal{S}}(\phi) \leq E_\phi$ and we see that $\widetilde{\mathcal{S}}(\phi) = E_\phi$.

If ϕ is a continuous function on X, then it is the limit of a decreasing sequence of bounded below continuous functions $\phi_m = \max\{\phi_m, -m\}$. It is easy to see that $E_\phi = \lim_{m\to\infty} E_{\phi_m}$ and that the sequence of the continuous functions E_{ϕ_m} is decreasing. Hence, the function E_ϕ is upper semicontinuous. Since all $E_{\phi_m} \in \widetilde{S}$, the function E_ϕ also belongs to $\widetilde{S}$ and we see that $E_\phi = \widetilde{S}(\phi)$.

If ϕ is upper semicontinuous then we consider the set A of all continuous functions on X greater or equal to ϕ. By Corollary 4.3 this set is non-empty and let $A_\phi = \inf\{E_\psi,\, \psi \in A\}$.

The function A_ϕ is upper semicontinuous and $A_\phi \leq \phi$ by Corollary 4.3. To see this just take $\mu = \delta_x$ in the corollary. Let us show that $A_\phi \in \widetilde{S}$. For this we take $x \in X$ and $\mu \in J_x^{\mathcal{S}}$ and by Corollary 4.3 for every $\varepsilon > 0$ find a continuous function f on X such that $f \geq A_\phi$ and $\mu(f) < \mu(A_\phi) + \varepsilon$. Since the support of μ is compact and the function $S(\psi)$ is upper semicontinuous for every $\psi \in A$, we can find functions $\psi_1, \dots, \psi_m$ in A such that

$$\min\{E_{\psi_1}, \dots, E_{\psi_m}\} \leq f + \varepsilon$$

on $\operatorname{supp}\mu$.

Let $\psi = \min\{\psi_1, \dots, \psi_m\}$. Since $E_{\psi_j} \geq E_\psi$, $1 \leq j \leq m$,

$$E_\psi \leq \min\{E_{\psi_1}, \dots, E_{\psi_m}\} \leq f + \varepsilon$$

on $\operatorname{supp}\mu$. Hence

$$\mu(A_\phi) + 2\varepsilon \geq \mu(E_\psi) \geq E_\psi(x) \geq A_\phi(x).$$

But ε can be as small as we want and, therefore, $A_\phi \in \widetilde{\mathcal{S}}$.

Consequently, $E_\phi \geq A_\phi$. But $E_\phi \leq E_\psi$ for every $\psi \in A$. Thus $E_\phi = A_\phi = \widetilde{\mathcal{S}}(\phi)$. □

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