On Transverse Triangulations

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December 20, 2010

Abstract

We show that every smooth manifold admits a smooth triangulation transverse to a given smooth map. This removes the properness assumption on the smooth map used in an essential way in Scharlemann's construction [5].

1 Introduction

For $l\in\mathbb{Z}^{\geq 0}$, let $\Delta^l\subset\mathbb{R}^l$ denote the standard l-simplex. If $|K|\subset\mathbb{R}^N$ is a geometric realization of a simplicial complex K in the sense of [4, Section 3], for each l-simplex σ of K there is an injective linear map $\iota_\sigma\colon\Delta^l\longrightarrow|K|$ taking Δ^l to $|\sigma|$.[1] If X is a smooth manifold, a topological embedding $\mu:\Delta^l\longrightarrow X$ is a smooth embedding if there exist an open neighborhood Δ^l_μ of Δ^l in $\mathbb{R}^l$ and a smooth embedding $\tilde\mu\colon\Delta^l_\mu\longrightarrow X$ so that $\tilde\mu|_{\Delta^l}=\mu$. A triangulation of a smooth manifold X is a pair $T=(K,\eta)$ consisting of a simplicial complex and a homeomorphism $\eta\colon|K|\longrightarrow X$ such that

$$\eta\circ\iota_\sigma\colon\Delta^l\longrightarrow X$$

is a smooth embedding for every l-simplex σ in K and $l\in\mathbb{Z}^{\geq 0}$. If $T=(K,\eta)$ is a triangulation of X and $\psi\colon X\longrightarrow X$ is a diffeomorphism, then $\psi_*T=(K,\psi\circ\eta)$ is also a triangulation of X.

Theorem 1 *If X,Y are smooth manifolds and $h:Y\longrightarrow X$ is a smooth map, there exists a triangulation (K,η) of X such that h is transverse to $\eta|_{\mathrm{Int}\,\sigma}$ for every simplex $\sigma\in K$.*

This theorem is stated in [7] as Lemma 2.3 and described as an obvious fact. As pointed out to the author by Matthias Kreck, Scharlemann [5] proves Theorem 1 under the assumption that the smooth map h is proper, and his argument makes use of this assumption in an essential way. For the purposes of [7], a transverse C^1-triangulation would suffice, and the existence of a such triangulation is fairly evident from the point of view of Sard-Smale Theorem [6, (1.3)]. In this note we give a detailed proof of Theorem 1 as stated above, using Sard's theorem [2, Section 2].

The author would like to thank M. Kreck for detailed comments and suggestions on [7] and earlier versions of this note, as well as D. McDuff and J. Milnor for related discussions.

*Partially supported by DMS grant 0846978

[1] i.e. ι_σ takes the vertices of Δ^l to the vertices of $|\sigma|$ and is linear between them, as in [7, Footnote 5]

2 Outline of proof of Theorem 1

If K is a simplicial complex, we denote by $\mathrm{sd}\,K$ the barycentric subdivision of K. For any non-negative integer l, let K_l be the l-th skeleton of K, i.e. the subcomplex of K consisting of the simplices in K of dimension at most l. If σ is a simplex in a simplicial complex K with geometric realization $|K|$, let

$$\mathrm{St}(\sigma, K) = \bigcup_{\sigma\subset\sigma'} \mathrm{Int}\,\sigma'$$

be the star of σ in K, as in [4, Section 62], and $b_\sigma \in \mathrm{sd}\,K$ the barycenter of σ. The main step in the proof of Lemma 2.3 is the following observation.

Proposition 2 *Let $h\colon Y \longrightarrow X$ be a smooth map between smooth manifolds. If (K,η) is a triangulation of X and σ is an l-simplex in K, there exists a diffeomorphism $\psi_\sigma\colon X \longrightarrow X$ restricting to the identity outside of $\eta(\mathrm{St}(b_\sigma, \mathrm{sd}\,K))$ so that $\psi_\sigma\circ\eta|_{\mathrm{Int}\ \sigma}$ is transverse to h.*

If σ and σ' are two distinct simplices in K of the same dimension l,

$$\mathrm{St}(b_\sigma, \mathrm{sd}\,K) \cap \mathrm{St}(b_{\sigma'}, \mathrm{sd}\,K) = \emptyset. \tag{1}$$

Since ψ_σ is the identity outside of $\eta(\mathrm{St}(b_\sigma, \mathrm{sd}\,K))$ and the collection $\{\mathrm{St}(b_\sigma, \mathrm{sd}\,K)\}$ is locally finite, the composition $\psi_l\colon X \longrightarrow X$ of all diffeomorphisms $\psi_\sigma\colon X \longrightarrow X$ taken over all l-simplices σ in K is a well-defined diffeomorphism of X.[2] Since $\psi_l\circ\eta|_{|\sigma|} = \psi_\sigma\circ\eta|_{|\sigma|}$ for every l-simplex σ in K, we obtain the following conclusion from Proposition 2.

Corollary 3 *Let $h\colon Y \longrightarrow X$ be a smooth map between smooth manifolds. If (K,η) is a triangulation of X, for every $l = 0, 1, \ldots, \dim X$, there exists a diffeomorphism $\psi_l\colon X \longrightarrow X$ restricting to the identity on $\eta(|K_{l-1}|)$ so that $\psi_l\circ\eta|_{\mathrm{Int}\ \sigma}$ is transverse to h for every l-simplex σ in K.*

This corollary implies Theorem 1. By [3, Chapter II], X admits a triangulation (K, η_{-1}). By induction and Corollary 3, for each $l = 0, 1, \ldots, \dim X - 1$ there exists a triangulation $(K, \eta_l) = (K, \psi_l\circ\eta_{l-1})$ of X which is transverse to h on every open simplex in K of dimension at most l.

3 Proof of Proposition 2

Lemma 4 *For every $l \in \mathbb{Z}^+$, there exists a smooth function $\rho_l\colon \mathbb{R}^l \longrightarrow \bar{\mathbb{R}}^+$ such that*

$$\rho_l^{-1}(\mathbb{R}^+) = \mathrm{Int}\,\Delta^l.$$

Proof: Let $\rho\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the smooth function given by

$$\rho(r) = \begin{cases} \mathrm{e}^{-1/r}, & \text{if } r > 0; \\ 0, & \text{if } r \le 0. \end{cases}$$

[2]The locally finite property implies that the composition of these diffeomorphisms in any order is a diffeomorphism; by (1), these diffeomorphisms commute and so the composition is independent of the order.

The smooth function $\rho_l\colon \mathbb{R}^l \longrightarrow \mathbb{R}$ given by

$$\rho_l(t_1,\ldots,t_n) = \rho\bigg(1-\sum_{i=1}^{i=l} t_i\bigg)\cdot\prod_{i=1}^{i=l}\rho(t_i)$$

then has the desired property.

Lemma 5 *Let (K,η) be a triangulation of a smooth manifold X and σ an l-simplex in K. If*

$$\tilde{\mu}_\sigma\colon \Delta_\sigma^l \times \mathbb{R}^{m-l} \longrightarrow U_\sigma \subset X$$

is a diffeomorphism onto an open neighborhood U_σ of $\eta(|\sigma|)$ in X such that $\tilde{\mu}_\sigma(t,0)=\eta(\iota_\sigma(t))$ for all $t\in\Delta_\sigma$, there exists $c_\sigma\in\mathbb{R}^+$ such that

$$\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}\colon\ |v|\le c_\sigma\rho_l(t)\big\} \subset \tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(b_\sigma,\mathrm{sd}\,K))\big).$$

Proof: It is sufficient to show that there exists $c_\sigma>0$ such that

$$\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}\colon\ |v|\le c_\sigma\rho_l(t)\big\} \subset \tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(\sigma,K))\big).[3]$$

We assume that $0<l<m$. Suppose $(t_p,v_p)\in(\mathrm{Int}\,\Delta^l)\times(\mathbb{R}^{m-l}-0)$ is a sequence such that

$$(t_p,v_p)\notin\tilde{\mu}_\sigma^{-1}\big(\eta(\mathrm{St}(\sigma,K))\big), \qquad |v_p|\le\frac{1}{p}\rho_l(t_p). \tag{2}$$

Since $\eta(\mathrm{St}(\sigma,K))$ is an open neighborhood of $\eta(\mathrm{Int}\,\sigma)$ in X, by shrinking v_p and passing to a subsequence we can assume that

$$(t_p,v_p)\in\tilde{\mu}_\sigma^{-1}\big(\eta(|\tau'|)\big)\subset\tilde{\mu}_\sigma^{-1}\big(\eta(|\tau|)\big) \tag{3}$$

for an m-simplex τ in K and a face τ' of τ so that $\sigma\not\subset\tau'$, $\tau'\not\subset\sigma$, and $\sigma\subset\tau$. Let $\iota_\tau:\Delta^m\longrightarrow|K|$ be an injective linear map taking Δ^m to $|\tau|$ so that

$$\iota_\tau^{-1}(|\sigma|)=\Delta^m\cap\mathbb{R}^l\times0\subset\mathbb{R}^l\times\mathbb{R}^{m-l}, \qquad \iota_\tau^{-1}(|\tau'|)=\Delta^m\cap0\times\mathbb{R}^{m-1}\subset\mathbb{R}^1\times\mathbb{R}^{m-1}. \tag{4}$$

Choose a smooth embedding $\mu_\tau:\Delta_\tau^m\longrightarrow X$ from an open neighborhood of Δ^m in $\mathbb{R}^m$ such that $\mu_\tau|_{\Delta^m}=\eta\circ\iota_\tau$. Let ϕ be the first component of the diffeomorphism

$$\mu_\tau^{-1}\circ\tilde{\mu}_\sigma\colon\tilde{\mu}_\sigma^{-1}\big(\mu_\tau(\Delta_\tau^m)\big)\longrightarrow\mu_\tau^{-1}\big(\mu_\sigma(\Delta_\sigma^l\times\mathbb{R}^{m-l})\big)\subset\mathbb{R}^1\times\mathbb{R}^{m-1}.$$

By (3), the second assumption in (4), the continuity of $\mathrm{d}\phi$, and the compactness of Δ^l,

$$\big|\phi(t_p,0)\big|=\big|\phi(t_p,0)-\phi(t_p,v_p)\big|\le C|v_p| \qquad \forall\,p, \tag{5}$$

for some $C>0$. On the other hand, by the first assumption in (4), the vanishing of ρ_l on $\mathrm{Bd}\,\Delta^l$, the continuity of $\mathrm{d}\rho_l$, and the compactness of Δ^l,

$$\big|\rho_l(t_p)\big|\le C\big|\phi(t_p,0)\big| \qquad \forall\,p, \tag{6}$$

for some $C>0$. The second assumption in (2), (5), and (6) give a contradiction for $p>C^2$.

[3] If K' is the subdivision of K obtained by adding the vertices $b_{\sigma'}$ with $\sigma'\supsetneq\sigma$, then $\mathrm{St}(b_\sigma,\mathrm{sd}\,K)=\mathrm{St}(\sigma,K')$.

Lemma 6 *Let $h: Y \longrightarrow X$ be a smooth map between smooth manifolds, (K,η) a triangulation of X, σ an l-simplex in K, and*

$$\tilde{\mu}_\sigma \colon \Delta^l_\sigma \times \mathbb{R}^{m-l} \longrightarrow U_\sigma \subset X$$

a diffeomorphism onto an open neighborhood U_σ of $\eta(|\sigma|)$ in X such that $\tilde{\mu}_\sigma(t,0)=\eta(\iota_\sigma(t))$ for all $t\in\Delta_\sigma$. For every $\epsilon>0$, there exists $s_\sigma\in C^\infty(\mathrm{Int}\,\Delta^l;\mathbb{R}^{m-l})$ so that the map

$$\tilde{\mu}_\sigma \circ (\mathrm{id}, s_\sigma) \colon \mathrm{Int}\,\Delta^l \longrightarrow X \tag{7}$$

is transverse to h,

$$\big|s_\sigma(t)\big| < \epsilon^2 \rho_l(t) \quad \forall\, t\in \mathrm{Int}\,\Delta^l, \qquad \lim_{t\longrightarrow \mathrm{Bd}\,\Delta^l} \rho_l(t)^{-i}\big|\nabla^j s_\sigma(t)\big| = 0 \quad \forall i,j\in\mathbb{Z}^{\ge 0}, \tag{8}$$

where $\nabla^j s_\sigma$ is the multi-linear functional determined by the j-th derivatives of s_σ.

Proof: The smooth map

$$\phi\colon \mathrm{Int}\,\Delta^l \times \mathbb{R}^{m-l} \longrightarrow X, \qquad \phi(t,v) = \tilde{\mu}_\sigma\big(t, \mathrm{e}^{-1/\rho_l(t)} v\big),$$

is a diffeomorphism onto an open neighborhood U'_σ of $\eta(\mathrm{Int}\,\sigma)$ in X. The smooth map (7) with $s_\sigma = \mathrm{e}^{-1/\rho_l(t)}v$ is transverse to h if and only if $v\in\mathbb{R}^{m-l}$ is a regular value of the smooth map

$$\pi_2 \circ \phi^{-1} \circ h \colon h^{-1}(U'_\sigma) \longrightarrow \mathbb{R}^{m-l},$$

where $\pi_2\colon \mathrm{Int}\,\Delta^l \times \mathbb{R}^{m-l} \longrightarrow \mathbb{R}^{m-l}$ is the projection onto the second component. By Sard's Theorem, the set of such regular values is dense in $\mathbb{R}^{m-l}$. Thus, the map (7) with $s_\sigma=\mathrm{e}^{-1/\rho_l(t)}v$ is transverse to h for some $v\in\mathbb{R}^{m-l}$ with $|v|<\epsilon^2$. The second statement in (8) follows from $\rho_l|_{\mathrm{Bd}\,\Delta^l}=0$.

Corollary 7 *Let $h\colon Y \longrightarrow X$ be a smooth map between smooth manifolds, (K,η) a triangulation of X, σ an l-simplex in K, and*

$$\tilde{\mu}_\sigma \colon \Delta^l_\sigma \times \mathbb{R}^{m-l} \longrightarrow U_\sigma \subset X$$

a diffeomorphism onto an open neighborhood U_σ of $\eta(|\sigma|)$ in X such that $\tilde{\mu}_\sigma(t,0)=\eta(\iota_\sigma(t))$ for all $t\in\Delta_\sigma$. For every $\epsilon>0$, there exists a diffeomorphism ψ'_σ of $\Delta^l_\sigma\times\mathbb{R}^{m-l}$ restricting to the identity outside of

$$\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l} \colon\ |v|\le\epsilon\rho_l(t)\big\}$$

so that the map $\tilde{\mu}_\sigma\circ\psi'_\sigma|_{\mathrm{Int}\,\Delta^l\times 0}$ is transverse to h.

Proof: Choose $\beta\in C^\infty(\mathbb{R};[0,1])$ so that

$$\beta(r) = \begin{cases} 1, & \text{if } r\le \frac{1}{2}; \\ 0, & \text{if } r\ge 1. \end{cases}$$

Let $C_\beta = \sup_{r\in\mathbb{R}} |\beta'(r)|$. With s_σ as provided by Lemma 6, define

$$\psi'_\sigma \colon \Delta^l_\sigma \times \mathbb{R}^{m-l} \longrightarrow \Delta^l_\sigma \times \mathbb{R}^{m-l} \qquad \text{by}$$

$$\psi'_\sigma(t,v) = \begin{cases} \left(t, v+\beta\left(\frac{|v|}{\epsilon\rho_l(t)}\right) s_\sigma(t)\right), & \text{if } t\in \mathrm{Int}\,\Delta^l; \\ (t,v), & \text{if } t\notin \mathrm{Int}\,\Delta^l. \end{cases}$$

The restriction of this map to $(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}$ is smooth and its Jacobian is

$$\mathcal{J}\psi'_\sigma\big|_{(t,v)} = \begin{pmatrix} \mathbb{I}_l & 0 \\ \beta\left(\frac{|v|}{\epsilon\rho_l(t)}\right)\nabla s_\sigma(t) - \beta'\left(\frac{|v|}{\epsilon\rho_l(t)}\right)\frac{|v|}{\epsilon\rho_l(t)}\frac{s_\sigma(t)}{\rho_l(t)}\nabla\rho_l & \mathbb{I}_{m-l} + \beta'\left(\frac{|v|}{\epsilon\rho_l(t)}\right)\frac{s_\sigma(t)}{\epsilon\rho_l(t)}\frac{v^{tr}}{|v|} \end{pmatrix}. \quad (9)$$

By the first property in (8), this matrix is non-singular if $\epsilon<1/C_\beta$. If W is any linear subspace of $\mathbb{R}^{m-l}$ containing $s_\sigma(t)$,

$$\psi'_\sigma(t\times W)\subset t\times W, \qquad \psi'_\sigma(t,v)=(t,v) \quad \forall\, v\in W \text{ s.t. } |v|\geq\epsilon\rho_l(t).$$

Thus, ψ'_σ is a bijection on $t\times W$, a diffeomorphism on $(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}$, and a bijection on $\Delta^l_\sigma\times\mathbb{R}^{m-l}$.

Since $\beta(r)=0$ for $r\geq1$, $\psi'_\sigma(t,v)=(t,v)$ unless $t\in\mathrm{Int}\,\Delta^l$ and $|v|<\epsilon\rho_l(t)$. It remains to show that ψ'_σ is smooth along

$$\overline{\big\{(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}:\ |v|\leq\epsilon\rho_l(t)\big\}} - (\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l} = (\mathrm{Bd}\,\Delta^l)\times 0.$$

Since $|s_\sigma(t)|\longrightarrow0$ as $t\longrightarrow\mathrm{Bd}\,\Delta^l$ by the first property in (8), ψ'_σ is continuous at all $(t,0)\in(\mathrm{Bd}\,\Delta^l)\times0$. By the first property in (8), ψ'_σ is also differentiable at all $(t,0)\in(\mathrm{Bd}\,\Delta^l)\times0$, with the Jacobian equal to $\mathbb{I}_m$. By (9) and the compactness of Δ^l,

$$\big|\mathcal{J}\psi'_\sigma|_{(t,v)}-\mathbb{I}_m\big| \leq C\big(|\nabla s_\sigma(t)|+\rho(t)^{-1}|s_\sigma(t)|\big) \quad \forall\,(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}$$

for some $C>0$. So $\mathcal{J}\psi'_\sigma$ is continuous at $(t,0)$ by the second statement in (8), as well as differentiable, with the differential of $\mathcal{J}\psi'_\sigma$ at $(t,0)$ equal to 0. For $i\geq2$, the i-th derivatives of the second component of ψ'_σ at $(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}$ are linear combinations of the terms

$$\beta^{\langle i_1\rangle}\left(\frac{|v|}{\epsilon\rho_l(t)}\right)\cdot\left(\frac{|v|}{\epsilon\rho_l(t)}\right)^{i_1}\cdot\prod_{k=1}^{k=j_1}\left(\frac{\nabla^{p_k}\rho_l}{\rho_l(t)}\right)\cdot\frac{v_J}{|v|^{2j_2}}\cdot\nabla^{i_2}s_\sigma(t)\,,$$

where $i_1,i_2,j_1,j_2\in\mathbb{Z}^{\geq0}$ and $p_1,\ldots,p_{j_1}\in\mathbb{Z}^+$ are such that

$$i_1+(p_1+p_2+\ldots+p_{j_1}-j_1)+i_2=i, \qquad j_1+j_2\leq i_1\,,$$

and v_J is a j_2-fold product of components of v. Such a term is nonzero only if $\epsilon\rho_l(t)/2<|v|<\epsilon\rho_l(t)$ or $i_1=0$ and $|v|<\epsilon\rho_l(t)$. Thus, the i-th derivatives of ψ'_σ at $(t,v)\in(\mathrm{Int}\,\Delta^l)\times\mathbb{R}^{m-l}$ are bounded by

$$C_i\sum_{i_1+i_2\leq i}\rho_l(t)^{-i_1}\big|\nabla^{i_2}s_\sigma(t)\big|$$

for some constant $C_i>0$. By the second statement in (8), the last expression approaches 0 as $t\longrightarrow\mathrm{Bd}\,\Delta^l$ and does so faster than ρ_l. It follows that ψ'_σ is smooth at all $(t,0)\in(\mathrm{Bd}\,\Delta^l)\times0$.

Proof of Proposition 2: Let Δ^l_σ be a contractible open neighborhood of Δ^l in $\mathbb{R}^l$ and $\mu_\sigma\colon\Delta^l_\sigma\longrightarrow X$ a smooth embedding so that $\mu_\sigma|_{\Delta^l}=\eta\circ\iota_\sigma$. By the Tubular Neighborhood Theorem [1, (12.11)], there exist an open neighborhood U_σ of $\mu_\sigma(\Delta^l_\sigma)$ in X and a diffeomorphism

$$\tilde\mu_\sigma\colon\Delta^l_\sigma\times\mathbb{R}^{m-l}\longrightarrow U_\sigma \qquad \text{s.t.} \qquad \tilde\mu_\sigma(t,0)=\mu_\sigma(t) \ \ \forall\,t\in\Delta^l_\sigma.\text{[4]}$$

[4]Since Δ^l_σ is contractible, the normal bundle to the embedding μ_σ is trivial.

Let $c_\sigma > 0$ be as in Lemma 5 and ψ'_σ as in Corollary 7 with $\epsilon = c_\sigma$. The diffeomorphism

$$\psi_\sigma = \tilde{\mu}_\sigma \circ \psi'_\sigma \circ \tilde{\mu}_\sigma^{-1} : U_\sigma \longrightarrow U_\sigma$$

is then the identity on $U_\sigma - \mathrm{St}(b_\sigma, \mathrm{sd}\, K)$. Since ψ_σ is also the identity outside of a compact subset of U_σ, it extends by identity to a diffeomorphism on all of X.

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