Preprint: September 26, 2009

IDENTITIES AND CONGRUENCES FOR A NEW SEQUENCE

ZHI-Hong Sun

School of the Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, PR China
Email: szh6174@yahoo.com
Homepage: http://www.hytc.edu.cn/xsjl/szh

Abstract. Let $[x]$ be the greatest integer not exceeding x. In the paper we introduce the sequence $\{U_n\}$ given by $U_0 = 1$ and $U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}\quad (n \ge 1)$, and establish many recursive formulas and congruences involving $\{U_n\}$.

MSC: Primary 11A07, Secondary 11B68
Keywords: congruence; Euler polynomial; identity; p-regular function

1. Introduction.

The Euler numbers $\{E_n\}$ are defined by

$$E_0 = 1 \quad \text{and} \quad E_n = -\sum_{k=1}^{[n/2]}\binom{n}{2k}E_{n-2k} \quad (n \ge 1),$$

where $[x]$ is the greatest integer not exceeding x. There are many well known identities and congruences involving Euler numbers. In the paper we introduce the sequence $\{U_n\}$ similar to Euler numbers as below:

$$U_0 = 1, \quad U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k} \quad (n \ge 1). \tag{1.1}$$

Clearly $U_{2n-1} = 0$ for $n \ge 1$. In Section 2 we establish many recursive relations for $\{U_n\}$. In Section 3, we deduce some congruences involving $\{U_n\}$. As examples, for a prime $p > 3$ and $k \in \{2, 4, \dots, p-3\}$ we have

$$\sum_{x=1}^{[p/6]}\frac{1}{x^k} \equiv \frac{6^k(2^k+1)}{4(2^{k-1}+1)}\Big(\frac{p}{3}\Big)U_{p-1-k} \pmod p,$$

where $(\frac{a}{m})$ is the Legendre-Jacobi-Kronecker symbol; for a prime $p \equiv 1 \pmod 4$ we have

$$U_{\frac{p-1}{2}} \equiv (1 + 2(-1)^{\frac{p-1}{4}})h(-3p) \pmod p,$$

The author is supported by the Natural Sciences Foundation of China (grant No. 10971078).

where $h(d)$ is the class number of the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d.

Let $\mathbb{N}$ be the set of positive integers. For $m \in \mathbb{N}$ let $\mathbb{Z}_m$ be the set of rational numbers whose denominator is coprime to m. For a prime p, in [S1] the author introduced the notion of p-regular functions. If $f(k) \in \mathbb{Z}_p$ for $k = 0, 1, 2, \dots$ and $\sum_{k=0}^{n} \binom{n}{k}(-1)^k f(k) \equiv 0 \pmod{p^n}$ for all $n \in \mathbb{N}$, then f is called a p-regular function. If f and g are p-regular functions, from [S1, Theorem 2.3] we know that $f \cdot g$ is also a p-regular function. Thus all p-regular functions form a ring.

Let p be an odd prime, and let $b \in \{0, 2, 4, \dots\}$. In Section 4 we show that $f(k) = (1 - (\frac{p}{3})p^{k(p-1)+b})U_{k(p-1)+b}$ is a p-regular function. Using the properties of p-regular functions in [S1,S3], we deduce many congruences for $\{U_{2n}\} \pmod{p^m}$. For example, if $\varphi(n)$ is Euler's totient function, for $k, m \in \mathbb{N}$ we have

$$U_{k\varphi(p^m)+b} \equiv (1 - (\frac{p}{3})p^b)U_b \pmod{p^m}.$$

In Section 4 we also show that $U_{2n} \equiv -16n - 42 \pmod{128}$ for $n \geq 3$.

In Section 5 we show that there is a set X and a map $T: \ X \to X$ such that $(-1)^n U_{2n}$ is the number of fixed points of T^n.

In addition to the above notation, we also use throughout this paper the following notation: $\mathbb{Z}$—the set of integers, $\{x\}$—the fractional part of x, $\text{ord}_p n$—the nonnegative integer α such that $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$ (that is $p^\alpha \parallel n$), $\mu(n)$—the Möbius function.

2. Some identities involving $\{U_n\}$.

Let $\{U_n\}$ be defined by (1.1). Then clearly $U_n \in \mathbb{Z}$. The first few values of U_{2n} are shown below:

$$U_2 = -2,\ U_4 = 22,\ U_6 = -602,\ U_8 = 30742,\ U_{10} = -2523002,$$
$$U_{12} = 303692662,\ U_{14} = -50402079002,\ U_{16} = 11030684333782.$$

Lemma 2.1. *We have*

$$\sum_{n=0}^{\infty} U_n \frac{t^n}{n!} = \frac{1}{e^t + e^{-t} - 1} \quad \big(|t| < \frac{\pi}{3}\big)$$

and

$$\sum_{n=0}^{\infty} (-1)^n U_{2n} \frac{t^{2n}}{(2n)!} = \frac{1}{2\cos t - 1} \quad \big(|t| < \frac{\pi}{3}\big).$$

Proof. By (1.1) we have

$$\begin{aligned}(e^t + e^{-t} - 1)\Big(\sum_{n=0}^{\infty} U_n \frac{t^n}{n!}\Big) &= \Big(1 + 2\sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!}\Big)\Big(\sum_{m=0}^{\infty} U_m \frac{t^m}{m!}\Big) \\ &= 1 + \sum_{n=1}^{\infty} \Big(U_n + 2\sum_{k=1}^{[n/2]} \binom{n}{2k} U_{n-2k}\Big)\frac{t^n}{n!} = 1.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} U_{2n}\frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} U_n\frac{t^n}{n!} = \frac{1}{e^t+e^{-t}-1}.$$

Replacing t with it and noting that $\mathrm{e}^{it}+\mathrm{e}^{-it}=2\cos t$ we deduce the remaining result.

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\ (n\ge 0).$$

The Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\ \ (|t|<\pi), \tag{2.1}$$

which is equivalent to (see [MOS])

$$E_n(x)+\sum_{r=0}^{n}\binom{n}{r}E_r(x)=2x^n\ \ (n\ge 0). \tag{2.2}$$

It is well known that ([MOS])

$$\begin{aligned}E_n(x)&=\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r\\&=\frac{2}{n+1}\Big(B_{n+1}(x)-2^{n+1}B_{n+1}\Big(\frac{x}{2}\Big)\Big)\\&=\frac{2^{n+1}}{n+1}\Big(B_{n+1}\Big(\frac{x+1}{2}\Big)-B_{n+1}\Big(\frac{x}{2}\Big)\Big).\end{aligned} \tag{2.3}$$

In particular,

$$E_n=2^nE_n\Big(\frac{1}{2}\Big)\quad\text{and}\quad E_n(0)=\frac{2(1-2^{n+1})B_{n+1}}{n+1}. \tag{2.4}$$

It is also known that ([MOS])

$$B_{2n+3}=0,\ B_n(1-x)=(-1)^nB_n(x)\quad\text{and}\quad E_n(1-x)=(-1)^nE_n(x). \tag{2.5}$$

Lemma 2.2. *For $n\in\mathbb{N}$ we have*

$$E_n\Big(\frac{1}{3}\Big)=\frac{2}{n+1}\big((-2)^{n+1}-1\big)\big)B_{n+1}\Big(\frac{1}{3}\Big)=\frac{2^{n+1}((-2)^{n+1}-1)}{(n+1)((-2)^n+1)}B_{n+1}\Big(\frac{1}{6}\Big).$$

Proof. By (2.3) we have $E_n(\frac13) = \frac{2}{n+1}\big(B_{n+1}(\frac13) - 2^{n+1}B_{n+1}(\frac16)\big)$. From Raabe's theorem (see [S3,(2.9)]) we have $B_{n+1}(\frac16)+B_{n+1}(\frac16+\frac12) = 2^{-n}B_{n+1}(\frac13)$. As $B_{n+1}(\frac16+\frac12) = B_{n+1}(\frac23) = (-1)^{n+1}B_{n+1}(\frac13)$, we see that

$$B_{n+1}\Big(\frac16\Big) = \big(2^{-n} - (-1)^{n+1}\big)B_{n+1}\Big(\frac13\Big).$$

Thus,

$$\begin{aligned} E_n\Big(\frac13\Big) &= \frac{2}{n+1}\Big(B_{n+1}\Big(\frac13\Big) - 2^{n+1}B_{n+1}\Big(\frac16\Big)\Big) \\ &= \frac{2}{n+1}\Big(1 - 2^{n+1}(2^{-n} - (-1)^{n+1})\Big)B_{n+1}\Big(\frac13\Big) \\ &= \frac{2}{n+1}\cdot\frac{(-2)^{n+1}-1}{2^{-n}+(-1)^n}B_{n+1}\Big(\frac16\Big). \end{aligned}$$

So the lemma is proved.

Theorem 2.1. *For $n \in \mathbb{N}$ we have*

$$U_{2n} = 3^{2n}E_{2n}\Big(\frac13\Big) = -2\big(2^{2n+1}+1\big)3^{2n}\frac{B_{2n+1}(\frac13)}{2n+1} = -\frac{2(2^{2n+1}+1)6^{2n}}{2^{2n}+1}\cdot\frac{B_{2n+1}(\frac16)}{2n+1}.$$

Proof. Using (2.1) and Lemma 2.1 we see that

$$\begin{aligned} 2\sum_{n=0}^{\infty}E_{2n}\Big(\frac13\Big)\frac{(3t)^{2n}}{(2n)!} &= \sum_{n=0}^{\infty}E_n\Big(\frac13\Big)\frac{(3t)^n}{n!} + \sum_{n=0}^{\infty}E_n\Big(\frac13\Big)\frac{(-3t)^n}{n!} \\ &= \frac{2e^t}{e^{3t}+1} + \frac{2e^{-t}}{e^{-3t}+1} = \frac{2e^t+2e^{2t}}{e^{3t}+1} = \frac{2e^t}{e^{2t}-e^t+1} \\ &= \frac{2}{e^t+e^{-t}-1} = 2\sum_{n=0}^{\infty}U_n\frac{t^n}{n!} = 2\sum_{n=0}^{\infty}U_{2n}\frac{t^{2n}}{(2n)!}. \end{aligned}$$

Thus $U_{2n} = 3^{2n}E_{2n}(\frac13)$. Now applying Lemma 2.2 we deduce the remaining result.

Theorem 2.2. *For two sequences $\{a_n\}$ and $\{b_n\}$ we have the following inversion formula:*

$$\begin{aligned} b_n &= 2\sum_{k=0}^{[n/2]}\binom{n}{2k}a_{n-2k} - a_n \quad (n=0,1,2,\dots) \\ \iff a_n &= \sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}b_{n-2k} \quad (n=0,1,2,\dots). \end{aligned}$$

Proof. It is clear that

$$(e^t+e^{-t}-1)\Big(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\Big)=\Big(-1+2\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\Big)\Big(\sum_{m=0}^{\infty}a_m\frac{t^m}{m!}\Big)$$
$$=\sum_{n=0}^{\infty}\Big(2\sum_{k=0}^{[n/2]}\binom{n}{2k}a_{n-2k}-a_n\Big)\frac{t^n}{n!}.$$

Thus, using Lemma 2.1 and the fact $U_{2n-1}=0$ we see that

$$b_n=2\sum_{k=0}^{[n/2]}\binom{n}{2k}a_{n-2k}-a_n\quad(n=0,1,2,\dots)$$
$$\iff (e^t+e^{-t}-1)\Big(\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\Big)=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}$$
$$\iff \sum_{n=0}^{\infty}a_n\frac{t^n}{n!}=\Big(\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\Big)\Big(\sum_{k=0}^{\infty}U_{2k}\frac{t^{2k}}{(2k)!}\Big)$$
$$\iff a_n=\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}b_{n-2k}\quad(n=0,1,2,\dots).$$

This proves the theorem.

Theorem 2.3. *Let n be a nonnegative integer. For any complex number x we have*

(i) $$\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}((x-1)^{n-2k}-x^{n-2k}+(x+1)^{n-2k})=x^n,$$

(ii) $$\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}(x^{n-2k}+(x+3)^{n-2k})=(x+1)^n+(x+2)^n,$$

(iii) $$\sum_{k=0}^{[n/2]}\binom{n}{2k}U_{2k}((x+3)^{n-2k}-(x-3)^{n-2k})$$
$$=(x+2)^n+(x+1)^n-(x-1)^n-(x-2)^n.$$

Proof. From the binomial theorem we see that

$$2\sum_{k=0}^{[n/2]}\binom{n}{2k}x^{n-2k}-x^n=(x-1)^n+(x+1)^n-x^n.$$

Thus, applying Theorem 2.2 we deduce (i). Since

$$x^m-(x+1)^m+(x+2)^m+(x+1)^m-(x+2)^m+(x+3)^m=x^m+(x+3)^m,$$

from (i) we deduce (ii). As $x^m+(x+3)^m-((x-3)^m+x^m)=(x+3)^m-(x-3)^m$, from (ii) we deduce (iii). So the theorem is proved.

Theorem 2.4. *For $n \in \mathbb{N}$ we have*

(i) $$\sum_{k=0}^{[n/2]} \binom{n}{2k}(2^{n-2k}-1)U_{2k} = 1 - U_n,$$

(ii) $$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k} 6^{n-2k} U_{2k} = 5^n + 4^n - 2^n - 1,$$

(iii) $$U_{2n} = 1 + 2^{2n} - \sum_{k=0}^{n} \binom{2n}{2k} 3^{2n-2k} U_{2k},$$

(iv) $$U_{2n} = 2(-1)^n - 4\sum_{k=1}^{[n/2]} \binom{2n}{4k}((-4)^k - 1)U_{2n-4k},$$

(v) $$U_{2n} = 4^{n-1} + \frac{1+V_{2n}}{4} - \frac{3}{4}\sum_{k=1}^{[n/3]} \binom{2n}{6k} 3^{6k} U_{2n-6k},$$

where V_m is given by $V_0 = 2,\ V_1 = 1$ and $V_{m+1} = V_m - 7V_{m-1}$ $(m \geq 1)$.

Proof. Taking $x = 1$ in Theorem 2.3(i) and noting that $U_n = 0$ for odd n we obtain (i). Taking $x = 3$ in Theorem 2.3(iii) we deduce (ii). Taking $x = 0$ in Theorem 2.3(ii) and then replacing n with $2n$ we derive (iii). Set $i = \sqrt{-1}$. By Theorem 2.3(i) we have

$$\sum_{k=0}^{n} \binom{2n}{2k} U_{2k}\big((i-1)^{2n-2k} - i^{2n-2k} + (i+1)^{2n-2k}\big) = i^{2n}.$$

That is,

$$\sum_{k=0}^{n} \binom{2n}{2k} U_{2k}\big((-2i)^{n-k} - (-1)^{n-k} + (2i)^{n-k}\big) = (-1)^n.$$

Hence

$$\sum_{\substack{k=0\\ 2\mid n-k}}^{n} \binom{2n}{2k} U_{2k}\big(2^{n+1-k}(-1)^{\frac{n-k}{2}} - 1\big) + \sum_{\substack{k=0\\ 2\nmid n-k}}^{n} \binom{2n}{2k} U_{2k} = (-1)^n.$$

Therefore,

$$\sum_{\substack{k=0\\ 2\mid n-k}}^{n} \binom{2n}{2k} U_{2k}\big(2^{n+1-k}(-1)^{\frac{n-k}{2}} - 2\big) = (-1)^n - \sum_{k=0}^{n} \binom{2n}{2k} U_{2k} = (-1)^n - \frac{1}{2}U_{2n}$$

and so

$$2\sum_{\substack{r=0\\ 2\mid r}}^{n} \binom{2n}{2r} U_{2n-2r}\big((-1)^{\frac{r}{2}} 2^r - 1\big) = (-1)^n - \frac{1}{2}U_{2n}.$$

This yields (iv).

Set $\omega = (-1+\sqrt{-3})/2$. From Theorem 2.3(ii) we have

$$\sum_{k=0}^{n}\binom{2n}{2k}U_{2k}\Big((3\omega)^{2n-2k}+(3\omega+3)^{2n-2k}\Big) = (3\omega+1)^{2n}+(3\omega+2)^{2n}.$$

It is easily seen that $V_m = \big(\frac{1+3\sqrt{-3}}{2}\big)^m + \big(\frac{1-3\sqrt{-3}}{2}\big)^m = (2+3\omega)^m + (-1-3\omega)^m$ and

$$\omega^{2n-2k}+(\omega+1)^{2n-2k} = \omega^{2n-2k}+(\omega^2)^{2n-2k} = \begin{cases} 2 & \text{if } 3 \mid n-k, \\ \omega+\omega^2 = -1 & \text{if } 3 \nmid n-k. \end{cases}$$

Thus

$$\begin{aligned} &3\sum_{\substack{k=0\\3\mid n-k}}^{n}\binom{2n}{2k}3^{2n-2k}U_{2k} - \sum_{k=0}^{n}\binom{2n}{2k}3^{2n-2k}U_{2k} \\ &= \sum_{k=0}^{n}\binom{2n}{2k}U_{2k}\Big((3\omega)^{2n-2k}+(3\omega+3)^{2n-2k}\Big) \\ &= (3\omega+1)^{2n}+(3\omega+2)^{2n} = V_{2n}. \end{aligned}$$

Hence, applying (iii) we deduce

$$\begin{aligned} &3\sum_{\substack{k=0\\3\mid k}}^{n}\binom{2n}{2k}3^{2k}U_{2n-2k} \\ &= 3\sum_{\substack{k=0\\3\mid n-k}}^{n}\binom{2n}{2k}3^{2n-2k}U_{2k} = \sum_{k=0}^{n}\binom{2n}{2k}3^{2n-2k}U_{2k} + V_{2n} \\ &= 1+2^{2n}-U_{2n}+V_{2n}. \end{aligned}$$

This yields (v). The proof is now complete.

Lemma 2.3 ([MOS, p.30]). *For $n \in \mathbb{N}$ and $0 \le x \le 1$ we have*

$$E_n(x) = 4\cdot\frac{n!}{\pi^{n+1}}\sum_{m=0}^{\infty}\frac{\sin((2m+1)\pi x - \frac{n\pi}{2})}{(2m+1)^{n+1}}.$$

Theorem 2.5. *Let $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{\infty}\Big(\frac{1}{(6k+1)^{2n+1}} - \frac{1}{(6k+5)^{2n+1}}\Big) = (-1)^n\frac{U_{2n}\cdot\pi^{2n+1}}{2\sqrt{3}\cdot 3^{2n}\cdot(2n)!}.$$

Proof. From Lemma 2.3 and Theorem 2.1 we see that

$$\begin{aligned}&(-1)^n\frac{U_{2n}\cdot\pi^{2n+1}}{4\cdot 3^{2n}\cdot(2n)!}\\&=(-1)^n\frac{E_{2n}(\frac13)\pi^{2n+1}}{4\cdot(2n)!}=(-1)^n\sum_{m=0}^{\infty}\frac{\sin(\frac{2m+1}{3}\pi-n\pi)}{(2m+1)^{2n+1}}\\&=\sum_{m=0}^{\infty}\frac{\sin\frac{2m+1}{3}\pi}{(2m+1)^{2n+1}}=\frac{\sqrt3}{2}\sum_{k=0}^{\infty}\Big(\frac{1}{(6k+1)^{2n+1}}-\frac{1}{(6k+5)^{2n+1}}\Big).\end{aligned}$$

This yields the result.

Corollary 2.1. *For $n\in\mathbb{N}$ we have $(-1)^nU_{2n}>0$.*

3. Congruences involving $\{U_{2n}\}$.

Theorem 3.1. *Let p be a prime of the form $4k+1$. Then*

$$U_{\frac{p-1}{2}}\equiv\big(1+2(-1)^{\frac{p-1}{4}}\big)h(-3p)\pmod p.$$

Proof. From Theorem 2.1 we see that

$$\begin{aligned}U_{\frac{p-1}{2}}&=-2(2^{\frac{p+1}{2}}+1)3^{\frac{p-1}{2}}\frac{B_{\frac{p+1}{2}}(\frac13)}{\frac{p+1}{2}}\equiv-4\Big(2\Big(\frac{2}{p}\Big)+1\Big)\Big(\frac{3}{p}\Big)B_{\frac{p+1}{2}}\Big(\frac13\Big)\\&\equiv\begin{cases}-12B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv1\pmod{24}\\ -4B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv5\pmod{24},\\ 4B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv13\pmod{24},\\ 12B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv17\pmod{24}.\end{cases}\end{aligned}$$

By [S3, Theorem 3.2(i)] we have

$$h(-3p)\equiv\begin{cases}-4B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv1\pmod{12},\\ 4B_{\frac{p+1}{2}}(\frac13)\pmod p & \text{if } p\equiv5\pmod{12}\end{cases}$$

Now combining the above we deduce the result.

Corollary 3.1. *Let p be a prime of the form $4k+1$. Then $p\nmid U_{\frac{p-1}{2}}$.*

Proof. From [UW, p.40] we know that $h(-3p)=2\sum_{a=1}^{[p/3]}\big(\frac{p}{a}\big)$. Thus $1\le h(-3p)<p$. Now the result follows from Theorem 3.1.

For an odd prime p and $a\in\mathbb{Z}$ with $p\nmid a$ let $q_p(a)=(a^{p-1}-1)/p$ denote the corresponding Fermat quotient.

Theorem 3.2. *Let p be a prime greater than 5. Then*

(i) $\sum_{k=1}^{[p/6]}\frac1k \equiv -2q_p(2)-\frac32 q_p(3)+p\big(q_p(2)^2+\frac34 q_p(3)^2\big)-\frac{5p}{2}\left(\frac p3\right)U_{p-3} \pmod{p^2}$,

(ii) $\sum_{k=1}^{[p/3]}\frac1k \equiv -\frac32 q_p(3)+\frac34 pq_p(3)^2-p\left(\frac p3\right)U_{p-3} \pmod{p^2}$,

(iii) $\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k} \equiv 9\sum_{\substack{k=1\\ 3\mid k+p}}^{p-1}\frac1k \equiv 3p\left(\frac p3\right)U_{p-3} \pmod{p^2}$.

(iv) *We have*

$$(-1)^{[\frac p6]}\binom{p-1}{[\frac p6]} \equiv 1+p\Big(2q_p(2)+\frac32 q_p(3)\Big)+p^2\Big(q_p(2)^2+3q_p(2)q_p(3)+\frac38 q_p(3)^2-5\Big(\frac p3\Big)U_{p-3}\Big) \pmod{p^3}$$

and

$$(-1)^{[\frac p3]}\binom{p-1}{[\frac p3]} \equiv 1+\frac32 pq_p(3)+\frac38 p^2q_p(3)^2-\frac{p^2}{2}\Big(\frac p3\Big)U_{p-3} \pmod{p^3}.$$

Proof. From Theorem 2.1 and Fermat's little theorem we have

$$U_{p-3}=-\frac{2(2^{p-2}+1)\cdot 6^{p-3}}{2^{p-3}+1}\cdot\frac{B_{p-2}(\frac16)}{p-2}\equiv\frac1{30}B_{p-2}\Big(\frac16\Big) \pmod p.$$

Now applying [S4, Theorem 3.9] we deduce the result.

Theorem 3.3. *Let $p>3$ be a prime and $k\in\{2,4,\dots,p-3\}$. Then*

$$\sum_{x=1}^{[p/6]}\frac1{x^k}\equiv 6^k\sum_{\substack{x=1\\ 6\mid x-p}}^{p-1}\frac1{x^k}\equiv\frac{6^k(2^k+1)}{4(2^{k-1}+1)}\Big(\frac p3\Big)U_{p-1-k} \pmod p$$

and

$$\sum_{x=1}^{[p/3]}\frac1{x^k}\equiv 3^k\sum_{\substack{x=1\\ 3\mid x-p}}^{p-1}\frac1{x^k}\equiv\frac{6^k}{4(2^{k-1}+1)}\Big(\frac p3\Big)U_{p-1-k} \pmod p.$$

Proof. Let $m\in\{3,6\}$. As $B_{p-k}(\frac{m-1}{m})=(-1)^{p-k}B_{p-k}(\frac1m)=-B_{p-k}(\frac1m)$, we see that $B_{p-k}(\{\frac pm\})=(\frac p3)B_{p-k}(\frac1m)$. Now putting $s=1$ and substituting k with $p-1-k$ in [S3, Corollary 2.2] we see that for $k\in\{2,4,\dots,p-3\}$,

$$\sum_{x=1}^{[p/m]}\frac1{x^k}\equiv\sum_{x=1}^{[p/m]}x^{p-1-k}\equiv\frac{B_{p-k}(0)-B_{p-k}(\{\frac pm\})}{p-k}=-\Big(\frac p3\Big)\frac{B_{p-k}(\frac1m)}{p-k} \pmod p.$$

By [S3, (2.6)] we have

$$\sum_{\substack{x=1\\ m|x-p}}^{p-1}\frac{1}{x^k}\equiv\sum_{\substack{x=1\\ m|x-p}}^{p-1}x^{p-1-k}\equiv(-m)^{p-1-k}\sum_{x=1}^{[p/m]}x^{p-1-k}\equiv\frac{1}{m^k}\sum_{x=1}^{[p/m]}\frac{1}{x^k}\pmod p.$$

From Theorem 2.1 we know that

$$\frac{B_{p-k}(\frac16)}{p-k}=-\frac{1+2^{p-1-k}}{2(2^{p-k}+1)6^{p-1-k}}U_{p-1-k}\equiv-\frac{1+2^{-k}}{2(2^{1-k}+1)6^{-k}}U_{p-1-k}\pmod p$$

and

$$\frac{B_{p-k}(\frac13)}{p-k}=-\frac{U_{p-1-k}}{2\cdot3^{p-1-k}(2^{p-k}+1)}\equiv-\frac{U_{p-1-k}}{2\cdot3^{-k}(2^{1-k}+1)}\pmod p.$$

Now putting all the above together we deduce the result.

Corollary 3.2. *Let $p>3$ be a prime and $k\in\{2,4,\dots,p-3\}$. Then*

$$\sum_{x=[p/6]+1}^{[p/3]}\frac{1}{x^k}\equiv-\frac{12^k}{4(2^{k-1}+1)}\Big(\frac{p}{3}\Big)U_{p-1-k}\pmod p$$

and

$$\sum_{x=1}^{[p/3]}\frac{1}{x^k}\equiv\frac{1}{2^k+1}\sum_{x=1}^{[p/6]}\frac{1}{x^k}\equiv-\frac{1}{2^k}\sum_{x=[p/6]+1}^{[p/3]}\frac{1}{x^k}\pmod p.$$

Remark 3.1 For a prime $p>5$ the congruence $\sum_{x=1}^{[p/3]}\frac{1}{x^2}\equiv\frac15\sum_{x=1}^{[p/6]}\frac{1}{x^2}\pmod p$ was first found by Schwindt. See [R].

Theorem 3.4. *Let $p>3$ be a prime and $k\in\{2,4,\dots,p-3\}$. Then*

$$\sum_{x=1}^{[p/3]}(-1)^{x-1}\frac{1}{x^k}\equiv-\frac{3^k}{2}\Big(\frac{p}{3}\Big)U_{p-1-k}\pmod p$$

and

$$\sum_{x=1}^{[\frac{p+3}{6}]}\frac{1}{(2x-1)^k}\equiv-\frac{3^k}{2^{k+1}+4}\Big(\frac{p}{3}\Big)U_{p-1-k}\pmod p.$$

Proof. Putting $m=3$ and $s=1$ in [S3, Corollary 2.2] and then substituting k with $p-1-k$ we see that

$$\begin{aligned}&E_{p-1-k}(0)-(-1)^{[\frac p3]}E_{p-1-k}\Big(\Big\{\frac p3\Big\}\Big)\\ &\equiv2(-1)^{p-1-k-1}\sum_{x=1}^{[p/3]}(-1)^x x^{p-1-k}\equiv2\sum_{x=1}^{[p/3]}(-1)^{x-1}\frac{1}{x^k}\pmod p.\end{aligned}$$

By (2.4) and (2.5) we have

$$E_{p-1-k}(0)=\frac{2(1-2^{p-k})B_{p-k}}{p-k}=0.$$

From (2.5) and Theorem 2.1 we have

$$E_{p-1-k}\Big(\Big\{\frac{p}{3}\Big\}\Big)=E_{p-1-k}\Big(\frac{1}{3}\Big)=3^{k+1-p}U_{p-1-k}\equiv 3^kU_{p-1-k} \pmod p.$$

Observe that $(-1)^{[\frac{p}{3}]}=(\frac{p}{3})$. From the above we deduce the first part. Since

$$\sum_{x=1}^{[\frac{p}{3}]}(-1)^{x-1}\frac{1}{x^k}=-\sum_{x=1}^{[\frac{p}{6}]}\frac{1}{(2x)^k}+\sum_{x=1}^{[\frac{p+3}{6}]}\frac{1}{(2x-1)^k},$$

applying the first part and Theorem 3.3 we deduce the remaining result.

Corollary 3.3. *Let p be a prime of the form $4k+1$. Then*

$$U_{\frac{p-1}{2}}\equiv -2\big(2+(-1)^{\frac{p-1}{4}}\big)\sum_{x=1}^{[\frac{p+3}{6}]}\Big(\frac{p}{2x-1}\Big) \pmod p.$$

Proof. Taking $k=(p-1)/2$ in Theorem 3.4 and applying Euler's criterion we obtain

$$\sum_{x=1}^{[\frac{p+3}{6}]}\Big(\frac{2x-1}{p}\Big)\equiv -\frac{(\frac{3}{p})(\frac{p}{3})}{4+2(\frac{2}{p})}U_{\frac{p-1}{2}}=-\frac{1}{4+2(-1)^{\frac{p-1}{4}}}U_{\frac{p-1}{2}} \pmod p.$$

This yields the result.

4. Congruences for $U_{k(p-1)+b} \pmod{p^n}$.

Theorem 4.1. *Let $n\in\mathbb{N}$ with $n\ge 3$, and let α be a nonnegative integer such that $2^\alpha\mid n$. Then $U_{2n}\equiv \frac{2}{3} \pmod{2^{\alpha+4}}$. Moreover,*

$$U_{2n}\equiv\begin{cases}48n+\frac{2}{3} \pmod{2^{\alpha+7}} & \text{if } 2\mid n,\\ 48n+22 \pmod{2^7} & \text{if } 2\nmid n.\end{cases}$$

Proof. From Theorem 2.4(i) we have

$$\sum_{k=0}^{n}\binom{2n}{2k}(2^{2n-2k}-1)U_{2k}=1-U_{2n}.$$

Thus, using (1.1) we see that

$$\sum_{k=0}^{n}\binom{2n}{2k}2^{2n-2k}U_{2k}=1+\sum_{k=0}^{n-1}\binom{2n}{2k}U_{2k}=1-\frac{1}{2}U_{2n}.$$

Hence

$$U_{2n}=2-2\sum_{r=0}^{n}\binom{2n}{2r}2^{2r}U_{2n-2r}$$

and so

$$U_{2n}=\frac{2}{3}\Big(1-\sum_{r=1}^{n}\binom{2n}{2r}4^rU_{2n-2r}\Big)=\frac{2}{3}-\frac{2n}{3}\sum_{r=1}^{n}\binom{2n-1}{2r-1}\frac{4^r}{r}U_{2n-2r}. \tag{4.1}$$

From the definition of U_{2n} we know that $2 \mid U_{2m}$ for $m\ge 1$. Thus, for $1\le r\le n$ and $n\ge 2$ we have $\frac{4^r}{r}U_{2n-2r}\equiv 0 \pmod 8$ and so $2n\cdot\frac{4^r}{r}U_{2n-2r}\equiv 0 \pmod{2^{\alpha+4}}$. Therefore, by (4.1) we have $U_{2n}\equiv \frac{2}{3} \pmod{2^{\alpha+4}}$ and hence $U_{2n}\equiv 6 \pmod{16}$ for $n\ge 2$.

Since $\frac{4^{n-3}}{n}\in\mathbb{Z}_2$ for $n\ge 3$, we see that $\frac{2n}{3}\cdot\frac{4^n}{n}=\frac{2^7n}{3}\cdot\frac{4^{n-3}}{n}\equiv 0 \pmod{2^{\alpha+7}}$. Thus, using (4.1) and the fact $U_{2m}\equiv 6 \pmod{16}$ for $m\ge 2$ we see that for $n\ge 3$,

$$\begin{aligned}U_{2n}-\frac{2}{3}&=-\frac{2n}{3}\Big(4\sum_{r=1}^{n-2}\binom{2n-1}{2r-1}\frac{4^{r-1}}{r}U_{2n-2r}-2\cdot 2^{2n-2}(2n-1)+\frac{4^n}{n}\Big)\\&\equiv -\frac{2n}{3}\cdot 4\sum_{r=1}^{n-1}\binom{2n-1}{2r-1}\frac{4^{r-1}}{r}\cdot 6\\&=-16n\Big(2n-1+2\binom{2n-1}{3}+\sum_{r=3}^{n-1}\binom{2n-1}{2r-1}\frac{4^{r-1}}{r}\Big)\\&\equiv -16n\Big(2n-1+2\binom{2n-1}{3}\Big) \pmod{2^{\alpha+7}}.\end{aligned}$$

It is clear that

$$\begin{aligned}2n-1+2\binom{2n-1}{3}&\equiv 9\Big(2n-1+2\binom{2n-1}{3}\Big)\\&=3(2n-1)(2n+(2n-3)^2)\equiv 3(2n-1)(2n+1)\\&=3(2n-1)^2+6(2n-1)\equiv 4n-3 \pmod 8.\end{aligned}$$

Thus,

$$U_{2n}-\frac{2}{3}\equiv -16n(4n-3)\equiv 48n+32(1-(-1)^n) \pmod{2^{\alpha+7}}.$$

This yields the result.

Corollary 4.1. *Let $n \in \mathbb{N}$ and $n \geq 3$. Then*

$$U_{2n} \equiv 6 \pmod{16} \quad \text{and} \quad U_{2n} \equiv \begin{cases} 48n - 42 \pmod{128} & \text{if } 2 \mid n, \\ -16n - 42 \pmod{128} & \text{if } 2 \nmid n. \end{cases}$$

Theorem 4.2. *Let p be an odd prime and $b \in \{0, 2, 4, \dots\}$. Then $f(k) = (1 - (\frac{p}{3})p^{k(p-1)+b})U_{k(p-1)+b}$ is a p-regular function.*

Proof. Suppose $n \in \mathbb{N}$. From Theorem 2.1 and (2.3) we have

$$\begin{aligned} 2^{2k+b}U_{2k+b} &= 2^{2k+b} \cdot 3^{2k+b}E_{2k+b}\Big(\frac{1}{3}\Big) = 3^{2k+b}\sum_{r=0}^{2k+b}\binom{2k+b}{r}\Big(-\frac{1}{3}\Big)^{2k+b-r}E_r \\ &= \sum_{r=0}^{2k+b}\binom{2k+b}{r}(-3)^r E_r \equiv \sum_{r=0}^{n-1}\binom{2k+b}{r}(-3)^r E_r \\ &= \sum_{r=0}^{n-1}(2k+b)(2k+b-1)\cdots(2k+b-r+1)\frac{(-3)^r}{r!}E_r \pmod{3^n}. \end{aligned}$$

Since $E_r \in \mathbb{Z}$ and $3^r/r! \in \mathbb{Z}_3$, there are $a_0, a_1, \dots, a_{n-1} \in \mathbb{Z}_3$ such that

$$2^{2k+b}U_{2k+b} \equiv a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 \pmod{3^n} \quad \text{for every } k = 0, 1, 2, \dots.$$

Hence, using [S1, Theorem 2.1] we see that $2^{2k+b}U_{2k+b}$ is a 3-regular function. As

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k 2^{-2k-b} = 2^{-b}\Big(1 - \frac{1}{4}\Big)^n \equiv 0 \pmod{3^n},$$

we see that 2^{-2k-b} is also a 3-regular function. Hence, using the above and the product theorem of p-regular functions (see [S1, Theorem 2.3]) we deduce that $f(k) = U_{2k+b}$ is a 3-regular function. Therefore, the result is true for $p = 3$.

Now let us consider the case $p > 3$. For $x \in \mathbb{Z}_p$ let $\langle x\rangle_p$ be the least nonnegative residue of x modulo p. Since $2 \mid b$ we have $p - 1 \nmid b + 1$. From [S1, Theorem 3.2] we know that

$$f_1(k) = \frac{B_{k(p-1)+b+1}(\frac{1}{3}) - p^{k(p-1)+b}B_{k(p-1)+b+1}\big(\frac{\frac{1}{3}+\langle -\frac{1}{3}\rangle_p}{p}\big)}{k(p-1)+b+1}$$

is a p-regular function. As

$$\frac{\frac{1}{3} + \langle -\frac{1}{3}\rangle_p}{p} = \begin{cases} \frac{\frac{1}{3}+\frac{p-1}{3}}{p} = \frac{1}{3} & \text{if } p \equiv 1 \pmod 3, \\ \frac{\frac{1}{3}+\frac{2p-1}{3}}{p} = \frac{2}{3} & \text{if } p \equiv 2 \pmod 3 \end{cases}$$

and $B_{k(p-1)+b+1}(\frac{2}{3}) = (-1)^{k(p-1)+b+1}B_{k(p-1)+b+1}(\frac{1}{3}) = -B_{k(p-1)+b+1}(\frac{1}{3})$, we see that

$$f_1(k) = \Big(1 - \Big(\frac{p}{3}\Big)p^{k(p-1)+b}\Big)\frac{B_{k(p-1)+b+1}(\frac{1}{3})}{k(p-1)+b+1}.$$

By Theorem 2.1 and the above we have

$$\begin{aligned} f(k) &= \Big(1 - \Big(\frac{p}{3}\Big)p^{k(p-1)+b}\Big) \cdot (-2)\big(2^{k(p-1)+b+1} + 1\big)3^{k(p-1)+b}\frac{B_{k(p-1)+b+1}(\frac{1}{3})}{k(p-1)+b+1} \\ &= -2\big(2^{k(p-1)+b+1} + 1\big)3^{k(p-1)+b}f_1(k). \end{aligned}$$

Since

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\big(2^{k(p-1)+b+1}+1\big)3^{k(p-1)+b} = 2\cdot 6^b(1-6^{p-1})^n+3^b(1-3^{p-1})^n \equiv 0 \pmod{p^n},$$

using the above and the product theorem of p-regular functions (see [S1, Theorem 2.3]) we deduce that $f(k)$ is a p-regular function, which completes the proof.

From Theorem 4.2 and [S3, Theorem 4.3 (with $t = 1$ and $d = 0$)] we deduce the following result.

Theorem 4.3. *Let p be an odd prime, $k, m, n \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. Then*

$$\begin{aligned} &\Big(1 - \Big(\frac{p}{3}\Big)p^{k\varphi(p^m)+b}\Big)U_{k\varphi(p^m)+b} \\ &\equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\Big(1 - \Big(\frac{p}{3}\Big)p^{r\varphi(p^m)+b}\Big)U_{r\varphi(p^m)+b} \pmod{p^{mn}}. \end{aligned}$$

In particular, for $n = 1$ we have $U_{k\varphi(p^m)+b} \equiv (1 - (\frac{p}{3})p^b)U_b \pmod{p^m}$.

From Theorem 4.2 and [S1, Theorem 2.1] we deduce the following result.

Theorem 4.4. *Let p be an odd prime, $n \in \mathbb{N}$, $p \ge n$ and $b \in \{0, 2, 4, \dots\}$. Then there are unique integers $a_0, a_1, \dots, a_{n-1} \in \{0, \pm1, \pm2, \dots, \pm\frac{p^n-1}{2}\}$ such that*

$$\Big(1 - \Big(\frac{p}{3}\Big)p^{k(p-1)+b}\Big)U_{k(p-1)+b} \equiv a_{n-1}k^{n-1} + \cdots + a_1k + a_0 \pmod{p^n}$$

for every $k = 0, 1, 2, \dots$. Moreover, $\mathrm{ord}_p a_s \ge s - \mathrm{ord}_p s!$ *for $s = 0, 1, \dots, n-1$.*

Corollary 4.2. *Let $k \in \mathbb{N}$. Then*

(i) $U_{2k} \equiv -3k + 1 \pmod{27}$;

(ii) $U_{4k} \equiv 1250k^4 + 500k^3 + 725k^2 - 1205k + 2 \pmod{3125}$ $(k \ge 2)$;

(iii) $U_{4k+2} \equiv 1250k^4 - 1125k^3 - 675k^2 - 52 \pmod{3125}$.

From Theorem 4.2 and [S3, Corollary 4.2(iv)] we deduce:

Theorem 4.5. *Let p be an odd prime, $k, m \in \mathbb{N}$ and $b \in \{0, 2, 4, \dots\}$. Then*

$$U_{k\varphi(p^m)+b} \equiv (1-kp^{m-1})\Big(1-\big(\frac{p}{3}\big)p^b\Big)U_b + kp^{m-1}\Big(1-\big(\frac{p}{3}\big)p^{p-1+b}\Big)U_{p-1+b} \pmod{p^{m+1}}.$$

5. $\{(-1)^n U_{2n}\}$ is realizable.

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences satisfying $a_1 = b_1$ and $b_n + a_1 b_{n-1} + \cdots + a_{n-1}b_1 = na_n$ $(n > 1)$, following [S2] we say that (a_n, b_n) is a Newton-Euler pair. If (a_n, b_n) is a Newton-Euler pair and $a_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \dots$, then we say that $\{b_n\}$ is a Newton-Euler sequence.

Let $\{b_n\}$ be a Newton-Euler sequence. Then clearly $b_n \in \mathbb{Z}$ for all $n = 1, 2, 3, \dots$. In [DHL], $\{-b_n\}$ is called a Newton sequence generated by $\{-a_n\}$.

Lemma 5.1. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. Then the following statements are equivalent:*

(i) *$\{b_n\}$ is a Newton-Euler sequence.*

(ii) *$\sum_{d\mid n} \mu(\frac{n}{d})b_d \equiv 0 \pmod n$ for every $n \in \mathbb{N}$.*

(iii) *For any prime p and $\alpha, m \in \mathbb{N}$ with $p \nmid m$ we have $b_{mp^{\alpha}} \equiv b_{mp^{\alpha-1}} \pmod{p^{\alpha}}$.*

(iv) *For any $n, t \in \mathbb{N}$ and prime p with $p^t \parallel n$ we have $b_n \equiv b_{\frac{n}{p}} \pmod{p^t}$.*

(v) *There exists a sequence $\{c_n\}$ of integers such that $b_n = \sum_{d\mid n} dc_d$ for any $n \in \mathbb{N}$.*

(vi) *For any $n \in \mathbb{N}$ we have*

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \frac{b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}}{1^{k_1}\cdot k_1! \cdot 2^{k_2} \cdot k_2! \cdots n^{k_n} \cdot k_n!} \in \mathbb{Z}.$$

(vii) *For any $n \in \mathbb{N}$ we have*

$$\frac{1}{n!}\begin{vmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ -1 & b_1 & b_2 & \dots & b_{n-1} \\ & -2 & b_1 & \dots & b_{n-2} \\ & & \ddots & \ddots & \vdots \\ & & & -(n-1) & b_1 \end{vmatrix} \in \mathbb{Z}.$$

Proof. From [A, Theorem 3] or [DHL] we know that (i), (ii) and (iii) are equivalent. Clearly (iii) is equivalent (iv). Using the Möbius inversion formula we see that (ii) and (v) are equivalent. By [S2, Theorems 2.2 and 2.3], (i),(vi) and (vii) are equivalent. So the lemma is proved.

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of nonnegative integers. If there is a set X and a map $T: X \to X$ such that b_n is the number of fixed points of T^n, following [PW] and [A] we say that $\{b_n\}$ is realizable.

In [PW], Puri and Ward proved that a sequence $\{b_n\}$ of nonnegative integers is realizable if and and only if for all $n \in \mathbb{N}$, $\frac{1}{n}\sum_{d\mid n} \mu(\frac{n}{d})b_d$ is a nonnegative integer. Thus, using the Möbius inversion formula we see that a sequence $\{b_n\}$ is realizable if and and only if there exists a sequence $\{c_n\}$ of nonnegative integers such that $b_n = \sum_{d\mid n} dc_d$ for any $n \in \mathbb{N}$. In [A] J. Arias de Reyna showed that $\{E_{2n}\}$ is a Newton-Euler sequence and $\{|E_{2n}|\}$ is realizable.

Now we state the following result.

Theorem 5.1. *$\{U_{2n}\}$ is a Newton-Euler sequence and $\{(-1)^n U_{2n}\}$ is realizable.*

Proof. Suppose $n \in \mathbb{N}$ and $\alpha = \text{ord}_2 n$. If $2 \mid n$, by Theorem 4.1 we have $U_{2n} \equiv \frac{2}{3} \pmod{2^{\alpha+4}}$ and $U_n \equiv \frac{2}{3} \pmod{2^{\alpha+3}}$ for $n \geq 6$. Thus $U_{2n} \equiv \frac{2}{3} \equiv U_n \pmod{2^\alpha}$ for $n \geq 6$. For $n = 2, 4$ we also have $U_{2n} \equiv U_n \pmod{2^\alpha}$. If $2 \nmid n$, by (1.1) we have $U_{2n} \equiv 0 = U_n \pmod{2^0}$.

Now assume that p is an odd prime divisor of n and $n = p^t n_0$ with $p \nmid n_0$. Using Theorem 4.3 and the fact $2n_0 p^{t-1} \geq t$ we see that

$$U_{2n} = U_{2n_0 p^t} = U_{2n_0 \varphi(p^t) + 2n_0 p^{t-1}} \equiv U_{2n_0 p^{t-1}} \pmod{p^t}.$$

By the above, for any prime divisor p of n we have $U_{2n} \equiv U_{2n/p} \pmod{p^t}$, where $p^t \parallel n$. Hence, it follows from Lemma 5.1 that $\{U_{2n}\}$ is a Newton-Euler sequence.

By Corollary 2.1 we have $(-1)^n U_{2n} > 0$. Suppose that p is a prime divisor of n and $p^t \parallel n$. If p is odd, then $(-1)^n = (-1)^{\frac{n}{p}}$. If $p = 2$ and $4 \mid n$, we have $(-1)^n = (-1)^{\frac{n}{2}}$. If $p = 2$ and $2 \parallel n$, then $(-1)^n \equiv (-1)^{\frac{n}{2}} \pmod 2$. Thus, we always have $(-1)^n \equiv (-1)^{\frac{n}{p}} \pmod{p^t}$. By the previous argument, we also have $U_{2n} \equiv U_{2n/p} \pmod{p^t}$. Therefore, $(-1)^n U_{2n} \equiv (-1)^{\frac{n}{p}} U_{2n/p} \pmod{p^t}$. Hence, by Lemma 5.1 we have $\frac{1}{n} \sum_{d \mid n} \mu(\frac{n}{d})(-1)^d U_{2d} \in \mathbb{Z}$. Now it remains to show that $\sum_{d \mid n} \mu(\frac{n}{d})(-1)^d U_{2d} \geq 0$.

For $m \in \mathbb{N}$, by Theorem 2.5 we have

$$(-1)^m U_{2m} = \frac{2\sqrt{3} \cdot 3^{2m} \cdot (2m)!}{\pi^{2m+1}} \sum_{k=0}^{\infty} \Big(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}} \Big).$$

Since

$$\sum_{k=0}^{\infty} \Big(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}} \Big) = 1 - \sum_{k=0}^{\infty} \Big(\frac{1}{(6k+5)^{2m+1}} - \frac{1}{(6k+7)^{2m+1}} \Big) < 1$$

and

$$\sum_{k=0}^{\infty} \Big(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}} \Big) > 1 - \frac{1}{5^{2m+1}} > 1 - \frac{1}{5} = \frac{4}{5},$$

we see that

$$\frac{4}{5} \cdot \frac{2\sqrt{3} \cdot 3^{2m} \cdot (2m)!}{\pi^{2m+1}} < (-1)^m U_{2m} < \frac{2\sqrt{3} \cdot 3^{2m} \cdot (2m)!}{\pi^{2m+1}}.$$

Hence

$$
\begin{aligned}
\sum_{d|n}\mu\Big(\frac{n}{d}\Big)(-1)^dU_{2d} &= (-1)^nU_{2n} + \sum_{d|n,d\le\frac{n}{2}}\mu\Big(\frac{n}{d}\Big)(-1)^dU_{2d} \\
&\ge (-1)^nU_{2n} - \sum_{1\le d\le\frac{n}{2}}(-1)^dU_{2d} \\
&> \frac{4}{5}\cdot\frac{2\sqrt{3}\cdot 3^{2n}\cdot(2n)!}{\pi^{2n+1}} - \sum_{1\le d\le\frac{n}{2}}\frac{2\sqrt{3}\cdot 3^{2d}\cdot(2d)!}{\pi^{2d+1}} \\
&> \frac{4}{5}\cdot\frac{2\sqrt{3}\cdot 3^{2n}\cdot(2n)!}{\pi^{2n+1}} - \sum_{d=1}^{\infty}\frac{2\sqrt{3}\cdot 3^{2d}\cdot n!}{\pi^{2d+1}} \\
&= \frac{8\sqrt{3}}{5\pi}\cdot n!\Big\{\Big(\frac{9}{\pi^2}\Big)^n(n+1)(n+2)\cdots(2n) - \frac{5}{4}\cdot\frac{9/\pi^2}{1-9/\pi^2}\Big\}.
\end{aligned}
$$

For $m\in\mathbb{N}$ it is clear that

$$
\begin{aligned}
\Big(\frac{9}{\pi^2}\Big)^{m+1}(m+2)(m+3)\cdots(2m+2) &= \frac{9}{\pi^2}(4m+2)\cdot\Big(\frac{9}{\pi^2}\Big)^m(m+1)(m+2)\cdots(2m) \\
&> \Big(\frac{9}{\pi^2}\Big)^m(m+1)(m+2)\cdots(2m).
\end{aligned}
$$

Thus, for $n\ge 3$ we have

$$
\Big(\frac{9}{\pi^2}\Big)^n(n+1)(n+2)\cdots(2n) \ge \Big(\frac{9}{\pi^2}\Big)^3\cdot 4\cdot 5\cdot 6 > \frac{5}{4}\cdot\frac{9/\pi^2}{1-9/\pi^2}
$$

and so $\sum_{d|n}\mu(\frac{n}{d})(-1)^dU_{2d} > 0$. This inequality is also true for $n=1,2$. Thus, $\{(-1)^nU_{2n}\}$ is realizable. This completes the proof.

Let $\{a_n\}$ be defined by

$$
a_1=-2 \quad\text{and}\quad na_n = U_{2n}+a_1U_{2n-2}+\cdots+a_{n-1}U_2 \quad (n=2,3,4,\ldots).
$$

By Theorem 5.1 we have $a_n\in\mathbb{Z}$ for all $n\in\mathbb{N}$. The first few values of a_n are shown below:

$$
a_2=13,\ a_3=-224,\ a_4=8170,\ a_5=-522716,\ a_6=51749722,\ a_7=-7309866728.
$$

References

[A] J. Arias de Reyna, *Dynamical zeta functions and Kummer congruences*, Acta Arith. **119** (2005), 39-52.

[DHL] B.S. Du, S.S. Huang and M.C. Li, *Generalized Fermat, double Fermat and Newton sequences*, J. Number Theory **98** (2003), 172-183.

[MOS] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (*3rd edition*), Springer, New York, 1966, pp. 25-32.

[PW] Y. Puri and T. Ward, *Arithmetic and growth of periodic ordits*, J. Integer Seq. 4(2001), Art. 01.2.1.

[R] P. Ribenboim, *Thirteen Lectures on Fermat's Last Theorem*, Springer, New York, 1979, pp. 139-159.

[S1] Z.H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), 193-223.

[S2] Z.H. Sun, *On the properties of Newton-Euler pairs*, J. Number Theory **114** (2005), 88-123.

[S3] Z.H. Sun, *Congruences involving Bernoulli polynomials*, Discrete Math. **308** (2008), 71-112.

[S4] Z.H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory **128** (2008), 280-312.

[UW] J. Urbanowicz and K.S. Williams, *Congruences for L-Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000, pp. 3-5, 40.