Propagation of analyticity for a class of nonlinear hyperbolic equations

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Abstract

We consider the hyperbolic semilinear equations of the form

$$\partial_t^m u \,+\, a_1(t)\,\partial_t^{m-1}\partial_x u \,+\, \cdots \,+\, a_m(t)\,\partial_x^m u \,=\, f(u),$$

$f(u)$ entire analytic, with characteristic roots satisfying the condition

$$\lambda_i^2(t) + \lambda_j^2(t) \leq M(\lambda_i(t) - \lambda_j(t))^2, \quad \text{for} \;\; i \neq j,$$

and we prove that, if the $a_h(t)$ are analytic, each solution bounded in $\mathcal{C}^\infty$ enjoys the propagation of analyticity; while if $a_h(t) \in \mathcal{C}^\infty$, this property holds for those solutions which are bounded in some Gevrey class.

1 Introduction

The linear operator

$$\mathcal{L}\,U \,=\, U_t \,+\, \sum_{h=1}^{n} A_h(t,x)\,U_{x_h} \quad \text{on} \;\; [0,T]\times\mathbb{R}^n, \tag{1}$$

where the A_h's are $N\times N$ matrices, $U \in \mathbb{R}^N$, is *hyperbolic* when, for all $\xi \in \mathbb{R}^n$, the matrix $\sum A_h(t,x)\,\xi_h$ has real eigenvalues $\lambda_j(t,x,\xi)$, $1 \leq j \leq N$.
Denoting by $\mu(\lambda)$ the multiplicity of the eigenvalue λ, we call *multiplicity* of (1) the integer $m = \max_{t,x,\xi}\, \max_j\{\mu(\lambda_j(t,x,\xi))\}$. The case $m = 1$ corresponds to the *strictly hyperbolic systems*.

We study the regularity of solutions to nonlinear weakly hyperbolic system, in particular, *semilinear systems*

$$\mathcal{L}\,U \,=\, f(t,x,U)\,, \tag{2}$$

where $U : [0,T] \times \mathbb{R}^n \to \mathbb{R}^N$, and $f(t,x,U)$ is a $\mathbb{R}^N$-valued, analytic function, typically a polynomial in the scalar components of U.

More precisely, assuming the coefficients of $\mathcal{L}$ analytic in x, we investigate under which additional assumptions a solution $U(t,x)$ of (2), analytic at the initial time, keeps its analyticity, i.e., satisfies

$$U(0,\cdot) \in \mathcal{A}(\mathbb{R}^n) \quad \Longrightarrow \quad U(t,\cdot) \in \mathcal{A}(\mathbb{R}^n) \quad \forall\, t \in [0,T] \tag{3}$$

Actualy, we consider two versions of (3), the first weaker and the second one stronger than (3):

$$U(0,\cdot) \in \mathcal{A}_{L^2}(\mathbb{R}^n) \quad \Longrightarrow \quad U(t,\cdot) \in \mathcal{A}_{L^2}(\mathbb{R}^n) \quad \forall\, t \in [0,T]\,, \tag{4}$$

$$U(0,\cdot) \in \mathcal{A}(\Gamma_0) \quad \Longrightarrow \quad U(t,\cdot) \in \mathcal{A}(\Gamma_t) \quad \forall\, t \in [0,T]\,, \tag{5}$$

where $\mathcal{A}_{L^2}(\mathbb{R}^n)$ is the class of (analytic) functions $\varphi(x) \in H^\infty$ such that $\|\partial^j \varphi\|_{L^2} \leq C \Lambda^j\, j!$, while Γ is a *cone of determinacy* for the operator $\mathcal{L}$ with base Γ_0 (at $t=0$) and sections $\{\Gamma_t\}$.

The propagation of analyticity is a natural property for nonlinear hyperbolic equations. Indeed, on one side, the theorem of Cauchy-Kovalewsky ensures the validity of (3) in some time interval $[0,\tau[$ (the problem is to prove that $\tau = T$), on the other side, by the Bony-Schapira's theorem, the Cauchy problem for any linear (weakly) hyperbolic system is globally well posed in the class of analytic functions.

The first results of analytic propagation goes back to Lax ([L], 1953) who considered (2) with $n=1$ in the strictly hyperbolic case, and proved (5) for those solutions which are a priori bounded in $\mathcal{C}^1$. Later on Alinhac and Métivier ([AM], 1984) extended this results to several space dimensions, but assuming that $U(t,\cdot)$ is bounded in $H^s(\mathbb{R}^n)$ for s greater than some $\bar{s}(n)$.

In the weakly hyperbolic (nonlinear) case, the first results were concerning a second order equation of the form

$$\mathcal{L}_0\, u \equiv \sum_{i,j}^{1,n} \partial_{x_i}(a_{ij}(t,x)\, \partial_{x_j} u) \;=\; f(u), \quad \sum a_{ij}\, \xi_i \xi_j \geq 0, \tag{6}$$

with $f(u), a_{ij}(t,x)$ analytic :

Theorem A ([S], 1989)
i) In the special case when $a_{ij} = \beta_0(t)\, \alpha_{ij}(x)$, a solution of (6) enjoys (5) as long as remains bounded in $\mathcal{C}^\infty$.
ii) In the general case, a solution $u(t,\cdot)$ enjoys (5) provided it is bounded in some Gevrey class γ^s with $s < 2$.

We recall that the Cauchy problem for any strictly hyperbolic linear system is globally wellposed in $\mathcal{C}^\infty$. On the other hand, the Cauchy problem for the linear equation $\mathcal{L}_0 u = 0$, i is globally wellposed in $\mathcal{C}^\infty$ n the special case (*i*), whereas it is only globally wellposed in γ^s for $s < 2$ in the general case (*ii*). Thus, it is natural to formulate the following

Conjecture *In order to get the analytic propagation for a given solution to a weakly hyperbolic system $\mathcal{L}\,U = f(t,x,U)$, it is sufficient to assume* a priori *that $U(t,\cdot)$ is bounded in some functional class $\mathcal{X}$ in which the Cauchy problem for the linear systems $\mathcal{L}U + B(t,x)U = f(t,x)$ is globally well posed.*

[Typically the space $\mathcal{X}$ is equal to $\mathcal{C}^\infty$ or to some Gevrey class γ^s]

In the case when $\mathcal{L}$ is a weakly hyperbolic operator of the *general type* (1), this Conjecture says that a solution $U(t,\cdot)$ enjoys the analytic propagation a long as remains bounded in some Gevrey class γ^s of order $s < m/(m-1)$, where m is the multiplicity of $\mathcal{L}$. Indeed, Bronshtein's Theorem ([B], 1979) states that, for any linear system $\mathcal{L}\,U + B(t,x)U = f(t,x)$ with analytic coefficients in x, the Cauchy problem is well-posed in these Gevrey classes.

Actually, this fact was proved in two special cases: time depending coefficients, and one space variable. More precisely:

Theorem B ([DS], 1999) *A solution of*

$$U_t \,+\, \textstyle\sum_{j=1}^n A_j(t)\, U_{x_j} \;=\; f(t,x,U), \qquad x \in \mathbb{R}^n,$$

satisfies (4) as long as $U(t,\cdot)$ remains bounded in some γ^s with $s < m/(m-1)$.

Theorem C ([ST], 2010) *A solution of*

$$U_t \,+\, A(t,x)\, U_x \;=\; f(t,x,U), \qquad x \in \mathbb{R},$$

satisfies (5) as long as $U(t,\cdot)$ remains bounded in some γ^s with $s < m/(m-1)$.

The study of the general case (coefficients depending on (t,x), and $n \geq 2$) is in progress.

Open Problem. To prove the sharpness of the bound $s < m/(m-1)$ in Theorems B or C. In particular: to construct a hyperbolic nonlinear system admitting a solution $U \in \mathcal{C}^\infty(\mathbb{R}^2)$ which is analytic on the halfplane $\{t < 0\}$ but non analytic at some point of the line $t = 0$. This kind of questions is related to the so called *Nonlinear Holmgren Theorem* (see [M]).

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2 Main results

Hence, we consider the scalar equations of the form

$$\mathcal{L}\, u \;\equiv\; \partial_t^m u \,+\, a_1(t)\, \partial_t^{m-1}\partial_x u \,+\, \cdots \,+\, a_m(t)\, \partial_x^m u \;=\; f(u)\,, \qquad (7)$$

on $[0,T]\times\mathbb{R}$, where $f(u)=\sum_{\nu=0}^{\infty} u^\nu$ is an entire analytic, real function on $\mathbb{R}$. We assume that the characteristic roots of the equation are real functions, say

$$\lambda_1(t)\le\lambda_2(t)\le\ldots\le\lambda_m(t)\,,$$

which satisfying the condition

$$\lambda_1^2(t)+\lambda_j^2(t)\;\le\; M\,(\lambda_i(t)-\lambda_j(t)^2, \qquad \forall\, t\in[0,T] \quad (i\neq j). \qquad (8)$$

Remark 1 *Due to its symmetry with respect to the roots λ_j, condition (8) can be rewritten in term of the coefficients $\{a_h\}$* (Newton's theorem. *In particular* (see [KS])*: for a second order equation, (8) reads (for some $c>0$)*

$$\Delta(t)\;\equiv\; a_1^2(t)-4\,a_2(t)\;\ge\; c\,a_1^2(t)\,;$$

while for a third order equation, it becomes

$$\Delta(t)\;\ge\; c\,(a_1(t)a_2(t)-9\,a_3(t))^2, \qquad (9)$$

the discriminant being now $\Delta\;=\;-4\,a_2^3\,-\,27\,a_3^2+a_1^2\,a_2^2-4\,a_1^3\,a_3+18\,a_1a_2a_3$. Particularly simple are the third order traceless *equations. i.e., when $a_1\equiv 0$: here $a_2=-(\lambda_1^2+\lambda_2^2+\lambda_3^2)/2\;\le\;0$, $\Delta=-4\,a_2^3\,-\,27\,a_3^2$, so that (8) becomes $\Delta\ge -c\,a_2^3$, or equivalently $\Delta\ge c\,a_3^2$.*

Condition (8) for the linear equation $\mathcal{L}u=0$ was introduced in [CO] as a sufficient (and almost necessary) condition for the wellposedness in $\mathcal{C}^\infty$. A different proof of such a result, based on the quasi-symmetrizer, was given in [KS], where, also the case of non-analytic coefficients was considered: it was proved that, if $a_h(t)\in\mathcal{C}^\infty([0,T])$ and (8) is fulfilled, then the Cauchy problem for $\mathcal{L}u=0$ is well posed in each Gevrey class γ^s, $s\ge 1$.

By these existence results, it is natural to expect some kind of analytic propagation for the solutions which are bounded in $\mathcal{C}^\infty$ in case of analytic coefficients, or for those which are bounded in some Gevrey class γ^s in case of $\mathcal{C}^\infty$ coefficients.

Actually, introducing the analytic, and Gevrey classes

$$\begin{aligned}\mathcal{A}_{L^2} \;&=\; \left\{\varphi(x)\in\mathcal{C}^\infty(\mathbb{R}) : \|\partial^j\varphi\|_{L^p(\mathbb{R})}\;\le\; C\,\Lambda^j\, j!\,\right\},\\ \gamma^s_{L^2} \;&=\; \left\{\varphi(x)\in\mathcal{C}^\infty(\mathbb{R}) : \|\partial^j\varphi\|_{L^p(\mathbb{R})}\;\le\; C\,\Lambda^j\, j!^s\right\},\end{aligned}$$

where $s\ge 1$, we prove:

Theorem 1 *Assume that the $a_j(t)$'s are analytic functions on $[0,T]$. Then, for any solution of (7) satisfying*

$$\sup_{0\le t\le T}\int_{\mathbb{R}} |\partial_t^h \partial_x^j u(t,x)|\,dx \;<\; \infty, \qquad \forall\, j\in\mathbb{N}, \tag{10}$$

$$\partial_t^h u(0,\cdot) \;\in\; \mathcal{A}_{L^2}, \tag{11}$$

for $h=0,1,\dots,m-1$, it holds

$$u \;\in\; \mathcal{C}^{m-1}([0,T],\mathcal{A}_{L^2})\,. \tag{12}$$

Under the same assumptions, we have also

$$u\in \mathcal{A}\left([0,T]\times\mathbb{R}\right). \tag{13}$$

Theorem 2 *If the $a_j(t)$'s are $\mathcal{C}^\infty$ functions on $[0,T]$, the implication* (11) $\Longrightarrow$ (12) *holds true for those solutions which belong to $\mathcal{C}^m\left([0,T],\gamma^s_{L^2}\right)$ for some $s\ge 1$.*

Proof of Theorem 1. For the sake of simplicity, we shall perform the proof only in the case when the nonlinear term $f(u)$ is a monomial function, the general case requiring only minor additional computations. Thus, for a given integer $\nu\ge 1$, we consider the equation

$$\partial_t^m u \,+\, a_1(t)\,\partial_t^{m-1}\partial_x u \,+\,\cdots\,+\, a_m(t)\,\partial_x^m u \;=\; u^\nu. \tag{14}$$

Putting

$$\widehat{u}(t,\xi) \;=\; \int_{-\infty}^{+\infty} e^{-i\xi x}\,u(t,x)\,dx,$$

$$V(t,\xi)=\begin{pmatrix}(i\xi)^{m-1}\,\widehat{u}\\ (i\xi)^{m-2}\,\widehat{u}\,'\\ \vdots\\ \widehat{u}^{(m-1)}\end{pmatrix},\quad F(t,\xi)=\begin{pmatrix}0\\0\\ \vdots\\ f(t,\xi)\end{pmatrix}, \tag{15}$$

and

$$A(t)=\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ a_m(t) & \cdots & a_2(t) & a_1(t)\end{pmatrix}, \tag{16}$$

we transform equation (14) into the ODE's system

$$V' + i\,\xi A(t)V \;=\; F(t,\xi), \tag{17}$$

where

$$f(t,\xi) \;=\; \underbrace{\widehat{u}*\cdots*\widehat{u}}_{\nu}. \tag{18}$$

Our target is to prove that, if

$$\int_{\mathbb{R}} |\xi|^j \, |V(t,\xi)| \, d\xi \;\le\; K_j < \infty \qquad \forall\, j\,, \quad \forall\, t \in [0,T], \tag{19}$$

$$\int_{\mathbb{R}} |\xi|^j \, |V(0,\xi)| \, d\xi \;\le\; C\, \Lambda^j j! \qquad \forall\, j, \tag{20}$$

then, for some new constants $\widetilde{C}$, $\widetilde{\Lambda}$, it holds

$$\int_{\mathbb{R}} |\xi|^j \, |V(t,\xi)| \, d\xi \;\le\; \widetilde{C}\, \widetilde{\Lambda}^j j!\,, \qquad \forall\, j\,, \quad \forall\, t \in [0,T]. \tag{21}$$

Indeed, (20) is an easy consequences of (11); while (10) implies that $\{\partial_t^h \partial_x^j u(t,\cdot)\}$ is bounded in $L^\infty(\mathbb{R})$ for all j, whence (19). Finally, taking into account that $|V(t,\xi)| \le K < \infty$ (by (10)), we see that (21) implies (12).

To get this target, we firstly prove an apriori estimate for the *linear system* (17), without taking (18) into account. We follow [KS], but some modifications are needed in order to get an estimate suitable to the nonlinear case. The main tool is the theory of quasi-symmetrizer developed in [J] and [DS].

Recalls on quasi-symmetrizer.

[DS] : For any matrix of the form (16) with real eigenvalues, we can find a family of Hermitian matrices

$$Q_\varepsilon(t) \;=\; \mathcal{Q}_0(t) + \varepsilon^2 \mathcal{Q}_1(t) + \cdots + \varepsilon^{2(m-1)} \mathcal{Q}_{m-1}(t) \tag{22}$$

such that the entries of the $\mathcal{Q}_r(t)$'s are polynomial functions of the coefficients $a_1(t), \ldots, a_m(t)$ (in particular inherit their regularity in t), and

$$C^{-1} \varepsilon^{2(m-1)}\, |V|^2 \;\le\; (Q_\varepsilon(t)V, V) \;\le\; C\, |V|^2 \tag{23}$$

$$\big| \big(Q_\varepsilon(t) A(t) - A(t) Q_\varepsilon(t) \big) V,\, V \big) \big| \;\le\; C\, \varepsilon^{1-m}\, (Q_\varepsilon(t)V, V). \tag{24}$$

for all $V \in \mathbb{R}^m$, $0 < \varepsilon \le 1$.

[KS] : If the eigenvalues of $A(t)$ satisfy the condition (8), then $Q_\varepsilon(t)$ is a *nearly diagonal matrix*, i.e., it satisfies, for some constant $c > 0$, independent on ε,

$$(Q_\varepsilon(t)V, V) \;\ge\; c \sum_{j=1}^{m} q_{\varepsilon,jj}(t) v_j^2\,, \qquad \forall\, V \in \mathbb{R}^m, \tag{25}$$

where $q_{\varepsilon,ij}$ are the entries of Q_ε, v_j the scalar components of V. □

In our assumptions, the $a_h(t)$'s are analytic functions on $[0,T]$, consequently also the entries $q_{r,ij}(t)$, $1 \le i,j \le m$ of the matrix $\mathcal{Q}_r(t)$ will be analytic.

Therefore, putting together all the isolated zeroes of these functions, we form a partition of $[0,T]$, independent on ε,

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, \tag{26}$$

such that, for each r, i, j, it holds:

$$\text{either} \;\; q_{r,ij} \equiv 0, \quad \text{or} \quad q_{r,ij}(t) \neq 0 \quad \forall t \in I_h = [t_{h-1}, t_h[\,.$$

Now, let us notice that, by Cauchy-Kovalewsky, if at some point t a solution to (14) satisfies $\partial_t^h u(t,\cdot) \in \mathcal{A}_{L^2}(\mathbb{R})$ for all $h \leq m-1$, then the same holds in a right neighborhood of t. Thus, it will be sufficient to put ourselves inside one of the intervals $I_1, \ldots, I_N$. In other words it is not restrictive to assume that, for each r, i, j,

$$\text{either} \;\; q_{r,ij} \equiv 0, \quad \text{or} \quad q_{r,ij}(t) \neq 0 \quad \text{for} \;\; 0 \leq t < T. \tag{27}$$

Therefore, by the analyticity of $q_{r,ij}(t)$ we easily derive that

$$|q'_{r,ij}(t)| \leq \frac{C}{T-t}\, |q_{r,ij}(t)| \quad \text{on} \;\; [0,T[. \tag{28}$$

Next, following [KS], for any fixed $\xi \in \mathbb{R}$ we prove two different apriori estimates for a solution $V(t,\xi)$ of (17): a *Kovalewskian* estimate in a (small) left neighborhood of T, $[T-\tau,\, T[$, and a *hyperbolic* estimate on $[0,\tau]$.
[In the following C, C_j will be constants depending on the coefficients of (14)]

Lemma 1 *Let $V(t,\xi)$ be a solution of (17) on $[0,T[$, and put*

$$E_\varepsilon(t,\xi) \,=\, (Q_\varepsilon(t)\, V(t,\xi), V(t,\xi)). \tag{29}$$

Then, for any fixed $\xi \in \mathbb{R}$, the following estimates hold:

$$\partial_t\, |V(t,\xi)| \leq \frac{C_0}{T}\, |\xi|\, |V(t,\xi)| + |F(t,\xi)|, \tag{30}$$

$$\partial_t \sqrt{E_\varepsilon(t,\xi)} \leq C_0 \Big(\frac{1}{T-t} + \varepsilon\, |\xi| \Big) \sqrt{E_\varepsilon(t,\xi)} \,+\, C_0\, |F(t,\xi)|, \tag{31}$$

C_0 a constant depending only on the coefficients of the equation, and on T.
In particular, putting

$$E_* \,=\, E_{\varepsilon_*}, \qquad \text{where} \;\; \varepsilon_* \,=\, \langle\xi\rangle^{-1}, \;\; \langle\xi\rangle \,=\, 1 + |\xi|\,, \tag{32}$$

(31) gives

$$(\sqrt{E_*})' \leq C_0 \Big(\frac{1}{T-t} + 1 \Big) \sqrt{E_*} \,+\, C_0\, |F(t,\xi)|. \tag{33}$$

Proof: As an easy consequence of (17), we get (30) with

$$C_0 \,\geq\, \max_{t\in[0,T]} \|A(t)\|\,, \quad C_0 \,\geq\, 1\,.$$

To prove (31) we differentiate (29) in time. Recalling (23) we find

$$\begin{aligned} E'_\varepsilon(t,\xi) &= (Q'_\varepsilon V, V) + (Q_\varepsilon V', V) + (Q_\varepsilon V, V') \\ &= (Q'_\varepsilon V, V) + i\,\xi\,((Q_\varepsilon A - A^* Q_\varepsilon)V, V) + 2\,\Re(Q_\varepsilon F, V) \\ &\leq K_\varepsilon(t,\xi)\,E_\varepsilon(t,\xi) + C_1\,|F(t,\xi)|\,\sqrt{E_\varepsilon(t,\xi)} \end{aligned}$$

where $V = V(t,\xi)$ and

$$K_\varepsilon(t,\xi) \,=\, \frac{|(Q'_\varepsilon V, V)|}{(Q_\varepsilon V, V)} \,+\, |\xi|\,\frac{|((Q_\varepsilon A - A^* Q_\varepsilon)V, V)|}{(Q_\varepsilon V, V)}. \tag{34}$$

We have to prove that

$$K_\varepsilon(t,\xi) \,\leq\, C\left(\frac{1}{T-t} + \varepsilon\,|\xi|\right) \qquad \forall\, t \in [0, T[\,. \tag{35}$$

Let us firstly note that the second quotient in (34) is estimated by $C\varepsilon$ by the property (24) of our quasi-symetrizer. To estimate the first quotient, apply to the nearly diagonality of the matrix $Q_\varepsilon(t)$, i.e., (25): recalling (22), and noting that $|q_{r,ij}| \leq \sqrt{q_{r,ii}\, q_{r,jj}}$ (since $\mathcal{Q}_r(t)$ is a symmetric matrix ≥ 0), it follows

$$\begin{aligned} |(Q'_\varepsilon V, V)| \,&\leq\, \sum_{r=0}^{m-1} \varepsilon^{2r} \sum_{ij}^{1,n} |q'_{r,ij}||v_i v_j| \,\leq\, C\,(T-t)^{-1} \sum_r \varepsilon^{2r} \sum_{ij} |q_{r,ij}||v_i v_j| \\ &\leq\, C\,(T-t)^{-1} \sum_r \varepsilon^{2r} \sum_j q_{r,jj}\, v_j^2 \,=\, C\,(T-t)^{-1} q_{\varepsilon,jj}\, v_j^2 \\ &\leq\, C_1\,(T-t)^{-1} (Q_\varepsilon V, V)\,. \end{aligned}$$

This completes the proof of (35), hence of (31). □

Next, we define

$$\tau(\xi) \,=\, T - |\xi|^{-1}\,, \tag{36}$$

$$\Phi(t,\xi) = C_0\,\min\left\{(T-t)^{-1} + 1, \langle\xi\rangle\right\} = \begin{cases} C_0\left\{(T-t)^{-1} + 1\right\} & \text{on } [\,0, \tau(\xi)] \\ \\ C_0\,\langle\xi\rangle & \text{on } [\tau(\xi), T\,[\end{cases} \tag{37}$$

$$\rho(t,\xi) \,=\, \textstyle\int_t^T \Phi(s,\xi)\,ds\,. \tag{38}$$

Therefore, by (30) and (33) it follows

$$\begin{aligned} \big\{|V(t,\xi)|\big\}' \;&\le\; \Phi(t,\xi)\,|V(t,\xi)| \;+ C_0\,|F(t,\xi)| \qquad \text{on } \;[\tau(\xi),T[\\ \big\{\sqrt{E_*(t,\xi)}\big\}' \;&\le\; \Phi(t,\xi)\,\sqrt{E_*(t,\xi)} \,+\, C_0\,|F(t,\xi)| \qquad \text{on } \;[\,0,\tau(\xi)\,[\end{aligned} \tag{39}$$

and thus, since $\rho' = -\Phi$,

$$\begin{aligned} \partial_t\Big\{e^{\rho(t,\xi)}\,|V(t,\xi)|\Big\} \;&\le\; C_0\,e^{\rho(t,\xi)}\,|F(t,\xi)| \qquad \text{for } \;\tau(\xi)\le t\le T \\ \partial_t\Big\{e^{\rho(t,\xi)}\sqrt{E_*(t,\xi)}\Big\} \;&\le\; C_0\,e^{\rho(t,\xi)}\,|F(t,\xi)| \qquad \text{for } \;0\le t\le \tau(\xi)\,. \end{aligned}$$

By integrating in time, we find (omitting ξ everywhere)

$$e^{\rho(t)}\,|V(t)| \;\le\; e^{\rho(\tau)}\,|V(\tau)| + C_0\int_\tau^t e^{\rho(s)}|F(s)|\,ds \tag{40}$$

$$e^{\rho(\tau)}\sqrt{E_*(\tau)} \;\le\; e^{\rho(0)}\sqrt{E_*(0)} + C_0\int_0^\tau e^{\rho(s)}|F(s)|\,ds \tag{41}$$

Now, by (23) with $\varepsilon = \langle\xi\rangle^{-1}$, we know that

$$C^{-1}\langle\xi\rangle^{-2(1-m)}\,|V(t,\xi)|^2 \;\le\; E_*(t,\xi) \;\le\; C\,|V(t,\xi)|^2,$$

hence we derive, form (40) and (41),

$$\begin{aligned} e^{\rho(t)}\,|V(t)| \;&=\; C_1\langle\xi\rangle^{m-1}\,e^{\rho(\tau)}\sqrt{E_*(\tau)} \,+\, C_0\int_\tau^t e^{\rho(s)}|F(s)|\,ds \\ &\le\; C_1\langle\xi\rangle^{m-1}\Big\{e^{\rho(0)}\sqrt{E_*(0)} + \int_0^\tau e^{\rho(s)}|F(s)|\,ds\Big\} + C_0\int_\tau^t e^{\rho(s)}|F(s)|\,ds \\ &\le\; C_2\,\langle\xi\rangle^{m-1}\Big\{e^{\rho(0)}\sqrt{E_*(0)} + C_0\int_0^t e^{\rho(s)}|F(s)|\,ds\Big\}. \end{aligned}$$

Recalling the definitions (37) and (38) of Φ and ρ, we get

$$\rho(0,\xi) \;=\; \int_0^T \Phi(s,\xi)\,ds \;\le\; C_0\int_0^{\tau(\xi)}\Big\{\frac{1}{T-t}+1\Big\}\,dt \,+\, (T-\tau(\xi))\langle\xi\rangle$$

and hence we derive, since $\partial_t\rho<0$ and $\tau(\xi) = T-|\xi|^{-1}$,

$$\rho(t,\xi) \;\le\; C\,(\log\langle\xi\rangle + 1) \qquad \text{for all } \; t\in[0,T]. \tag{42}$$

Therefore we obtain, for some integer N,

$$e^{\rho(t,\xi)}|V(t,\xi)| \;\le\; C\,\langle\xi\rangle^N\,|V(0,\xi)| \,+\, C\,\langle\xi\rangle^{m-1}\int_0^t e^{\rho(s,\xi)}|F(s,\xi)|\,ds. \tag{43}$$

By the way, we note that the last inequality ensures the wellposedness in $\mathcal{C}^\infty$ of the Cauchy problem for the linear system (17).

Let us go back to the nonlinear equation $\mathcal{L}u = u^\nu$. For our purpose we must consider a more general equation, namely

$$\mathcal{L}u \,=\, u_1 \cdots u_\nu,$$

where the $u_j = u_j(t,x)$ are given functions (actually, some x-derivatives of u). In such a case, the function F in (17) is

$$F(t,\xi) \,=\, \widehat{u}_1 * \cdots * \widehat{u}_\nu, \tag{44}$$

where the convolutions are effected w.r. to ξ, and thus

$$|F(t,\xi)| \,\le\, \int_{\xi_1+\cdots+\xi_\nu=\xi} |\widehat{u}_1(t,\xi_1)\cdots\widehat{u}_\nu(t,\xi_\nu)|\, d\sigma_{(\xi_1,\dots,\xi_\nu)}\,.$$

We notice that the function $\xi \mapsto \min\{C,|\xi|\}$ is a sub-additive; consequently for each fixed t (see (37),(38)) the function $\Phi(t,\xi)$, hence also $\rho(t,\xi)$, is sub-additive in ξ. On the other hand, $\xi \to \langle\xi\rangle$ is sub-multiplicative.
Thus one has, for $\xi = \xi_1 + \cdots + \xi_\nu$,

$$\rho(t,\xi) \,\le\, \rho(t,\xi_1) + \cdots + \rho(t,\xi_\nu), \qquad \langle\xi\rangle^{m-1} \le \langle\xi_1\rangle^{m-1}\cdots\langle\xi_\nu\rangle^{m-1},$$
$$e^{\rho(t,\xi)}\,\langle\xi\rangle^{m-1} \,\le\, e^{\rho(t,\xi_1)}\,\langle\xi_1\rangle^{m-1}\cdots\, e^{\rho(t,\xi_\nu)}\,\langle\xi_\nu\rangle^{m-1},$$

whence, by (44), it follows the pointwise estimate

$$e^\rho\,\langle\xi\rangle^{m-1}|F| \,\le\, \big(e^\rho\,\langle\xi\rangle^{m-1}|\widehat{u}_1|\big) * \cdots * \big(e^\rho\,\langle\xi\rangle^{m-1}|\widehat{u}_\nu|\big).$$

Now, if $V_j(t,\xi)$ are the vectors formed as $V(t,\xi)$ (see (15)), with u_j in place of u, we have

$$\langle\xi\rangle^{m-1}\,|\widehat{u}_j(t,\xi)| \,\le\, |V_j(t,\xi)|\,, \qquad j = 1,\dots,\nu,$$

and thus, going back to (43), we obtain

$$e^{\rho(t,\xi)}|V(t,\xi)| \,\le\, C\langle\xi\rangle^N\,|V(0,\xi)| \,+\, C\int_0^t \big(e^\rho\,|V_1| * \cdots * e^\rho\,|V_\nu|\big)(s,\xi)\,ds.$$

Finally, we integrate in $\xi \in \mathbb{R}$ to get

$$\mathcal{E}(t,u) \,\le\, C\int_{\mathbb{R}} |V(0,\xi)|\,\langle\xi\rangle^N d\xi \,+\, C\int_0^t \mathcal{E}(s,u_1)\cdots\mathcal{E}(s,u_\nu)\,ds\,, \tag{45}$$

where we define the $\mathcal{C}^\infty$-*energy*

$$\mathcal{E}(t,u) \,=\, \int_{\mathbb{R}} e^{\rho(t,\xi)}|V(t,\xi)|\,d\xi. \tag{46}$$

We emphasize that, by virtue of our assumption (19), and (42), we have

$$\mathcal{E}(t,u) \;\leq\; M_0 \;<\; \infty \qquad (0 \leq t \leq T). \tag{47}$$

Differentiating j times in x the equation $\mathcal{L}u = u^\nu$, we get

$$\mathcal{L}(\partial^j u) \;=\; j! \sum_{h_1+\cdots+h_\nu=j} \frac{\partial^{h_1}u\cdots\partial^{h_\nu}u}{h_1!\cdots h_\nu!} \qquad (\text{where } \;\partial = \partial_x),$$

and to this equation we apply the estimate (45) with $u_j = \partial^j u$. We obtain:

$$\frac{\mathcal{E}_j(t)}{j!} \;\leq\; C\int_{\mathbb{R}} \frac{|V_j(0,\xi)|}{j!}\,\langle\xi\rangle^N d\xi \;+\; C\sum_{|h|=j}\int_0^t \frac{\mathcal{E}_{h_1}(s)}{h_1!}\cdots\frac{\mathcal{E}_{h_\nu}(s)}{h_\nu!}\,ds, \tag{48}$$

where $V_j(t,\xi)$ is the vector associated to $u_j \equiv \partial^j u$, and

$$\mathcal{E}_j(t) \;=\; \mathcal{E}(t,\partial^j u).$$

Putting

$$\alpha_j(t) \;=\; \int_{\mathbb{R}} |V_j(0,\xi)|\,\langle\xi\rangle^N d\xi \;+\; j!\sum_{|h|=j}\int_0^t \frac{\mathcal{E}_{h_1}(s)}{h_1!}\cdots\frac{\mathcal{E}_{h_\nu}(s)}{h_\nu!}\,ds\,,$$

we rewrite (48) as

$$\mathcal{E}_j(t) \;\leq\; C\,\alpha_j(t)\,. \tag{49}$$

Next, we introduce the *super-energies*

$$\mathcal{F}(t,u) \;=\; \sum_0^\infty \mathcal{E}_j(t)\,\frac{r(t)^j}{j!}\,, \tag{50}$$

$$\mathcal{G}(t,u) \;=\; \sum_0^\infty \alpha_j(t)\,\frac{r(t)^j}{j!}\;, \quad \mathcal{G}^1(t,u) \;=\; \sum_1^\infty \alpha_j(t)\,\frac{r(t)^{j-1}}{(j-1)!}\;, \tag{51}$$

where $r(t)$ is a decreasing, positive function on [0,T] to be defined later.
By differentiating in time, we find

$$\mathcal{G}' \;=\; \sum_0^\infty \alpha_j'\,\frac{r^j}{j!} + \sum_1^\infty \alpha_j\,\frac{r^{j-1}}{(j-1)!}\,r' \;=\; \sum_{j=0}^\infty\sum_{|h|=j} \mathcal{E}_{h_1}\frac{r^{h_1}}{h_1!}\cdots\mathcal{E}_{h_\nu}\frac{r^{h_\nu}}{h_\nu!} + r'\,\mathcal{G}^1$$

$$=\; \Big\{\sum_{h=0}^\infty \mathcal{E}_h\frac{r^h}{h!}\Big\}^\nu + r'\,\mathcal{G}^1 \;=\; \mathcal{F}^\nu \;+\; r'\,\mathcal{G}^1,$$

and hence, noting that $\mathcal{F}(t) \le C\,\mathcal{G}(t)$ by (49),

$$\mathcal{G}' \;\le\; C^{\nu}\,\mathcal{G}^{\nu} \,+\, r'\,\mathcal{G}^{1}. \tag{52}$$

Now, noting that (by (19) and (47))

$$\alpha_0(t) \;=\; \int_R |V(0,\xi)|\,\langle\xi\rangle^N d\xi \,+\, \int_0^t \mathcal{E}(s)\,ds \;\le\; K_N + M_0 \;\equiv\; M,$$

by the definition (51) of $\mathcal{G}(t)$ it follows

$$\mathcal{G}(t) \;\le\; \alpha_0(t) \,+\, r(t)\,\mathcal{G}^1(t) \;\le\; M \,+\, r(t)\,\mathcal{G}^1(t)\,.$$

From this inequality it follows, arguing by induction w.r. to ν,

$$\mathcal{G}^{\nu} \;\le\; M^{\nu} \,+\, r\,\mathcal{G}^1\,\big(\mathcal{G} + M\big)^{\nu-1};$$

consequently (52) gives (putting $\phi(\mathcal{G}) = C^{\nu}(M+\mathcal{G})^{\nu-1}$)

$$\mathcal{G}' \;\le\; \mathcal{G}^1\big\{r' + r\,\phi(\mathcal{G})\big\} \,+\, (CM)^{\nu}. \tag{53}$$

On the other hand, by virtue of our assumption (20), we see that

$$\mathcal{G}(0,u) \;=\; \sum_{j=0}^{\infty} \Big\{ \int_{\mathbb{R}} |V_j(0,\xi)|\,\langle\xi\rangle^N\,d\xi \Big\}\,\frac{r(0)^j}{j!} \;<\; \infty\,.$$

provided $r(0) \equiv r_0$ is small enough. Therefore, taking

$$L \;=\; \mathcal{G}(0,u) + (CM)^{\nu}\,T\,, \qquad r(t) \;=\; r_0\,e^{-\phi(L)\,t}\,, \tag{54}$$

we can derive from (53) the estimate

$$\mathcal{G}(t,u) \;<\; L \qquad \text{for all } \; t \in [0,T]. \tag{55}$$

Proof of (55). Since $L > \mathcal{G}(0)$, this estimate holds true in a right neighborhood of $t = 0$ by Cauchy-Kovalewsky. Then assuming that, for some $\tau_* < T$, (55) holds for all $t < \tau_*$ but not at $t = \tau_*$, we have $\mathcal{G}(\tau_*) = L$, and hence, with $r(t)$ as in (54),

$$r'(t) + r(t)\,\phi(\mathcal{G}(t)) \;\le\; r'(t) + r(t)\,\phi(L) \;\le\; 0 \qquad \text{on } \; [0,\tau_*[\,.$$

This yelds a contradiction; indeed, by (53),

$$\mathcal{G}(t) \;\le\; \mathcal{G}(0) + (CM)^{\nu}\tau_* \;<\; L \qquad \text{on } \; [0,\tau_*]\,.$$

Conclusion of the Proof of Theorem 1. Recalling that $\mathcal{F}(t,u) \le C\,\mathcal{G}(t,u)$, (55) says that $\mathcal{F}(t,u) < CL$ on $[0,T]$. Therefore, by (50), we get our goal (21):

$$\begin{aligned}\int_{\mathbb{R}} |V(t,\xi)|\,|\xi|^j\,d\xi \;&\le\; \int_{\mathbb{R}} e^{\rho(t,\xi)}|V(t,\xi)|\,|\xi|^j d\xi \;=\; \mathcal{E}(t,\partial^j u) \;\le\; \mathcal{F}(t)\,r(t)^{-j}\,j! \\ &\le\; CL\left\{r_0\,e^{\phi(L)T}\right\}^j j! \;=\; \widetilde{C}\,\widetilde{\Lambda}^{j+1}\,j!\,.\end{aligned}$$

To prove (13), i.e., the global analyticity of the solution u in (t,x), it is sufficient to resort to Cauchy-Kovalewski. □

Remark 2 *The previos proof of (55) is somewhat formal, since it assumes not only that $\mathcal{G}(t) < \infty$, but also that $\mathcal{G}^1(t) < \infty$ on $[0,\tau_*[$. To make the proof more precise we must replace the radius function $r(t)$ by $r_\eta(t) = \eta\exp(-\phi(L)t)$, $\eta < 1$, and apply the previous computation to the corresponding functions $\mathcal{G}_\eta(t)$ and $\mathcal{G}^1_\eta(t)$. Finally let $\eta \to 1$* (see [ST] for the details)*.*

Proof of Theorem 2. The proof is not very different from that of Thm.1, thus we give only a sketch of it.

The main difference is that the entries $q_{r,ij}(t)$ are no more analytic, but only $\mathcal{C}^\infty$, hence (28) fails. However, for any function $f \in \mathcal{C}^k([0,T])$ it holds

$$|f'(t)| \;\le\; \Lambda(t)\,|f(t)|^{1-1/k}\,\|f\|_{\mathcal{C}^k([0,T])},$$

for some $\Lambda \in L^1(0,T)$ [this was proved in [CJS] in the case $f(t) \ge 0$, and in [T] in the general case]. Therefore, recalling that $Q_\varepsilon(t)$ is a nearly diagonal matrix, and proceeeding as in [KS], we get, for all integer $k \ge 1$,

$$\left|\left(Q'_\varepsilon(t)V(t,\xi),V(t,\xi)\right)\right| \;\le\; \Lambda_k(t)\left(Q_\varepsilon(t)V(t,\xi),V(t,\xi)\right)^{1-1/k}|V(t,\xi)|^{2/k} \quad (56)$$

for some $\Lambda_k \in L^1(0,T)$, independent of ε. Differently from Thm. 1, we need now to consider only the *hyperbolic energy*

$$E_*(t,\xi) \;=\; (Q_{\varepsilon^*}(t)V,V) \quad \text{with } \; \varepsilon = |\xi|^{-1}.$$

Thanks to (56), we prove (for every integer $k \ge 1$) the estimate

$$\left\{\sqrt{E_*(t,\xi)}\right\}' \;\le\; C_0\,\Phi(t,\xi)\,\sqrt{E_*(t,\xi)} \;+\; C_0\,|F(t,\xi)|$$

on all the interval $[0,T]$, where

$$\Phi(t,\xi) \;=\; \Lambda_k(t)|\xi|^{2(m-1)/k} \;+\; 1$$

Note that Φ is sub-additive w.r. to ξ as soon as $k \ge 2(m-1)$.

Next, putting

$$\rho(t,\xi) \,=\, \int_t^T \Phi(t,\xi)\, d\xi \,\equiv\, |\xi|^{2(m-1)/k} \int_t^T \Lambda_k(s)\, ds \,+\, (T-t)\,,$$

we define the *Gevrey-energy*

$$\mathcal{E}(t,u) \,=\, \int_{\mathbb{R}} e^{\rho(t,\xi)} \sqrt{E_*(t,\xi)}\, d\xi.$$

We conclude as in the proof of Thm.1.

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