EQUATIONS IN SIMPLE LIE ALGEBRAS

TATIANA BANDMAN, NIKOLAI GORDEEV, BORIS KUNYAVSKIĬ, EUGENE PLOTKIN

Abstract. Given an element $P(X_1,\dots,X_d)$ of the free Lie K-algebra $\mathcal{L}_d$, for any Lie algebra $\mathfrak{g}$ we can consider the induced polynomial map $P\colon \mathfrak{g}^d \to \mathfrak{g}$. Assuming that K is an arbitrary field of characteristic $\neq 2$, we prove that if P is not an identity in $\mathfrak{sl}(2,K)$, then this map is dominant for any Chevalley algebra $\mathfrak{g}$. This result can be viewed as a weak infinitesimal counterpart of Borel's theorem on the dominancy of the word map on connected semisimple algebraic groups.

We prove that for the Engel monomials $[[[X,Y],Y],\dots,Y]$ and, more generally, for their linear combinations, this map is, moreover, surjective onto the set of noncentral elements of $\mathfrak{g}$ provided that the ground field K is big enough, and show that for monomials of large degree the image of this map contains no nonzero central elements.

We also discuss consequences of these results for polynomial maps of associative matrix algebras.

1. Introduction

For a given element $P(X_1,\dots,X_d)$ of the free Lie K-algebra $\mathcal{L}_d$ on the finite set $\{X_1,\dots,X_d\}$ over a given field K, and a given Lie algebra $\mathfrak{g}$ over K, one can ask the following question:

Question 1.1. Is the equation

$$P(X_1,\dots,X_d)=A$$

solvable

a) for all $A\in\mathfrak{g}$,

or, at least,

b) for a generic $A\in\mathfrak{g}$?

In the present paper we consider the following case. Let R be a root system and let Π be a simple root system corresponding to R. Further, let $L(\mathsf{R},\mathbb{C})$ be a semisimple complex Lie algebra. Then there exists a Chevalley basis $\{h_\alpha\}_{\alpha\in\Pi}\cup\{e_\beta\}_{\beta\in\mathsf{R}}$ of $L(\mathsf{R},\mathbb{C})$ such that

1) $[e_\alpha,e_{-\alpha}]=h_\alpha$ for every $\alpha\in\Pi$;
2) $h_\beta:=[e_\beta,e_{-\beta}]\in\sum_{\alpha\in\Pi}\mathbb{Z}h_\alpha$ for every $\beta\in\mathsf{R}$;
3) $[h_\beta,h_\gamma]=0$ for every $\beta,\gamma\in\mathsf{R}$;
4) $[h_\beta,e_\gamma]=\frac{2(\beta,\gamma)}{(\beta,\beta)}e_\gamma$ for every $\beta,\gamma\in\mathsf{R}$ (note that $\frac{2(\beta,\gamma)}{(\beta,\beta)}=0,\pm1,\pm2,\pm3$);
5) $[e_\beta,e_\gamma]=0$ if $\beta+\gamma\notin\mathsf{R}$;

6) $[e_\beta, e_\gamma] = p_{\beta,\gamma} e_{\beta+\gamma}$ if $\beta+\gamma \in \mathsf{R}$ (note that $p_{\beta,\gamma} = \pm 1, \pm 2, \pm 3$).

One can now define the corresponding Lie algebra over any prime field F using the same basis and relations 1)–6) in the case $F = \mathbb{Q}$ or the same basis and relations 1)-6) modulo p in the case $F = \mathbb{F}_p$. Then one can define the Lie algebra $L(\mathsf{R}, K)$ over any field K using the same basis and relations induced by 1.-6. We will denote such an algebra by $L(\mathsf{R}, K)$ and call it a *Chevalley* algebra. The Chevalley algebra $L(\mathsf{R}, K)$ decomposes into the sum $\sum_i L(\mathsf{R}_i, K)$ where $\mathsf{R} = \cup_i \mathsf{R}_i$ is the decomposition of the root system R into the disjoint union of irreducible root subsystems. The Lie algebras $L(\mathsf{R}_i, K)$ are not simple if the characteristic of K is not a "very good prime" [Ca]. However, if $\mathsf{R}_i \neq \mathsf{A}_1, \mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4$ when $\mathrm{char}(K) = 2$ and $\mathsf{R}_i \neq \mathsf{G}_2$ when $\mathrm{char}(K) = 3$, the algebra $L(\mathsf{R}_i, K)/\mathfrak{z}_i$ is simple (here $\mathfrak{z}_i$ is the centre of $L(\mathsf{R}_i, K)$). Thus, if the Lie algebra $L(\mathsf{R}_i, K)$ has no components pointed out above, the quotient $L(\mathsf{R}, K)/\mathfrak{z}$ (where $\mathfrak{z}$ is the centre of $L(\mathsf{R}, K)$) is a semisimple Lie algebra.

In this paper we consider maps $P(X_1, \dots, X_d)\colon \mathfrak{g}^d \to \mathfrak{g}$ for semisimple Lie algebras $\mathfrak{g} = L(\mathsf{R}, K)/\mathfrak{z}$ which are quotients of Chevalley algebras modulo the centre. Such algebras are called "classical" semisimple Lie algebras (abusing the terminology accepted in the characteristic zero case in order to distinguish from Lie algebras of Cartan type appearing in positive characteristics).

Our motivation lies in widely discussed group-theoretic analogues of Question 1.1:

Question 1.2. Let $w(x_1, \dots, x_d)$ be an element of the free group $\mathcal{F}_d$ on the finite set $\{x_1, \dots, x_d\}$ (i.e., a word in x_i and x_i^{-1}) and a group G be given. Is the equation

$$w(x_1, \dots, x_d) = g$$

solvable

a) for all $g \in G$,

or, at least,

b) for a generic $g \in G$?

If G is a connected semisimple algebraic K-group, a theorem of Borel [Bo2], stating that the word map $G^d \to G$ is dominant whenever $w \neq 1$, gives a positive answer to part b). One can, however, easily produce examples where the word map is not surjective and so the answer to part a) is negative (see [Bo2] and references therein). Some particular words have been extensively studied, and Question 1.2a has been answered in the affirmative. Say, if $d = 2$ and $w(x, y) = [x, y]$ (the commutator), the positive answer is known long ago for the connected semisimple compact topological groups [Got], connected complex semisimple Lie groups [PW] and algebraic groups defined over an algebraically closed field [Ree], as well as for some simple groups over reals [Dj] and more general fields [Th1], [Th2]. In the case where G is a finite (nonabelian) simple group, Question 1.2 for this word constitutes an old problem posed by Ore in 1950s and solved very recently in [LOST]; on the way to this solution, part b) was thoroughly investigated, and several different approaches to the definition of a "generic" element have been tried, see, e.g., [Gow], [EG], [Sh1] (the last paper contains a survey of some recent developments). Note that the simplicity assumption is essential: even for groups very close to simple, the answer to Question 1.2a may be negative; say, some quasisimple groups contain central elements that are not commutators, see [Th1], [Dj] for infinite groups and [Bl] for finite groups. One has to note that the question on the existence of a simple group not every element of which is a commutator remained open for a long time. First examples of such groups appeared in geometric context [BG], where the groups under consideration were infinitely generated; later on there were constructed finitely generated groups with the same property [Mu]. These are counter-examples in very strong sense: the so-called commutator width, defined as supremum of the minimal number of commutators needed for a representation of a given element as a product of commutators, may be arbitrarily large or even infinite [Mu, Theorems 4 and 5].

Similar questions for words more complicated than the commutator remain widely open (see, however, [GS], [Sh2] for new approaches towards part b), and [BGG], where a particular case $G = PSL(2, q)$ and w an Engel word is treated).

It is worth noting that analogues of Questions 1.1 and 1.2 for associative algebras have also been intensely investigated (first questions of such kind go back to Kaplansky), see [KBMR] and references therein. In a sense, the case of Lie algebras treated in the present paper may be viewed as a sort of "bridge" between groups and associative algebras in what concerns dominancy and surjectivity of polynomial maps; see Remark 5.1 below.

Going over to infinitesimal analogues, one can first mention that the equation $[X, Y] = A$ is solvable for all A in any classical split semisimple Lie algebra $\mathfrak{g}$, under the assumption that the ground field K is sufficiently large. (Here, of course, brackets stand for the Lie product.) This fact was established by Brown [Br], and in [Hi] estimates on the size of K were improved.

Our first results (Section 3) concern the general case where we are given an element $P(X_1, \dots, X_d)$ of the free Lie K-algebra $\mathcal{L}_d$. Then for any Lie algebra $\mathfrak{g}$ we can consider the induced polynomial map $P\colon \mathfrak{g}^d \to \mathfrak{g}$. Assuming that K is an arbitrary field of characteristic $\neq 2$, we prove that if P is not an identity in $\mathfrak{sl}(2, K)$, then this map is dominant for any Chevalley algebra $\mathfrak{g}$. This result can be viewed as a weak infinitesimal counterpart of Borel's theorem on the dominancy of the word map on connected semisimple algebraic groups.

Going over from dominancy to surjectivity (Section 4), we prove that for the Engel monomials $[[[X, Y], Y], \dots, Y]$ and, more generally, for their linear combinations, the image of the corresponding map contains the set of noncentral elements of $\mathfrak{g}$ provided that the ground field K is big enough. We show that for Engel monomials of large degree this image contains no nonzero central elements.

We also discuss consequences of these results for polynomial maps of associative matrix algebras as well as some other possible generalizations (Section 5).

2. Preliminaries. Chevalley and classical Lie algebras

2.1. First recall that a K-morphism $f\colon V \to W$ of algebraic K-varieties (=reduced K-schemes of finite type) is called *dominant* if its image $f(V)$ is Zariski dense in W. We will mostly deal with the case where V and W are irreducible. In such a case $f(V)$ contains a non-empty open set U (see, e.g., [Pe, Th. IV.3.7]). If L/K is a field extension, then f is dominant if and only if the L-morphism $f_L\colon V_L \to W_L$ obtained by extension of scalars is dominant.

2.2. Let $L(\mathsf{R}, K)$ be a Chevalley algebra over a field K which corresponds to an irreducible reduced root system R. Denote by R^+ (resp. R^-) the set of positive (resp. negative) roots and put

$$H = \sum_{\alpha\in\Pi} K h_\alpha, \;\; U^{\pm} = \sum_{\beta\in\mathsf{R}^{\pm}} K e_\beta, \;\; U = U^- + U^+.$$

Then

$$L(\mathsf{R}, K) = H + U = H + U^- + U^+.$$

The number $r = \dim H = |\Pi|$ is called the *rank* of $L(\mathsf{R}, K)$.

Let now $i\colon L(\mathsf{R}, K) \to \mathrm{End}(V)$ be a linear representation. Then one can construct the corresponding Chevalley group $G(\mathsf{R}, K) \leq GL(V)$, which is generated by the so-called root subgroups $x_\beta(t)$ (see [St2], [Bo1]), and a homomorphism $j\colon G(\mathsf{R}, K) \to \mathrm{Aut}(i(L(\mathsf{R}, K)))$.

Suppose K is an algebraically closed field and i is a representation of $L(\mathsf{R}, K)$ such that the group of weights of i coincides with the group generated by fundamental weights. Then $G(\mathsf{R}, K)$ is a simple, simply connected algebraic group, $i(L(\mathsf{R}, K))$ is the Lie algebra of $G(\mathsf{R}, K)$, and the homomorphism j defines the adjoint action of $G(\mathsf{R}, K)$ on its Lie algebra $i(L(\mathsf{R}, K))$ [Bo1, 3.3].

Below we will always consider the Chevalley group $G(\mathsf{R}, K)$ constructed through a faithful representation i such that $G(\mathsf{R}, K)$ is simply connected. We also identify the Lie algebra $i(L(\mathsf{R}, K))$ with $L(\mathsf{R}, K)$. The group $j(G(\mathsf{R}, K)) \leq \mathrm{Aut}(L(\mathsf{R}, K))$ will be denoted by G. Note that G is the group generated by the images $j(x_\beta(t))$ of the root subgroups which will also be denoted by $x_\beta(t)$.

An element $x \in L(\mathsf{R}, K)$ is called *semisimple* (resp. *nilpotent*) if for a faithful linear representation $\rho\colon L(\mathsf{R}, K) \to \mathrm{End}(V)$ the operator $\rho(x)$ is semisimple (resp. nilpotent). Every $x \in L(\mathsf{R}, K)$ has the Jordan decomposition $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent, $[x_s, x_n] = 0$.

Let K be an algebraically closed field. Then:

a. *Every element of the Lie algebra $L(\mathsf{R}, K)$ is G-conjugate to an element $x = x_s + x_n$ such that $x_s \in H$, $x_n \in U^+$, $[x_s, x_n] = 0$.*

b. [SS, II.3.20] *The G-orbit of an element $x \in L(\mathsf{R}, K)$ is closed if and only if x is semisimple.*

c. *Suppose there is a regular element $h \in H$, i.e., $\beta(h) \neq 0$ for every $\beta \in \mathsf{R}$. Then the set of all elements in $L(\mathsf{R}, K)$ which are G-conjugate to elements from H is dense in $L(\mathsf{R}, K)$.*

d. *There is a G-equivariant dominant morphism*

$$\pi\colon L(\mathsf{R}, K) \to \mathsf{Q}$$

where Q is an affine variety and the map

$$\bar{\pi} = \pi_{|H}\colon H \to \mathsf{Q}$$

satisfies the following condition:

$$\bar{\pi}^{-1}(\pi(h)) = Wh$$

where W is the Weyl group, which acts naturally on H.

Proof. Put $L = L(\mathsf{R}, K)$, and let $S = K[L]$ be the algebra of polynomial functions on L. Since G is a simple algebraic group, $R = S^G$ is finitely generated (see, e.g., [Sp, Cor. 2.4.10]), say, by $f_1, \dots, f_k$. Consider the map

$$\pi\colon L \to \mathbb{A}^k$$

given by the formula $\pi(x) = (f_1(x), \dots, f_k(x))$. If $x = x_s + x_n$ is the Jordan decomposition then $\pi(x) = \pi(x_s)$. (Indeed, $x_s \in \overline{O_x}$. Since π is a regular map constant on the orbit O_x, it is constant on $\overline{O_x}$.) Hence $\mathsf{Q} := \overline{\mathrm{Im}\ \pi} = \overline{\mathrm{Im}\ \bar{\pi}}$. Further, functions from R separate closed orbits in L (see, e.g., [Po, Chapter 1, § 1.2]). Hence $\bar{\pi}^{-1}(\pi(h)) = H \cap O_h$ where O_h is the orbit of h. Since $H \cap O_h = Wh$ [SS, 3.16], we are done. □

Remark 2.1. If $\mathrm{char}(K)$ is not a torsion prime for $G(\mathsf{R}, K)$, then there is an isomorphism $\pi'\colon L(\mathsf{R}, K)/G \xrightarrow{\sim} H/W$, and the quotient H/W is isomorphic to $\mathbb{A}^r$ [Sl, 3.12]. Hence in this case $\mathsf{Q} \cong H/W \cong \mathbb{A}^r$.

2.3. *For every root $\beta \in \mathsf{R}$ there is a linear map $\beta\colon H \to K$ defined by the formula $[h, e_\beta] = \beta(h)e_\beta$.*

$$\beta \equiv 0 \Leftrightarrow \mathsf{R} = \mathsf{C}_r, r \geq 1, \mathrm{char}(K) = 2, \beta \text{ is a long root} \tag{2.1}$$

(*here* $\mathsf{C}_1 = \mathsf{A}_1$, $\mathsf{C}_2 = \mathsf{B}_2$). *Thus, if we are not in the case* $\mathsf{R} = \mathsf{C}_r$, $r \geq 1$, $\mathrm{char}(K) = 2$, *the subalgebra H is a Cartan subalgebra, that* *is, a nilpotent subalgebra coinciding with its normalizer. In the case* $\mathsf{R} = \mathsf{C}_r$, $r \geq 1$, $\mathrm{char}(K) = 2$, *the subalgebra* H *is a Cartan subalgebra of* $[L(\mathsf{R}, K), L(\mathsf{R}, K)] \cong L(\mathsf{D}_r, K)$ *(here* $L(\mathsf{D}_1, K) = K, L(\mathsf{D}_2, K) = \mathfrak{sl}(2, K) \times \mathfrak{sl}(2, K)$*).*

e. *Suppose we are not in the case* $\mathsf{R} = \mathsf{C}_r$, $r \geq 1$, $\mathrm{char}(K) = 2$. *Then if* $|K| \geq |\mathsf{R}^+|$, *the subalgebra* H *contains a regular element. Moreover, if* $|K| > m|\mathsf{R}|$ *for some* $m \in \mathbb{N}$, *then for every subset* $S \subset K$ *of size* m *there exists* $h \in H$ *such that* $\beta(h) \notin S$ *for every* $\beta \in \mathsf{R}$.

Proof. For infinite fields the statement is trivial. If K is a finite field, then $|H| = |K|^r$, and the hyperplane $H_{x,\beta}$ of H defined by the equation $[h, e_\beta] = x$, $x \in S$, consists of $|K|^{r-1}$ points. Thus,

$$| \bigcup_{x \in S, \beta \in \mathsf{R}} H_{x,\beta}| \leq |S| \cdot |H_{x,\beta}| = m|\mathsf{R}| \cdot |K|^{r-1} < |H|,$$

and therefore we can take $h \in H \setminus \cup_{x \in S, \beta \in \mathsf{R}} H_{x,\beta}$. The first statement can be proved by the same arguments for $S = \{0\}$ using the fact that $0 \in H_{0,\beta} = H_{0,-\beta}$ for every $\beta \in \mathsf{R}$. □

2.4. The Chevalley algebra $L(\mathsf{R}, K)$ modulo the centre is not simple in the following cases:

$$\mathsf{R} = \mathsf{A}_1, \mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4 \text{ if } \mathrm{char}(K) = 2, \ \ \mathsf{R} = \mathsf{G}_2 \text{ if } \mathrm{char}(K) = 3. \tag{2.2}$$

Namely:

1) Let $\mathsf{R} = \mathsf{A}_1$ and $\mathrm{char}(K) = 2$. Then $L(\mathsf{A}_1, K) \cong \mathfrak{sl}(2, K)$ is a nilpotent algebra satisfying the identity $[[X, Y], Z] \equiv 0$.

2) Let $\mathsf{R} = \mathsf{B}_2$ and $\mathrm{char}(K) = 2$. Then $L(\mathsf{B}_2, K) \cong \mathfrak{so}(5, K)$ is a solvable algebra satisfying the identity $[[X, Y], [Z, T]] \equiv 0$.

3) Let $\mathsf{R} = \mathsf{B}_r, r > 2$ and $\mathrm{char}(K) = 2$. Then $L(\mathsf{B}_r, K)$ contains the nilpotent ideal I generated by $\{e_\beta \mid \ \beta \text{ is a short root}\}$ and $L(\mathsf{B}_r, K)/I \cong L(\mathsf{D}_r, K)/Z'$ where $Z' \leq Z(L(\mathsf{D}_r, K))$.

4) Let $\mathsf{R} = \mathsf{F}_4$ and $\mathrm{char}(K) = 2$. Then $L(\mathsf{F}_4, K)$ contains the ideal I generated by $\{e_\beta \mid \ \beta \text{ is a short root}\}$ and $L(\mathsf{F}_4, K)/I \cong L(\mathsf{D}_4, K)/Z'$ with $Z' \leq Z(L(\mathsf{D}_4, K))$ where $Z(L(\mathsf{D}_4, K))$ is the centre.

5) Let $\mathsf{R} = \mathsf{G}_2$ and $\mathrm{char}(K) = 3$. Then $L(\mathsf{G}_2, K)$ contains the ideal $I \cong \mathfrak{sl}(3, K)$ generated by $\{e_\beta \mid \ \beta \text{ is a short root}\}$ and $L(\mathsf{G}_2, K)/I \cong \mathfrak{sl}(3, K)/Z(\mathfrak{sl}(3, K))$.

6) Let $\mathsf{R} = \mathsf{C}_r, r > 2$ and $\mathrm{char}(K) = 2$. Then $L(\mathsf{C}_r, K)$ contains the ideal $I \cong L(\mathsf{D}_r, K)$ generated by $\{e_\beta \mid \ \beta \text{ is a short root}\}$ and the algebra $L(\mathsf{C}_r, K)/I$ is abelian.

The other Chevalley algebras $L(\mathsf{R}, K)$ corresponding to irreducible root systems R are simple modulo the center [St1]. The simple algebras $\mathfrak{g} = L(\mathsf{R}, K)/Z(L(\mathsf{R}, K))$ are classical. The classical semisimple Lie algebras and the corresponding Chevalley algebras form a natural class to consider polynomial maps $P(X_1, \dots, X_d)$ on its products. However, note that the algebras appearing in "bad cases" 3)–5) are perfect, i.e., satisfy the condition $[L(\mathsf{R},K),L(\mathsf{R},K)] = L(\mathsf{R},K)$, and therefore we can also raise the question on dominancy of polynomial maps on such algebras.

f. *Suppose we are not in the cases appearing in list* (2.2). *Fix an arbitrary non-central element $h \in H$. Then for every non-central element $l \in L(\mathsf{R},K)$ there is $g \in G$ such that $g(l) \in h + U$* [Gorde, Proposition 1] (actually we mostly need below a particular case $h = 0$ treated in [Br, Lemma II]).

3. Dominancy of polynomial maps on Chevalley algebras

In this section K is an algebraically closed field.

3.1. We are interested in the following analogue of the Borel dominancy theorem for semisimple Lie algebras:

Question 3.1. For a given element $P(X_1,\dots,X_d)$ of the free Lie K-algebra $\mathcal{L}_d$ on the finite set $\{X_1,\dots,X_d\}$ over a given algebraically closed field K, and a given semisimple Lie algebra $\mathfrak{g}$ over K, is the map

$$P(X_1,\dots,X_d)\colon \mathfrak{g}^d \to \mathfrak{g}$$

dominant under the condition that $P(X_1,\dots,X_d)$ is not an identity on $\mathfrak{g}$?

We do not know the answer to this question. However, we can get it under some additional assumption. Our main result is

Theorem 3.2. *Let $L(\mathsf{R},K)$ be a Chevalley algebra. If* $\mathrm{char}(K) = 2$, *assume that* R *does not contain irreducible components of type* C_r, $r \geq 1$ (*here* $\mathsf{C}_1 = \mathsf{A}_1, \mathsf{C}_2 = \mathsf{B}_2$).

Suppose $P(X_1,\dots,X_d)$ is not an identity of the Lie algebra $\mathfrak{sl}(2,K)$. Then the induced map $P\colon L(\mathsf{R},K)^d \to L(\mathsf{R},K)$ is dominant.

Below we repeatedly use the following construction. Put

$$\mathfrak{I} = \overline{P(L(\mathsf{R},K)^d)}.$$

Then $\mathfrak{I}$ is an irreducible affine variety, and therefore $\overline{\pi(\mathfrak{I})} \hookrightarrow \mathsf{Q}$ is an irreducible closed subset of an r-dimensional affine variety Q (see 2.3.**d**). If $\overline{\pi(\mathfrak{I})} = \mathsf{Q}$, then, by 2.3.**d**, the set $\mathfrak{I}$ contains all elements which are G-conjugate to elements of H. It implies, by 2.3.**c**, that $\mathfrak{I} = L(\mathsf{R},K)$. Thus,

$$P \text{ is dominant} \Leftrightarrow \overline{\pi(\mathfrak{I})} = \mathsf{Q}. \tag{3.1}$$

Lemma 3.3. *Let $M \subset L(\mathsf{R},K)$ be an irreducible closed subset such that*

(i) $\dim \overline{\pi(P(M))} = r - 1$;
(ii) $\overline{\pi(P(M))} \neq \overline{\pi(\mathfrak{I})}$.

Then P is dominant.

Proof. Since M is irreducible and $\dim \pi(P(M)) = r-1$, we conclude that $\overline{\pi(P(M))}$ is an irreducible hypersurface in Q. The assertion of the lemma now follows from (ii) and (3.1). □

We can now start the proof of Theorem 3.2.

First of all, the statement is obviously reduced to the case where R is irreducible. Indeed, if R is a disjoint union of R_i, then $\mathfrak{g} = L(\mathsf{R}, K)$ is a direct sum of $\mathfrak{g}_i = L(\mathsf{R}_i, K)$, and the image Im P of the map $P = P(X_1, \dots, X_d)\colon \mathfrak{g}^d \to \mathfrak{g}$ is equal to $\oplus_i \mathrm{Im}\, P_i$ where P_i is the restriction of P to $\mathfrak{g}_i$.

Note that

$$P(X_1, \dots, X_d) = \sum_i a_i X_i + \sum (\text{monomials of } \ \mathcal{L}_d \text{ of degree} > 1)$$

where $a_i \in K$. If $a_i \neq 0$ for some i, the statement is trivial. Thus we may and will assume $a_i = 0$ for every i.

First we prove the assertion of the theorem for the case $\mathsf{R} = \mathsf{A}_r$.

By (3.1), it is enough to prove $\overline{\pi(\mathfrak{I})} = \mathsf{Q}$. Note that in the case $\mathrm{char}(K) = 2$ the statement of the theorem fails for $r = 1$. However, as we will see below, the equality $\overline{\pi(\mathfrak{I})} = \mathsf{Q}$ holds even in this case. Thus we can prove that $\overline{\pi(\mathfrak{I})} = \mathsf{Q}$ by induction on the rank r starting at $r = 1$.

We identify $L(\mathsf{A_r}, K)$ with $\mathfrak{sl}(r+1, K)$, the algebra of $(r+1) \times (r+1)$-matrices with zero trace. We fix the chain of subalgebras $L_1 \subset L_2 \subset \dots \subset L_r = \mathfrak{sl}(r+1, K)$ where $L_{i-1} = \mathfrak{sl}(i, K)$ is the subalgebra embedded in the $i \times i$ upper left corner of the matrix algebra $L_i = \mathfrak{sl}(i+1, K)$. We also fix, for each i, the subalgebra $H_i \subset L_i$ of diagonal matrices in L_i. Further, let $P_i = P_{|L_i}$.

Induction base: we prove that $\dim \overline{\pi(\mathrm{Im}\ P_1)} = 1$.

Let first $\mathrm{char}(K) = 2$. Then according to case 1) of Section 2.4, we have

$$P_1 = \sum_{i,j} a_{ij} [X_i, X_j]$$

where $a_{ij} \in K$. On putting $X = X_{i_0}$, $Y = a_{i_0, j_0} X_{j_0}$, $X_i = 0$ for appropriate $i \neq i_0, j_0$, we can get the map $P_1' = [X, Y]$. Then Im $P_1' = H_1 \subset$ Im P_1, and therefore $\dim \overline{\pi(\mathrm{Im}\ P_1)} = 1$.

Let us now assume $\mathrm{char}(K) \neq 2$.

As the map P is not identically zero, we may apply Lemma 3.3 with $M = 0^d$. We obtain the dominancy of $P_1\colon L_1^d \to L_1$ which implies $\dim \overline{\pi(\mathrm{Im}\ P_1)} = 1$.

Inductive step: assume

$$\dim \overline{\pi(\mathrm{Im}\ P_{r-1})} = r - 1 \tag{3.2}$$

and prove

$$\dim \overline{\pi(\mathrm{Im}\ P_r)} = r. \tag{3.3}$$

We have

$$H_{r-1} = \{x = \mathrm{diag}(\alpha_1, \dots, \alpha_r, 0) \in M_{r+1}(K) \mid \ \mathrm{tr}\, x = 0\}.$$

(Here $M_{r+1}(K)$ is the algebra of $(r+1)\times(r+1)$-matrices over K.) We have

$$\overline{\pi(\mathrm{Im}\ P_{r-1})} = \overline{\bar{\pi}(H_{r-1})}. \tag{3.4}$$

Suppose that

$$\mathfrak{I} \cap H_r \neq WH_{r-1}. \tag{3.5}$$

Then

$$\overline{\pi(\mathfrak{I})} \supseteq \bar{\pi}(\mathfrak{I} \cap H_r) \neq \overline{\bar{\pi}(H_{r-1})}. \tag{3.6}$$

Condition (3.2) is condition 1) from Lemma 3.3 with $M = L_{r-1}$. Conditions (3.4) and (3.6) give us condition 2) from the same lemma. Note that the dominancy of P implies $\dim \overline{\pi(\mathfrak{I})} = r$. Hence we have to prove that condition (3.5) holds.

We may assume that the transcendence degree of K is sufficiently large because this does not have any influence on dominancy of P. Then we may also assume that there exist a subfield $F \subset K$ and a division algebra $D_{r+1} \subset M_{r+1}(K)$ with centre F such that $D_{r+1} \otimes_F K = M_{r+1}(K)$ [DS], [Bo2]. The algebra D_{r+1} is dense in $M_{r+1}(K)$. Hence the set $[D_{r+1}, D_{r+1}]$ is dense in $[M_{r+1}(K), M_{r+1}(K)] = \mathfrak{sl}(r+1, K)$. On the other hand, $[D_{r+1}, D_{r+1}] \subset D_{r+1}$. Thus the set $S_{r+1} = D_{r+1} \cap \mathfrak{sl}(r+1, K)$ is dense in $\mathfrak{sl}(r+1, K)$, and therefore the restriction of P to S_{r+1}^d is not the zero map. Then there exist $s_1, \dots, s_d \in S_{r+1}$ such that $s = P(s_1, \dots, s_d) \neq 0$. Since $s_1, \dots, s_d \in D_{r+1}$, we have $s \in D_{r+1}$. As there are no nonzero nilpotent elements in division algebras, all elements of D_{r+1} are semisimple, so we may assume $s \in H_r$. Since s has no zero eigenvalues, $s \notin WH_{r-1}$, and we get (3.5). Thus (3.3) is proven, and the assertion of the theorem for $\mathfrak{sl}_r$ is established.

The general case is a consequence of the following observation [Bo2]: every irreducible root system R has a subsystem R' which has the same rank as R and decomposes into a disjoint union of irreducible subsystems $\mathsf{R}' = \bigcup_i \mathsf{R}'_i$ where each R'_i is a system of type A_{r_i}. Hence

$$L' = \bigoplus_i L(\mathsf{A}_{r_i}, K) \subset L(\mathsf{R}, K), \ \sum_i r_i = r.$$

Thus

$$\dim \overline{\pi(P(L'))} = r \Rightarrow \overline{\pi(\mathfrak{I})} = \mathsf{Q},$$

and we get the statement from (3.1).

Theorem 3.2 is proved. □

Corollary 3.4. *Let $\mathfrak{g}$ be a classical semisimple Lie algebra. Suppose $P(X_1, \dots, X_d)$ is not an identity of the Lie algebra $\mathfrak{sl}(2, K)$. Then the induced map $P\colon \mathfrak{g}^d \to \mathfrak{g}$ is dominant.*

Proof. Let R be the root system corresponding to $\mathfrak{g}$. If the Chevalley algebra $L(\mathsf{R}, K)$ is semisimple, we have $\mathfrak{g} = L(\mathsf{R}, K)$, and there is nothing to prove. If $\mathfrak{g} = L(\mathsf{R}, K)/\mathfrak{z}$, where $\mathfrak{z}$ is the centre, the assertion is an immediate consequence of the following obvious observation: if the Lie polynomial $P(X_1, \dots, X_d)$ does not contain terms of degree 1, then the map $P\colon L(\mathsf{R}, K)^d \to L(\mathsf{R}, K)$ is trivial on $\mathfrak{z}$. □

3.2. Theorem 3.2 reduces the problem of dominancy to the class of maps P which are identically zero on $\mathfrak{sl}(2, K)$. The following theorem gives another possibility to reduce the problem of dominancy.

Theorem 3.5. *Let $L(\mathsf{R}, K)$ be a Chevalley algebra corresponding to an irreducible root system R, and suppose that $R \neq \mathsf{C}_r$ if $\mathrm{char}(K) = 2$.*

Suppose that the map $P\colon L(\mathsf{R}, K)^d \to L(\mathsf{R}, K)$ is dominant for $\mathsf{R} = \mathsf{A}_2$ and B_2. Then P is dominant for every $L(\mathsf{R}, K)$, $r > 1$.

Proof. We prove the theorem by induction on r. Let first $r = 2$. The cases $\mathsf{R} = \mathsf{A}_2, \mathsf{B}_2$ are included in the hypothesis, and the case $\mathsf{R} = \mathsf{G}_2$ is established by the same argument as at the end of the proof of Theorem 3.2 because G_2 contains A_2.

Let now $r > 2$, and make the induction hypothesis:
the map P is dominant for every $L(\mathsf{R}, K)$ where $1 < \mathrm{rank}\ \mathsf{R} < r$.

We proceed case by case.

$\mathsf{R} = \mathsf{A}_r$. The induction step is the same as in the proof of Theorem 3.2.

$\mathsf{R} = \mathsf{C}_r$ $(r \geq 3)$ or D_r $(r \geq 4)$. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the simple root system numerated as in Bourbaki [Bou]. Let $\Pi_1 = \{\alpha_1, \dots, \alpha_{r-1}\}$, $\Pi_2 = \{\alpha_2, \dots, \alpha_r\}$. Then $\mathsf{R}_1 = \langle \Pi_1 \rangle = \mathsf{A}_{r-1}$, $\mathsf{R}_2 = \langle \Pi_2 \rangle = \mathsf{C}_{r-1}$ or D_{r-1}, respectively. Let $H_i = H \cap L(\mathsf{R}_i, K)$. There exists $h \in H_1$ such that $h \notin WH_2$.

Indeed, let $\epsilon_i\colon H \to K$ be the weights given by the formula $\epsilon_i(h_{\alpha_k}) = \frac{2(\epsilon_i, \alpha_k)}{(\alpha_k, \alpha_k)}$. Then $\epsilon_1(H_2) = 0$, and therefore for every $h' \in WH_2$ we have $\epsilon_i(h') = 0$ for some i. On the other hand, since $\mathsf{R}_1 = \mathsf{A}_{r-1}$, we can find $h \in H_1$ such that $\epsilon_i(h) \neq 0$ for every i, and therefore $h \notin WH_2$.

Note that $h \in \overline{P(L(\mathsf{R}_1, K)^d)}$ because P is dominant on $L(\mathsf{R}_1, K)$ (see the proof of Theorem 3.2). Then $h \in \mathfrak{I}$. On the other hand, $\pi(h) \notin \overline{\pi(P(L(\mathsf{R}_2, K)^d))}$ because $h \notin WH_2$. Hence we can apply Lemma 3.3 with $M = L(\mathsf{R}_2, K)$.

$\mathsf{R} = \mathsf{B}_r, \mathsf{F}_4$. Here we have $\mathsf{D}_r \subset \mathsf{R}$. (The respective embeddings are as follows: $\mathfrak{so}(2r) \subset \mathfrak{so}(2r+1)$ is a natural inclusion, and D_4 embeds into F_4 as the subsystem consisting of the long roots.)

Then $H \subset \overline{P(L(\mathsf{D}_r, K))}$, and therefore P is dominant on $L(\mathsf{R}, K)$.

$\mathsf{R} = \mathsf{E}_r$. Consider the extended Dynkin diagram. We obtain a needed subsystem of type A by removing one of its vertices. In each case the diagram is a trident. We remove the 3-valent vertex in the case $r = 6$, and the tooth of length 1 (the lower vertex α_2 in the Bourbaki notation) in the cases $r = 7, 8$. We obtain subsystems of types $\mathsf{A}_2 \times \mathsf{A}_2 \times \mathsf{A}_2$, A_7, and A_8, respectively. Then we use the same argument as above. □

3.3. To use the theorems proven above for practical purposes, the following simple remarks may be useful.

If $P(X_1, \dots, X_d) \in \mathcal{L}_d$ is a polynomial containing a monomial of degree < 5, then the map $P\colon L(\mathsf{R}, K)^d \to L(\mathsf{R}, K)$ $(\mathrm{char}(K) \neq 2)$ is dominant.

The reason is that such a polynomial cannot be an identity in $\mathfrak{sl}(2, K)$. Indeed, if it were an identity, so would be its homogeneous component of the lowest degree (because any homogeneous component of any polynomial identity of any algebra of any signature over any infinite field is an identity, see [Ro, 6.4.14]). On the other hand, any identity of the Lie algebra $\mathfrak{sl}(2, K)$ $(\mathrm{char}(K) \neq 2)$, is an identity of $\mathfrak{gl}(2, K)$ (because every matrix is a sum of a trace zero matrix and a scalar matrix, and such an identity lifts to an identity of the associative matrix algebra $M_2(K)$. The latter one does not contain identities of degree less than 4 (which is the smallest degree of the so-called standard identity satisfied in M_2), hence the same is true for $\mathfrak{gl}(2)$ (see, e.g., [Ro, Remark 6.1.18] or [Ba, Exercise 2.8.1]). Moreover, a little subtler argument allows one to show that $\mathfrak{sl}(2, K)$ does not contain identities of degree 4 (see, e.g., [Ba, Section 5.6.2]).

Note that Razmyslov [Ra] found a finite basis for identities in this algebra (assuming K to be of characteristic zero). Moreover, it turned out that all such identities are consequence of the single identity [Fi]:

$$P = [[[Y, Z], [T, X]], X] + [[[Y, X], [Z, X]], T],$$

and this result remains true for any infinite field K, $\mathrm{char}(K) \neq 2$ [Va].

Below we illustrate how one can apply Theorem 3.5 using one of the identities appearing in Razmyslov's basis (the reader willing to deduce this identity from Filippov's one mentioned above is referred to Section 2 of [Fi]).

Example 3.6. The polynomial $[[[[Z, Y], Y], X], Y] - [[[[Z, Y], X], Y], Y]$ appears in [Ra] as one of the elements of a finite basis of identities in $\mathfrak{sl}(2, K)$ $(\mathrm{char}(K) = 0)$. Clearly, the polynomial

$$P(X, Y, Z) = [[[[[Z, Y], Y], X], Y], [[[[Z, Y], X], Y], Y]]$$

is also identically zero in $\mathfrak{sl}(2, K)$. We check dominancy of the map

$$P\colon L(\mathsf{R}, K)^3 \to L(\mathsf{R}, K)$$

using computations by MAGMA. In view of Theorem 3.5, we have to check dominancy only for $\mathsf{R} = \mathsf{A}_2, \mathsf{B}_2$.

Consider the map $\pi\colon L(\mathsf{R},K)\to\mathsf{Q}$ defined in Section 2.2.**d**. Since $\mathrm{char}(K)=0$, we have $\mathsf{Q}\cong H/W\cong\mathbb{A}^r$, and $\pi=(f_1,f_2,\dots,f_r)$ where $f_1,f_2,\dots,f_r$ are G-invariant homogeneous polynomials on $L(\mathsf{R},K)$ which generate the invariant algebra $K[L(\mathsf{R},K)]^G\cong K[H]^W$. Moreover, $\deg f_1\deg f_2\cdots\deg f_r=|W|$ (see Section 2.2.**d**). In our cases, $r=2$ and we have $\deg f_1=2$, $\deg f_2=3$ for $\mathsf{R}=\mathsf{A}_2$ and $\deg f_1=2$, $\deg f_2=4$ for $\mathsf{R}=\mathsf{B}_2$.

Let

$$0\neq D_1=P(A,B,C),D_2=P(A',B',C')\in\mathfrak{I}=\overline{\mathrm{Im}\,P(L(\mathsf{R},K))^3}.$$

Since P is a homogeneous map with respect to X,Y,Z, the lines $l_j:=KD_j$, $j=1,2$, also lie in $\mathfrak{I}$, and the curves $\pi(l_j)$ in the affine space $\mathbb{A}^2$ with coordinates (x_1,x_2) are defined by equations of the form

$$x_1^{m_1}/x_2^{m_2}=c_j,\quad\text{where } m_1=\deg f_2, m_2=\deg f_1, c_j=\mathrm{const}\,. \tag{3.7}$$

Put

$$\theta:=f_1^{m_1}/f_2^{m_2}. \tag{3.8}$$

From (3.7) and (3.8) we get

$$\theta(D_1)\neq\theta(D_2)\Rightarrow\pi(l_1)\neq\pi(l_2). \tag{3.9}$$

By Lemma 3.3 with $M=l_1$, from (3.9) we see that the inequality

$$\theta(P(A,B,C))\neq\theta(P(A',B',C')) \tag{3.10}$$

implies the dominancy of P.

Case $\mathsf{R}=\mathsf{A}_2$. We may identify $L(\mathsf{A}_2,K)=\mathfrak{sl}(3,K)$. The characteristic polynomial is $\chi(t)=t^3+pt+q$ where p and q can be viewed as $SL_3(K)$-invariant homogeneous polynomials of degrees 2 and 3, respectively. Therefore $p=f_1$, $q=f_2$. We point out triples (A,B,C), (A',B',C') satisfying inequality (3.10) which were found by MAGMA:

$$A=\begin{pmatrix}3&1&0\\1&-1&1\\0&1&-2\end{pmatrix},\quad B=\begin{pmatrix}2&1&5\\0&4&-3\\1&0&-6\end{pmatrix},\quad C=\begin{pmatrix}0&0&2\\1&0&3\\0&1&0\end{pmatrix},$$
$$A'=\begin{pmatrix}8&1&0\\1&-1&1\\0&1&-7\end{pmatrix},\quad B'=\begin{pmatrix}2&3&5\\0&4&-3\\1&0&-6\end{pmatrix},\quad C'=\begin{pmatrix}0&0&2\\1&0&3\\0&1&0\end{pmatrix}.$$

Case $\mathsf{R}=\mathsf{B}_2$. We may identify $L(\mathsf{R},K)=\mathfrak{so}(5,K)$. Consider the embedding $\mathfrak{so}(5,K)\hookrightarrow\mathfrak{sl}(5,K)$ given by identification of $\mathfrak{so}(5,K)$ with matrices of the form

$$\begin{pmatrix}0&b&c\\-c^t&m&n\\-b^t&p&-m^t\end{pmatrix}$$

where m, n, p are 2×2-matrices, and n, p are skew-symmetric (see, e.g., [Hu, 1.2]). The characteristic polynomial is $\chi(t)=t^5+pt^3+qt$ where

p and q can be viewed as $SO_5(K)$-invariant homogeneous polynomials on $\mathfrak{so}(5,K)$ of degrees 2 and 4, respectively. Hence $f_1 = p$, $f_2 = q$. We point out triples (A,B,C), (A',B',C') satisfying inequality (3.10) which were found by MAGMA:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -3 & 5 & 6 & 0 & 9 \\ -4 & 7 & 8 & -9 & 0 \\ -1 & 0 & 10 & -5 & -7 \\ -2 & -10 & 0 & -6 & -8 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 5 & -6 & -7 & 8 \\ 7 & -1 & 2 & 0 & 2 \\ -8 & 3 & 4 & -2 & 0 \\ -5 & 0 & -3 & 1 & -3 \\ 6 & 3 & 0 & -2 & -4 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 4 & 1 & 2 & 3 \\ -2 & -8 & 6 & 0 & -9 \\ -3 & -9 & 7 & 9 & 0 \\ -4 & 0 & 10 & 8 & 9 \\ -1 & -10 & 0 & -6 & -7 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 6 & -7 & 10 & -3 \\ -10 & -8 & -6 & 0 & 5 \\ 3 & 1 & 2 & -5 & 0 \\ -6 & 0 & -4 & 8 & -1 \\ 7 & 4 & 0 & 6 & -2 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 & 2 & -3 & 4 \\ 3 & -5 & -6 & 0 & 10 \\ -4 & 7 & 8 & -10 & 0 \\ 1 & 0 & 9 & 5 & -7 \\ -2 & -9 & 0 & 6 & -8 \end{pmatrix}, C' = \begin{pmatrix} 0 & -6 & 6 & -3 & 8 \\ 3 & 7 & 6 & 0 & 11 \\ -8 & 7 & 3 & -11 & 0 \\ 6 & 0 & -2 & -7 & -7 \\ -6 & 2 & 0 & -6 & -3 \end{pmatrix}.$$

4. From dominancy to surjectivity

For some polynomials $P \in \mathcal{L}_d$ we can say more than in the preceding section. Namely, we present here several cases where the map $P\colon \mathfrak{g}^d \to \mathfrak{g}$ is surjective.

We start with the following simple observation (parallel to Remark 3 in [Bo2, §1]).

Proposition 4.1. *Let $P_1(X_1,\dots,X_{d_1})$, $P_2(Y_1,\dots,Y_{d_2})$ be Lie polynomials. Let $\mathfrak{g}$ be a Lie algebra. Suppose that each of the maps $P_i\colon \mathfrak{g}^{d_i} \to \mathfrak{g}$ is dominant. Let $d = d_1 + d_2$,*

$$P(X_1,\dots X_{d_1}, Y_1,\dots Y_{d_2}) = P_1(X_1,\dots,X_{d_1}) + P_2(Y_1,\dots Y_{d_2}).$$

Then the map $P\colon \mathfrak{g}^d \to \mathfrak{g}$ is surjective.

Proof. We may assume the ground field to be algebraically closed. As the underlying variety of $\mathfrak{g}$ is irreducible, the image of each of the dominant morphisms P_i $(i = 1,2)$ contains a non-empty open subset U_i. It remains to notice that $U_1 + U_2 = \mathfrak{g}$ (see, e.g., [Bo3, Chapter I, § 1, 1.3]). □

Let us now prove surjectivity for some special maps, which are linear in one variable.

Definition 4.2. We call

$$E_m(X,Y) = \underbrace{[[\dots[}_{m \text{ times}} X,Y],Y],\dots,Y] \in \mathcal{L}_2$$

an Engel polynomial of degree $(m+1)$. We call

$$\sum_{i=1}^{m} a_i E_i(X,Y) \in \mathcal{L}_2,$$

where $a_i \in K$, a generalized Engel polynomial.

Theorem 4.3. *Let $P(X,Y) \in \mathcal{L}_2$ be a generalized Engel polynomial of degree $(m+1)$, and let $P\colon L(\mathsf{R},K)^2 \to L(\mathsf{R},K)$ be the corresponding map of Chevalley algebras. If R does not contain irreducible components of types listed in* (2.2) *and $|K| > m|R|$, then the image of P contains*

$$(L(\mathsf{R},K) \setminus Z(L(\mathsf{R},K)) \cup \{0\}.$$

Moreover, if P is an Engel polynomial, then the same is true under the assumption $|K| > |R^+|$.

Proof. Since $|K| > m|R|$, for any chosen $S \subset K$ of size m there is $h \in H$ such that $\beta(h) \notin S$ for all $\beta \in \mathsf{R}$ (see 2.3.**e**). Further, for every $h \in H$ the map $P_h\colon L(\mathsf{R},K) \to L(\mathsf{R},K)$, given by $X \mapsto P(X,h)$, is a semisimple linear operator on $L(\mathsf{R},K)$ which is diagonalizable in the Chevalley basis. Each h_α is its eigenvector with zero eigenvalue. Further, there is a degree m polynomial $f \in K[t]$ such that $P(e_\beta,h) = f(\beta(h))e_\beta$ for every $\beta \in \mathsf{R}$. (Explicitly, one can take $f = \sum_{i=1}^{m}(-1)^i a_i t^i$.) Define S as the set of roots of f in K. Then $f(\beta(h)) \neq 0$ for every $\beta \in \mathsf{R}$, and therefore $\operatorname{Im}(P_h) = U$. Now the statement follows from 2.4.**f.**

If P is an Engel polynomial of degree $(m+1)$, then one can take $f = x^m$, and therefore $S = \{0\}$, that is, h is a regular element. Once again, we can use 2.4.**f.** □

Corollary 4.4. *Let $P = P(X,Y) \in \mathcal{L}_2$ be a generalized Engel polynomial of degree $(m+1)$, and let $\mathfrak{g}$ be a simple classical Lie algebra corresponding to the root system R. If $|K| > m|R|$, then the map $P\colon \mathfrak{g}^2 \to \mathfrak{g}$ is surjective. Moreover, if P is an Engel polynomial, the same is true under the assumption $|K| > |R^+|$.*

Remark 4.5. Corollary 4.4 generalizes Theorem 7 of [Th3] where Question 1.1a was answered in the affirmative for the words in three variables $P(X,Y,Z)$ of the form $[X,Y,\dots,Y,Z]$ and $\mathfrak{g} = \mathfrak{sl}(n)$.

Our next result shows that one cannot hope to extend surjectivity to central elements.

Proposition 4.6. *Let $P_m(X,Y) \in \mathcal{L}_2$ be an Engel polynomial of degree m, and let $P\colon L(\mathsf{R},K)^2 \to L(\mathsf{R},K)$ be the corresponding map of Chevalley algebras. Then for m big enough the image of P contains no nonzero elements of $Z(L(\mathsf{R},K))$.*

Proof. The idea is as follows: if X and Y centralize the same element of a Cartan subalgebra of $L(\mathsf{R}, K)$, then $P_m(X, Y) = 0$ for m big enough. Otherwise, the term corresponding to the "shortest" nontrivial root which does not vanish on X never goes to zero after multiplication by Y. Here is a detailed argument.

We may assume K algebraically closed. Then, by "bringing to the Jordan form", we may assume $Y = h + y$ where $h \in H$, $y \in U^+$, $[h, y] = 0$. Further, let $X = h' + x$, $h' \in H$, $x \in U$.

For brevity, for every n denote $z_n = P_n(X, Y)$.

Case I. $[h, x] = 0$.

Let us prove that $z_n = P_n(X, y)$ and $[z_n, h] = 0$. We use induction on n. For $n = 1$ we have $z_1 = [X, h + y] = [X, h] + [X, y]$. Since $[x, h] = [h', h] = 0$, we have $z_1 = [X, y]$. Further, $[z_1, h] = [[X, y], h] = [[h', y], h] + [[x, y], h]$. Since $[h', h] = [x, h] = [y, h] = 0$, each summand equals zero by the Jacobi identity, so $[z_1, h] = 0$.

Assume $z_{n-1} = P_{n-1}(X, y)$ and $[z_{n-1}, h] = 0$. We have

$$z_n = [z_{n-1}, Y] = [z_{n-1}, h + y] = [z_{n-1}, y] = [P_{n-1}(X, y), y] = P_n(X, y)$$

and $[z_n, h] = [[z_{n-1}, y], h] = 0$ by the Jacobi identity (because $[z_{n-1}, h] = [y, h] = 0$).

Thus we have $P_n(X, Y) = [[X, y], y, \dots, y]$ which is zero for n big enough because y is nilpotent.

Case II. $[h, x] \neq 0$.

First suppose that $y = 0$, i.e. $Y = h$ is semisimple. As $x \neq 0$, we can write

$$x = \sum_{\beta \in \mathsf{R}} f_\beta e_\beta,$$

where $f_\beta \in K$. Since $[h, x] \neq 0$, there exists β such that $[h, e_\beta] \neq 0$. We now observe that if $f_\beta \neq 0$ then for every m the term of $P_m(X, Y)$ containing e_β enters with nonzero coefficient, so $P_m(X, Y)$ belongs to U and thus does not belong to the centre.

So assume $y \neq 0$ and write

$$y = \sum_{\beta \in \mathsf{R}^+} p_\beta e_\beta,$$

where $p_\beta \in K$.

Put

$$\mathsf{R}_h = \{\beta \in \mathsf{R} \mid \beta(h) \neq 0\}, \ \hat{\mathsf{R}}_h = \{\beta \in \mathsf{R} \mid \beta(h) = 0\},$$

$$\mathsf{R}_x = \{\beta \in \mathsf{R} \mid f_\beta \neq 0\}, \ \mathsf{R}_y = \{\beta \in \mathsf{R} \mid p_\beta \neq 0\}.$$

All these sets are non-empty, and

$$\mathsf{R}_{h,x} = \mathsf{R}_h \cap \mathsf{R}_x \neq \emptyset.$$

We have $\mathsf{R}_y \subseteq \mathsf{R}^+$, $\mathsf{R}_y \subseteq \hat{\mathsf{R}}_h$.

Let $\prec$ be the (partial) order on R induced by height. Recall that by definition $\alpha \prec \beta$ if and only if $\beta - \alpha$ is a sum of positive roots. We fix some minimal γ in $\mathsf{R}_{h,x}$.

Further, write

$$P_n(X,Y) = z_n = \sum_{\beta \in \mathsf{R}} d_{n,\beta} e_\beta + h_n$$

where $h_n \in H$, $d_{n,\beta} \in K$.

Claim:

a) $d_{n,\gamma} \neq 0$;

b) if $d_{n,\delta} \neq 0$ and $\delta \neq \gamma$, then either $\delta \in \hat{\mathsf{R}}_h$ or $\delta \not\prec \gamma$.

Evidently, a) is enough to establish the assertion of the proposition.

Let us prove the claim by induction on n. Let first $n = 1$. We have

$$[h', y] = \sum_{\beta \in \hat{\mathsf{R}}_h} a_\beta e_\beta, \ [x, y] = \sum_{\beta \in \mathsf{R}} b_\beta e_\beta + h_1, \ [x, h] = \sum_{\beta \in \mathsf{R}_h} c_\beta e_\beta, \quad (4.1)$$

where $h_1 \in H$, and we have $d_{1,\beta} = a_\beta + b_\beta$ or $d_{1,\beta} = b_\beta + c_\beta$.

a) We have $a_\gamma = 0$, $c_\gamma \neq 0$ because $\gamma \in \mathsf{R}_{h,x} \subseteq \mathsf{R}_h$. Let us prove that $b_\gamma = 0$. Assume to the contrary that $b_\gamma \neq 0$. Then from the middle equality in (4.1) it follows that there are roots $\alpha \in \mathsf{R}_x$ and $\beta \in \mathsf{R}_y$ such that $[e_\alpha, e_\beta] = e_\gamma$ (and so $\gamma = \alpha + \beta$). Since $[h, e_\beta] = 0$ and $[h, e_\gamma] \neq 0$, we have $[h, e_\alpha] \neq 0$. Hence $\alpha \in \mathsf{R}_h$ and therefore $\alpha \in \mathsf{R}_{h,x} = \mathsf{R}_h \cap \mathsf{R}_x$. Since $\gamma = \alpha + \beta$, we have the inequality $\alpha \prec \gamma$ because β is a positive root. This is a contradiction with the choice of γ (recall that γ is a minimal root in $\mathsf{R}_{h,x}$ with respect to the partial order $\prec$). Thus $b_\gamma = 0$ and $d_{1,\gamma} = c_\gamma \neq 0$.

b) Suppose $d_{1,\delta} \neq 0$ and $\delta \notin \hat{\mathsf{R}}_h$. Then $\delta \in \mathsf{R}_h$ and $d_{1,\delta} = b_\delta + c_\delta$. If $c_\delta \neq 0$, then $\delta \in \mathsf{R}_x$. Hence $\delta \in \mathsf{R}_{h,x}$ and $\delta \not\prec \gamma$ because of the choice of γ. If $c_\delta = 0$, then $d_{1,\delta} = b_\delta \neq 0$. Then $e_\delta = [e_\alpha, e_\beta]$ for some $\alpha \in \mathsf{R}_x$, $\beta \in \mathsf{R}_y$. Since $\delta \in \mathsf{R}_h$ (and so $[h, e_\delta] \neq 0$) and $\beta \in \mathsf{R}_y \subseteq \hat{\mathsf{R}}_h$ (and so $[h, e_\beta] = 0$), we have $[h, e_\alpha] \neq 0 \Rightarrow \alpha \in \mathsf{R}_h \Rightarrow \alpha \in \mathsf{R}_{h,x}$. Suppose that $\delta = \alpha + \beta \prec \gamma$. Then $\alpha \prec \gamma$ which is again a contradiction with the choice of γ. Hence $\delta \not\prec \gamma$.

Let us now assume

a) $d_{n-1,\gamma} \neq 0$;

b) if $d_{n-1,\delta} \neq 0$ and $\delta \neq \gamma$, then either $\delta \in \hat{\mathsf{R}}_h$ or $\delta \not\prec \gamma$,

and prove the same assertions for n.

Consider

$$z_n = [z_{n-1}, h + y] = \underbrace{\sum_{\beta \in \mathsf{R}} d_{n-1,\beta} [e_\beta, h]}_{I} + \underbrace{\sum_{\beta \in \mathsf{R}} d_{n-1,\beta} [e_\beta, y]}_{II} + \underbrace{[h_{n-1}, y]}_{III}.$$

The induction hypotheses imply that

$$z_n = \underbrace{\sum_{\delta\in\hat{\mathsf{R}}_h} q_\delta e_\delta}_{\spadesuit} + s_\gamma e_\gamma + \underbrace{\sum_{\delta\in\mathsf{R}_h,\delta\not\succ\gamma} s_\delta e_\delta}_{\heartsuit} + h_n,$$

where $h_n \in H$ and $s_\gamma \neq 0$. Indeed, sum I has only terms of types $\heartsuit$ and the term $s_\gamma e_\gamma \neq 0$. Further, sum II has terms of types $\spadesuit$ and $\heartsuit$ and elements of H. Sum III has only terms of type $\spadesuit$ because $\mathsf{R}_y \subseteq \hat{\mathsf{R}}_h$. Thus conditions a) and b) hold for z_n, and $z_n = P(X, Y) \notin Z(L(\mathsf{R}, K))$. □

Remark 4.7. Suppose we are in one of the exceptional cases listed in (2.2). Let us exclude abelian and solvable cases 1), 2) of Section 2.4. Also in case 6) in Theorem 4.3 we may consider the Lie algebra $[L(\mathsf{R}, K), L(\mathsf{R}, K)]$ instead of $L(\mathsf{R}, K)$. In cases 3),4), 5) the algebra $L(\mathsf{R}, K)$ contains an ideal I (generated by short roots) such that the quotient $\bar{L} = L(\mathsf{R}, K)/I$ is not on list (2.2), and therefore the assertion of Theorem 4.3 on surjectivity of P holds for $\bar{L}$.

Example 4.8. In the following example we show that non-Engel maps are not necessarily surjective. Let

$$P = P(X, Y) = [[[X, Y], X], [X, Y], Y]]\colon \mathfrak{sl}(2, K) \times \mathfrak{sl}(2, K) \to \mathfrak{sl}(2, K)$$

where $\mathrm{char}(K) \neq 2$ and K is an algebraically closed field. Note that if either X or Y is nilpotent, then either $[[X, Y], X] = 0$ or $[X, Y], Y] = 0$. So we may assume that both X and Y are semisimple and $X = h \in H$. Then $P(X, Y) = P(X, Y + X)$, and therefore, by subtracting scalar multiples of X, we may assume $Y = v + u$ where $0 \neq v \in U^-$, $0 \neq u \in U^+$. Then $[X, Y] = av - au$ for some $0 \neq a \in K$ and

$$[X, Y], X] = -a^2v+a^2u, [X, Y], Y] = [av-au, v+u] = 2a[v, u] = h' \in H,$$

and so

$$[[[X, Y], X], [X, Y], Y]] = a^2dv - a^2du$$

for some $d \neq 0$. Thus $P(X, Y)$ is a semisimple element. Hence in $\mathrm{Im}(P)$ there are no nilpotent elements.

5. Possible generalizations

Remark 5.1. The method used in the proof of Theorem 3.2 (which goes back to [DS] and [Bo2]) is applicable to the problem of dominancy of polynomial maps on associative matrix algebras (which is attributed to Kaplansky, see [KBMR] and references therein). More precisely, let $P(X_1, \ldots, X_d) \in K\langle X_1, \ldots, X_d\rangle$ be an associative, noncommutative polynomial (i.e., an element of the free associative algebra on d variables over K), and let $P\colon M_n(K)^d \to M_n(K)$ denote the corresponding map. Then the same inductive argument as in the proof of Theorem 3.2 shows that if $P(X_1, \ldots, X_d)$ is not identically zero on K^d then the map

P is dominant for all n. In the situation where $P(X_1, \dots X_d)$ is identically zero on K^d, one can consider the induction base $n = 2$ and prove that if the restriction of P to $M_2(K)^d$ is dominant then so is P. The assumption made above holds, for instance, for any semi-homogeneous, non-central polynomial having at least one 2×2-matrix with nonzero trace among its values [KBMR, Theorem 1]. If, under the same assumptions on P, $\mathrm{Im}(P)$ lies in $\mathfrak{sl}(n, K)$, then $\overline{\mathrm{Im}(P)} = \mathfrak{sl}(n, K)$.

Remark 5.2. It would be interesting to consider maps P with some fixed $X_i = A_i$. Then one could find an approach to the dominancy calculating the differential map of P.

Remark 5.3. It would be interesting to consider a more general set-up when we have a polynomial map $P\colon L^d \to L^s$. In [GR] some dominancy results were obtained for the multiple commutator map $P\colon L \times L^d \to L^d$ given by the formula $P(X, X_1, \dots, X_d) = ([X, X_1], \dots, [X, X_d])$.

Remark 5.4. In a similar spirit, one can consider generalized word maps $w\colon G^d \to G^s$ on simple groups. Apart from [GR], see also a discussion of a particular case $w = (w_1, w_2)\colon G^2 \to G^2$ in [BGGT, Problem 1].

Remark 5.5. One could try to extend some of results of this paper to the case where the ground field is replaced with some sufficiently good ring. One has to be careful in view of [RR]: there are rings R such that not every element of $\mathfrak{sl}(n, R)$ is a commutator.

Remark 5.6. One can ask questions similar to Questions 1.1 and 1.2 for other classes of algebras (beyond groups, Lie algebras and associative algebras). The interested reader may refer to [Gordo] for the case of values of commutators and associators on alternative and Jordan algebras.

Acknowledgements. A substantial part of this work was done during the visits of the first three coauthors to the MPIM (Bonn) in 2010. Bandman, Kunyavskiĭ and Plotkin were supported in part by the Minerva foundation through the Emmy Noether Research Institute. Gordeev was supported in part by RFFI research grants 08-01-00756-a, 10-01-90016-Bel-a.

The support of these institutions is gratefully appreciated.

We thank M. Agranovsky, M. Gorelik, A. Joseph, and A. Kanel-Belov for helpful discussions.

References

[Ba] Yu. A. Bahturin, *Identical Relations in Lie Algebras*, VNU Science Press BV, Utrecht, 1987.

[BGG] T. Bandman, S. Garion, F. Grunewald, *On the surjectivity of Engel words on $PSL(2, q)$*, preprint `arxiv:1008.1397`.

[BG] J. Barge, É. Ghys, *Cocycles d'Euler et de Maslov*, Math. Ann. **264** (1992) 235–265.

[Bl] H. I. Blau, *A fixed-point theorem for central elements in quasisimple groups*, Proc. Amer. Math. Soc. **122** (1994) 79–84.

[Bo1] A. Borel, *Properties and linear representations of Chevalley groups*, in: *Seminar on Algebraic Groups and Related Finite Groups* (A. Borel *et al.*, eds.), Lecture Notes Math., vol. 131, Springer-Verlag, Berlin–Heidelberg–New York, 1970, pp. 1–55.

[Bo2] A. Borel, *On free subgroups of semisimple groups*, Enseign. Math. **29** (1983) 151–164; reproduced in Œuvres - Collected Papers, vol. IV, Springer-Verlag, Berlin–Heidelberg, 2001, pp. 41–54.

[Bo3] A. Borel, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Math., vol. 126, Springer-Verlag, New York, 1991.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV, V, VI, Hermann, Paris, 1968.

[BGGT] E. Breuillard, B. Green, R. Guralnick, T. Tao, *Strongly dense free subgroups of semisimple algebraic groups*, preprint `arXiv:1010.4259`.

[Br] G. Brown, *On commutators in a simple Lie algebra*, Proc. Amer. Math. Soc. **14** (1963) 763–767.

[Ca] R. W. Carter, *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters*, John Wiley & Sons, Chichester et al., 1985.

[DS] P. Deligne, D. Sullivan, *Division algebras and the Hausdorff–Banach–Tarski paradox*, Enseign. Math. **29** (1983) 145–150.

[Dj] D. Ž. Đoković, *On commutators in real semisimple Lie groups*, Osaka J. Math. **23** (1986) 223–228.

[EG] E. W. Ellers, N. Gordeev, *On the conjectures of J. Thompson and O. Ore*, Trans. Amer. Math. Soc. **350** (1998) 3657–3671.

[Fi] V. T. Filippov, *On a variety of Mal'tsev algebras*, Algebra i Logika **20** (1981) 300–314; English transl. in: Algebra and Logic **20** (1981) 200–210.

[GS] S. Garion, A. Shalev, *Commutator maps, measure preservation, and T-systems*, Trans. Amer. Math. Soc. **361** (2009) 4631–4651.

[Gorde] N. Gordeev, *Sums of orbits of algebraic groups. I*, J. Algebra **295** (2006) 62–80.

[GR] N. Gordeev, U. Rehmann, *On multicommutators for simple algebraic groups*, J. Algebra **245** (2001) 275–296.

[Gordo] S. R. Gordon, *Associators in simple Lie algebras*, Pacific J. Math. **51** (1974) 131–141.

[Got] M. Gotô, *A theorem on compact semi-simple groups*, J. Math. Soc. Japan **1** (1949) 270–272.

[Gow] R. Gow, *Commutators in finite simple groups of Lie type*, Bull. London Math. Soc. **32** (2000) 311–315.

[Hi] R. Hirschbühl, *Commutators in classical Lie algebras*, Linear Algebra Appl. **142** (1990) 91–111.

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 2nd printing, Springer-Verlag, New York–Berlin, 1978.

[KBMR] A. Kanel-Belov, S. Malev, L. Rowen, *The images of non-commutative polynomials evaluated on* 2×2 *matrices*, preprint `arXiv:1005:0191`, to appear in Proc. Amer. Math. Soc.

[LOST] M. W. Liebeck, E. A. O'Brien, A. Shalev, P. H. Tiep, *The Ore conjecture*, J. Europ. Math. Soc. **12** (2010) 939–1008.

[Mu] A. Muranov, *Finitely generated infinite simple groups of infinite commutator width*, Internat. J. Algebra Computation **17** (2007) 607–659.

[PW] S. Pasiencier, H.-C. Wang, *Commutators in a semi-simple Lie group*, Proc. Amer. Math. Soc. **13** (1962) 907–913.

[Pe] D. Perrin, *Géométrie algébrique. Une introduction*, 2ième éd., EDP Sciences, CNRS Éditions, Paris, 2001.

[Po] V. L. Popov, *Groups, generators, syzygies, and orbits in invariant theory*, Transl. Math. Monographs, vol. 100, Amer. Math. Soc., Providence, RI, 1992.

[Ra] Yu. P. Razmyslov, *The existence of a finite basis for the identities of the matrix algebra of order two over a field of characteristic zero*, Algebra i Logika **12** (1973) 83–113; English transl. in: Algebra and Logic **12** (1973) 47–63.

[Ree] R. Ree, *Commutators in semi-simple algebraic groups*, Proc. Amer. Math. Soc. **15** (1964) 457–460.

[RR] M. Rosset, S. Rosset, *Elements of trace zero that are not commutators*, Comm. Algebra **28** (2000) 3059–3072.

[Ro] L. H. Rowen, *Ring Theory*, vol. II, Pure Appl. Math. **128**, Academic Press, Inc., Boston, MA, 1988.

[Sh1] A. Shalev, *Commutators, words, conjugacy classes and character methods*, Turkish J. Math. **31** (2007) 131–148.

[Sh2] A. Shalev, *Word maps, conjugacy classes, and a non-commutative Waring-type theorem*, Ann. Math. **170** (2009) 1383–1416.

[Sl] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Lecture Notes Math., vol. 815, Springer-Verlag, Berlin–Heidelberg–New York, 1980.

[Sp] T. A. Springer, *Invariant Theory*, Lecture Notes Math., vol. 585, Springer-Verlag, Berlin-New York, 1977.

[SS] T. A. Springer, R. Steinberg, *Conjugacy classes*, in: *Seminar on Algebraic Groups and Related Finite Groups* (A. Borel *et al.*, eds.), Lecture Notes Math., vol. 131, Springer-Verlag, Berlin–Heidelberg–New York, 1970, pp. 67–266.

[St1] R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific J. Math. **11** (1961) 1119–1129.

[St2] R. Steinberg, *Lectures on Chevalley Groups*, Yale Univ., 1967.

[Th1] R. C. Thompson, *Commutators in the special and general linear groups*, Trans. Amer. Math. Soc. **101** (1961) 16–33.

[Th2] R. C. Thompson, *Commutators of matrices with coefficients from the field of two elements*, Duke Math. J. **29** (1962) 367–373.

[Th3] R. C. Thompson, *Matrices with zero trace*, Israel J. Math. **4** (1966) 33–42.

[Va] S. Yu. Vasilovskiĭ, *The basis of identities of a three-dimensional simple Lie algebra over an infinite field*, Algebra i Logika **28** (1989) 534–554; English transl. in: Algebra and Logic **28** (1989) 355-368.

Bandman: Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, ISRAEL
E-mail address: `bandman@macs.biu.ac.il`

Gordeev: Department of Mathematics, Herzen State Pedagogical University, 48 Moika Embankment, 191186, St.Petersburg, RUSSIA
E-mail address: `nickgordeev@mail.ru`

Kunyavskiĭ: Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, ISRAEL
E-mail address: `kunyav@macs.biu.ac.il`

Plotkin: Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, ISRAEL
E-mail address: plotkin@macs.biu.ac.il