ON TRIPLY EVEN BINARY CODES

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Abstract. A triply even code is a binary linear code in which the weight of every codeword is divisible by 8. We show how two doubly even codes of lengths m_1 and m_2 can be combined to make a triply even code of length $m_1 + m_2$, and then prove that every maximal triply even code of length 48 can be obtained by combining two doubly even codes of length 24 in a certain way. Using this result, we show that there are exactly 10 maximal triply even codes of length 48 up to equivalence.

1. Introduction

For the past few decades, extensive research of doubly even binary linear codes has been done. These codes turned out to be connected with objects in various areas, for example, sphere packing problem, combinatorial designs, finite groups, integral lattices, modular forms and so on [4, 17]. In this paper, we are concerned with a subclass of the class of doubly even codes, called triply even binary codes. A triply even code is a binary linear code in which every codeword has weight divisible by 8, in other words, a binary divisible code of level 3 in the sense of [12]. Dong, Griess and Höhn [5] pointed out that a certain triply even binary code of length 48 arose naturally from a Virasoro frame of the moonshine vertex operator algebra $V^\natural$. Subsequently, Miyamoto [15] found a construction method of $V^\natural$ from that code. Lam and Yamauchi [11] formulated this construction for the class of framed vertex operator algebras. To be precise, a framed vertex operator algebra of central charge n is constructed from a triply even code of length $2n$ whose dual is even. Unlike doubly even codes, the classification of all triply even codes of modest lengths has not been established yet.

The purpose of this paper is to develop a basic theory of maximal triply even codes, and to give a classification of maximal triply even codes of length 48. Since any triply even code of length up to 48 can

Date: December 18, 2010.
2000 *Mathematics Subject Classification.* 05E20, 05E30, 94B05.
Key words and phrases. binary code, divisible code, doubly even code, triangular graph, framed vertex operator algebra.

be regarded as a subcode of some maximal triply even codes of length 48, one can derive easily the classification of all triply even codes of lengths up to 48. It turns out that every maximal triply even code of length n with $n \equiv 0 \pmod 8$ and $n \leq 40$ is obtained as the generalized doubling $\tilde{\mathfrak{D}}(C)$ of a maximal doubly even code C (see Definition 7), and $n = 48$ is the smallest length with $n \equiv 0 \pmod 8$ for which there exists a triply even code not equivalent to $\tilde{\mathfrak{D}}(C)$ for any doubly even self-dual code C. The unique maximal triply even code $\hat{C}(T_{10})$ of length 48 not equivalent to generalized doublings is obtained by augmenting the code $C(T_{10})$ of length 45 generated by the adjacency matrix of the triangular graph T_{10}.

By Lam and Yamauchi [11], every triply even code of length a multiple of 16 containg the all-ones vector is the structure code of some framed vertex operator algebra. So it is natural to ask which framed vertex operator algebra of central charge 24 has $\hat{C}(T_{10})$ as its structure code. Since $\hat{C}(T_{10})^{\perp}$ has minimum weight 2, $\hat{C}(T_{10})$ cannot be any structure code of the moonshine vertex operator algebra by [7, Proposition 3.2]. Also this implies that every structure code of moonshine vertex operator algebra lies in the generalized doubling $\tilde{\mathfrak{D}}(C)$ of a doubly even self-dual code C of length 24. We note that Lam [10] recently constructed 10 vertex operator algebras which correspond to conformal field theories predicted to exist by Schellekens [18], using subcodes of $\hat{C}(T_{10})$.

This paper is organized as follows. In Section 2, properties and some construction methods of triply even codes are given. In Section 3, we prove that some maximal triply even codes can be constructed from doubly even self-dual codes by the doubling process. In Section 4, an infinite series of maximal triply even codes is constructed by triangular graphs and some properties of the codes in this class are given. In Section 5, a method for constructing a triply even code from a pair of doubly even codes is given. The main result in Section 5 states that every maximal triply even code is obtained from a pair of doubly even codes containing their radicals. In Section 6, an efficient method is described for determining whether a given doubly even code contains its radical. In Section 7, we show that the method described in Section 5 gives all maximal triply even codes of length 48 and, as a result, a classification of maximal triply even codes of length 48 is given. In Section 8, a classification of maximal triply even codes of lengths 8, 16, 24, 32 and 40 is given. Appendix gives a complete program in Magma [1] needed to produce the result. The result is also available electronically from [2].

2. Basic constructions for triply even codes

Throughout the paper, a code will mean a binary linear code, or equivalently, a linear subspace of the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$ of two elements. The support of a vector $u = (u_1, \dots, u_n) \in \mathbb{F}_2^n$ is the set $\mathrm{supp}(u) = \{i \mid u_i = 1\}$, and the weight of u is $\mathrm{wt}(u) = |\mathrm{supp}(u)|$. A *triply even* code is a code in which every codeword has weight divisible by 8. A *doubly even* code is a code in which every codeword has weight divisible by 4.

In this section, we give basic properties of triply even codes, and construction methods of triply even codes from doubly even codes. An $[n, k]$ code is a code $C \subset \mathbb{F}_2^n$ with $\dim C = k$, and n is called the length of C. For codes C and D of length n, C is *equivalent* to D if $C = D^\sigma$ for some coordinate permutation $\sigma \in S_n$. The *automorphism group* $\mathrm{Aut}(C)$ of C is defined as $\{\sigma \in S_n \mid C = C^\sigma\}$. The linear span of a subset $S \subset \mathbb{F}_2^n$ over $\mathbb{F}_2$ is denoted by $\langle S \rangle$. For $u, v \in \mathbb{F}_2^n$, we define $u * v$ to be the vector in $\mathbb{F}_2^n$ with $\mathrm{supp}(u * v) = \mathrm{supp}(u) \cap \mathrm{supp}(v)$. For $C, D \subset \mathbb{F}_2^n$, we define $C * D := \langle u * v \mid u \in C, v \in D \rangle$. For vectors $u \in \mathbb{F}_2^m$ and $v \in \mathbb{F}_2^n$, we denote by $(u \mid v) \in \mathbb{F}_2^{m+n}$ the vector obtained by concatenating u and v. For subsets $C \subset \mathbb{F}_2^{n_1}$, $D \subset \mathbb{F}_2^{n_2}$, we define the *direct sum* of C and D as

$$C \oplus D = \{(u \mid v) \in \mathbb{F}_2^{n_1+n_2} \mid u \in C,\ v \in D\}.$$

If C and C' (resp. D and D') are codes of length n_1 (resp. n_2) then $(C \oplus D) * (C' \oplus D') = (C * C') \oplus (D * D')$. A code C is said to be *decomposable* if it is a direct sum of two codes. We denote by $\mathbf{1}_n \in \mathbb{F}_2^n$ and $\mathbf{0}_n \in \mathbb{F}_2^n$, the all-ones vector, the zero vector, respectively. We will omit the subscript if there is no confusion.

For vectors $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{F}_2^n$, we denote by $u \cdot v$ the standard inner product $\sum_{i=1}^n u_i v_i$. The dual code of a code C is defined as $\{u \in \mathbb{F}_2^n \mid u \cdot v = 0 \text{ for any } v \in C\}$ and is denoted by $C^\perp$. A code C is *self-dual* (resp. *self-orthogonal*) if $C = C^\perp$ (resp. $C \subset C^\perp$). There exists a doubly even self-dual code of length n, if and only if n is divisible by 8. If C and D are codes, then $(C \oplus D)^\perp = C^\perp \oplus D^\perp$.

The following lemma is a special case of [20, Theorem 5.3] (see also [13, Proposition 2.1]).

Lemma 1. *Let $C = \langle S \rangle$ be a code generated by a set S. Then C is a triply even code if and only if the following conditions hold for any*

$u, v, w \in S$:

(1) $$\mathrm{wt}(u) \equiv 0 \pmod 8,$$

(2) $$\mathrm{wt}(u * v) \equiv 0 \pmod 4,$$

(3) $$\mathrm{wt}(u * v * w) \equiv 0 \pmod 2.$$

Definition 2. Let C be a doubly even code of length n. We define functions

$$Q : C \longrightarrow \mathbb{F}_2, \qquad u \mapsto \frac{\mathrm{wt}(u)}{4} \bmod 2,$$
$$B : C \times C^{\perp} \longrightarrow \mathbb{F}_2, \qquad (v, u) \mapsto \frac{\mathrm{wt}(v * u)}{2} \bmod 2,$$
$$T : \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2, \qquad (u, v, w) \mapsto \mathrm{wt}(u * v * w) \bmod 2.$$

Clearly, the following equalities hold:

(4) $$Q(x + y) = Q(x) + Q(y) + B(x, y) \quad (x, y \in C),$$

(5) $$B(x, y + z) = B(x, y) + B(x, z) + T(x, y, z) \quad (x \in C,\ y, z \in C^{\perp}),$$

(6) $$B(x + y, z) = B(x, z) + B(y, z) + T(x, y, z) \quad (x, y \in C,\ z \in C^{\perp}),$$

(7) $$T(x, y, z) = 0 \quad (x, y \in C, z \in (C * C)^{\perp}).$$

The *doubly even radical* $\mathrm{rad}\, C$, and the *triply even radical* $\mathrm{Rad}\, C$ are defined as

$$\mathrm{rad}\, C = \{y \in C^{\perp} \mid B(x, y) = 0\ (\forall x \in C)\},$$
$$\mathrm{Rad}\, C = \{x \in \mathrm{rad}\, C \mid Q(x) = 0\}.$$

Clearly

(8) $$\mathrm{rad}(C \oplus D) = \mathrm{rad}\, C \oplus \mathrm{rad}\, D,$$

(9) $$\mathrm{Rad}(C \oplus D) \supset \mathrm{Rad}\, C \oplus \mathrm{Rad}\, D$$

hold.

In general, the radicals $\mathrm{rad}\, C$, $\mathrm{Rad}\, C$ are not linear and not necessarily contained in C, even if C is triply even. An example is $C = \langle \mathbf{1}_8 \rangle \oplus \langle \mathbf{1}_8 \rangle$. However, the following holds.

Lemma 3. *Let C be a doubly even code. Then $\mathrm{rad}\, C \subset (C * C)^{\perp}$.*

Proof. We note that $(C * C)^{\perp} = \{z \in C^{\perp} \mid T(x, y, z) = 0$ for any $x, y \in C\}$. Suppose $x, y \in C$ and $z \in \mathrm{rad}\, C$. Since $x + y \in C$, we have $T(x, y, z) = 0$ by (6). Thus $z \in (C * C)^{\perp}$, and the result follows. □

Lemma 4. *Let C be a doubly even code, and suppose $x, y \in \mathrm{rad}\, C$.*

(i) *If* $y \in C$*, then* $x + y \in \operatorname{rad} C$*.*
(ii) *If* $x + y \in C$*, then* $x + y \in \operatorname{rad} C$*.*

Proof. Observe that, by Lemma 3, $x \in (C * C)^{\perp}$ holds. For any $z \in C$, we have $B(z, x + y) = T(x, y, z)$ by (5). If $y \in C$, then $T(x, y, z) = 0$. Thus (i) holds. If $x + y \in C$, then $T(x, y, z) = T(x, x + y, z) + T(x, x, z) = 0$. Thus (ii) holds. □

Lemma 5. *Let* C *be a doubly even code, and suppose* $x \in \operatorname{rad} C$ *and* $z \in \operatorname{Rad} C$*. Then*

$$x + C \cap \operatorname{rad} C = (x + C \cap (C * C)^{\perp}) \cap \operatorname{rad} C, \tag{10}$$

$$z + C \cap \operatorname{Rad} C = (z + C \cap (C * C)^{\perp}) \cap \operatorname{Rad} C. \tag{11}$$

Proof. The containment $x + C \cap \operatorname{rad} C \subset (x + C \cap (C * C)^{\perp}) \cap \operatorname{rad} C$ follows from Lemma 3 and Lemma 4(i). As for the reverse containment, suppose $y \in C \cap (C * C)^{\perp}$ and $x + y \in \operatorname{rad} C$. Since $x \in \operatorname{rad} C$, we have $y \in \operatorname{rad} C$ by Lemma 4(ii). Thus $x + y \in x + C \cap \operatorname{rad} C$ and (10) holds.

From (10),

$$(z + C \cap (C * C)^{\perp}) \cap \operatorname{Rad} C = (z + C \cap \operatorname{rad} C) \cap \operatorname{Rad} C.$$

Suppose $y \in C \cap \operatorname{rad} C$. Since $\operatorname{wt}(z) \equiv 0 \pmod 8$, $z + y \in \operatorname{Rad} C$ if and only if $\operatorname{wt}(y) \equiv 0 \pmod 8$. Therefore

$$(z + C \cap (C * C)^{\perp}) \cap \operatorname{Rad} C = z + C \cap \operatorname{Rad} C.$$

Thus (ii) holds. □

Lemma 6. *Let* C *be a doubly even code and* $D = (C * C)^{\perp} \cap C$*. Then the restriction* $B|_{C \times D}$ *of* B *to* $C \times D$ *is a bilinear pairing and* $Q|_D$ *is a quadratic form with associated bilinear form* $B|_{D \times D}$*. Moreover,* $C \cap \operatorname{rad} C$ *and* $C \cap \operatorname{Rad} C$ *are linear subcodes of* C*. In particular, if* $\operatorname{rad} C \subset C$ *(resp.* $\operatorname{Rad} C \subset C$*), then* $\operatorname{rad} C$ *(resp.* $\operatorname{Rad} C$*) is linear.*

Proof. First, note that since $C \subset C * C$, we have $D \subset (C * C)^{\perp} \subset C^{\perp}$. For any $x, y \in C$ and $z \in D$, we have $T(x, y, z) = 0$ by (7), hence $B(x+y, z) = B(x, z) + B(y, z)$ by (6). Also, for any $x \in C$ and $y, z \in D$, we have $T(x, y, z) = 0$ by (7), hence $B(x, y + z) = B(x, y) + B(x, z)$ by (5). Therefore, B is a bilinear pairing on $C \times D$, and $Q|_D$ is a quadratic form with associated bilinear form $B|_{D \times D}$ by (4).

By Lemma 3, $C \cap \operatorname{rad} C = \{y \in D \mid B(x, y) = 0 \text{ for any } x \in C\}$. Since $B|_{C \times D}$ is linear in the second variable, $C \cap \operatorname{rad} C$ is a linear subcode of C.

Also, by (4), Q is linear on $C \cap \operatorname{rad} C$. Then, $C \cap \operatorname{Rad} C = \{x \in C \cap \operatorname{rad} C \mid Q(x) = 0\}$ is a linear subcode of C. □

Definition 7. Let C be a code of length n and set $R = C \cap \operatorname{Rad} C$. We define the extended doubling $\mathcal{D}(C)$ and the generalized doubling $\tilde{\mathcal{D}}(C)$ as

$$\mathcal{D}(C) = \langle (\mathbf{1}_n|\mathbf{0}_n), (\mathbf{0}_n|\mathbf{1}_n), \{(x|x) \mid x \in C\}\rangle, \tag{12}$$

$$\tilde{\mathcal{D}}(C) = \langle R \oplus \mathbf{0}_n, \{(x|x) \mid x \in C\}\rangle. \tag{13}$$

We note that if C is a doubly even code, then $\tilde{\mathcal{D}}(C)$ is a triply even code and

$$\dim \tilde{\mathcal{D}}(C) = \dim C + \dim(C \cap \operatorname{Rad} C). \tag{14}$$

Note also that if C is a doubly even $[n, d]$ code and $n \equiv 0 \pmod 8$, then $\mathcal{D}(C)$ is a triply even code of length $2n$, dimension $d+1$ or $d+2$, depending on $\mathbf{1} \in C$ or not. In particular, if C is a doubly even self-dual code of length n, then $\mathcal{D}(C)$ is a triply even $[2n, n+1]$ code. This is a particularly important construction in connection with framed vertex operator algebras and lattices (see [7]). In the next section, we give a sufficient condition for C under which $\tilde{\mathcal{D}}(C)$ is a maximal triply even code.

3. Maximality of triply even codes

In this section, we discuss *maximal* triply even codes, that is, triply even codes not contained in any larger triply even code.

Lemma 8. *If C is a triply even code, then $C \subset \operatorname{Rad} C$. Moreover, equality holds if and only if C is a maximal triply even code.*

Proof. The first part is immediate from Lemma 1. For a vector x, Lemma 1 implies that $\langle C, x\rangle$ is a triply even code if and only if $x \in (C * C)^{\perp} \cap \operatorname{Rad} C = \operatorname{Rad} C$ by Lemma 3. Thus the result follows. □

Lemma 9. *Let $C = \bigoplus_{i=1}^{k} C_i$ be a maximal doubly even code where C_i is an indecomposable component of length n_i for $i = 1, \dots, k$. Then*

$$\operatorname{rad} C = \bigoplus_{i=1}^{k} \langle \mathbf{s}_i \rangle, \tag{15}$$

$$\operatorname{Rad} C = \bigoplus_{i=1}^{k} \langle \mathbf{t}_i \rangle. \tag{16}$$

where

$$\mathbf{s}_i = \begin{cases} \mathbf{1}_{n_i} & n_i \equiv 0 \pmod 4 \\ \mathbf{0}_{n_i} & n_i \not\equiv 0 \pmod 4, \end{cases}$$

$$\mathbf{t}_i = \begin{cases} \mathbf{1}_{n_i} & n_i \equiv 0 \pmod 8 \\ \mathbf{0}_{n_i} & n_i \not\equiv 0 \pmod 8. \end{cases}$$

In particular, if C is a doubly even self-dual code, then

$$\operatorname{rad} C = \operatorname{Rad} C = \bigoplus_{i=1}^{k} \langle \mathbf{1}_{n_i} \rangle, \tag{17}$$

$$\tilde{\mathcal{D}}(C) \cong \bigoplus_{i=1}^{k} \tilde{\mathcal{D}}(C_i) = \bigoplus_{i=1}^{k} \mathcal{D}(C_i). \tag{18}$$

Proof. By (8), it suffices to prove (15) when C is indecomposable. Suppose $v \in \operatorname{rad} C$ and $x \in C$. Then

$$\operatorname{wt}(v * x) \equiv 0 \pmod 4 \tag{19}$$

By Lemma 3, $v \in (C*C)^\perp$. Then for any $y \in C$, $0 = v\cdot(x*y) = (v*x)\cdot y$. Hence

$$v * x \in C^\perp. \tag{20}$$

By (19) and (20), $\langle C, v * x \rangle$ is a doubly even code. By maximality, $v * x \in C$. Also since $x \in C$ was arbitrary, C is the direct sum of codes supported by $\operatorname{supp}(v)$ and its complement. Since C is indecomposable, we obtain $v \in \langle \mathbf{1} \rangle$. Hence $\operatorname{rad} C \subset \langle \mathbf{1} \rangle$. Therefore (15) holds.

We claim that there is at most one i such that $n_i \equiv 4 \pmod 8$. If there are distinct i, j such that $i \equiv j \equiv 4 \pmod 8$, then $C_i \oplus C_j$ is not a maximal doubly even code. This contradicts maximality of C. Therefore (16) follows from (15).

If C is a doubly even self-dual code, then each C_i is a doubly even self-dual code, hence n_i is divisible by 8. Now, (17) follows from (15) and (16).

By (17), we have

$$\begin{aligned} C \cap \operatorname{Rad} C &= \bigoplus_{i=1}^{k} \langle \mathbf{1}_{n_i} \rangle \\ &= \bigoplus_{i=1}^{k} C_i \cap \operatorname{Rad} C_i \end{aligned}$$

and hence $\tilde{\mathcal{D}}(\bigoplus_{i=1}^{k} C_i) = \bigoplus_{i=1}^{k} \tilde{\mathcal{D}}(C_i)$. Since C_i is indecomposable, (17) implies $\tilde{\mathcal{D}}(C_i) = \mathcal{D}(C_i)$. This proves (18). □

Proposition 10. *For any doubly even self-dual code C, $(\tilde{\mathcal{D}}(C)*\tilde{\mathcal{D}}(C))^{\perp} = \tilde{\mathcal{D}}(C)$. In particular $\tilde{\mathcal{D}}(C)$ a maximal triply even code.*

Proof. Suppose that C is an indecomposable doubly even self-dual code of length $2n$. Then (18) implies $\tilde{\mathcal{D}}(C) * \tilde{\mathcal{D}}(C) = \mathcal{D}(C) * \mathcal{D}(C) = C \oplus C + \mathcal{D}(C * C)$, hence $\dim(\tilde{\mathcal{D}}(C) * \tilde{\mathcal{D}}(C)) = 3n - 1 = 4n - \dim \tilde{\mathcal{D}}(C)$. This implies that $(\tilde{\mathcal{D}}(C) * \tilde{\mathcal{D}}(C))^{\perp} = \tilde{\mathcal{D}}(C)$. By (18), the identity holds also for decomposable double even self-dual codes C. Now $\operatorname{rad} \tilde{\mathcal{D}}(C) \subset \tilde{\mathcal{D}}(C)$ by Lemma 3, and hence $\tilde{\mathcal{D}}(C)$ a maximal triply even code by Lemma 8. □

Example 11. It is known that the $[8, 4, 4]$ Hamming code $e_8 = \mathcal{D}(\langle \mathbf{1}_4 \rangle^{\perp})$ is the unique doubly even self-dual codes of length 8, up to equivalence. Also, $d_{16}^{+} = \mathcal{D}(\langle \mathbf{1}_8 \rangle^{\perp})$ and $e_8 \oplus e_8$ are the only doubly even self-dual codes of length 16, up to equivalence. By Proposition 10, $\tilde{\mathcal{D}}(e_8)$, $\tilde{\mathcal{D}}(d_{16}^{+})$, $\tilde{\mathcal{D}}(e_8 \oplus e_8)$ are maximal triply even code of dimension 5, 9 and 10 respectively. In particular $\tilde{\mathcal{D}}(e_8) = \mathcal{D}(e_8)$ is the Reed–Muller code $\mathrm{RM}(1, 4)$ and $\tilde{\mathcal{D}}(e_8 \oplus e_8) = \mathrm{RM}(1, 4)^{\oplus 2}$.

Example 12. It is known [16] that there are precisely 9 doubly even self-dual codes of length 24. Two of these 9 codes are decomposable, and they are $d_{16}^{+} \oplus e_8$ and $e_8^{\oplus 3}$. The remaining 7 codes are indecomposable and they are denoted by $g_{24}, d_{24}^{+}, d_{12}^{2+}, (d_{10}e_7^2)^{+}, d_8^{3+}, d_6^{4+}, d_4^{6+}$. By Proposition 10, $\tilde{\mathcal{D}}(C)$ is a maximal triply even code for any of the 9 doubly even self-dual codes C. We note from (17) that $\operatorname{Rad} C \subset C$ and $\dim \operatorname{Rad} C$ is the number of indecomposable components. Thus, for indecomposable doubly even self-dual codes C of length 24, $\dim \tilde{\mathcal{D}}(C) = 13$ holds. Also, $\tilde{\mathcal{D}}(d_{16}^{+} \oplus e_8) = \mathcal{D}(\mathcal{D}(\langle \mathbf{1}_8 \rangle^{\perp})) \oplus \mathrm{RM}(1, 4)$ has dimension 14, while $\tilde{\mathcal{D}}(e_8^{\oplus 3}) = \mathrm{RM}(1, 4)^{\oplus 3}$ has dimension 15.

Remark 13. As shown in Example 12, the dimension of maximal triply even codes varies even if the length is fixed. The largest possible dimension of triply even codes, however, has been determined in [21], and the codes achieving the largest dimension have been determined in [13].

4. Triply even codes constructed from triangular graphs

Let n be a positive integer with $n \geq 4$, and let Ω be a set of n elements. We denote by $\binom{\Omega}{2}$ the set of two-element subsets of Ω. The triangular graph T_n has the set of vertices $\binom{\Omega}{2}$, and two vertices α, β

are adjacent whenever $|\alpha \cap \beta| = 1$. It is known [8] that the graph T_n is a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = \left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right).$$

Let A_n denote the adjacency matrix of T_n. Then every row of A_n has weight $2(n-2)$, and for any two distinct rows of A_n, the size of the intersection of their supports is either $n-2$ or 4. Let $C(T_n)$ be the binary code with generator matrix A_n.

It is clear that the code $C(T_n)$ is triply even only if $n \equiv 2 \pmod 4$. The converse also holds by the following lemma.

Lemma 14 (Haemers, Peeters and van Rijckevorsel [6, Subsection 4.1])**.** *If* $n \equiv 2 \pmod 4$*, the weight enumerator of* $C(T_n)$ *is*

$$\mathrm{we}_{C(T_n)}(x) = \sum_{l=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{2l} x^{2l(n-2l)}.$$

In particular, $C(T_n)$ *is a triply even of dimension* $n-2$*.*

Let $\alpha_i = \{i, n\} \in \binom{\Omega}{2}$, and we denote by r_i the row of A_n indexed by α_i i.e., $\{k, l\} \in \mathrm{supp}(r_i)$ if and only if $|\alpha_i \cap \{k, l\}| = 1$. Then the following lemma holds.

Lemma 15 (Key, Moori and Rodrigues [9, Lemma 3.5])**.** *If* n *is even, then* $\{r_i \mid i = 1, 2, \dots, n-2\}$ *is a basis of* $C(T_n)$*.*

We note that the dimension of $C(T_n)$ has already been determined by Tonchev [19, Lemma 3.6.6] and Brouwer and Van Eijl [3]. An explicit basis of $C(T_n)$ is needed in the sequel to establish maximality of $C(T_n)$. The weight enumerator given in Lemma 14 can also be derived from the basis.

Lemma 16. *If* n *is even, then* $\{r_i * r_j \mid 1 \le i \le j \le n-2\}$ *is a basis of* $C(T_n) * C(T_n)$*. In particular,*

$$\dim(C(T_n) * C(T_n)) = \frac{(n-1)(n-2)}{2}.$$

Proof. Observe that, for $1 \le i < j < n$, we have

$$\mathrm{supp}(r_i * r_j) = \{\{i, j\}\} \cup \{\alpha_k \mid 1 \le k < n,\ k \ne i, j\}. \tag{21}$$

Suppose

$$\sum_{i=1}^{n-2} c_i r_i + \sum_{1 \le i < j \le n-2} c_{i,j} r_i * r_j = 0,$$

where $c_i, c_{i,j} \in \mathbb{F}_2$. Then $c_i = 0$ for $i = 1, \dots, n-2$, because $|\alpha_i \cap \{j, n-1\}| = 1$ if and only if $i = j$. Thus

$$\sum_{1 \le i < j \le n-2} c_{i,j} r_i * r_j = 0.$$

For $i, j, k, l \in \{1, \dots, n-2\}$ with $i \neq j$, $k \neq l$, (21) implies $\{k, l\} \in \operatorname{supp}(r_i * r_j)$ if and only if $\{k, l\} = \{i, j\}$. This implies $c_{i,j} = 0$. □

Lemma 17. *If $n \equiv 2 \pmod 4$, then $(C(T_n) * C(T_n))^\perp = C(T_n) + \langle \mathbf{1} \rangle$. In particular, $C(T_n)$ is a maximal triply even code.*

Proof. By $(C(T_n) * C(T_n))^\perp \supset C(T_n) + \langle \mathbf{1} \rangle$ and comparing the dimensions using Lemmas 15 and 16, we obtain $(C(T_n) * C(T_n))^\perp = C(T_n) + \langle \mathbf{1} \rangle$. Since $\operatorname{wt}(\mathbf{1}) = \frac{n(n-1)}{2} \equiv 1 \pmod 2$, Lemma 3 implies $\operatorname{Rad} C(T_n) \subset C(T_n)$. Thus $C(T_n)$ is a maximal triply even code by Lemma 8. □

We define $\hat{C}(T_n)$ to be the code of length $l = 8\lceil \frac{1}{8} \frac{n(n-1)}{2} \rceil$ constructed from $C(T_n)$ together with the all-ones vector of length l, i.e., $\hat{C}(T_n) = \langle \mathbf{1}_l \rangle + C(T_n) \oplus \mathbf{0}_{l'}$ where $l' = l - \frac{n(n-1)}{2}$.

Theorem 18. *If $n \equiv 2 \pmod 4$, then $\hat{C}(T_n)$ is a maximal triply even code.*

Proof. Let $l = 8\lceil \frac{1}{8} \frac{n(n-1)}{2} \rceil$. Then

$$\begin{aligned}
(\hat{C}(T_n) * \hat{C}(T_n))^\perp &= (\langle \mathbf{1}_l \rangle + (C(T_n) * C(T_n)) \oplus \mathbf{0})^\perp \\
&= \langle \mathbf{1}_l \rangle^\perp \cap ((C(T_n) * C(T_n))^\perp \oplus \mathbb{F}_2^{l'}) \\
&= \langle \mathbf{1}_l \rangle^\perp \cap ((C(T_n) + \langle \mathbf{1} \rangle) \oplus \mathbb{F}_2^{l'}) \qquad \text{(by Lemma 17)} \\
&= C(T_n) \oplus \langle \mathbf{1}_{l'} \rangle^\perp + \langle \mathbf{1}_l \rangle \\
&= \hat{C}(T_n) + \mathbf{0} \oplus \langle \mathbf{1}_{l'} \rangle^\perp.
\end{aligned}$$

Since $l' < 8$, Lemma 3 implies $\operatorname{Rad} C(T_n) \subset C(T_n)$. The result follows from Lemma 8. □

5. Triply even codes constructed from pairs of doubly even codes with isometries

In Section 2, we gave construction methods for a triply even code from a doubly even code. In this section, we give a generalization of these construction methods for a pair of doubly even codes.

For a set of coordinates $\{i_1, i_2, \dots, i_t\} \subset \{1, 2, \dots, n\}$, let $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^t$, $\pi' : \mathbb{F}_2^n \to \mathbb{F}_2^{n-t}$ be the projection to the set of coordinates $\{i_1, i_2, \dots, i_t\}$, $\{j_1, \dots, j_{n-t}\}$, respectively, where $\{j_1, \dots, j_{n-t}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_t\}$.

For a code C of length n, the *punctured code* and the *shortened code* of C on a set of coordinates $\{i_1, i_2, \ldots, i_t\}$ are the codes $\pi'(C)$, $\{\pi'(c) \mid c \in C,\ \pi(c) = \mathbf{0}\}$, respectively.

Let C_1 and C_2 be doubly even codes and R_i be a subcode of $C_i \cap \operatorname{Rad} C_i$ for $i = 1, 2$. A bijective linear map

$$f : C_1/R_1 \to C_2/R_2 \tag{22}$$

is called an *isometry* if $\mathrm{wt}(x_1) \equiv \mathrm{wt}(x_2) \pmod 8$ for any $x_1 + R_1 \in C_1/R_1$ and $x_2 + R_2 \in f(x_1 + R_1)$. We note that if $x + R_1 = y + R_1$ with $x, y \in C_1$, then $\mathrm{wt}(x) \equiv \mathrm{wt}(y) \pmod 8$. The set of isometries (22) is denoted by $\Phi(C_1/R_1, C_2/R_2)$.

For an isometry $f \in \Phi(C_1/R_1, C_2/R_2)$, we define a code

$$D(C_1, C_2, R_1, R_2, f) = \{(x_1|x_2) \mid x_1 \in C_1,\ x_2 \in f(x_1 + R_1)\}. \tag{23}$$

Since f is a bijective linear map,

$$D(C_1, C_2, R_1, R_2, f) = \{(x_1|x_2) \mid x_2 \in C_2,\ x_1 \in f^{-1}(x_2 + R_2)\}. \tag{24}$$

Proposition 19. *Let C_i be a doubly even code of length m_i for $i = 1, 2$ and R_i be a subcode of $C_i \cap \operatorname{Rad} C_i$. If $f \in \Phi(C_1/R_1, C_2/R_2)$, then the code $D(C_1, C_2, R_1, R_2, f)$ is a triply even code of length $m_1 + m_2$ of dimension $\dim C_1 + \dim R_2 = \dim R_1 + \dim C_2$.*

Proof. Fix C_1, C_2, R_1, R_2 and f. We abbreviate $D(C_1, C_2, R_1, R_2, f)$ as D. Since f is a linear map, D is linear. Since C_1 and C_2 are doubly even codes and f is an isometry, all the weights of elements of D are multiple of 8, that is, D is a triply even code. Moreover

$$|D(C_1, C_2, R_1, R_2, f)| = |C_1| \times |f(R_1)| = |C_1| \times |R_2|.$$

Therefore $\dim D = \dim C_1 + \dim R_2 = \dim R_1 + \dim C_2$. □

Remark that the construction method in Proposition 19 contains the constructions $\mathcal{D}(C)$ in (12) and $\tilde{\mathcal{D}}(C)$ in (13) as special cases. Indeed, let C be a doubly even code of length n. Then we have

$$\tilde{\mathcal{D}}(C) = D(C, C, C \cap \operatorname{Rad} C, C \cap \operatorname{Rad} C, \mathrm{id}) \tag{25}$$

and if, moreover, $n \equiv 0 \pmod 8$, then

$$\mathcal{D}(C) = D(C + \langle \mathbf{1} \rangle, C + \langle \mathbf{1} \rangle, \langle \mathbf{1} \rangle, \langle \mathbf{1} \rangle, \mathrm{id}).$$

Note that, given doubly even codes C_1, C_2 and subcodes $R_1 \subset C_1 \cap \operatorname{Rad} C_1$, $R_2 \subset C_2 \cap \operatorname{Rad} C_2$, the set $\Phi(C_1/R_1, C_2/R_2)$ may be empty, and in this case Proposition 19 produces no triply even codes. We shall give a necessary and sufficient condition for the set $\Phi(C_1/R_1, C_2/R_2)$ to be non-empty in Proposition 21 below. First we need to introduce some terminology.

Definition 20. Let C be a doubly even code, and let R be a subcode of $C \cap \operatorname{Rad} C$. Let

$$X = \{x + R \in C/R \mid \operatorname{wt}(x) \equiv 0 \pmod 8\}.$$

We call the elements of the set X *singular points* of C/R. Then the group $\mathcal{G}_1(C, R)$ forms the setwise stabilizer of X in $\operatorname{GL}(C/R)$. The *triply even check code* $\mathcal{C}(C, R)$ of (C, R) is defined as

$$\mathcal{C}(C, R) = \{c = (c_x \in \mathbb{F}_2 \mid x \in X) \in \mathbb{F}_2^X \mid \sum_{x \in X} c_x x \in R\}.$$

By the definition, $\mathcal{G}_1(C, R)$ acts on $\mathcal{C}(C, R)$ as automorphisms, but the action is not necessarily faithful. Indeed, X may not span C/R.

Proposition 21. *Let C_i be a doubly even code for $i = 1, 2$ and R_i be a subcode of $C_i \cap \operatorname{Rad} C_i$. Suppose that $\dim C_1/R_1 = \dim C_2/R_2$. Then $\mathcal{C}(C_1, R_1) \cong \mathcal{C}(C_2, R_2)$ if and only if $\Phi(C_1/R_1, C_2/R_2) \neq \emptyset$.*

Proof. If $\mathcal{C}(C_1, R_1) \cong \mathcal{C}(C_2, R_2)$, then there exists a bijection f from the set X_1 of singular points of C_1/R_1 to the set X_2 of those of C_2/R_2 which induces an equivalence from $\mathcal{C}(C_1, R_1)$ to $\mathcal{C}(C_2, R_2)$. It follows from the definition of the triply even check code that the bijection f extends to a linear mapping $\langle X_1 \rangle \to \langle X_2 \rangle$. Extending further to C_1/R_1 in an arbitrary manner, we obtain an isometry from C_1/R_1 to C_2/R_2. The proof of the converse is immediate. □

The next proposition shows that every triply even code can be constructed by means of the construction described in Proposition 19.

Proposition 22. *Let D be a triply even code of length n. Fix a codeword $x \in D$ of weight m_1 with $0 < m_1 < n$. Let $S_1 = \operatorname{supp}(\mathbf{1} + x)$ and $S_2 = \operatorname{supp}(x)$ and let C_i and R_i be the punctured code and the shortened code of D on S_i, respectively, for $i = 1, 2$. Then C_i is doubly even, $R_i \subset C_i \cap \operatorname{Rad} C_i$ for $i = 1, 2$, and*

$$D \cong D(C_1, C_2, R_1, R_2, f)$$

for some $f \in \Phi(C_1/R_1, C_2/R_2)$.

Moreover, If D is a maximal, then $\operatorname{Rad} C_i = R_i$ for $i = 1, 2$.

Proof. All the statement except on the last one follows easily from Lemma 1. Let $\pi_1 : \mathbb{F}_2^n \to \mathbb{F}_2^{m_1}$, $\pi_2 : \mathbb{F}_2^n \to \mathbb{F}_2^{n-m_1}$ be the projection to the set of coordinates $\operatorname{supp}(x)$, $\operatorname{supp}(\mathbf{1} + x)$, respectively. Define $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^n$ by $\pi(x) = (\pi_1(x) | \pi_2(x))$ $(x \in \mathbb{F}_2^n)$. Then $C_i = \pi_i(D)$ $(i = 1, 2)$ and D is equivalent to $\pi(D)$. It is clear that the mapping

$$\begin{aligned} f : C_1/R_1 &\longrightarrow C_2/R_2 \\ c_1 + R_1 &\mapsto \{x \in C_2 \mid (c_1|x) \in \pi(D)\} \end{aligned}$$

is a well-defined isometry and $\pi(D) = D(C_1, C_2, R_1, R_2, f)$.

If D is a maximal triply even code, then so is $\pi(D)$. This implies $(r_1|\mathbf{0})$, $(\mathbf{0}|r_2) \in \operatorname{Rad}\pi(D)$ for $r_1 \in \operatorname{Rad} C_1$ and $r_2 \in \operatorname{Rad} C_2$. By Lemma 8, $\operatorname{Rad}\pi(D) \subset \pi(D)$. Therefore $R_i \subset \operatorname{Rad} C_i$ for $i = 1, 2$. Hence the result follows. □

Proposition 22 indicates that every triply even code of length n containing a codeword of weight m_1 can be constructed from a pair of doubly even codes of lengths m_1 and $n - m_1$. We will classify maximal triply even codes of length 48 by setting $m_1 = 24$ in Section 7.

For fixed codes C_1, C_2 and $R_1 \subset C_1 \cap \operatorname{Rad} C_1$, $R_2 \subset C_2 \cap \operatorname{Rad} C_2$ the resulting code

$$D(C_1, C_2, R_1, R_2, f)$$

depends on the choice of the isometry f. However, some of these codes are equivalent to each other. The first algorithm is to check this, that is, we will give a sufficient condition for two resulting codes to be equivalent. We need this algorithm to reduce the amount of calculation to be reasonable.

First, we define some groups. For a code C and a subcode $R \subset C \cap \operatorname{Rad} C$, we denote by $\mathcal{G}_0(C, R)$ the subgroup of $\mathrm{GL}(C/R)$ induced by the action of $\operatorname{Aut}(C) \cap \operatorname{Aut}(R)$ on C/R and denote by $\mathcal{G}_1(C, R)$ the subgroup $\Phi(C/R, C/R)$ of $\mathrm{GL}(C/R)$. By the definition, the group $\mathcal{G}_0(C, R)$ is a subgroup of $\mathcal{G}_1(C, R)$. If $R = C \cap \operatorname{Rad} C$, then we abbreviate $\mathcal{G}_0(C, R)$, $\mathcal{G}_1(C, R)$ as $\mathcal{G}_0(C)$, $\mathcal{G}_1(C)$, respectively. If $f \in \Phi(C_1/R_1, C_2/R_2)$, then

$$\mathcal{G}_1(C_1, R_1) = f^{-1} \circ \mathcal{G}_1(C_2, R_2) \circ f \tag{26}$$

and

$$\Phi(C_1/R_1, C_2/R_2) = f \circ \mathcal{G}_1(C_1, R_1) = \mathcal{G}_1(C_2, R_2) \circ f. \tag{27}$$

If we replace f by $\sigma_2 \circ f \circ \sigma_1$, where $\sigma_i \in \mathcal{G}_0(C_i, R_i)$, then the resulting codes are equivalent, that is,

$$D(C_1, C_2, R_1, R_2, f) \cong D(C_1, C_2, R_1, R_2, \sigma_2 \circ f \circ \sigma_1).$$

This means that, in order to enumerate

$$\{D(C_1, C_2, R_1, R_2, h) \mid h \in \Phi(C_1/R_1, C_2/R_2)\}$$

up to equivalence, we first fix $f \in \Phi(C_1/R_1, C_2/R_2)$, and it suffices to enumerate the codes $D(C_1, C_2, R_1, R_2, f \circ g)$ where g runs through a set of representatives for the double cosets

$$(f^{-1} \circ \mathcal{G}_0(C_2, R_2) \circ f)\backslash \mathcal{G}_1(C_1, R_1)/\mathcal{G}_0(C_1, R_1).$$

6. Doubly even codes containing their radicals

In view of Propsition 22, it will be necessary to extract only those doubly even codes C which satisfy $\mathrm{Rad}\, C \subset C$, in order to enumerate maximal triply even codes. In this section, we will give a criteria to verify whether a doubly even code C contains its triply even radical i.e., $\mathrm{Rad}\, C \subset C$.

Throughout this section, let C be a doubly even code containg $\mathbf{1}$, and we denote $(C * C)^\perp \cap C$ by D. For $x \in C^\perp$, one can define a mapping $B_x : C \to \mathbb{F}_2$ by $B_x(c) = B(c, x)$ $(c \in C)$. By (6), B_x is linear when $x \in (C * C)^\perp$. Thus we obtain a map

$$\begin{aligned} \phi : (C * C)^\perp &\to \mathrm{Hom}(C, \mathbb{F}_2) \\ x &\mapsto B_x. \end{aligned}$$

By Lemma 3, we can write

$$\phi^{-1}(0) = \mathrm{rad}\, C. \tag{28}$$

We remark that the map ϕ is not linear in general. More precisely, if we define a bilinear map δ as

$$\begin{aligned} \delta : \mathbb{F}_2^n \times \mathbb{F}_2^n &\to \mathrm{Hom}(C, \mathbb{F}_2) \\ (x, y) &\mapsto (v \mapsto T(x, y, v)), \end{aligned}$$

then for $x, y \in (C * C)^\perp$,

$$\phi(x + y) = \phi(x) + \phi(y) + \delta(x, y)$$

holds by (5). In particular, (7) implies

$$\phi(x + y) = \phi(x) + \phi(y) \quad (x \in (C * C)^\perp,\ y \in D), \tag{29}$$

and ϕ is linear on D.

The function Q from Definition 2 can also be defined on $\mathrm{rad}\, C$, so we denote it by the same Q as follows.

$$Q : \mathrm{rad}\, C \to \mathbb{F}_2, \quad u \mapsto \frac{\mathrm{wt}(u)}{4} \bmod 2.$$

Then $\mathrm{Rad}\, C = Q^{-1}(0)$, and

$$Q(x + y) = Q(x) + Q(y) \quad (x \in C \cap \mathrm{rad}\, C,\ y \in \mathrm{rad}\, C). \tag{30}$$

Lemma 23. *For a coset $M \in (C*C)^\perp/D$, the following are equivalent.*

(i) $\phi(M) \cap \phi(D) \neq \emptyset$,
(ii) $\phi(M) = \phi(D)$,
(iii) $M \cap \mathrm{rad}\, C \neq \emptyset$.

Moreover, if $C \cap \mathrm{rad}\, C \neq C \cap \mathrm{Rad}\, C$, then each of (i)–(iii) *is equivalent to*

(iv) $M \cap \operatorname{Rad} C \neq \emptyset$.

Proof. Equivalence of (i)–(iii) follows immediately from (28) and (29). Suppose $C \cap \operatorname{rad} C \neq C \cap \operatorname{Rad} C$. It suffices to show that (iii) implies (iv).

Suppose $x \in M \cap \operatorname{rad} C$. If $Q(x) = 0$, then clearly (iv) holds, so suppose $Q(x) = 1$. By assumption, there exists $y \in C \cap \operatorname{rad} C$ such that $Q(y) = 1$. Then $x + y \in M$, $\phi(x + y) = \phi(x) + \phi(y) = 0$ by (29), hence $x + y \in \operatorname{rad} C$. Moreover, $Q(x + y) = Q(x) + Q(y) = 0$ by (30). Thus $x + y \in \operatorname{Rad} C$, and hence (iv) holds. □

Proposition 24. *Let C be a doubly even code of length a multiple of eight, containing* $\mathbf{1}$. *Suppose $C \cap \operatorname{rad} C \neq C \cap \operatorname{Rad} C$. Then* $\operatorname{Rad} C \not\subset C$ *if and only if there exists a coset $M \in (C * C)^{\perp}/D$ satisfying $\phi(M) \cap \phi(D) \neq \emptyset$ and $M \neq D$.*

Proof. Since $\operatorname{Rad} C \subset \operatorname{rad} C \subset (C * C)^{\perp}$ by Lemma 3, $\operatorname{Rad} C \not\subset C$ if and only if $M \cap \operatorname{Rad} C \neq \emptyset$ for some coset $M \in (C * C)^{\perp}/D$ different from D. The resut then follows from Lemma 23. □

In view of equivalence of (i) and (ii) in Lemma 23, one can check the condition $\phi(M) \cap \phi(D) \neq \emptyset$ by testing whether an arbitrarily chosen element $x \in M$ satisfies $\phi(x) \in \phi(D)$. Thus, the above proposition gives a convenient criterion for $\operatorname{Rad} C \subset C$ in terms of coset representatives for $(C * C)^{\perp}/D$, provided $C \cap \operatorname{rad} C \neq C \cap \operatorname{Rad} C$. In the case where $C \cap \operatorname{rad} C = C \cap \operatorname{Rad} C$, the situation is slightly more complicated.

Lemma 25. *Suppose $C \cap \operatorname{rad} C = C \cap \operatorname{Rad} C$, and $M \in (C * C)^{\perp}/D$. If $M \cap \operatorname{Rad} C \neq \emptyset$, then*

$$M \cap \operatorname{rad} C = M \cap \operatorname{Rad} C. \tag{31}$$

Proof. By assumption, there exists $x \in M \cap \operatorname{Rad} C$. Then by Lemma 5, we have

$$M \cap \operatorname{rad} C = x + C \cap \operatorname{Rad} C. \tag{32}$$

Since $x \in \operatorname{Rad} C$, (11) implies $x + C \cap \operatorname{Rad} C \subset \operatorname{Rad} C$, hence $M \cap \operatorname{rad} C \subset \operatorname{Rad} C$ by (32). This proves $M \cap \operatorname{rad} C \subset M \cap \operatorname{Rad} C$, and the reverse containment is trivial. □

Proposition 26. *Let C be a doubly even code of length a multiple of eight, containing* $\mathbf{1}$. *Suppose $C \cap \operatorname{rad} C = C \cap \operatorname{Rad} C$. Let $\{x_1, \dots, x_t\} \subset (C * C)^{\perp}$ be a set of coset representatives for the cosets $M \in (C * C)^{\perp}/D$ satisfying $\phi(M) \cap \phi(D) \neq \emptyset$ and $M \neq D$. For each $i \in \{1, \dots, t\}$, choose $y_i \in D$ in such a way that $\phi(x_i) = \phi(y_i)$. Then the following are equivalent.*

(i) $\operatorname{Rad} C \not\subset C$,
(ii) $\operatorname{wt}(x_i + y_i) \equiv 0 \pmod 8$ *for some* $i \in \{1, \dots, t\}$.

Proof. First, we note that $\phi(M) \cap \phi(D) \neq \emptyset$ implies that $\phi(M) = \phi(D)$ by Lemma 23. Thus there exists $y_i \in D$ such that $\phi(x_i) = \phi(y_i)$, no matter how we choose a representative x_i for the coset $x_i + D$.

Suppose (i) holds. Take $x \in \operatorname{Rad} C \setminus C$ and set $M = x + D$. Then $x \in M \cap \operatorname{Rad} C$, and hence (31) holds by Lemma 25. Also, as $x \notin D$ and $\phi(x) = 0$ by (28), $M = x_i + D$ holds for some $i \in \{1, \dots, t\}$. Thus $x_i + y_i \in M$, while $\phi(x_i + y_i) = \phi(x_i) + \phi(y_i) = 0$ by (29). Therefore, $x_i + y_i \in M \cap \operatorname{rad} C \subset \operatorname{Rad} C$ by (31). This implies $\operatorname{wt}(x_i + y_i) \equiv 0 \pmod 8$.

Conversely, if (ii) holds, then $x_i + y_i \in \operatorname{Rad} C \setminus C$, and hence (i) holds. □

7. Classification of maximal triply even codes of length 48

In this section, we aim to give a classification of maximal triply even codes of length 48. In Section 3 and Section 4, we gave 10 distinct maximal triply even codes of length 48. Now we show that the list is complete for a classification up to equivalence applying Proposition 22 and 19 for $n = 48$ and $m_1 = m_2 = 24$. To do this, we first need to establish the existence of a codeword of weight 24 in any maximal triply even code of length 48.

Lemma 27. *Let* D *be a maximal triply even code of length* n. *Let* Γ *be the graph with vertex set* $\{1, \dots, n\}$ *and edge set*

$$\{\operatorname{supp}(x) \mid x \in D^{\perp},\ \operatorname{wt}(x) = 2\}.$$

Then the following hold:

(i) *every connected component of* Γ *is a complete graph with at most* 8 *vertices,*
(ii) *if there is a connected component of* Γ *with more than* 4 *vertices, then any other connected component has at most* 3 *vertices.*

Proof. Since $D^{\perp}$ is a linear code, it is clear that every connected component of Γ is a complete graph. Suppose that there is a connected component K of Γ with $|K| > 8$. Then there exists a vector $x \in \mathbb{F}_2^n$ with $\operatorname{wt}(x) = 8$ and $\operatorname{supp}(x) \subset K$. Since the restriction of y to K is $\mathbf{0}$ or $\mathbf{1}$ for any $y \in D$, we have $\operatorname{wt}(x * y) = 0$ or 8. This implies that $\langle D, x \rangle$ is triply even. Taking $i \in \operatorname{supp}(x)$ and $j \in K \setminus \operatorname{supp}(x)$, the vector with support $\{i, j\}$ belongs to $D^{\perp}$ and is not orthogonal to x.

Thus $x \notin D$. This contradicts the fact that D is maximal, and the proof of (i) is complete.

To prove (ii), suppose that there are distinct connected components K, K' of Γ with $|K| > 4$ and $|K'| \geq 4$. Then there exists a vector $x \in \mathbb{F}_2^n$ with $\mathrm{wt}(x) = 8$, $|\operatorname{supp}(x) \cap K| = |\operatorname{supp}(x) \cap K'| = 4$. Since the restriction of y to K or K' is $\mathbf{0}$ or $\mathbf{1}$ for any $y \in D$, we have $\mathrm{wt}(x * y) = 0, 4$ or 8. This implies that $\langle D, x\rangle$ is triply even. The rest of the proof is exactly the same as (i). □

Lemma 28. *Let D be a maximal triply even code of length* 48 *containing* $\mathbf{1}$. *Then D has at least one codeword of weight* 24.

Proof. By Lemma 27, the number of codewords of $D^\perp$ with weight 2 is of the form

$$\sum_K \binom{|K|}{2},$$

where the summation is taken over the set of connected components of the graph Γ defined in Lemma 27. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the partition of 48 associated with the decomposition of the vertex set of Γ into connected components. Lemma 27 implies that one of the following holds:

(i) $4 < \lambda_1 \leq 8$ and $\lambda_i \leq 3$ for all $i \geq 2$,
(ii) $\lambda_i \leq 4$ for all $i \geq 1$.

It is not difficult to show that the maximum value of $\sum_i \binom{\lambda_i}{2}$ is $\binom{8}{2} + 13\binom{3}{2} = 67$ for the case (i), and $12\binom{4}{2} = 72$ for the case (ii). Therefore, we conclude that $D^\perp$ has at most 72 codewords of weight 2.

Now suppose that D has no codeword of weight 24, so that its weight enumerator is

$$X^{48} + aX^{40}Y^8 + (2^{k-1} - (1 + a))(X^{32}Y^{16} + X^{16}Y^{32}) + aX^8Y^{40} + Y^{48},$$

where $k = \dim D$. It follows from the MacWilliams identities that the number of codewords of weight 2 in $D^\perp$ is

$$3 \cdot 2^{8-k}a + 104 + 2^{11-k}$$

which is certainly greater than 72. This is a contradiction. □

In order to construct all maximal triply even codes of length 48 by means of Proposition 19 and 22 for $n = 48$ and $m_1 = m_2 = 24$, it suffices to consider the codes of length 24 satisfying $\operatorname{Rad} C_i \subset C_i$ as candidates for C_1 and C_2. This is because, if a resulting code $D(C_1, C_2, R_1, R_2, f)$ is maximal, then $R_i = \operatorname{Rad} C_i$ for $i = 1, 2$ as we mentioned in Proposition 22.

We are now ready to describe our enumeration using Magma system [1].

As the first step, we enumerate all doubly even codes of length 24 containing its triply even radical. Since there is a database of doubly even codes [14], we could make use of it and extract only those which contain the triply even radical. However, since every doubly even code is equivalent to a subcode of the nine doubly even self-dual codes of length 24 [16], we can find all the desired doubly even codes by successively taking subcodes of codimension one starting from the doubly even self-dual codes. This approach has an advantage that once we encounter a doubly even code C with $\operatorname{Rad} C \not\subset C$, then $\operatorname{Rad} C' \not\subset C'$ for any subcode C' of C, so that it is no longer necessary to consider subcodes of C by Lemma 8. Table 1 gives the numbers of doubly even codes of length 24 containing its triply even radical with each given dimension and dimension of its triply even radical.

Table 1. The numbers of doubly even code C of length 24 with $\operatorname{Rad} C \subset C$

$\dim C \setminus \dim \operatorname{Rad} C$	1	2	3	4	5	6
12	7	1	1	0	0	0
11	33	6	3	0	0	0
10	130	19	10	1	0	0
9	308	40	23	5	0	1
8	363	37	25	10	1	1
7	180	16	10	11	2	1
6	27	2	0	4	2	1
5	0	0	0	0	1	0

As the second step, we enumerate all resulting codes

$$D(C_1, C_2, \operatorname{Rad} C_1, \operatorname{Rad} C_2, f \circ g)$$

obtained from the all combinations of doubly even codes C_1, C_2 above and a representative $g \in \bar{g}$ for each double coset

$$\bar{g} \in (f^{-1} \circ \mathcal{G}_0(C_2, R_2) \circ f) \backslash \mathcal{G}_1(C_1, R_1) / \mathcal{G}_0(C_1, R_1),$$

where f is a fixed element of $\Phi(C_1 / \operatorname{Rad} C_1, C_2 / \operatorname{Rad} C_2)$ by the procedure given in Proposition 19

We denote the set of doubly even codes of length 24 by

$$\Delta = \{g_{24}, d_{24}^{+}, d_{12}^{2+}, (d_{10}e_7^2)^+, d_8^{3+}, d_6^{4+}, d_4^{6+}, d_{16}^{+} \oplus e_8, e_8^{\oplus 3}\},$$

in accordance with the notation of [16]. From the combinations with $C_1 = C_2$, we obtain 1482 triply even codes. However, many codes of them of the form (25) turn out not to be maximal. This is because, if there is a doubly even code C' such that $C \subsetneq C'$ and $\operatorname{Rad} C = \operatorname{Rad} C'$, then $\tilde{\mathfrak{D}}(C) \subsetneq \tilde{\mathfrak{D}}(C')$. Therefore, we find that only 216 codes among the 1482 codes are possibly maximal. Then we use Lemma 8 to check maximality, and we are able to confirm that only 30 codes among them are maximal. Each of these 30 codes turns out to be equivalent to $\tilde{\mathfrak{D}}(C)$ for some $C \in \Delta$.

From the combinations with $C_1 \not\cong C_2$, we obtain 225 triply even codes, and 5 codes among them are maximal. One code is equivalent to $\hat{C}(T_{10})$. The other codes are equivalent to a member of $\{\tilde{\mathfrak{D}}(C) \mid C \in \Delta\}$. Therefore we obtain the following theorem.

Theorem 29. *Every maximal triply even code of length* 48 *is equivalent to* $\tilde{\mathfrak{D}}(C)$ *for some* $C \in \Delta$ *or* $\hat{C}(T_{10})$.

8. Classification of maximal triply even codes of lengths 8, 16, 24, 32 and 40

In this section, we give a classification of maximal triply even codes of lengths 8, 16, 24, 32 and 40 by using a shortening process from the results of maximal triply even codes of length 48 in the previous sections.

It is easy to see that every maximal triply even code of length n is a shortened code of a maximal triply even code of length $n+1$. From the list of maximal triply even codes of length 48, we can derive the list of all maximal triply even codes of shorter lengths by the shortening process. The shortened code of $\tilde{\mathfrak{D}}(C)$ on one coordinate has an odd length, so it cannot be of the form $\tilde{\mathfrak{D}}(C')$ for any C'. However, for lengths divisible by 8, the following holds.

Theorem 30. *For* $n = 4, 8, 12, 16$ *and* 20*, every triply even code of length* $2n$ *is of the form* $\tilde{\mathfrak{D}}(C)$ *for some maximal doubly even code* C *of length* n.

Table 2 gives the numbers of the maximal triply even codes of lengths $8, 16, 24, 32$ and 40, up to equivalence.

In Table 2, the first and fifth columns indicate the length of each doubly even code and each triply even code, respectively. The second and sixth columns indicate the dimension as well. The third column indicates the number of indecomposable components of the doubly even code. The fourth and seventh columns indicate the number of codes satisfying the condition. The eighth column gives the other construction method to obtain it.

Note that if C is some maximal doubly even code and k is the number of self-dual indecomposable components of C, then $\dim \tilde{\mathfrak{D}}(C) = \dim C + k$ by (14) and (16). For example, there is a unique doubly even $[20, 9]$ code C which is the direct sum of three indecomposable codes, two of which are self-dual. Then $\tilde{\mathfrak{D}}(C)$ is a triply even $[40, 11]$ code. Similarly, there is a unique doubly even $[24, 12]$ code C which is the direct sum of three indecomposable self-dual codes. Then $\tilde{\mathfrak{D}}(C)$ is a triply even $[48, 15]$ code.

Table 2. The numbers of maximal triply even codes of lengths multiple of 8 up to 48

maximal doubly even codes				maximal triply even codes			
len	dim	#compos	#codes	len	dim	#codes	remark
4	1	1	1	8	1	1	
8	4	1	1	16	5	1	$\hat{C}(T_6)$
12	5	1	1	24	5	1	
		2	1		6	1	
16	8	1	1	32	9	1	
		2	1		10	1	
20	9	1	7	40	9	7	
		2	2		10	2	
		3	1		11	1	
24	12	1	7	48	13	7	
		2	1		14	1	
		3	1		15	1	
				48	9	1	$\hat{C}(T_{10})$

Acknowledgments. The authors would like to thank Ching Hung Lam, Masaaki Harada and Hiroki Shimakura for helpful discussions, and Jon-Lark Kim for bringing the paper [13] to the authors' attention.

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Appendix A. A Magma program for classification

Enumeration of doubly even codes of length 24. This appendix gives Magma scripts to verify Theorem 29 in Section 7.

It is known that there are precisely 9 doubly even self-dual codes of length 24 up to equivalence [16]. The object `desd24genmats` is the list of generator matrices in the hexadecimal expression. Also the object `desd24` is the list of the codes

$$\Delta = \{g_{24}, d_{24}^{+}, d_{12}^{2+}, (d_{10}e_7^2)^+, d_8^{3+}, d_6^{4+}, d_4^{6+}, d_{16}^{+} \oplus e_8, e_8^{\oplus 3}\}.$$

```
desd24genmats:=[
[ 0xC75001, 0x49F002, 0xD4B004, 0x6E3008, 0x9B3010, 0xB66020,
  0xECC040, 0x1ED080, 0x3DA100, 0x7B4200, 0xB1D400, 0xE3A800 ],
[ 0x7FE801, 0x802802, 0x804804, 0x808808, 0x810810, 0x820820,
  0x840840, 0x880880, 0x900900, 0xA00A00, 0xC00C00, 0xFFF000 ],
[ 0x7E0F81, 0xFC0082, 0xFC0104, 0xFC0208, 0xFC0410, 0xFC0820,
  0x820FC0, 0x861000, 0x8A2000, 0x924000, 0xA28000, 0xC30000 ],
[ 0xD003C1, 0xD1A042, 0xD1A084, 0xD1A108, 0xD1A210, 0x01A3E0,
  0x00E400, 0x01C800, 0x017000, 0x720000, 0xE40000, 0xB80000 ],
[ 0x7800E1, 0x88F022, 0x88F044, 0x88F088, 0xF0F0F0, 0x78E100,
  0x78D200, 0x78B400, 0x787800, 0x990000, 0xAA0000, 0xCC0000 ],
[ 0xE24031, 0x738012, 0x738024, 0x91C038, 0x938C40, 0xE1C480,
  0xE1C900, 0x724E00, 0x02D000, 0x036000, 0xB40000, 0xD80000 ],
[ 0xCC6009, 0x66A00A, 0xAAC00C, 0xC6C090, 0x6A60A0, 0xACA0C0,
  0x6CC900, 0xA66A00, 0xCAAC00, 0x00F000, 0x0F0000, 0xF00000 ],
[ 0x0000B1, 0x0000E2, 0x000074, 0x0000D8, 0x7E8100, 0x828200,
  0x848400, 0x888800, 0x909000, 0xA0A000, 0xC0C000, 0xFF0000 ],
[ 0x0000B1, 0x0000E2, 0x000074, 0x0000D8, 0x00B100, 0x00E200,
  0x007400, 0x00D800, 0xB10000, 0xE20000, 0x740000, 0xD80000 ]];

desd24:=
   [LinearCode<GF(2),24|[Prune(Intseq(n+0x1000000,2)) : n in code]>
     : code in desd24genmats];
```

The function `subcodes` takes a doubly even code C containing R as an argument, and returns the list of subcodes of codimension 1 of C satisfying $C \supset R$ up to the action of $\mathrm{Aut}(C)$.

```
subcodes:=function(C,R)
   A:=AutomorphismGroup(C);
   P:=PermutationModule(A,GF(2));
   DC:=Dual(C);
   DR:=Dual(R);
   PDC:=sub<P | VectorSpace(DC)>;
   PDR,e:=sub<P | VectorSpace(DR)>;
   M,p:=quo<PDR | PDC>;
   G:=MatrixGroup(M);
   X:=[DR| o[1] @@ p @ e : o in Orbits(G) | not 0 in o];
   overcodes:=[sub<DR|DC,x> : x in X];
   return [Dual(CC) : CC in overcodes];
```

```
end function;
```

Given a sequence of pairs of a code and a number, the function `uptoequivalenceDE` returns a subsequence of complete representatives of codes up to equivalence with the largest numbers appearing in the second components.

```
uptoequivalenceDE:=function(Ds)
   Css:=[];
   for D in Ds do
      ord := #AutomorphismGroup(D[1]);
      we  := WeightEnumerator(D[1]);
      if not exists(v){i:i in [1..#Css]|
           Css[i][1] eq ord and
           Css[i][2] eq we and
           IsEquivalent(Css[i][3], D[1])} then
         Append(~Css, <ord, we, D[1], D[2]>);
      else
         Css[v][4]:=Max([Css[v][4], D[2]]);
      end if;
   end for;
   return [<D[3],D[4]>: D in Css ];
end function;
```

Basic operation for codes. Given a pair of vectors, the functions `entrywiseProduct` and `CstarC` return $c_1 * c_2 = c_1 \cap c_2$ as the support and $C * C = \langle c_1 * c_2 \mid c_1, c_2 \in C \rangle$ respectively.

```
entrywiseProduct:=func<x,y|
   CharacteristicVector(Parent(x), Support(x) meet Support(y))>;

CstarC:=function(D)
   k:=Dimension(D);
   CC:=LinearCode<GF(2), Length(D)|
      [entrywiseProduct(D.i,D.j):i,j in [1..k] | i lt j] cat
      [D.i : i in [1..k]]>;
   return CC;
end function;
```

Given codewords x, y of a doubly even code C, the functions `QForm` and `BForm` return $Q(x)$ and $B(x, y)$ respectively.

```
QForm:=func<u|GF(2)!(Weight(u) div 4)>;
BForm:=func<u,v|GF(2)!(#(Support(u) meet Support(v)) div 2)>;
```

Given a vector x and a doubly even code D with a basis $\{u_1, u_2, \ldots, u_k\}$, the function `BFormArray` returns an array $(B(x, u_i))_i$. Given doubly even codes C, D with respective bases $\{u_1, u_2, \ldots, u_k\}$ and $\{v_1, v_2, \ldots, v_l\}$, the function `BFormMatrix` returns a matrix $(B(u_i, v_i))_{i,j}$.

```
BFormArray:=function(x,D)
   kD:=Dimension(D);
   return [BForm(x,D.j) : j in [1..kD]];
end function;
```

```

BFormMatrix:=function(C,D)
  kC:=Dimension(C);
  kD:=Dimension(D);
  M:=Matrix(GF(2), kC, kD,
             [BFormArray(C.i, D) : i in [1..kC]]);
  return M;
end function;
```

Given a doubly even code C, the functions `Cmeetrad` and `CmeetRad` return the subcode $C \cap \operatorname{rad} C$ and $C \cap \operatorname{Rad} C$ respectively, applying Lemma 6.

```
Cmeetrad:=function(C)
  D:=Dual(CstarC(C)) meet C;
  H:=VectorSpace(GF(2),Dimension(C));
  VD:=VectorSpace(D);
  g:=hom<VD->H|BFormMatrix(D,C)>;
  rad:=sub<D|Kernel(g)>;
  return rad;
end function;

CmeetRad:=function(C)
  rad:=Cmeetrad(C);
  k:=Dimension(rad);
  H:=VectorSpace(GF(2),1);
  VD:=VectorSpace(rad);
  g:=hom<VD->H| [[QForm(rad.i)]:i in [1..k]]>;
  Rad:=sub<rad|Kernel(g)>;
  return Rad;
end function;
```

Doubly even codes which contain each triply even radical. The function `outsideVectors` returns a complete list of representatives of cosets $(C * C)^{\perp}/((C * C)^{\perp} \cap C)$ up to the action of $\mathrm{Aut}(C)$. Given a doubly even code C, the function `existsOutsideRad` returns true if and only if $\operatorname{Rad} C \not\subset C$, applying Lemma 23 and Lemma 25.

```
outsideVectors:=function(C)
  U:=Generic(C);
  A:=AutomorphismGroup(C);
  P:=PermutationModule(A,GF(2));
  D:=Dual(CstarC(C));
  E:=C meet D;
  PD,e:=sub<P | VectorSpace(D)>;
  PDC:=sub<P | VectorSpace(E)>;
  M,p:=quo<PD | PDC>;
  G:=MatrixGroup(M);
  return {D!(o[1] @@ p @ e) : o in Orbits(G) | not 0 in o};
end function;

existsOutsideRad:=function(C)
```

```
   H:=VectorSpace(GF(2),Dimension(C));
   D:=Dual(CstarC(C)) meet C;
   VD:=VectorSpace(D);
   g:=hom<VD->H| BFormMatrix(VD,C)>;
   Im:=Image(g);
   rad := Kernel(g);
   b1:=exists(u){ i : i in [1..Dimension(rad)] | QForm(rad.i) ne 0};
   X:=outsideVectors(C);
   b2:=exists(v){x : x in X |
      imgx in Im and (b1 or QForm(x+imgx @@ g) eq 0)
      where imgx:= H!BFormArray(x,C)};
   return b2;
end function;
```

The record `RF` equips the following objects for a doubly even code of length 24.

```
RF:=recformat<
   C,    // the original code
   R,    // the triply even radical of C
   prd,  // the max dim of radical of supcode of codim = 1
   CR,   // the quotient space C/R
   p,    // the projection C -> C/R
   X,    // the array [ x in CR | Q(x) = 0 ]
   px,   // the projection V(X)->C/R
   CC,   // the triply even check code
   AutCR // Aut(C) meet Aut(R)
>;
```

Given a doubly even code C and its triply even radical R, the procedure `profiles` constructs the quotient C/R, the projection $p : C \to C/R$, the singular points X, the automorphism group $\mathrm{Aut}(C) \cap \mathrm{Aut}(R)$ and the triply even check code, and then returns a record containing them.

```
profiles:=function(C, prd)
   s:=rec<RF | C:=C, prd:=prd>;
   s`R:=CmeetRad(C);
   s`CR,s`p:=VectorSpace(C)/VectorSpace(s`R);
   s`X:=[x:x in s`CR|QForm(x @@ s`p) eq 0];
   M:=Matrix(GF(2), #s`X, Dimension(s`CR), s`X);
   s`px:=hom<VectorSpace(GF(2),#s`X)->s`CR|M>;
   s`AutCR:=AutomorphismGroup(C)
             meet AutomorphismGroup(s`R);
   s`CC:=LinearCode(Kernel(s`px));
   return s;
end function;
```

The procedure `constAllSubcodeContainsRad` constructs the list of all doubly even codes of length 24 containing its triply even radical.

```
constAllSubcodeContainsRad:=function(maxcodes24)
   codes:=[[ profiles(D, 0) : D in maxcodes24]];
```

```
  print "=> Now, constructing all admissible doubly even codes of length
   24...";
  for i in [1..9] do
     d:=12-i;
     ovcodes:=codes[#codes];
     reps:=&cat[[<C,Dimension(s`R)>:C in subcodes(s`C, s`R)]: s in ovcodes];
     reps:=uptoequivalenceDE(reps);
     reps:=[S : S in reps | not existsOutsideRad(S[1])];
     printf "=> Completed for dim=%3o, the number of codes=%4o.\n", d, #reps
   ;
     Append(~codes, [profiles(D[1], D[2]) :D in reps]);
     if IsEmpty(reps) then
           break i;
     end if;
  end for;
  printf "=> This is the expected result : %o.\n",
          [#x:x in codes] eq [9,42,160,377,437,220,36,1, 0];
  return &cat(codes);
end function;
```

Identification of maximal triply even codes. Given a triply even code, the function `isMaximal` returns true if and only if the code is a maximal triply even code.

```
isMaximal:=function(C)
  D:=CodeComplement(Dual(CstarC(C)), C);
  t:=exists(u){x:x in D| x ne 0 and QForm(x) eq 0
     and forall(v){i:i in [1..Dimension(C)] |  BForm(x,C.i) eq 0}};
  return not t;
end function;
```

The procedure `appendCode` appends a new maximal triply even code to the list of codes.

```
appendCode:=procedure(~codenum, ~maxcodes, reps, Ds, id)
  codenum:=codenum + #Ds;
  for D in Ds do
     if isMaximal(D) then
        Append(~maxcodes, D);
        invt:=<Dimension(D),NumberOfWords(D,8)>;
        id0:=Position(reps[2],invt);
        if id0 ne 0 and not IsEquivalent(D,reps[1][id0]) then
            id0 := 0;
        end if;
        printf
         "Found a MTE code = Rep.%2o of dim=%2o from DE code No.%o : %o.\n",
         id0, Dimension(D), id, reps[3][id0+1];
     end if;
  end for;
end procedure;
```

Triply even codes constructed from the combinations with $C_1 = C_2$. We enumerate all codes obtained from the method in Proposition 19 with $C_1 = C_2$.

Given a doubly even code and its triply even radical, the function `constDoubleCosetsCC` returns the representatives of double cosets

$$\mathcal{G}_0(C,R)\backslash\mathcal{G}_1(C,R)/\mathcal{G}_0(C,R). \tag{33}$$

```
constDoubleCosetsCC:=function(s)
  CRs:={@x:x in s`CR|x ne 0@};
  SCRs:=Sym(CRs);
  G0:=sub<SCRs|{[((x @@ s`p)^g) @ s`p:x in CRs]:
             g in Generators(s`AutCR)}>;
  GLCR:=sub<SCRs|{[x^g:x in CRs]:
             g in Generators(GL(s`CR))}>;
  G1:=Stabilizer(GLCR, {x : x in s`X | x ne 0});
  return DoubleCosetRepresentatives(G1, G0, G0);
end function;
```

Given a doubly even code C and the double cosets (33), the function `resultingCodesCC` returns triply even codes constructed from the code C using the method in Proposition 19 with $C_1 = C_2 = C$.

```
resultingCodesCC:=function(s, dc)
  D:=DirectSum(s`R, s`R);
  k:=Dimension(s`CR);
  M1:=Matrix([s`CR.i @@ s`p : i in [1..k]]);
  codes:=[D+LinearCode(HorizontalJoin(M1, M2)) where
    M2:=Matrix([((s`CR.i)^g) @@ s`p : i in [1..k]])
         : g in dc];
  return codes;
end function;
```

The object `partsDB` is the set of doubly even codes of length 24 containing its triply even radical. The function `duplextype` returns the list of all maximal triply even codes and the number of triply even codes of length 48 constructed from `partsDB` with $C_1 = C_2$.

```
duplextype:=function(partsDB, reps)
  maxcodes:=[];
  codenum:=0;
  excodenum:=0;
  for id in [1..#partsDB] do
    s:=partsDB[id];
    k:=Dimension(s`CR);
    if k eq 0 then
      if Dimension(s`R) eq s`prd then
        excodenum:=excodenum+1;
      else
        D:=DirectSum(s`R, s`R);
        appendCode(~codenum, ~maxcodes, reps, [D], id);
      end if;
```

```
      else
         doubleCosets:=constDoubleCosetsCC(s);
         if Dimension(s`R) eq s`prd then
            Remove(~doubleCosets, 1);
            excodenum:=excodenum+1;
         end if;
         if not IsEmpty(doubleCosets) then
            list:=resultingCodesCC(s, doubleCosets);
            appendCode(~codenum, ~maxcodes, reps, list, id);
         end if;
      end if;
   end for;
   return maxcodes, codenum, excodenum;
end function;
```

Triply even codes constructed from the combinations with $C_1 \not\cong C_2$. We enumerate all codes obtained from the method in Proposition 19 with $C_1 \not\cong C_2$.

Given a pair of doubly even codes and an isomorphism between their triply even check codes, the function `isometry` returns an isometry between them.

```
isometry:=function(s1, s2, g)
   CX:=Image(s1`px);
   bCR1:=ExtendBasis(Basis(CX), s1`CR);
   bCX2:=[bCR1[i] @@ s1`px @ g @ s2`px : i in [1..Dimension(CX)]];
   bCR2:=ExtendBasis(bCX2, s2`CR);
   return  hom<s1`CR->s2`CR | [bCR1[i]->bCR2[i] : i in [1..#bCR1]]>;
end function;
```

Given the object `partsDB`, which is the set of doubly even codes of length 24 containing its triply even radical, the function `isometricPairsC1C2` returns the list of isometric pairs of distinct doubly even codes and an isometry between them.

```
isometricPairsC1C2:=function(ss)
   C1C2s:=&cat[[<i, j, isometry(ss[i],ss[j],g)>
           : j in [i+1..#ss]
              | Dimension(ss[i]`CR) eq Dimension(ss[j]`CR)
                and #ss[i]`X eq #ss[j]`X and isEq
                where isEq, g := IsEquivalent(ss[i]`CC, ss[j]`CC)]
         : i in [1..#ss]];
   printf "The number of hybrid pairs = 125: %o.\n", #C1C2s eq 125;
   return C1C2s;
end function;
```

Given a pair of doubly even codes and an isometry, the function `constDoubleCosets` returns the double cosets

$$h^{-1}\mathcal{G}_0(C_2, R_2)h\backslash\mathcal{G}_1(C_1, R_1)/\mathcal{G}_0(C_1, R_1). \tag{34}$$

```
constDoubleCosetsC1C2:=function(s1, s2, h)
  CRs:={@ x : x in s1`CR  | x ne 0 @};
  SCRs:=Sym(CRs);
  G01:=sub<SCRs | {[((x @@ s1`p)^g) @ s1`p
          : x in CRs] : g in Generators(s1`AutCR)}>;
  G02:=sub<SCRs | {[((x @ h @@ s2`p)^g) @ s2`p @@ h
          : x in CRs] : g in Generators(s2`AutCR)}>;
  GLCR:=sub<SCRs | {[x^g : x in CRs] : g in Generators(GL(s1`CR))}>;
  G1:=Stabilizer(GLCR, {x : x in s1`X | x ne 0});
  return DoubleCosetRepresentatives(G1, G01, G02);
end function;
```

Given a pair of doubly even codes C_1 and C_2, an isometry h from C_1/R_1 to C_2/R_2 and the double cosets (34), the function `resultingCodesC1C2` returns triply even codes constructed from the pair of codes using the method in Proposition 19.

```
resultingCodesC1C2:=function(s1, s2, h, dc)
  k:=Dimension(s1`CR);
  D:=DirectSum(s1`R, s2`R);
  M1:=Matrix([s1`CR.i @@ s1`p : i in [1..k]]);
  codes:=[D+LinearCode(HorizontalJoin(M1, M2)) where
    M2:=Matrix([((s1`CR.i)^g) @ h @@ s2`p : i in [1..k]])
                                          : g in dc];
  return codes;
end function;
```

Recall that the object `partsDB` is the set of doubly even codes of length 24 containing its triply even radical. The function `hybridtype` returns the list of all maximal triply even codes and number of triply even codes of length 48 constructed from `partsDB` with $C_1 \not\cong C_2$.

```
hybridtype:=function(partsDB, reps)
  maxcodes:=[];
  codenum:=0;
  c1c2s:=isometricPairsC1C2(partsDB);
  for id in c1c2s do
     s1:=partsDB[id[1]];
     s2:=partsDB[id[2]];
     h:=id[3];
     k:=Dimension(s1`CR);
     if k eq 0 then
        D:=DirectSum(s1`R, s2`R);
        appendCode(~codenum, ~maxcodes, reps, [D], <id[1],id[2]>);
     else
        doubleCosets:=constDoubleCosetsC1C2(s1, s2, h);
        list:=resultingCodesC1C2(s1, s2, h, doubleCosets);
        appendCode(~codenum, ~maxcodes, reps, list, <id[1], id[2]>);
     end if;
  end for;
  return maxcodes, codenum;
end function;
```

Representative examples of maximal triply even codes. We give 10 maximal triply even codes $\{\tilde{\mathcal{D}}(C) \mid C \in \Delta\}$ and $\hat{C}(T_{10})$ of length 48. These codes are constructed by the functions `tildeD` and `TriangularGraphCode`.

```
tildeD := function(C)
  R := CmeetRad(C);
  return Juxtaposition(C,C)+DirectSum(R,R);
end function;

TriangularGraph:=function(v)
  X:=SetToIndexedSet(Subsets({1..v},2));
  return #X, Matrix(GF(2), #X, #X, [[#(x meet y):y in X] : x in X]);
end function;

TriangularGraphCode:=function(v)
  n, M:=TriangularGraph(v);
  r:=(-n) mod 8;
  return PadCode(LinearCode(M),r) + RepetitionCode(GF(2), n+r);
end function;
```

The object `repMTECodes` is the list of 10 maximal triply even codes equipped with their dimensions and the numbers of their codewords of weight 8.

```
repMTECodes1:=[ tildeD(C) : C in desd24 ] cat [TriangularGraphCode(10)];
repMTECodes2:=[<Dimension(C),NumberOfWords(C,8)> : C in repMTECodes1];
repMTECodes3:=["New!", "tD( g_{24} )", "tD( d_{24}^{+} )",
               "tD( d_{12}^{2+} )", "tD( (d_{10}e_7^2)^{+} )",
               "tD( d_8^{3+} )", "tD( d_6^{4+} )", "tD( d_4^{6+} )",
               "tD( d_{16}^{+}\oplus e_8 )", "tD( e_8^{\oplus3}\} )",
               "tT_{10}"];
dim_repMTECodes:={* Dimension(C) : C in repMTECodes1*};
printf "Representative codes are inequivalent each other: %o.\n",
        #repMTECodes1 eq #Seqset(repMTECodes2) and
        dim_repMTECodes eq {* 9^^1, 13^^7, 14^^1, 15^^1 *};
repMTECodes:=<repMTECodes1,repMTECodes2,repMTECodes3>;
```

Non existence of the other maximal triply even code. In this subsection, we aim to ensure that there does not exist any maximal triply even code of length 48 except for the representative examples in the previous subsection up to equivalence.

First, we enumerate all doubly even codes of length 24 which contain their triply even radicals.

```
partsDB:=constAllSubcodeContainsRad(desd24);
table:=[[Integers()!0: j in [1..13-k]]:k in [1..9]];
for k in [1..#partsDB] do
   i:=13-Dimension(partsDB[k]`C);
   j:=Dimension(partsDB[k]`R);
   table[i][j]+:=1;
```

```
end for;
printf "The number of admissible codes is same as expected: %o.\n",
table eq
[
   [   7,  1,  1,  0, 0, 0, 0, 0, 0, 0, 0, 0 ],
   [  33,  6,  3,  0, 0, 0, 0, 0, 0, 0, 0 ],
   [ 130, 19, 10,  1, 0, 0, 0, 0, 0, 0 ],
   [ 308, 40, 23,  5, 0, 1, 0, 0, 0 ],
   [ 363, 37, 25, 10, 1, 1, 0, 0 ],
   [ 180, 16, 10, 11, 2, 1, 0 ],
   [  27,  2,  0,  4, 2, 1 ],
   [   0,  0,  0,  0, 1 ],
   [   0,  0,  0,  0 ]
];
```

Second, we check the maximality of triply even codes constructed from all the doubly even codes in duplicate.

```
duplex_max, duplex_num, exduplex_num
        :=duplextype(partsDB, repMTECodes);
printf "%3o maximal codes of duplex type found.\n",#duplex_max;
printf "This is the expected result: %o.\n",
      <#duplex_max, duplex_num, exduplex_num> eq <30,214,1268>;
```

Next, we check the maximality of triply even codes constructed from all the pairs of distinct doubly even codes.

```
hybrid_max, hybrid_num:=hybridtype(partsDB, repMTECodes);
printf "%3o maximal codes of hybrid type found.\n",#hybrid_max;
printf "This is the expected result: %o.\n",
      <#hybrid_max, hybrid_num> eq <5,225>;
```

Result. A classification of triply even codes of length 48 has been completed into the 10 codes. This calculation has been completed in the total time: 650.240 seconds, the total memory usage: 534.91MB under the environment using “Intel® Core™ 2 Duo CPU T7500 @ 2.20GHz”.

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