A FOCK SPACE MODEL FOR ADDITION AND MULTIPLICATION OF C-FREE RANDOM VARIABLES

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Abstract. The paper presents a Fock space model suitable for constructions of c-free algebras. Immediate applications are direct proofs for the properties of the c-free R- and S-transforms.

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1. Introduction

Two important tools in Free Probability theory are the R- and S-transforms, that play similar role to Fourier, respectively Mellin transform. More precisely, besides strong regularity properties, if X and Y are two free non-commutative random variables, then $R_{X+Y}(z) = R_X(z) + R_Y(z)$ and $S_{XY}(z) = S_X(z) \cdot S_Y(z)$ if X, Y have non-zero first order moments.

In literature there are two main techniques to prove the additive, respectively multiplicative properties of the R- and S-transforms. The proofs given by D.-V. Voiculescu ([14], [15]) and U. Haagerup ([7]) based on functional analysis techniques, namely on the properties of the annihilation and creation operators on the full Fock space, while the proofs of R. Speicher and A. Nica ([8]) are based on combinatorial techniques on the lattice of non-crossing partitions (also non-crossing linked partitions appear in the proofs for the multiplicative property of the S-transform in [5], [11]).

In early '90's, M. Bozejko, M. Leinert and R. Speicher introduced the notion of *c-freeness*, which extends the notion of freeness to the framework of and algebra endowed with two (ϕ, ψ), rather than one, normalized linear functionals (see Section 2 for the exact definitions). (A more general approach to c-freeness, considering pairs of completely positive maps and conditional expectations have been pursued by F. Boca ([2]), K. Dykema and E. Blanchard ([6]), M. Popa, V. Vinnikov ([9], [12]) etc). Addition of c-free random variables is studied in [4], where is constructed a c-free version of the R-transform, the ${}^cR$-transform, with similar additivity and analytic properties (for $\phi = \psi$, the two transforms coincide); multiplication of c-free random variables was studied in [13], where is constructed a c-free extension of the S-transform. In both cases, the proofs of the key properties (addition for the ${}^cR$- and multiplication for the ${}^cS$-transform) are combinatorial, much like the proofs from [8], heavily relaying on the properties on non-crossing partitions. The present material gives a new approach to c-free random variables, in the spirit of the construction from [7]. Particularly, we give a more direct proof of the additive and multiplicative properties of the ${}^cR$- and ${}^cS$-transforms, based on the properties of the creation and annihilation operators on a certain type of Fock space.

Besides the Introduction, the paper is organized in 3 sections. Section 2 presents basic definitions, the construction of the space $\mathcal{E}(\mathfrak{H}, \mathcal{T}(\mathcal{K}))$ and an operator algebras model for c-free algebras. Section 3 presents the construction of some operators of prescribed ${}^cR$ and ${}^cS$-transforms and Section 4 gives the proof for the additive, respective multiplicative properties of ${}^cR$ and ${}^cS$.

In this paper, rather than the S- or ${}^cS$-transforms, we use, to simplify the notations their multiplicative inverses, the so-called T- and ${}^cT$-transforms – i. e. $T_X(z) \cdot S_X(z) = 1$ (see [5], [11]), respectively ${}^cT(z) \cdot {}^cS(z) = 1$ (see [13]).

2. A construction of c-free algebras

§1. Suppose $\mathcal{A}$ is a complex unital algebra endowed and $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear map such that $\varphi(1) = 1$. A family of unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ of $\mathcal{A}$ is said to be *free* (with respect to ψ) if

$$\psi(x_1 \cdots x_n) = 0$$

whenever $x_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(k) \neq \epsilon(k+1)$ and $\psi(x_j) = 0$ for $1 \leq j \leq n$ and $1 \leq k < n$.

If $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is another linear map with $\phi(1) = 1$, the family $\{\mathcal{A}_i\}_{i \in I}$ of unital subalgebras of $\mathcal{A}$ is said to be *c-free* with respect to (ϕ, ψ) if $\{A_i\}_{i \in I}$ are free with respect to ψ and

$$\phi(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n)$$

whenever $x_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(k) \neq \epsilon(k+1)$ and $\psi(x_j) = 0$ for $1 \leq j \leq n$ and $1 \leq k < n$.

Take X an element from $\mathcal{A}$ and let $m_X(z) = \sum_{k=1}^{\infty} \psi(X^k)$ denote the moment generating series of X with respct to ψ. As formal power series, the transforms $R_X(z)$ and, if $\psi(X) \neq 0$, $T_X(z)$ are defined by the equations

$$\begin{aligned} m_X(z) &= R_X(z[1 + m_X(z)]) \\ \frac{1}{z} m_X(z) &= [T_X(m_X(z))] \cdot (1 + m_X(z)). \end{aligned} \tag{1}$$

We warn the reader that the version of the R-trasform that is used in the present material differs from the original definition of D.-V. Voiculescu (that we will call here $\mathcal{R}$) by a multiplication with the variable z: $R_X(z) = z \cdot \mathcal{R}_X(z)$. As also seen in [13], [8], this shift of coefficients is simplifying the notations in several recurrence relations from §2.

For $\mathcal{H}$ a complex Hilbert space, we define $\mathcal{T}^0(\mathcal{H}) = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \ldots$ and $\mathcal{T}(\mathcal{H}) = \mathbb{C}\omega \oplus \mathcal{T}^0(\mathcal{H})$, where $\|\omega\| = 1$. For $e \in \mathcal{H}$ a nonzero vector, the creation operator over e, $a_e^* \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$, is given by the relations

$$\begin{aligned} &a_e^* \omega = e \\ &a_e^* v_1 \otimes v_2 \otimes \cdots v_k = e \otimes v_1 \otimes v_2 \otimes \cdots v_k, \text{ for } v_1, \ldots, v_k \in \mathcal{H} \end{aligned}$$

while the annihilation operator over e, $a_e \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$, is given by

$$\begin{aligned} &a_e \omega = 0 \\ &a_e v_1 \otimes v_2 \otimes \cdots v_k = \langle v_1, \xi \rangle \cdot v_2 \otimes \cdots \otimes v_k \end{aligned}$$

We remind the following result (see [16] for (1), [7], Theorem 2.2 and Theorem 2.3, for (2) and (3)):

Theorem 2.1. *Let $\mathcal{H}$ be a complex Hilbert space, e_1 and e_2 be two orthogonal vectors from $\mathcal{H}$, and f_1, f_2 be polynomials with complex coefficients. For $e \in \mathcal{H} \setminus \{0\}$ we will denote by $\mathcal{A}(e)$ the algebra generated by the creation and annihilation operators over e.*

(1) *The algebras $\mathcal{A}(e_1)$ and $\mathcal{A}(e_2)$ are free with respect to the vacuum state $T \mapsto \langle T\omega, \omega\rangle$.*
(2) *If $\alpha_i = a^*_{e_i} + f(a_{e_i})$, $(i = 1, 2)$, then $R_{\alpha_i}(z) = z \cdot f_i(z)$ and $R_{\alpha_1+\alpha_2}(z) = z \cdot f_1(z) + z \cdot f_2(z)$.*
(3) *If $f_i(0) \neq 0$, and $\beta_i = [\boldsymbol{Id}_{\mathcal{T}(\mathcal{H})} + a^*_{e_i}]f(a_{e_i})$, $(i = 1, 2)$, then $T_{\beta_i}(z) = f_i(z)$ and $T_{\beta_1\beta_2}(z) = f_1(z) \cdot f_2(z)$.*

§2. Consider now two complex Hilbert spaces $\mathcal{K}$ and $\mathfrak{H}$ and ω a distinguished unit vector in $\mathfrak{H}$. Take $\mathcal{T}(\mathcal{K}) = \omega_1 \oplus \mathcal{T}^0(\mathcal{K})$, where again $\|\omega_1\| = 1$ and

$$\mathcal{E}(\mathfrak{H}, \mathcal{K}) = \mathfrak{H} \oplus (\mathfrak{H} \otimes \mathcal{T}(\mathcal{K})).$$

Later, in Sections 2 and 4, we will consider $\mathcal{E}(\mathfrak{H}, \mathcal{K})$ for a particular $\mathfrak{H}$; when there is no possibility of confusion, to simplify the writting, we will use $\mathcal{E}$ for $\mathcal{E}(\mathfrak{H}, \mathcal{K})$. Put $\mathfrak{H}^0 = \mathfrak{H} \ominus \mathbb{C}\omega$, $\Omega = \omega \otimes \omega_1$, $\mathcal{E}^0 = \mathcal{E} \ominus \mathbb{C}\Omega$. We define the following embedding $\pi : \mathcal{L}(\mathcal{K}) \longrightarrow \mathcal{L}(\mathcal{T}(\mathcal{H}))$:

$$\pi(a) = a \oplus a \otimes (\mathbf{Id}_{\mathfrak{H}\otimes\mathcal{T}^0(\mathcal{K})} \oplus 0_{\mathbb{C}\Omega}).$$

Note that $\pi(\mathcal{L}(\mathfrak{H}))$ has unit $\pi(\mathbf{Id}_{\mathfrak{H}}) = \mathbf{Id}_{\mathcal{E}_0} \neq \mathbf{Id}_{\mathcal{E}}$.

For a nonzero vector $\eta \in \mathcal{K}$ we define the operators A^*_η and $\{A_{\eta,n}\}_{n\geq 0}$ from $\mathcal{L}(\mathcal{E})$ as follows:

$$\begin{aligned} A^*_\eta \zeta &= 0, \text{ if } \zeta \in \mathfrak{H} \oplus \mathfrak{H}^0 \otimes \mathcal{T}(\mathcal{K}) \\ A^*_\eta \omega \otimes \omega_1 &= \omega \otimes \eta \\ A^*_\eta \omega \otimes \zeta &= \omega \otimes (\eta \otimes \zeta) \text{ for all } \zeta \in \mathcal{T}^0(\mathcal{K}). \end{aligned}$$

we put $A_{\eta,0} = \mathbf{Id}_{\mathbb{C}\Omega}$ and, for $n \leq 1$, we define $A_{\eta,n}$ via

$$\begin{aligned} &A_{\eta,n}\omega \otimes (\eta^{\otimes n}) = \omega \otimes \omega_1 \\ &A_{\eta,n}\zeta = 0, \text{ if } \zeta \notin \mathbb{C}\omega_1 \otimes (\eta^{\otimes n}), \text{ where } \eta^{\otimes n} = \underbrace{\eta \otimes \cdots \otimes \eta}_{n \text{ times}}. \end{aligned}$$

We will use the notation $\mathcal{D}(\eta)$ for the algebra generated by A^*_η and $\{A_{\eta,n}\}_{n\geq 0}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two subalgebras of $\mathcal{L}(\mathcal{E})$, then the notation $\mathcal{A}_1 \vee \mathcal{A}_2$ will stand for the algebra generated by them in $\mathcal{L}(\mathcal{E})$.

Remark 2.2. *Fix $\eta, \eta_0 \in \mathcal{K}$ unit vectors. From the definitions of $\pi, A^*_\eta, A_{\eta,n}$, trivial verifications give that that*

$$A_{\eta,n}(A^*_\eta)^p = \begin{cases} A_{\eta,n-p} & \text{if } n \geq p \\ 0 & \text{if } n < p \end{cases}$$

If $x \in \pi(\mathcal{L}(\mathfrak{H}))$ and $n \geq 0$, then $xA_{\eta,n} = 0$. Also, if $m, n > 0$, then

$$\begin{aligned} &A_{\eta,n}A_{\eta_0,m} = 0 \\ &\boldsymbol{Id}_{\mathbb{C}\Omega}A_{\eta,n} = A_{\eta,n}; \quad A_{\eta,n}\boldsymbol{Id}_{\mathbb{C}\Omega} = 0 \\ &A_{\eta,n}\boldsymbol{Id}_{\mathcal{E}_0} = 0; \quad \boldsymbol{Id}_{\mathcal{E}_0}A^*_\eta = A^*_\eta\boldsymbol{Id}_{\mathcal{E}_0} = A^*_\eta. \end{aligned}$$

Remark 2.3. *For $\eta_1, \eta_2 \in \mathcal{K}$, we have that*

$$Range\Big((A^*_{\eta_1})^p A^*_{\eta_2}\Big) = Span\{\omega \otimes \eta_2 \otimes \eta_1^{\otimes p}, \omega \otimes \zeta \otimes \eta_2 \otimes \eta_1^{\otimes p} : \ \zeta \in \mathcal{T}^0(\mathcal{K})\},$$

therefore, if $\eta_1 \perp \eta_2$, then

$$A_{\eta_1,n}(A^*_{\eta_1})^p A^*_{\eta_2} = 0.$$

On $\mathcal{L}(\mathcal{E})$ we consider the functionals $\phi(\cdot) = \langle \cdot \Omega, \Omega \rangle$ and $\psi(\cdot) = \langle \cdot \Omega_1, \Omega_1 \rangle$.

Lemma 2.4. *Suppose $\eta_1, \eta_2 \in \mathcal{K}$ and $x \in \pi(\mathcal{L}(\mathfrak{H}))$. Then*

$$\begin{aligned} A^*_{\eta_1} x A^*_{\eta_2} &= A^*_{\eta_1} \psi(x) A^*_{\eta_2} \\ A_{\eta_1,n} x A^*_{\eta_2} &= A_{\eta_1,n} \psi(x) A^*_{\eta_2} \end{aligned}$$

Proof. Since $A_{\eta_2}(\mathcal{E}) = \omega_1 \otimes \mathcal{T}^0(\mathcal{K})$, for any $\zeta \in \mathcal{E}$ we have that:

$$A_{\eta_2}\zeta = \omega_1 \otimes \zeta' \text{ for some } \zeta' \in \mathcal{T}^0(\mathcal{K})$$
$$x A_{\eta_2}\zeta = \psi(x)\omega_1 \otimes \zeta' + v \otimes \zeta' \text{ for some } v \in \mathcal{T}^0(\mathcal{H}).$$

But $\mathcal{T}^0(\mathcal{H}) \otimes \mathcal{T}^0(\mathcal{K}) \subset \ker(A^*_{\eta_1}), \ker(A_{\eta_1,n})$ hence the conclusion. □

In the proof of the main result of this section, Theorem 2.6, we will use the following lemma:

Lemma 2.5. *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two free independent subalgebras of an algebra $\mathcal{A}$ with respect to some linear map φ and $a_0, a_1, \dots, a_{n+1} \in \mathcal{A}_1$, $b_1, \dots, b_n \in \mathcal{A}_2$ are such that $\varphi(a_k) = \varphi(b_k) = 0$ for all $1 \leq k \leq n$. Then*

$$\varphi(a_0 b_1 \cdots b_n a_{n+1}) = 0.$$

Proof. Take $d_j = a_j - \varphi(a_j)$, $j \in \{0, n+1\}$. Then $\varphi(d_j) = 0$ and $a_j = \varphi(a_j) + d_j$, therefore

$$\begin{aligned} \varphi(a_0 b_1 \cdots b_n a_{n+1}) &= \varphi(a_0)\varphi(a_0 b_1 \cdots b_n d_{n+1}) + \varphi(a_0)\varphi(a_0 b_1 \cdots b_n)\varphi(a_{n+1}) \\ &\quad + \varphi(d_0 b_1 \cdots b_n d_{n+1}) + \varphi(d_0 b_1 \cdots b_n)\varphi(a_{n+1}) \end{aligned}$$

and all the above four terms cancel from the definition of free independence. □

Theorem 2.6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be two subalgebras of $\mathcal{L}(\mathfrak{H})$ which are free independent with respect to ψ and let η_1, η_2 be two orthogonal unit vectors $\mathcal{K}$. Then the algebras $\mathfrak{A}_1 = \pi(\mathcal{A}_1) \vee \mathcal{D}(\eta_1)$ and $\mathfrak{A}_2 = \pi(\mathcal{A}_2) \vee \mathcal{D}(\eta_2)$ are c-free with respect to (ϕ, ψ).*

Proof. It suffices to prove that for $x_1, \dots, x_m$ such that $x_j \in \mathcal{A}(e_{\epsilon(j)}, \eta_{\epsilon(j)})$ with $\epsilon(i) \neq \epsilon(i+1)$ and $\psi(x_k) = 0$, we have

$$\psi(x_m \cdots x_2 x_1) = 0 \tag{2}$$

$$\phi(x_m \cdots x_2 x_1) = \phi(x_m) \cdots \phi(x_2)\phi(x_1) \tag{3}$$

Note that $\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}) \perp \Omega_1$ and

$$\begin{aligned} \mathcal{D}(\eta_i)(\mathcal{E}) &\subseteq \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}) \\ \pi(\mathcal{A}_i)\left(\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K})\right) &\subseteq \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}), \end{aligned}$$

hence ψ cancels on all reduced products from $\mathfrak{A}_1 \cup \mathfrak{A}_2$ that contain factors from $\mathcal{D}(\eta_1)$ or $\mathcal{D}(\eta_2)$. It follows that we only need to prove the relation (2) for $x_1, \dots, x_m \in \pi(\mathcal{A}_1) \cup \pi(\mathcal{A}_2)$, statement which is equivalent to the free independence of $\mathcal{A}_1$ and $\mathcal{A}_2$.

We will prove (3) by induction on n. For $n = 1$, the assertion is trivial. For the induction step, it suffices to prove that

$$\phi(x_n \cdots x_1) = \phi(x_n)\phi(x_{n-1} \cdots x_1). \tag{4}$$

Taking $x'_n = x_n - \phi(x_n)\mathbf{Id}_{\mathbb{C}\Omega}$, we have that $\phi(x'_n) = 0$ hence (4) is equivalent to $\phi(x_n \cdots x_1) = 0$ whenever $\phi(x_n) = 0$.

Suppose $x_n \in \mathcal{A}_1 \vee \mathcal{D}(\eta_1)$. then x_n is a linear combination of monomials in elements from $\mathcal{A}_1$ and $\mathcal{D}(\eta_1)$. From Lemma 2.4, we can suppose that all factors from $\mathcal{D}(\eta_1)$ are consecutive, so x_n is a sum of elements from $\mathcal{A}_1$ and monomials of the types $y'_1(A^*_{\eta_1})^p A_{\eta_1,m} y_1$ or $y'_1(A^*_{\eta_1})^p y_1$, with $y'_1, y_1 \in \mathcal{A}_1 \cup \mathbf{Id}$ and $p \geq 0$. If $y'_1 \neq \mathbf{Id}$ or $p \neq 0$, then $x_n(\mathcal{E}) \perp \Omega$, hence $\phi(x_n \cdots x_1) = 0$. Also, if $m = 0$, then either $y_1 = \mathbf{Id}$ and $\phi(x_n) \neq 0$ or $y_1 \in \mathcal{A}_1$ and $x_n = 0$. Therefore we can suppose that $x_n = A_{\eta_1,m} y_1$ for some $m > 0$ and $y_1 \in \mathcal{A}_1$ and all other x_j are either elements of $\mathcal{A}_{\epsilon(j)}$ or monomials as above.

Let $k = \max\{j : x_j \text{ contains } A^*_{\epsilon(j)}\}$ and $p = \max\{j : x_j \text{ contains } A_{\epsilon(j)}\}$. If $p > k$, then $x_n \cdots x_p = A_{\eta_1,m} y A_{\eta_{\epsilon(p)}} y'$, for some $y \in \mathcal{A}_1 \vee \mathcal{A}_2$ and $y' \in \mathcal{A}_{\epsilon(p)} \vee \mathcal{D}(\eta_{\epsilon(j)})$. From Lemma 2.4 and Remark 2.2, $A_{\eta_1,m} y A_{\eta_{\epsilon(p)}} = A_{\eta_1,m}\psi(y) A_{\eta_{\epsilon(p)}} = 0$.

Suppose that $p \leq k$. If $\epsilon(k) = 2$, then $x_n \cdots x_k = A_{\eta_1,m} y A^*_{\eta_2} y'$, for some $y \in \mathcal{A}_1 \vee \mathcal{A}_2$ and $y' \in \mathcal{A}_2 \vee \mathcal{D}(\eta_2)$. Applying Lemma 2.4 and Remark 2.3, we have

$$x_n \cdots x_k = A_{\eta_1,m}\psi(y) A^*_{\eta_2} y' = 0.$$

If $\epsilon(j) = 1$, then, from Remark 2.3,

$$\begin{aligned} x_n \cdots x_k &= A_{\eta_1,m} y_1 x_{n-1} \cdots x_{k+1} y A^*_{\eta_1} y' \\ &= A_{\eta_1,m}\psi(y_1 x_{n-1} \cdots x_{k+1} y) A^*_{\eta_1} y' \end{aligned}$$

but $\psi(y_1 x_{n-1} \cdots x_{k+1} y) = 0$ from Lemma 2.5, so q.e.d..

□

Corollary 2.7. *With the notations from §1, take $\mathfrak{H} = \mathbb{C}\omega \oplus \mathcal{T}^0(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space of dimension at least 2.*

Let e_1, e_2, respectively η_1, η_2 be two pairs of orthogonal unit vector from $\mathcal{H}$, respectively $\mathcal{K}$. Then the algebras $\mathcal{D}(\eta_1) \vee \pi(\mathcal{A}(e_1))$ and $\mathcal{D}(\eta_2) \vee \pi(\mathcal{A}(e_2))$ are c-free with respect to the maps ϕ and ψ considered above.

Proof. From Theorem 2.1(1), the algebras $\mathcal{A}(e_1)$ and $\mathcal{A}(e_2)$ are free in $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ with respect to $\langle \cdot \omega, \omega \rangle$, and the conclusion follows from Theorem 2.6.

□

3. The ${}^cR$- and ${}^cT$- transforms

Consider an algebra $\mathcal{A}$ with two states $\phi, \psi : \mathcal{A} \longrightarrow \mathbb{C}$ and $X \in \mathcal{A}$. Let $m_X(z) = \sum_{k=1}^{\infty} \psi(X^k)$, respectively $M_X(z) = \sum_{k=1}^{\infty} \phi(X^k)$ be the moment-generating series of X with respect to ψ, respectively ϕ. We define the ${}^cR$-, and, if $\psi(X) \neq 0$, the ${}^cT$-transforms of X by the following equations:

$$ {}^cR_X(z[1 + m_X(z)]) \cdot (1 + M_X(z)) = M_X(z)[1 + m_X(z)] \tag{5}$$

$$ [{}^cT_X(m_X(z))] \cdot (1 + M_X(z)) = \frac{M_X(z)}{z} \tag{6}$$

With the notations from Section 2, for $\eta \in \mathcal{K}$ a non-zero vector and $f = \sum_{k=0}^{N} f_k X^k$ a polynomial with complex coefficients, we define $A_{\eta, f^{\otimes}}$ via:

$$A_{\eta, f^{\otimes}} = \sum_{k=0}^{N} f_k \cdot A_{\eta, k}$$

3.1. **The ${}^cR$-transform.**

Theorem 3.1. *Let η be a unit vector from $\mathcal{K}$, $b \in \pi(\mathcal{L}(\mathfrak{H}))$ and $f = \sum_{p=0}^{M} g_p \cdot z^p$ be a polynomial with complex coefficients. Consider $\alpha \in \mathcal{L}(\mathcal{E})$ given by:*

$$\alpha = b + A_\eta^* + A_{\eta, f^{\otimes}}$$

Then ${}^cR_\alpha(z) = zf(z)$.

Proof. It suffices to show that $zf(z)$ satisfies the equation (5), which is equivalent to the following recurrence

$$\phi(\alpha^n) = \sum_{0 \le p \le n} \sum_{\substack{q_1, \dots, q_p \ge 0 \\ n \ge 1 + p + q_1 + \cdots q_p}} \phi\left(\alpha^{n-1-(p+q_1+\cdots+q_p)}\right) \cdot f_p \cdot \psi(\alpha^{q_1}) \cdots \psi(\alpha^{q_p}) \tag{7}$$

for all $n > 0$.

Let us denote $A = A_\eta^*$ and $B = A_{\eta, f^{\otimes}}$. The triple (b, A, B) satisfies the following relations:

$$b\Omega = 0, B(\mathcal{E}) = \mathbb{C}\Omega \tag{8}$$

$$Ab^qA = A\psi(b^q)A, Bb^qA = B\psi(b^q)A \text{ for all } q > 0 \tag{9}$$

$$\phi(BA^n) = f_n, \text{ for all } n \ge 0 \tag{10}$$

(equations (8) and (9) are consequences of the relations from Remark 2.2, and (10) follows from Lemma 2.4.)

Let $I = \{b, A, B\}$. Since $\alpha = \sum_{x \in I} x$, we have that

$$\phi(\alpha^n) = \sum_{(x_1, \dots, x_n) \in I^n} \phi(x_n x_{n-1} \cdots x_1) \tag{11}$$

To further simplify the writting, we introduce the following notations

$$I[n, j] = \Big\{(x_1, \dots, x_n) \in I^n, \min\{k : \ x_k = B\} = j\Big\}.$$

Since $b\Omega = 0$ and $A(\mathcal{E}) \perp \Omega$, we have that $\phi(x_n \cdots x_1) = 0$ unless $x_n = B$, hence $(x_n, \dots, x_1) \in I[n, j]$ for some j. Also, for $(x_n, \dots, x_1) \in I[n, j]$, since $B(\mathcal{E}) = \mathbb{C}\Omega$, we have that $x_j \cdots x_1 \Omega = \phi(x_j \cdots x_1)$, so $\phi(x_n \cdots x_1) = \phi(x_n \cdots x_{j+1})\phi(x_j \cdots x_1)$, therefore (11) becomes

$$\begin{aligned} \phi(\alpha^n) &= \sum_{j=1}^{n} \sum_{(x_1, \dots, x_n) \in I[n,j]} \phi(x_n \cdots x_1) \\ &= \sum_{j=1}^{n} \sum_{(x_1, \dots, x_n) \in I[n,j]} \phi(x_n \cdots x_{j+1})\phi(x_j \cdots x_1) \\ &= \sum_{j=1}^{n} \sum_{(x_1, \dots, x_j) \in I[j,j]} \phi(\alpha^{n-j})\phi(x_j \cdots x_1). \end{aligned} \tag{12}$$

Consider $(x_1, \ldots, x_n) \in I[n,n]$. If $n = 1$, then $\phi(x_n \cdots x_1) = \phi(B) = f_0$. If $n > 1$, then $x_1\Omega = 0$ unless $x_1 = A$. Let $1 = k_1 < \cdots < k_p < n$ be the set of all indices k such that $x_k = A$. Letting $q_j = k_{j+1} - k_j - 1$, $q_p = n - k_p - 1$ and applying property (10), we obtain:

$$
\begin{aligned}
\sum_{(x_1,\ldots,x_n)\in I[n,n]} \phi(x_n \cdots x_1) &= \sum_{p=1}^{n-1} \sum_{\substack{0\le q_1,\ldots,q_p \\ q_1+\ldots q_p<n-p}} \phi\left(B \cdot \psi(b^{q_p}) \cdot A \cdots \psi(b^{q_1})A\right) \\
&= \sum_{\substack{0\le q_1,\ldots,q_p \\ q_1+\ldots q_p<n-p}} f_p \cdot \psi(b^{q_p}) \cdots \psi(b^{q_1}). \qquad (13)
\end{aligned}
$$

Finally, the equality $\psi(b^q) = \psi(\alpha^q)$ and equations (12), (15) imply (7), so q.e.d.. □

3.2. The ${}^cT$-transform.

Theorem 3.2. *Let η be a unit vector from $\mathcal{K}$, $d \in \pi(\mathcal{L}(\mathfrak{H}))$ and $f(z) = \sum_{k=0}^{M} f_k \cdot z^k$ be a polynomial with complex coefficients such that $\psi(b), f_0 \neq 0$. Consider $\beta \in \mathcal{L}(\mathcal{E})$ given by:*

$$\beta = d + dA_\eta^* + A_{\eta,f^\otimes}.$$

Then ${}^cT_\beta(z) = f(z)$.

Proof. The proof is similar to the one of Theorem 3.1. Denote again $A = A_\eta^*$, $B = A_{\eta,f^\otimes}$ and consider the sets

$$
\begin{aligned}
&J = \{d, dA, B\} \\
&J[n,l] = \{(x_1, \ldots, x_n) \in J^n, min\{k : \ x_k = B\} = l\}.
\end{aligned}
$$

For $(x_1, \ldots, x_n) \in J^n$, we have that $\phi(x_n \ldots x_1) = 0$ unless $x_n = B$, hence $(x_1, \ldots, x_n) \in J[n,l]$ for some $1 \le l \le n$. Also, note that equation (12) holds true if we replace I with J, therefore

$$\phi(\beta^n) = \sum_{l=1}^{n} \phi(\beta^{n-l}) \sum_{(x_1,\ldots,x_l)\in J[l,l]} \phi(x_l \cdots x_1). \qquad (14)$$

Fix $n > 0$ and let $(x_1, \ldots, x_1) \in J[n.n]$. If $n = 1$, then $\phi(x_n, \ldots, x_1) = \phi(B) = f_0$. If $n = 1$, then $\phi(x_n \cdots x_1)$ cancels unless $x_1 = dA$. Let $1 = k_1 < k_2 < \cdots < k_p \le n - 1$ be the set of indices k such that $x_k = dA$. Taking $q_j = k_{j+1} - k_j - 1$ for $1 < j < n - 1$ and $q_p = n - k_p - 1$, and applying property (10), we obtain:

$$
\begin{aligned}
\sum_{(x_1,\ldots,x_n)\in J[n,n]} \phi(x_n \cdots x_1) &= \sum_{p=1}^{n-1} \sum_{\substack{0\le q_1,\ldots,q_p \\ q_1+\ldots q_p<n}} \phi\left(B \cdot (b^{q_p-1}) \cdot dA \cdot b^{q_{p-1}-1} \cdots b^{q_1-1} \cdot dA\right) \\
&= \sum_{p=1}^{n-1} \sum_{\substack{0\le q_1,\ldots,q_p \\ q_1+\ldots q_p<n}} \left(B \cdot \psi(d^{q_p}) \cdot A \cdot \psi(b^{q_{p-1}}) \cdots \psi(d^{q_1})A\right) \\
&= \sum_{p=1}^{n-1} \sum_{\substack{0\le q_1,\ldots,q_p \\ q_1+\ldots q_p<n}} f_p \cdot \psi(d^{q_p}) \cdots \psi(d^{q_1}). \qquad (15)
\end{aligned}
$$

And the conclusion follows, since (15), (14) and the identity $\psi(\beta^q)=\psi(d^q)$ imply the $f(z)$ satisfies (6).

$\square$

4. Addition and multiplication of c-free random variables

Theorem 4.1. *Let η_1, η_2 be orthogonal unit vectors from $\mathcal{K}$, let b_1, b_2, d_1, d_2 be some elements from $\pi(\mathcal{L}(\mathfrak{H}))$ and let f_1, f_2, F_1, F_2 be polynomials with complex coefficients, such that $\psi(d_i \neq 0 \neq F_i(0)$. Define ($i=1,2$):*

$$\begin{aligned}\alpha_i &= b_i + A^*_{\eta_i} + A_{\eta_i, f_i^{\otimes}}\\ \beta_i &= d_i + d_i\cdot A^*_{\eta_i} + A_{\eta_i, F_i^{\otimes}}.\end{aligned}$$

Then ${}^cR_{\alpha_1+\alpha_2}(z) = {}^cR_{\alpha_1}(z) + {}^cR_{\alpha_2}(z)$ and ${}^cT_{\beta_1\cdot\beta_2}(z) = {}^cT_{\beta_1}(z)\cdot {}^cT_{\beta_2}(z)$.

Proof. Suppose that $F_1(z)=\sum_{k=0}^M h_k\cdot z^k$ and $F_2(z)=\sum_{k=0}^M l_k\cdot z^k$ (eventually h_M or l_M are zero).

To prove the first equality, we introduce the notations

$$\begin{aligned}\widetilde{b} &= b_1+b_2\\ \widetilde{A} &= A^*_{\eta_1} + A^*_{\eta_2}\\ \widetilde{B} &= A_{\eta_1, F_1^{\otimes}} + A_{\eta_2, F_2^{\otimes}}\end{aligned}$$

Trivial verifications show that the triple $(\widetilde{b}, \widetilde{A}, \widetilde{B})$ verify the conditions (8)–(10), therefore the reccurrence (7) holds true for $\alpha = \widetilde{b}+\widetilde{A}+\widetilde{B} = \alpha_1+\alpha_2$ and $\{g_k\}_{k=1}^M$ the coefficients of $f_1(z)+f_2(z)$, so ${}^cR_{\alpha_1+\alpha_2}(z) = z[f_1(z)+f_2(z)]$, q.e.d..

For the second equality, we need to prove that $F_1(z)\cdot F_2(z)$ satisfies (6) for $\beta=\beta_1\cdot\beta_2$, that is the recurrence formula:

$$\begin{aligned}\phi(\beta^n) &= \sum_{p=0}^{n-1} \sum_{\substack{q_1,\dots,q_p>0\\ q_1+\dots+q_p\leq n}} \phi(\beta^{n-1-(q_1+\dots+q_p)})\cdot[g_p\cdot\psi(\beta^{q_p})\cdots\psi(\beta^{q_1})]\\ &= \sum_{m=1}^{n}\phi(\beta^{n-m})\cdot\Big[\sum_{p=1}^{m-1}\sum_{\substack{q_1,\dots,q_p>0\\ q_1+\dots+q_p<m}} g_p\cdot\psi(\beta^{q_p})\cdots\psi(\beta^{q_1})\Big]\end{aligned}\tag{16}$$

is verified for $\beta=\beta_1\cdot\beta_2$ and g_m the coefficient of z^m in $F_1(z)\cdot F_2(z)$.

We introduce the notations

$$\begin{aligned}&b=d_1;\quad d=d_2\\ &A_1 = A^*_{\eta_1};\quad A_2 = A^*_{\eta_2}\\ &B_1 = A_{\eta_1, F_1^{\otimes}};\quad B_2 = B_{\eta_2, F_2^{\otimes}}\end{aligned}$$

Then $B_iA_j = 0$ whenever $i\neq j$ and $\beta_1 = b+bA_1+B_1$, $\beta_2 = d+dA_2+B_2$. For $\beta=\beta_1\cdot\beta_2$, we have that

$$\begin{aligned}\beta &= (b+bA_1+B_1)(d+dA_2+B_2)\\ &= bd+bdA_2+bB_2+bA_1d+bA_1dA_2+bA_1B_2+B_1d+B_1dA_2+B_1B_2\end{aligned}$$

Consider the sets

$$\mathcal{J} = \{bd, bdA_2, bA_1d, bA_1dA_2, bA_1B_2, B_1d, B_1dA_2, B_1B_2\}$$
$$\mathcal{J}[n,m] = \Big\{(x_1,\dots,x_n)\in\mathcal{J}^n, \min_k\{x_k\in\{B_1d, B_1B_2\}\} = m\Big\}$$
$$\overline{\mathcal{J}} = \mathcal{J}\cup\{bA_1dA_2\}.$$

Note first that $bB_2 = 0$; also, since Lemma 2.4 implies $B_1dA_2 = B_1\psi(d)A_2 = 0$ we have that $\beta = \sum_{x\in\overline{\mathcal{J}}} x$, hence

$$\phi(\beta^n) = \sum_{(x_1,\dots,x_n)\in\overline{\mathcal{J}}} \phi(x_n\dots x_1)$$

If some x_k is bA_1dA_2, then $\phi(x_n\cdots x_1)$ cancels, since the vectors from $\mathcal{E}$ with the $\mathcal{T}(\mathcal{K})$ component containing mixed tensors in η_1 and η_2 are cancelled by any B_i $(i=1,2)$ and are also orthogonal to Ω. On the other hand, $\phi(x_n\cdots x_1)$ also cancels if $x_n\cdots x_1\perp\Omega$, that is if x_n does not start with some B_i. It follows that only terms having $x_n\in\{B_1d, B_1B_2\}$ contribute to the sum, that is the sum can be taken only for $(x_1,\dots,x_n)\in\mathcal{J}[n,m]$, $(1\le m\le n)$.

Consider now $(x_1,\dots,x_n)\in\mathcal{J}[n,m]$. Then $x_m\cdots x_1\Omega = \phi(x_m\cdots x_1)\Omega$, hence $\phi(x_n\cdots x_1) = \phi(x_n\cdots x_m)\phi(x_m\cdots x_1)$, and

$$\begin{aligned}\sum_{(x_1,\dots,x_n)\in\mathcal{J}[n,m]}\phi(x_n\dots x_1) &= \sum_{(x_1,\dots,x_n)\in\mathcal{J}[n,m]}\phi(x_n\cdots x_{m+1})\cdot\phi(x_m\cdots x_1)\\ &= \phi(\beta^{n-m})\cdot\Big[\sum_{(x_1,\dots,x_m)\in\mathcal{J}[m,m]}\phi(x_m\cdots x_1)\Big],\end{aligned}\tag{17}$$

therefore it suffices to prove that the second factors from the right hand sides of (17) coincides to the second factor of the m-th summand in (16), that is (we use that $g_m = \sum_{p+k=m} l_p\cdot h_k$):

$$\begin{aligned}\sum_{(x_1,\dots,x_n)\in\mathcal{J}[n,n]}\phi(x_n\cdots x_1) &= \sum_{m=0}^{n-1} g_m\cdot\sum_{\substack{q_1,\dots,q_m>0\\ q_1+\cdots+q_m<n}}\psi(\beta^{q_m})\cdots\psi(\beta^{q_1})\\ &= \sum_{\substack{p,k\ge 0\\ p+q<n}} E(p,k)\end{aligned}\tag{18}$$

for $E(p,k) = [l_p\psi(\beta^{q_m})\cdots\psi(\beta^{q_1})]\cdot[h_k\psi(\beta^{s_k})\cdots\psi(\beta^{s_1})$ where the summation is done over al $p,q\ge 0$ such that $p+q<n$ and all $q_1,\dots,q_p,s_1,\dots s_k>0$ with $q_1+\dots q_p+s_1+\dots+s_k = n-1$.

Consider $(x_1,\dots,x_n)\in\mathcal{J}[n,n]$ such that $\phi(x_n\cdots x_1)\ne 0$; then $x_1\Omega\ne 0$, so $x_1\in\{B_1B_2, bA_1B_2, bdA_2\}$.

Case I: $x_1 = B_2B_2$; this imply that $n=1$ (since x_1 already starts with a B_i) and $\phi(x_1) = \langle B_1B_2\Omega,\Omega\rangle = l_0h_0$, that is (18) for $n=1$.

Case II: $x_1 = bA_1B_2$.
In this case $x_1\Omega = bA_1B_2\Omega = bA_1\Omega\cdot l_0$.

Let $j = \min\{k:\ x_k$ contains B_1, i. e. $x_k\in\{B_1B_2, B_1d\}\}$ (since $x_n\in\{B_1B_2, B_1d\}$, the set is not void). Also, $j\ne n$ will contradict the definition of $\mathcal{J}[n,n]$, so $j=n$.

Tensors containing η_1 are canceled by B_2 and, as seen earlier, summands with A_2 and A_1 not separated by B_1 do not contribute to the sum, therefore it follows that $x_2, \dots, x_{n-1}$ do not contain A_2 nor B_2, so they can be only of the types bd and bA_1d.

Let $1 < k_2 < \cdots < k_p < n$ be the indices of the factors of type bA_1d and put $k_1 = 1$ and $k_{p+1} = n$. For $q_i = k_{i+1} - k_i$, applying Lemma 2.4, we have

$$\begin{aligned} x_n \cdots x_2 bA_1\Omega &= B_1 d(bd)^{q_p - 1} bA_1 d(bd)^{q_{p-1}-1} \cdots (bd)^{q_1 - 1} bA_1\Omega \\ &= B_1 \psi((db)^{q_p}) A_1 \psi((db)^{q_{p-1}}) \cdots \psi((db)^{q_1}) A_1 \Omega \\ &= (B_1 A_1^p \Omega) \psi((db)^{q_p}) \cdots \psi((db)^{q_1}) \\ &= h_p \Omega \cdot \psi(\beta^{q_p}) \cdots \psi(\beta^{q_1}) \end{aligned} \tag{19}$$

since $B_1 A_1^p \Omega = h_p \Omega$ and $\psi(db) = \psi(bd) = \psi(\beta)$ due to the traciality of the vector states. Multiplying with l_0 and summing, we obtain

$$\sum_{\substack{(x_1,\dots,x_n)\mathcal{J}[n,n] \\ x_1 = dA_1B_2}} \phi(x_n \cdots x_1) = \sum_{p=1}^{n-1} E(p, 0) \tag{20}$$

Case III: $x_1 = bdA_2$.

Let x_j be the factor of the smallest index that contains B_2. Since $x_j \in \mathcal{J}$, we have that $x_j = y \cdot B_2$, with $y \in \{bA_1, B_1\}$.

None of the factors $x_2, \dots, x_{j-1}$ contains B_1 (otherwise it will contradict the definition of $J[n,n]$); if some of them will contain A_1, then $yB_2 x_{j-1} \cdots x_2 = 0$, since B_2 cancels all the tensors mixing η_1 and η_2. Hence $x_2, \dots, x_{j-1} \in \{bd, bdA_2\}$. Again, let $1 = j_1 < \cdots < j_k < n$ be the indices of the factors of type bdA_2 and put $j_{k+1} = j$. For $s_i = k_{i+1} - k_i$, applying Lemma 2.4, we have

$$\begin{aligned} x_j \cdots x_1 \Omega &= y \cdot B_2 (bd)^{s_k} A_2 (bd)^{s_{k-1}} A_2 \cdots (bd)^{s_1} A_2 \Omega \\ &= y\Omega \cdot l_k \psi(\beta)^{s_k} \cdots \psi(\beta^{s_1}) \end{aligned} \tag{21}$$

(we used that $B_2 A_2^k \Omega = l_k \Omega$ and that $\psi(\beta^s) = \psi((bd)^s)$.

If $y = B_1$ then $y\Omega = h_0$; also, the minimality of n implies $j = n$.

If $y = bA_1$, since $x_n \in \{B_1B_2, B_1d\}$, the minimality of n implies that $x_{j+1}, \dots, x_{n-1}$ do not contain B_1. If they will contain A_2 or B_2, then $\phi(x_n \cdots x_1) = 0$ as seen earlier, so they must be of the types bd or bA_1d. Also, if $x_n = B_1B_2$, then again $\phi(x_n \cdots x_1) = 0$, so we can suppose $x_n = B_1 d$. In this case, $x_n \cdots x_{j=1} y\Omega$ is in the setting of formula (19), so it is computed accordingly to it.

Summing, we obtain

$$\sum_{\substack{(x_1,\dots,x_n)\in\mathcal{J}[n,n] \\ x_1 = bA_2}} \phi(x_n \cdots x_1) = \sum_{\substack{p, k-1 > 0 \\ p+k < n}} E(p, k) \tag{22}$$

and the conclusion follows, since (22) and (20) imply (18).

□

Corollary 4.2. *Let $\mathcal{A}$ be a unital algebra, $\Phi, \Psi : \mathcal{A} \longrightarrow \mathbb{C}$ be two linear maps with $\Phi(1) = \Psi(1) = 1$ and let X, Y be two c-free (with respect to the maps Φ, Ψ) elements from $\mathcal{A}$.*

(i) *$R_{X+Y} = R_X + R_X$ and ${}^cR_{X=Y} = {}^cR_X + {}^cR_Y$ as formal power series.*
(ii) *If $\Psi(X), \Psi(Y)$ are nonzero, then $T_{XY} = T_X \cdot T_Y$ and ${}^cT_{XY} = {}^cT_X \cdot {}^cT_Y$ as formal power series.*

Proof. The equalities for R- and T-transforms are basic properities in the Free Probability Theory (see [16], [8]). We need to prove (i) and (ii) for the ${}^cR$- and ${}^cT$-transforms.

As in Corollary 2.7, we will consider two complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ of dimension at least two and $\mathcal{E} = \mathcal{T}(\mathcal{H}) \oplus [\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K})]$, where

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}\omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \dots$$
$$\mathcal{T}(\mathcal{K}) = \mathbb{C}\omega \oplus \mathcal{K} \oplus (\mathcal{K} \otimes \mathcal{K}) \oplus \dots$$

We fix e_1, e_2, respectively η_1, η_2 two pairs of orthogonal unit vectors from $\mathcal{H}$, respectively $\mathcal{K}$. From Corollary 2.7, the algebras $\mathfrak{A}_1 = \mathcal{D}(\eta_1) \vee \pi(\mathcal{A}(e_1))$ and $\mathfrak{A}_2 = \mathcal{D}(\eta_2) \vee \pi(\mathcal{A}(e_2))$ are c-free with respect to $\phi(\cdot) = \langle \cdot \omega \otimes \omega, \omega \otimes \omega \rangle$ and $\psi(\cdot) = \langle \cdot \omega, \omega \rangle$.

Also note that, from the relations defining the free, respectively c-free independence (see Section 2, §1), the moments up to order N of $X + Y$ and XY with respect to Φ and Ψ are uniquelly determined by the moments of order up to N of X and Y.

For (i), consider $f_1(z)$, $F_1(z)$, repectively $f_2(z)$, $F_2(z)$ be the polynomials obtained by the trucation of order N of R_X, ${}^cR_X$, respectively R_Y, ${}^cR_Y$ (i. e. if $cR_X(z) = \sum_{k=1}^{\infty} l_k \cdot z^k$, then $F_1(z) = \sum_{k=1}^{N} l_k \cdot z^k$ and the analogues).

With the notations from Section 2, take $(i = 1, 2)$

$$\alpha_i = \pi(a^*_{e_i} + f_i(a_{e_i})) + A^*_{\eta_i} + A_{\eta_i, F_i^{\otimes}}$$

We have that $\alpha_i \in \mathfrak{A}_i$, so α_1, α_2 are c-free with respect to ϕ and ψ, hence, from Theorem 4.1,

$$ {}^cR_{\alpha_1+\alpha_2}(z) = {}^cR_{\alpha_1}(z) + {}^cR_{\alpha_2}(z) \tag{23}$$

From Theorem 2.1(2) and Theorem 3.1, we have that $R_{\alpha_i}(z) = f_i(z)$ and ${}^cR_{\alpha_i}(z) = F_i(z)$, therefore R_{α_1}, ${}^cR_{\alpha_1}$, and R_X, ${}^cR_X$ coincide up to order N. Then equations (1) and (5) imply that the moments up to order N of X with respect to Φ, respectively Ψ, coincide to the moments up to order N of α_1 with respect to ϕ and ψ. The same holds true for Y and α_2, therefore the moments up to order N of $X + Y$ and $\alpha_1 + \alpha_2$ do coincide. Henceforth, from equation (5), the first N coefficients of ${}^cR_{\alpha_1+\alpha_2}$ and ${}^cR_{X+Y}$ do coincide. Since N is arbitrary, equation (23) gives the conclusion.

The proof for (ii) is similar, taking $(i = 1, 2)$

$$\beta_i = \pi\Big((\mathbf{Id} + a^*_{e_i})f_i(a_{e_i})\Big) + \pi\Big((\mathbf{Id} + a^*_{e_i})f_i(a_{e_i})\Big)A^*_{\eta_i} + A_{\eta_i, F_i^{\otimes}}$$

where f_1, F_1, respectively f_2, F_2 are now the polynomials given by the truncation of order N of T_X and ${}^cT_X$, respectively T_Y and ${}^cT_Y$. □

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