Enumeration of closed random walks in the square lattice according to their areas

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Abstract

We study the area distribution of closed walks of length n, beginning and ending at the origin. The concept of area of a walk in the square lattice is generalized and the usefulness of the new concept is demonstrated through a simple argument. It is concluded that the number of walks of length n and area s equals to the coefficient of z^s in the expression $(x + x^{-1} + y + y^{-1})^n$, where the calculations are performed in a special group ring $R[x, y, z]$. A polynomial time algorithm for calculating these values, is then concluded. Finally, the provided algorithm and the results of implementation are compared with previous works.

Subject classification: 40.09, 70.04

1 Introduction

The problem of finding the area distribution of random walks of a given length is an interesting problem which has many applications, for instance in conformations of polymers and proteins (see [6, 1] and the references there in). When $n \to \infty$ this distribution was computed first by Lévy using Brownian paths. In [3], non-commutative geometry techniques are applied to the Harper equation and the asymptotic distribution of the area, enclosed by a random walk in the square lattice, is provided. This approach is also studied in [2]. In [7] the authors used a more complicated and more informative method to derive the above asymptotic distribution. They have used combinatorial arguments, in which, the enumeration of up-down permutations and the exponential formula for cycles of permutations play fundamental roles. The asymptotic formula in [3] is compared with exact results obtained by computers: For this purpose, the closed walks of lengthes $n = 16$ $n = 18$ and $n = 20$ are enumerated according to their areas. These computations are then extended in [1] to $n = 28$ by using algorithmic techniques and a DSP processor. However, the algorithm used there, is based on generating walks, and thus the required time grows exponentially with respect to the length of walks. Here, we present a polynomial time algorithm for the same enumeration problem. As examples we present the results of implementation for $n = 32, 64, 128$ in the appendix.

This paper is organized as follows: In Section 2, we generalize the concept of area for a desired (not necessarily closed) walk in the square lattice. We see that the area of composition of two walks satisfies a nice relation and this approves the usefulness of the generalized concept. In Section 3, we show that previous generalization naturally lead to algebraic tools, namely a generating function $\mathfrak{w}^n = (x + x^{-1} + y + y^{-1})^n$ and a group ring $R[x, y, z]$ which together simplify the process of computation of the area distribution. This is explained in Theorem 3.2 and the details following it. The algorithm is then discussed in Section 4 and some selected results obtained from the implementation are given in tables in the appendix.

2 Area of walks in the square lattice

It is well-known that the algebraic area corresponding to a polygon $\mathbf{P} = A_0 A_1 \ldots A_{n-1}$ with coordinates $A_i = (x_i, y_i), 0 \leq i \leq n-1$ and $(x_n, y_n) = (x_0, y_0)$, is as follows

$$S(\mathbf{P}) = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i). \tag{1}$$

If moreover, the sides of the polygon are parallel to the axes x or y i.e. if the identity

$$(x_i - x_{i+1})(y_i - y_{i+1}) = 0, \tag{2}$$

holds for $0 \leq i \leq n-1$, then

$$S(\mathbf{P}) = \sum_{i=0}^{n-1} x_i (y_{i+1} - y_i) \ . \tag{3}$$

By setting $\epsilon_i = y_{i+1} - y_i$ and $\delta_j = x_{j+1} - x_j$, it is obtained that

$$S(\mathbf{P}) = \sum_{0\le j<i\le n-1} \epsilon_i \delta_j \ . \tag{4}$$

Now, consider a more general case where we have just a sequence of vertices $\mathbf{Q} = A_0 A_1 \dots A_n$ in which the first and the last points are not necessarily the same, but the coordinates satisfy condition (2) for $0 \le i \le n-1$.
In this case (4) can be defined as the area of $\mathbf{Q}$. To rewrite this formula in terms of coordinates x_i and y_i, we do as follows:

$$\begin{aligned} S(\mathbf{Q}) &= \sum_{i=0}^{n-1} \epsilon_i \sum_{j=0}^{i-1} \delta_j \\ &= \sum_{i=0}^{n-1} \epsilon_i (x_i - x_0) \end{aligned}$$

and since $\sum_{i=0}^{n-1} \epsilon_i = y_n - y_0$, the area is obtained as

$$S(\mathbf{Q}) = \sum_{i=0}^{n-1} x_i(y_{i+1} - y_i) + x_0(y_0 - y_n). \tag{5}$$

This equation has a simple geometric interpretation: The area of $\mathbf{Q}$ equals the area of the following polygon

$$\mathbf{Q}_c = [(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n), (x_0, y_n), (x_0, y_0)].$$

Now, consider a walk beginning at (x_0, y_0) in the square lattice. Each step is just moving one unit to right, left, up or down. We may denote these moves simply by r, $\bar{r}$, u, and $\bar{u}$. Thus a walk of length n may be demonstrated by its first point (x_0, y_0) and a word $w = w_0 \dots w_{n-1}$ of length n over the alphabet $\mathcal{A} = \{r, \bar{r}, u, \bar{u}\}$. This walk can alternatively be demonstrated by corresponding sequence of vertices

$$W = [(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)],$$

which not only satisfies condition 2, but also the following equalities for $0 \le i \le n-1$:

$$\{|x_{i+1} - x_i|, |y_{i+1} - y_i|\} = \{0, 1\}. \tag{6}$$

Thus the equation (5), can naturally be used for the area of a walk in $\mathbb{Z}^2$. The area of such a walk is obviously independent from the beginning point (x_0, y_0) and depends only on the word w.

Two walks can simply be composed as follows: Let

$$W = [(x_0, y_0), (x_1, y_1) \dots, (x_n, y_n)],$$

$$W' = [(x'_0, y'_0), (x'_1, y'_1), \dots, (x'_m, y'_m)].$$

Then we define the composed walk WW' as

$$WW' = [(x_0, y_0), \dots, (x_n, y_n), (x_{n+1}, y_{n+1}), \dots, (x_{n+m}, y_{n+m})],$$

where $x_{n+i} = x_n + x'_i - x'_0$ and $y_{n+i} = y_n + y'_i - y'_0$ for $1 \le i \le m$. This composition corresponds to concatenation of words w and w'. It is easy to prove that

$$S(WW') = S(W) + S(W') + (x_n - x_0)(y'_m - y'_0).$$

Let $(x_0, y_0) = (x'_0, y'_0) = (0, 0), (x_n, y_n) = (i, j), (x_m, y_m) = (i', j')$. Then

$$S(WW') = S(W) + S(W') + ij'. \tag{7}$$

The geometric interpretation of this fact is demonstrated in Figure 1.

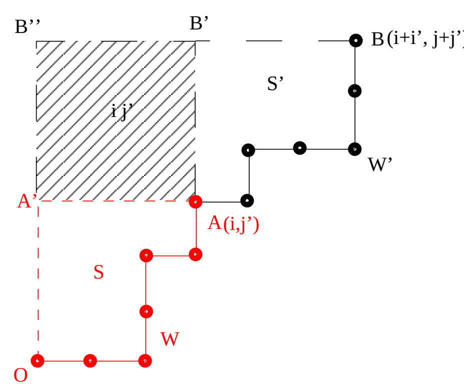

Figure 1

3 A generating function, a new multiplication and algorithmic results

The following enumerative result about the number of walks in the square lattice is well-known and easy to prove :

Lemma 3.1 *Let $\mathfrak{u} = \alpha + \alpha^{-1} + \beta + \beta^{-1}$ and denote the number of walks in the square lattice which begin at the origin and end in (p, q) by $a_n(p, q)$. Then*

(i) The number $a_n(p, q)$ equals the coefficient of $\alpha^p \beta^q$ in $\mathfrak{u}^n$.

(ii) We have $a_n(p, q) = \binom{n}{\frac{n+p+q}{2}}\binom{n}{\frac{n-p+q}{2}}$.

Notation. In this paper, we use the notation $[x^i y^j z^s]F(x, y, z)$, to show the coefficient of $x^i y^j z^s$ in $F(x, y, z)$. Thus Lemma 3.1(i), states that

$$a_n(p, q) = [\alpha^p \beta^q]\mathfrak{u}^n.$$

We remark that part (ii) of the above lemma can be proved either as an immediate consequence of part (i) or independently through a combinatorial argument by projecting the walks on axes $y = x$ and $y = -x$ (This technique is well-known, see for instance Proposition 2.3 of [7] and page 451 of [5]).

Now, let $w_n(i, j, s)$ be the number of walks beginning at the origin and ending in (i, j) and having algebraic area s. Then by using identity (7) we have the following enumerative identity:

$$w_{n+m}(i, j, s) = \sum_{I_{ijs}} w_n(i_1, j_1, s_1) w_m(i_2, j_2, s_2), \tag{8}$$

where I_{ijs} consists of the set of pairs of integer triples (i_1, j_1, s_1) and (i_2, j_2, s_2) which satisfy the following set of equations:

$$i_1 + i_2 = i, \; j_1 + j_2 = j, \; s_1 + s_2 + i_1 j_2 = s. \tag{9}$$

Equations (8) and (9) lead us to study the multiplication of two monomials $X = x^i y^j z^s$ and $X' = x^{i'} y^{j'} z^{s'}$ defined as

$$x^i y^j z^s . x^{i'} y^{j'} z^{s'} = x^{i+i'} y^{j+j'} z^{s+s'+ij'} \tag{10}$$

It is easy to check that the set of all monomials $x^i y^j z^s$ with this multiplication construct a non-commutative group. Note that the element $x^i y^j z^s$ is just a representation of an element of a group (which may also be represented just by the triple (i, j, s)) and in general does not equal $x^i . y^j . z^s$ (for instance $x.y.z = xyz^2$). This leads to construct a group ring with elements $\sum a(i, j, s) x^i y^j z^s$ with finitely many nonzero coefficients which come from a ring, say $\mathbb{R}$. Now considering the group ring $R[x, y, z]$, the following theorem is a generalization of Lemma 3.1:

Theorem 3.2 *Let $\mathfrak{w} = x+x^{-1}+y+y^{-1}$ and denote the number of walks in the square lattice which begin at the origin and end in (p,q) and has area s by $w_n(p,q,s)$. Then The number $w_n(p,q,s)$ equals the coefficient of $x^p y^q z^s$ in $\mathfrak{w}^n$, where computations are performed in $R[x,y,z]$:*

$$w_n(p,q,s) = [x^p y^q z^s]\mathfrak{w}^n$$

.

Obviously the mapping $\tau : R[x,y,z] \to R[\alpha,\beta]$ defined by $\tau(x^i y^j z^s) = \alpha^i \beta^j$ constructs a ring homomorphism. Note that calculating an expression of the form $(x+x^{-1}+y+y^{-1})^n$ is an idea, which is not only common between theorem 3.2 and Lemma 3.1, but also it is used in [3]. The important point here, is that in our approach, the function $\mathfrak{w}$ and the group ring $R[x,y,z]$, both appear as a conclusion of a purely combinatorial discussion by a proper extension of the concept of area for open walks.

Example 1 It is easily checked that

$$\begin{aligned}\mathfrak{w}^2 = &(x+x^{-1}+y+y^{-1}).(x+x^{-1}+y+y^{-1})\\ =&x^2+1+xy+xy^{-1}+1+x^{-2}+x^{-1}y+x^{-1}y^{-1}\\ &+xyz+x^{-1}yz^{-1}+y^2+1+xy^{-1}z^{-1}+x^{-1}y^{-1}z+1+y^{-2}\\ =&x^2+x^{-2}+y^2+y^{-2}+4+xy+xy^{-1}+x^{-1}y+x^{-1}y^{-1}\\ &+xyz+x^{-1}yz^{-1}+xy^{-1}z^{-1}+x^{-1}y^{-1}z.\end{aligned}$$

The terms of the form $axyz^i$ are xy and xyz. This means that there are just two walks of length 2 which begin from the origin and end in $(1,1)$: one walk with area 0 and another one with area 1.

Example 2 Again consider the previous example. If we want to answer the same question about walks of length 4, it is enough to calculate the terms $axyz^i$ in $\mathfrak{w}^4$. But using $\mathfrak{w}^4 = (\mathfrak{w}^2)^2$ and applying the above result, these terms are as follows:

$$\begin{aligned}&x^2.x^{-1}y+x^2.x^{-1}yz^{-1}+y^2.xy^{-1}+y^2.xy^{-1}z^{-1}+4xy+4xyz+\\ &4xy+xy^{-1}.y^2+x^{-1}y.x^2+4xyz+x^{-1}yz^{-1}.x^2+xy^{-1}z^{-1}.y^2\\ =\ &xy(2z^2+10z+10+2z^{-1}).\end{aligned}$$

4 An algorithm to enumerate walks

Similar to examples 1 and 2 one can compute the expression $\mathfrak{w}^n$ for given values of n to obtain values $w_n(i,j,s)$. Of course this can be done for any positive integer n (not only powers of 2,) by calculating expressions of form $\mathfrak{w}^{n_1}.\mathfrak{w}^{n_2}$ at most $\lg(n)$ times. To analyze the provided algorithm, first note that if $|i|+|j|>n$ or if $n+i+j$ is odd or if $|s|>\frac{n^2}{4}$, then $w_n(i,j,s)=0$ (More generally, if $|s|>\frac{(n+|i|+|j|)^2}{16}$ then $w_n(i,j,s)=0$). Thus the expression $\mathfrak{w}^n$ has $O(n^4)$ nonzero terms and computation of the expression $\mathfrak{w}^{n_1}.\mathfrak{w}^{n_2}$ needs totally $O({n_1}^4{n_2}^4)$ multiplications. Thus for calculating $\mathfrak{w}^n$, the number of required integer multiplications is $O(n^8\lg(n))$. However, as n grows larger, the coefficients $w_n(i,j,s)$ grow exponentially and should be considered as "large integers" (instead of integers) and the integer multiplication should not be considered as a unitary operation. (For computation with large integers, see for instance [4]). This problem is resolved by using modular arithmetic as follows: Choose k and distinct prime integers $p_1,\cdots,p_k$ such that $w_n(i,j,s)<p_1\cdots p_k$ (It is enough to select these numbers such that $p_1\cdots p_k>4^n$). For any i, $1\le i\le k$, calculate the coefficients of $\mathfrak{w}^n \mod p_i$. Finally for each s with $s\le n^2/16$ a sequence $t_1,\cdots,t_k$ with $w_n(0,0,s)\equiv t_i \pmod{p_i}$ is obtained. Thus the values $w_n(0,0,s)$ can easily be reconstructed using Chinese reminder theorem. Since $k=O(\lg(n))$, the complexity of our algorithm in this case (i.e. when 4^n is a large integer), is computed as $O(n^8\lg^2(n))$.

We have implemented this for $n=8,16,32,64$; moreover we have obtained the terms $w_{128}(0,0,s)$ in the expression $\mathfrak{w}^{128}$ (As mentioned before, since the coefficients are large for $n=64,128$, we have used some modular arithmetic to simplify the calculations). For any even integer n, $w_n(0,0,s)$ is a unimodal sequence which is symmetric with respect to the axis $s=0$ and takes its maximum at $s=0$. The results of computations are briefly demonstrated in tables 1.1, 1.2, 2.1, 2.2 (Due to the symmetry, negative values of s are omitted from these tables). Histograms of the number of closed walks with corresponding areas are demonstrated in table 1.1 for $n=16,32,64$ and in table 1.2 for $n=128$. We have used sampling of areas to estimate the whole distribution of walks with respect to their area in tables 2.1 for $n=32,64$ and in table 2.2 for $n=128$. We remark that it is possible to improve the complexity of this algorithm to $O(n^6\lg^3(n))$ by some modifications. However, we do not need to implement this modified version for $n\le 128$.

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Table 1.1. Histogram of the number of closed walks for $n = 16, 32, 64$.

Area s	$n = 16$	$n = 32$	$n = 64$
0	33820044	3581690, 9974343308	165545, 3003285874, 5794673311, 4483378060
1	28133728	3444028, 9607452416	163951, 1042472083, 1102818268, 2298389120
2	18569808	3073077, 4567275040	159289, 6955895232, 9652405885, 9706370944
3	10127744	2565083, 0257099008	151905, 3681689474, 6366542956, 7889122560
4	5015108	2025070, 2695492528	142312, 9428052661, 5202704645, 6952968352
5	2289760	1529170, 0875844800	131125, 1853389455, 8748104447, 7490274432
6	1036368	1116254, 1356438464	118976, 8131598077, 3546151378, 9923455744
7	435040	794324, 3235665408	106459, 4734512548, 5432356922, 6277637888
8	184104	554823, 8812436036	94076, 4049114445, 2830982857, 7473179632
9	73056	382197, 6073766784	82219, 0388885868, 0062962560, 1052407296
10	28064	260613, 1312522976	71162, 7986902081, 4394878378, 0740454720
11	10336	176316, 8848622336	61076, 6841799750, 4289859727, 5624957312
12	3760	118576, 4173049648	52040, 7460865277, 2477622381, 7129090240
13	1088	79335, 0438261504	44066, 4566930747, 9914521591, 0107701504
14	352	52859, 9386326560	37116, 4869321275, 0343143165, 4198093312
15	96	35083, 2035517248	31121, 8811662498, 3702411709, 7801894016
16	16	23202, 5420494728	25995, 8037652707, 6596386739, 4450277316
17	0	15290, 8309279936	21643, 8097824910, 3762445371, 2586987264
18	0	10044, 4272588768	17971, 0315275440, 8547703288, 0937254400
19	0	6574, 8927845440	14886, 8517633991, 1765417370, 3945559808
20	0	4289, 6632260736	12307, 6599368221, 5520959165, 4379143840
21	0	2788, 4857169280	10158, 2235010504, 6499042413, 6544733440
22	0	1806, 4537994848	8372, 1122544237, 1269186309, 3162306688
23	0	1165, 8943874752	6891, 5113696382, 9340030299, 8512645376
24	0	749, 7160572048	5666, 6701469563, 7412442711, 0537836464
25	0	480, 1211445312	4655, 1587087675, 5085653450, 8420517888
26	0	306, 2945599680	3821, 0488492849, 7035370375, 8706331264
27	0	194, 5755843520	3134, 0926012667, 5420527449, 4192796416
28	0	123, 0912937696	2568, 9430707074, 7787699462, 7648267040
29	0	77, 5044394624	2104, 4412234929, 2181969190, 3419356416
30	0	48, 5883898144	1722, 9792841968, 7831969876, 7886027968
31	0	30, 3067180160	1409, 9425700569, 3405222424, 1279150592
32	0	18, 8158770672	1153, 2267976836, 0220074562, 9419251848
33	0	11, 6190755520	942, 8248707710, 9783233726, 8937696128
34	0	7, 1372120768	770, 4760703796, 4923852241, 8443915072
35	0	4, 3588560640	629, 3700506407, 3095007510, 8661437568
36	0	2, 6468754368	513, 8983895305, 2165176788, 5718002464
37	0	1, 5971326400	419, 4468635943, 1128763468, 3935688320
38	0	9580227072	342, 2223782721, 8792023636, 1272721856
39	0	5704976448	279, 1091340556, 7526383551, 8177452288
40	0	3374362720	227, 5493796993, 4261990343, 2651390528
41	0	1979897600	185, 4447127237, 3591685480, 3318881920
42	0	1153531776	151, 0745274528, 2500466045, 4542766592
43	0	665930496	123, 0287071199, 9349261843, 0645507456
44	0	381403552	100, 1521446247, 9515175280, 4221836704
45	0	216272192	81, 4990481851, 9987650971, 6280904448
46	0	121397120	66, 2953446385, 4157940925, 1272815296
47	0	67391168	53, 9077623054, 4736632512, 9024635136
48	0	37007392	43, 8184278757, 0688296731, 2469700240
49	0	20046912	35, 6040025857, 9755520652, 8140275072
50	0	10730048	28, 9185589829, 9232653138, 8592603584

Table 1.2. Histogram of the number of closed walks for $n = 128$.

Area s	$n = 128$
0	1410, 7033892003, 4556275957, 3855536443, 1713372583, 8745556276, 5835946782, 8699656588
1	1407, 3024540489, 7561178225, 1492421016, 5903644384, 0838340709, 2111460482, 4937387520
2	1397, 1649733707, 8547470736, 0499774263, 8920388680, 8359080398, 3163318225, 4162311040
3	1380, 4842678848, 7302789379, 1645965215, 5579137301, 3499661185, 3647061656, 3763726848
4	1357, 5738773060, 9946518605, 0585703091, 4103655038, 8810690309, 8680556206, 4476333696
5	1328, 8551923721, 7677597822, 7209730464, 4139815783, 0161065810, 1704581134, 1916138496
6	1294, 8413214797, 3192840866, 6424244840, 7914453824, 1990913897, 2602303118, 5542646784
7	1256, 1182279806, 0973554574, 6222544230, 8237684539, 6052335327, 6433750005, 5423763456
8	1213, 3242855140, 0028030693, 5376370151, 6590699200, 7160036130, 3615131804, 7547629472
9	1167, 1294078548, 3705936097, 4700188169, 9134665437, 9964875422, 2126644178, 1188844032
10	1118, 2148257931, 4876464340, 5178198327, 6272565882, 5003428615, 7423056334, 4507678592
11	1067, 2544253010, 0491533514, 7894490482, 6439471597, 5277883548, 2780506254, 9148335104
12	1014, 8983533104, 4634867110, 1206819214, 1193382594, 8253734997, 0143386378, 9088034176
13	961, 7593661773, 7189043046, 3337315981, 8652756444, 2697345733, 0322031818, 4291592704
14	908, 4021657996, 3631929848, 5392789504, 0553785827, 1399264505, 9083287686, 8407903488
15	855, 3357592722, 7904570583, 4071833016, 3005561629, 6188755045, 2681666035, 3268214784
16	803, 0087037290, 4609496133, 5353158915, 5592171685, 1608553021, 1235191565, 6504449136
17	751, 8069661057, 2813072696, 6629583064, 1148708036, 7436477079, 4895846268, 6542646784
18	702, 0540396862, 5146881526, 9860430186, 5491459096, 2468812491, 6045345873, 6116161536
19	654, 0129126309, 4424638675, 1129960649, 6515635171, 2836056527, 8268318381, 3914975488
20	607, 8894723270, 7517045571, 1069505624, 9654909825, 9691568380, 4579410665, 3886321856
21	563, 8369457848, 1328923268, 0251406724, 5624510978, 4465364091, 8338626540, 2448514048
22	521, 9610124403, 1971690875, 7285266312, 1361048967, 8464554560, 1550600804, 2947782528
23	482, 3252741185, 5977063374, 9204570737, 0096681055, 7983960514, 9055938353, 6273292544
24	444, 9568211123, 2677528406, 7913521950, 2168637496, 3514310616, 1020629339, 0696154560
25	409, 8516882597, 8820945433, 0086445538, 6875648834, 3075383182, 7515093856, 2496782336
26	376, 9800468596, 7878523294, 6468408515, 6838053818, 2117951574, 8956155126, 1898202880
27	346, 2910248525, 7910212051, 9918713267, 6285219824, 3417736320, 7150365881, 6662952192
28	317, 7170876122, 6649370409, 8769310919, 3372955110, 9885406167, 7375795751, 0850541184
29	291, 1779444806, 8780735863, 2682558559, 3943314167, 7299990195, 1784320302, 7017344000
30	266, 5839720083, 3217144183, 2330534321, 5631455953, 9593549729, 2939952449, 8739436928
31	243, 8391642983, 0152229604, 1800095915, 7149399101, 9378288422, 7765033861, 2981857280
32	222, 8436346973, 8653725051, 1841404543, 0931745709, 4233102150, 3754802180, 2932444612
33	203, 4957022281, 8650274258, 8409511283, 0795000912, 6351287804, 8241811701, 0044212480
34	185, 6936015120, 1442388100, 3756227758, 1997286304, 1641800578, 0316304872, 5931503104
35	169, 3368573323, 1966937758, 8587669558, 1677928523, 5336728042, 7009350227, 8754297344
36	154, 3273651764, 1673709820, 8254120503, 0091312669, 4620532769, 1964334067, 6582675712
37	140, 5702177017, 1811870033, 4744950358, 1126325851, 8097400944, 1897482027, 1138028544
38	127, 9743146199, 7216121242, 6590547944, 5795679103, 3517158052, 5703219655, 5346880768
39	116, 4527903905, 2565093234, 5991490752, 6931680721, 8392377764, 6639015944, 4253611008
40	105, 9232906770, 0813934257, 9389349588, 9826312713, 6299361111, 8150382831, 9241859104
41	96, 3081249848, 4003726572, 9985651294, 9767291595, 5105092725, 2887050944, 2661204480
42	87, 5343194257, 0689809870, 9084153396, 1633523208, 5765141173, 2963363894, 5553767168
43	79, 5335902615, 1001088786, 2229069228, 4944928161, 2471305895, 5160229025, 3471878400
44	72, 2422558339, 9992080512, 9735364612, 5415047397, 5072102242, 8523062539, 9947401984
45	65, 6011017251, 4946098123, 3635319649, 9707716860, 2623541436, 0238379304, 1379316736
46	59, 5552115318, 2629696048, 9600650535, 3649769161, 7005888983, 0456617439, 1624406784
47	54, 0537734745, 4545929513, 5620500429, 2190688647, 1817158178, 7000139728, 0834058496
48	49, 0498711810, 5005814800, 2302396044, 8526299819, 6640254673, 3055286303, 8011851696
49	44, 5002653713, 1010783245, 2854022296, 7683798570, 5477576649, 4078144220, 8843469312
50	40, 3651717975, 4268971940, 8275974742, 9828479525, 6161271917, 9464626860, 0827549440

Table 2.1. Values of $w_{32}(0,0,2k)$ and $w_{64}(0,0,8k)$

Area s	$n = 32$	Area s	$n = 64$
0	35816909974343308	0	165545, 3003285874, 5794673311, 4483378060
2	30730774567275040	8	94076, 4049114445, 2830982857, 7473179632
4	20250702695492528	16	25995, 8037652707, 6596386739, 4450277316
6	11162541356438464	24	5666, 6701469563, 7412442711, 0537836464
8	5548238812436036	32	1153, 2267976836, 0220074562, 9419251848
10	2606131312522976	40	227, 5493796993, 4261990343, 2651390528
12	1185764173049648	48	43, 8184278757, 0688296731, 2469700240
14	528599386326560	56	8, 2365472231, 4016874589, 9460134624
16	232025420494728	64	1, 5096688485, 6768973162, 6864590576
18	100444272588768	72	2694437402, 9152872362, 8592916864
20	42896632260736	80	467552007, 4041838163, 0604620576
22	18064537994848	88	78744016, 9083977884, 8133019296
24	7497160572048	96	12846824, 7250409523, 5350912480
26	3062945599680	104	2025952, 5183890308, 8567109088
28	1230912937696	112	308080, 5223088698, 5702239872
30	485883898144	120	45051, 1484837085, 3522061280
32	188158770672	128	6315, 2306732457, 4256973920
34	71372120768	136	845, 5556839591, 3528941504
36	26468754368	144	107, 6805280447, 7651259776
38	9580227072	152	12, 9785607608, 7995391104
40	3374362720	160	1, 4718233876, 4286499776
42	1153531776	168	1559345348, 1572357568
44	381403552	176	153004502, 2583865088
46	121397120	184	13753291, 0981167232
48	37007392	192	1116802, 6179713536
50	10730048	200	80423, 8839635904
52	2932896	208	5007, 3152157248
54	743168	216	259, 8435002240
56	172224	224	10, 6187399552
58	35392	232	3102345664
60	5984	240	53445952
62	704	248	337280
64	32	256	64

Table 2.2. Values of $w_{128}(0,0,32k)$

Area s	$n = 128$
0	1410, 7033892003, 4556275957, 3855536443, 1713372583, 8745556276, 5835946782, 8699656588
32	222, 8436346973, 8653725051, 1841404543, 0931745709, 4233102150, 3754802180, 2932444612
64	10, 1648073494, 4923419172, 3106847671, 7720107384, 7236429835, 9359307733, 6892311304
96	4138993608, 0697485547, 8360086674, 4923420517, 9089245541, 4042051861, 9613801232
128	161039050, 5304195851, 5419275033, 5083133903, 1254858120, 1415241780, 7554992752
160	5990655, 6605275424, 0970303298, 7170349389, 5743855708, 8743098925, 7949721120
192	212607, 0814985186, 5951934575, 3598563447, 3878472768, 9106908764, 3180434272
224	7180, 6854685731, 4031517786, 4997582279, 9706234195, 6090873096, 6015299584
256	230, 1750942525, 0140859435, 7574600133, 2584273010, 4614639894, 3043516640
288	6, 9816042224, 1138277250, 0582452617, 1692631908, 7011142507, 9215100160
320	1997190758, 6553543306, 5729674881, 2193385181, 4767886062, 4658510144
352	53685272, 9803391380, 6122089350, 6109205977, 4658578703, 0650139264
384	1350455, 5535411518, 0818176830, 9757666811, 6769796180, 3565932544
416	31644, 2527945050, 2410030323, 1585016453, 7594319542, 2645555776
448	687, 1315164172, 5008635859, 1936569312, 0498550867, 2721950208
480	13, 7450595212, 7015217478, 1091307692, 8909430481, 5008168896
512	2515774906, 7368382551, 6728163527, 5448943450, 0831010624
544	41803611, 2117105753, 9735632910, 5539623620, 6960007680
576	624889, 6425097710, 2464041926, 3546679600, 4985488768
608	8312, 9826788959, 9447485149, 7366840822, 7746173056
640	97, 1577615295, 3509138500, 6659985912, 5226822016
672	9821475660, 0178800451, 7922679241, 8657940480
704	84231795, 4751454140, 9292439100, 2628160000
736	598159, 1733186708, 6431903566, 6217660160
768	3408, 4164367454, 1023022248, 2423726336
800	14, 9447433526, 3720269236, 4107405952
832	475760748, 0371846789, 7834797824
864	1010220, 8082355332, 7727527680
896	1253, 0843615061, 8578411264
928	7227374040, 4149925760
960	1233125, 5335552896
992	19, 8894005504
1024	128