FORBIDDEN INDUCED SUBGRAPHS OF DOUBLE-SPLIT GRAPHS

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ABSTRACT. In the course of proving the strong perfect graph theorem, Chudnovsky, Robertson, Seymour, and Thomas showed that every perfect graph either belongs to one of five basic classes or admits one of several decompositions. Four of the basic classes are closed under taking induced subgraphs (and have known forbidden subgraph characterizations), while the fifth one, consisting of double-split graphs, is not.

A graph is doubled if it is an induced subgraph of a double-split graph. We find the forbidden induced subgraph characterization of doubled graphs; it contains 44 graphs.

1. INTRODUCTION

A key ingredient in the proof of the strong perfect graph theorem by Chudnovsky, Robertson, Seymour, and Thomas [CRST06] is a decomposition theorem for all perfect graphs. This decomposition theorem states that all perfect graphs either belong to one of five basic classes or admit one of several decompositions. The five basic classes are bipartite graphs, complements of bipartite graphs, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and double-split graphs. The first four classes are closed under taking induced subgraphs and have known characterizations in terms of minimal forbidden induced subgraphs. Indeed, a forbidden induced subgraph characterization is known for the union of these four classes [ZZ05]. However, double-split graphs are not closed under taking induced subgraphs, and hence do not have such a characterization.

In this paper, we consider the downward closure of double-split graphs under induced subgraphs (that is, double-split graphs and all of their induced subgraphs) and we characterize this class in terms of minimal forbidden induced subgraphs. Unlike the lists for the other four basic classes, the one for this class of graphs is finite.

All graphs considered in this paper are finite and have no loops or multiple edges. For a graph G we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The complement of G is denoted by $\overline{G}$. A *clique* in a graph G is a set of vertices all pairwise adjacent and a *stable set* is a clique in $\overline{G}$. For $A \subseteq V(G)$, we denote the subgraph of G induced on A by $G|A$, sometimes further abbreviating $G|\{u, v, w\}$ by $G|uvw$. The notation $G \cong H$ means G is isomorphic to H. For $v \in V(G)$, we denote the set of neighbors of v in G by $N_G(v)$ and for $X \subseteq V(G)$, we denote by $N_X(v)$ the set of neighbors of v in $G|X$.

Let $X, Y \subseteq V(G)$ with $X \cap Y = \varnothing$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* if no vertex of X is adjacent to a member of Y. For an integer $i \geq 0$, let P_i, C_i denote the path and cycle with i edges, respectively.

For integers $a, b \geq 0$, let $M_{a,b}$ be the graph on $2a + b$ vertices consisting of the disjoint union of a edges and b isolated vertices. We say that a graph G is *semi-matched* if it is isomorphic to $M_{a,b}$ for some $a, b \geq 0$ and we say that it is *matched* if in addition $b = 0$. Similarly, we say that G is *semi-antimatched* if it is isomorphic to some $\overline{M_{a,b}}$ and *antimatched* if in addition $b = 0$.

Let $A, B \subseteq V(G)$ such that $A \cap B = \varnothing$, A is semi-matched, and B is semi-antimatched. We say that A and B are *aligned* if the following holds:
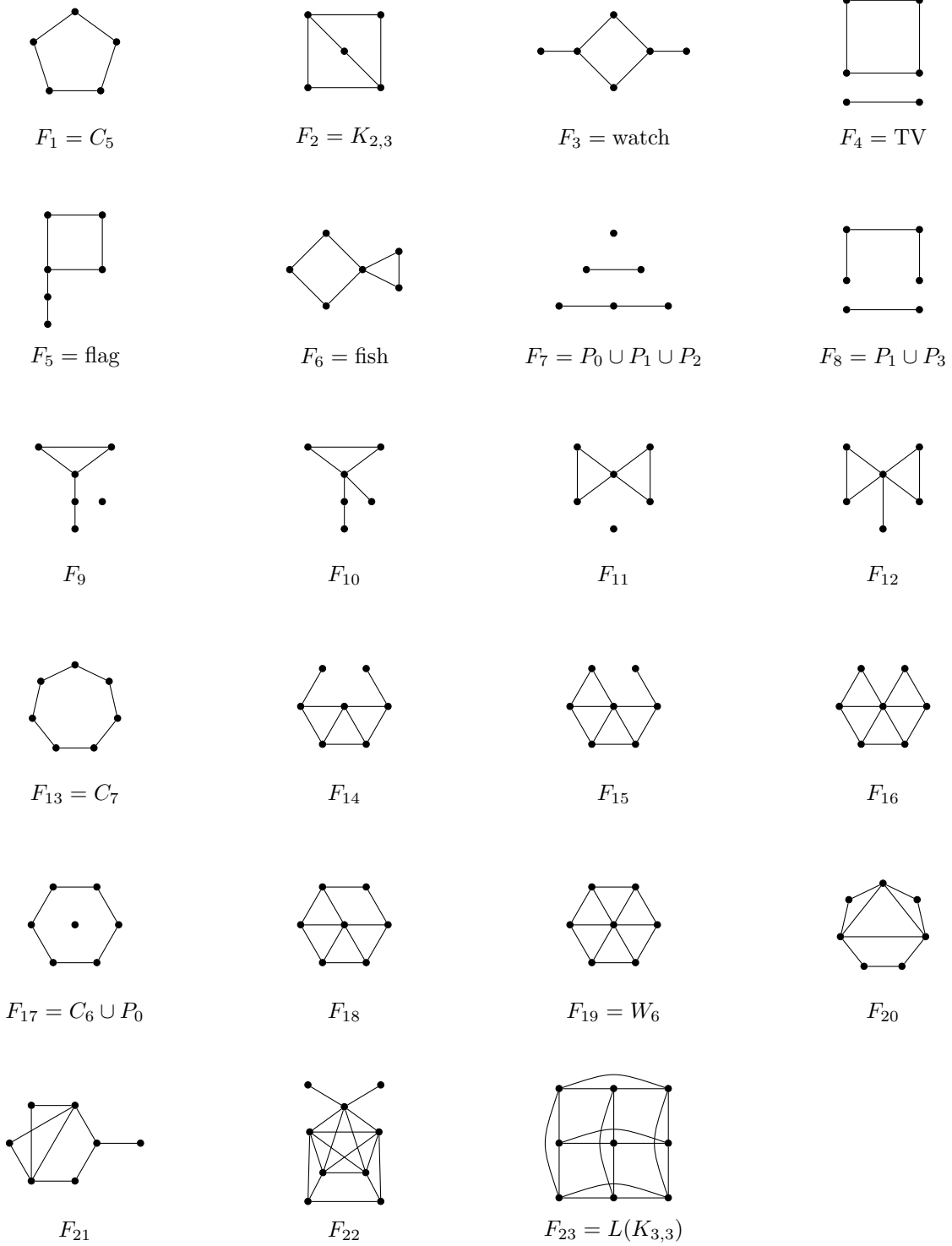
$F_1 = C_5$ $F_2 = K_{2,3}$ $F_3 = \text{watch}$ $F_4 = \text{TV}$

$F_5 = \text{flag}$ $F_6 = \text{fish}$ $F_7 = P_0 \cup P_1 \cup P_2$ $F_8 = P_1 \cup P_3$

F_9 F_{10} F_{11} F_{12}

$F_{13} = C_7$ F_{14} F_{15} F_{16}

$F_{17} = C_6 \cup P_0$ F_{18} $F_{19} = W_6$ F_{20}

F_{21} F_{22} $F_{23} = L(K_{3,3})$

FIGURE 1. The family $\mathcal{F}$: these 23 graphs, and their complements, are the minimal forbidden induced subgraphs for double-split graphs. Only $F_1 = C_5$ and $F_{23} = L(K_{3,3})$ are self-complementary.

- for all adjacent $u, v \in A$ and all $w \in B$, w is adjacent to exactly one of u and v
- for all $u \in A$ and non-adjacent $x, y \in B$, u is adjacent to exactly one of x and y.

A graph G is *split* if its vertex set $V(G)$ can be partitioned into a clique and a stable set. A graph G is *double-split* if its vertex set $V(G)$ can be partitioned into two sets, A and B, such that the following holds:

- $G|A$ is matched,
- $G|B$ is antimatched, and
- A and B are aligned.

It is easy to see that every split graph is an induced subgraph of many double-split graphs. Also, every induced subgraph of a split graph is also split. Split graphs have a well-known forbidden induced subgraphs characterization:

1.1. *[Foldes and Hammer* [FH77]*] A graph is split if and only if it does not contain* $C_4, \overline{C_4}$, *or* C_5 *as an induced subgraph.*

In this paper we consider a class of graphs that includes both split and double-split graphs. We say a graph G is *doubled* if there exists a double-split graph H that contains G as an induced subgraph. Notice that a graph G is double-split if and only if $\overline{G}$ is double-split, and hence a graph G is doubled if and only if $\overline{G}$ is doubled. The main result of this paper is the following:

1.2. *A graph is doubled if and only if it does not contain any graphs in* $\mathcal{F}$, *the family of graphs illustrated in Figure 1.*

It follows that $\mathcal{F}$ is the list of minimal forbidden induced subgraphs for double-split graphs. The idea for our proof of 1.2 is as follows. To prove the "if" part of 1.2, we assume that G is not split, hence contains one of C_4, $\overline{C_4}$, and C_5. Since C_5 is in $\mathcal{F}$ and the class of doubled graphs is self complementary, we may assume that G has C_4 as an induced subgraph. However, since C_4 is a doubled graph in two different ways (all four vertices can appear on the anti-matched side or 2 vertices can appear on the matched side and the other 2 vertices on the semi-antimatched side), there is no easy procedure to partition the remaining vertices of the graph. To avoid this obstacle, we introduce another class of graphs that lies inbetween the class of split graphs and the class of doubled graphs. In section 2, we find the forbidden induced subgraph characterization for this class and we use this characterization to prove 1.2 in section 3.

2. Almost-split graphs

We say a graph G is *almost-split* if G is doubled and there exists $v \in V(G)$ such that $G|(V(G) \setminus \{v\})$ is split. In other words, G is almost-split if there is at most one pair matched or antimatched. Note that every split graph is almost-split and every almost-split graph is doubled. In this section we present the list of forbidden induced subgraphs for the class of almost-split graphs.

2.1. *A graph is almost-split if and only if it does not contain any graphs in the circus, the list of graphs illustrated in Figure 2 along with their complements.*

Proof. The "only if" part is clear, as it is easy to check that none of the graphs in the circus are almost-split. For the "if" part, suppose that G does not contain any graphs in the circus. By 1.1, we may assume that G contains C_4 or $\overline{C_4}$ since split graphs are almost-split. Furthermore, since the statement is self-complementary,
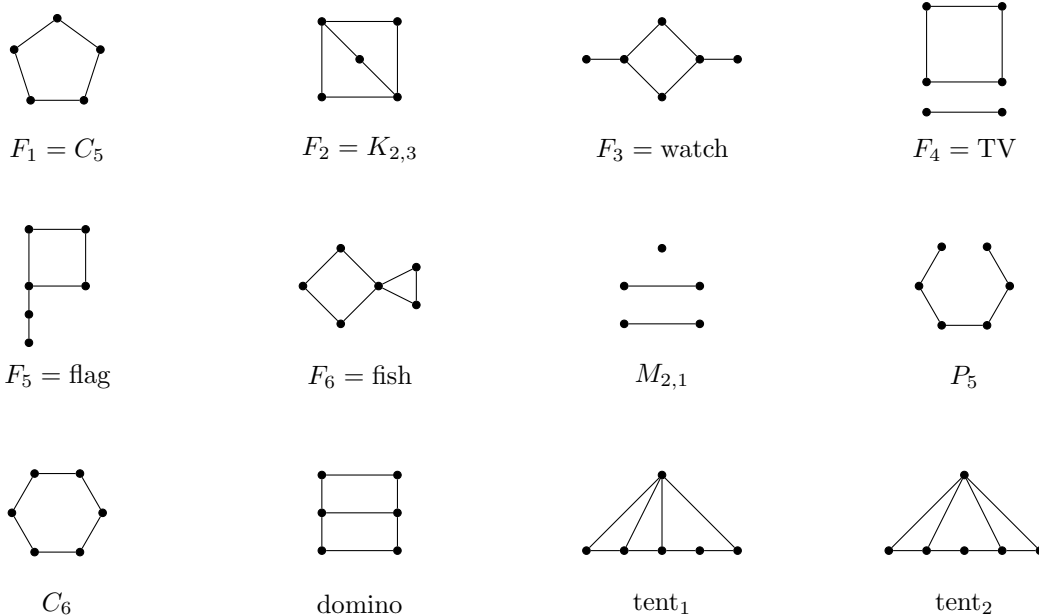
$F_1 = C_5$ $F_2 = K_{2,3}$ $F_3 = $ watch $F_4 = $ TV

$F_5 = $ flag $F_6 = $ fish $M_{2,1}$ P_5

C_6 domino   tent$_1$ tent$_2$

FIGURE 2. The "circus": these 12 graphs, and their complements, are the minimal forbidden induced subgraphs for almost-split graphs.

we may assume that G contains C_4. Let $a, b, c, d \in V(G)$ be such that $G|abcd \cong C_4$ and a is adjacent to b and d. Let $S = \{a, b, c, d\}$.

Since $W_4 \cong \overline{M_{2,1}}$ is in the circus, it follows that for all $v \in V(G)$, v is not complete to S. For $0 \le i \le 3$, let $A_i \subseteq V(G) \setminus S$ denote the set of vertices that have i neighbors in S. Our goal is to show that there exist adjacent $x, y \in S$ such that:

- $A_0 \cup A_1 \cup A_2 \cup \{x, y\}$ contains only one edge (namely xy), and
- $A_3 \cup (S \setminus \{x, y\})$ is a clique, and
- every vertex of $A_3 \cup (S \setminus \{x, y\})$ is adjacent to exactly one of x and y.

(1) If $A_2 \ne \varnothing$, then there exist $x, y \in S$ such that A_2 is complete to $\{x, y\}$. Moreover, A_2 is a stable set.

Let $A_{ab} \subseteq A_2$ be those vertices that are adjacent to a and b, and define $A_{ac}, A_{ad}, A_{bc}, A_{bd}, A_{cd}$ similarly. First suppose that $u \in A_{ac} \cup A_{bd}$; then $G|abcdu \cong K_{2,3}$. Hence, both A_{ac} and A_{bd} are empty. Next suppose there exists $u \in A_{ab}$ and $v \in A_{bc}$. Then either $G|abcduv \cong $ tent$_2$ or $G|acduv \cong C_5$, depending on the adjacency between u and v. Therefore, at least one of A_{ab} and A_{bc} is empty, and from symmetry the same is true for the pairs $\{A_{bc}, A_{cd}\}$, $\{A_{cd}, A_{ad}\}$, and $\{A_{ab}, A_{ad}\}$. We claim that at least one of A_{ab} and A_{cd} is empty. For suppose $u \in A_{ab}$ and $v \in A_{cd}$. Then $G|abcduv \cong \overline{C_6}$ or $G|abcduv \cong \overline{\text{domino}}$, depending on the adjacency between u and v. Similarly, at least one of A_{bc} and A_{ad} is empty. We conclude that at most one of $A_{ab}, A_{ac}, A_{ad}, A_{bc}, A_{bd}$, and A_{cd} is non-empty. Finally suppose that $u, v \in A_2$ are adjacent. Then $G|abcduv \cong \overline{\text{watch}}$. Hence, A_2 is a stable set. This proves (1).

(2) There exist adjacent $x, y \in S$ such that $N_S(A_1) \subseteq \{x, y\}$. Moreover, if $A_2 \ne \varnothing$, then $N_S(A_1) \subseteq N_S(A_2)$.

Let $A_a \subseteq A_1$ be those vertices that are adjacent to a, and define A_b, A_c and A_d similarly. We show that at least one of A_a and A_c is empty. For suppose that $u \in A_a$ and $v \in A_c$. Then either $G|abcduv \cong$ watch or $G|abcuv \cong C_5$, depending on the adjacency between u and v. Similarly, at least one of A_b and A_d is empty. This proves the first part of (2).

Next, let $u \in A_1$ and $v \in A_2$. Suppose that $N_S(A_1) \not\subseteq N_S(A_2)$. From symmetry, we may assume that $u \in A_a$ and $v \in A_{bc}$. But then either $G|abcduv \cong \overline{\text{tent}_1}$ or $G|acduv \cong C_5$, depending on the adjacency between u and v. This proves (2).

(3) $A_0 \cup A_1 \cup A_2$ is a stable set.

First, let $u, v \in A_0$ and suppose that they are adjacent. Then $G|abcduv \cong$ TV. Hence, A_0 is a stable set. Next, suppose $u, v \in A_1$ and suppose that they are adjacent. If u, v have a common neighbor in S then $G|abcduv \cong$ fish. If u, v have different neighbors in S, then by (2) their neighbors are adjacent and so $G|abcduv \cong$ domino. This proves that A_1 is a stable set. Recall that A_2 is a stable set by (1).

Now we show that A_0, A_1, and A_2 are pairwise anticomplete to each other. Let $u \in A_0$, $v \in A_1$ and suppose that u and v are adjacent. Then $G|abcduv \cong$ flag. Next, let $u \in A_0$ and $v \in A_2$ and again suppose that u and v are adjacent. Then $G|abcduv \cong \overline{\text{tent}_2}$. Finally, let $u \in A_1$ and $v \in A_2$ and suppose that they are adjacent. Then $G|abcduv \cong$ tent$_1$. Therefore, we have shown that $A_0 \cup A_1 \cup A_2$ is stable. This proves (3).

(4) There exist adjacent $x, y \in S$ such that A_3 is complete to x, y. Moreover, for all $u \in A_1 \cup A_2$ and $v \in A_3$, $N_S(u) \subseteq N_S(v)$.

Let $A_{abc} \subseteq A_3$ be the set of vertices that are adjacent to a, b and c, and define A_{abd}, A_{acd} and A_{bcd} similarly. We claim that at least one of A_{abc} and A_{acd} is empty. For suppose that $u \in A_{abc}$ and $v \in A_{acd}$. Then either $G|(S \cup \{u, v\}) \cong \overline{\text{TV}}$ or $G|acduv \cong W_4$, depending on the adjacency between u and v. This proves the claim. By a similar argument, at least one of A_{abd} and A_{bcd} is empty. Therefore, there exist (at least) 2 adjacent vertices of S that are complete to A_3.

Next, let $u \in A_1 \cup A_2$ and $v \in A_3$ and suppose that $N_S(u) \not\subseteq N_S(v)$. From symmetry, we may assume that $v \in A_{abc}$. If $u \in A_1$, then $u \in A_d$ and so either $G|abcduv \cong \overline{\text{fish}}$ or $G|acduv \cong K_{2,3}$. So we may assume that $u \in A_2$. Again from symmetry, we may assume that $u \in A_{cd}$. But then either $G|abcduv \cong \overline{\text{flag}}$ or $\overline{P_5}$, depending on the adjacency between u and v. This proves (4).

(5) A_3 is a clique.

Let $u, v \in A_3$ and suppose that they are not adjacent. By (4), there exist adjacent $x, y \in S$ such that A_3 is complete to $\{x, y\}$, and from symmetry we may assume $\{x, y\} = \{a, b\}$. First suppose that $u, v \in A_{abc}$. Then $G|acduv \cong K_{2,3}$. Therefore, A_{abc} is a clique, and similarly so is A_{abd}. Next suppose that $u \in A_{abc}$ and $v \in A_{abd}$. Then $G|abcduv \cong \overline{P_5}$. Hence, A_3 is a clique, and this proves (5).

From (1), (2), and (4), it follows that there exist adjacent $x, y \in S$ such that $A_3 \cup A_2$ is complete to $\{x, y\}$ and $N_S(A_1) \subseteq \{x, y\}$. From symmetry, we may assume that $\{x, y\} = \{a, b\}$. Hence, $A_0 \cup A_1 \cup A_2$ is

anticomplete to $\{c,d\}$. Therefore, by (3), $A_0 \cup A_1 \cup A_2 \cup \{c,d\}$ contains exactly one edge (namely cd). By (4) and (5), $A_3 \cup \{a,b\}$ is a clique. Also, since every member of A_3 is adjacent to exactly 3 members of S, it follows that for all $u \in A_3 \cup \{a,b\}$, u is adjacent to exactly one of c,d. Hence, we have shown that G is almost-split and this proves 2.1. $\qquad\square$

3. Excluding 6 graphs

In the previous section, we have seen the 12 minimal forbidden induced subgraphs (up to taking complements) for almost-split graphs. Six of them are doubled and the other six are not. In this section, we prove that if a graph contains one of these six doubled graphs but no graphs in $\mathcal{F}$, then it is doubled.

3.1. *A graph containing $M_{2,1}$ but no graphs in $\mathcal{F}$ is doubled.*

Proof. Let G be a graph containing $M_{2,1}$ but no graphs in $\mathcal{F}$. Let $G|abcde \cong M_{2,1}$, where bc and de are the two edges; let $S = \{a,b,c,d,e\}$. For $0 \le i \le 4$, let $A_i \subseteq V(G) \setminus S$ denote the set of vertices that have i neighbors in $\{b,c,d,e\}$. Our goal is to show the following:

- $A_1 = A_3 = A_4 = \varnothing$, and
- $G|(A_0 \cup S)$ is semi-matched, and
- $G|A_2$ is semi-antimatched, and
- $A_0 \cup S$ and A_2 are aligned.

Together, these statements imply that G is doubled.

(1) $A_1 = A_3 = A_4 = \varnothing$. *Also, if $v \in A_2$, then v is adjacent to exactly one of b and c, and to exactly one of d and e.*

If $v \in A_1$, then $G|abcdev \cong F_7$ or F_8, depending on the adjacency between v and a. Therefore A_1 is empty. If $v \in A_3$, then $G|abcdev \cong F_9$ or F_{10}, depending on the adjacency between v and a. Therefore A_3 is empty. And if $v \in A_4$, then $G|abcdev \cong F_{11}$ or $G|abcdev \cong F_{12}$, depending on the adjacency between v and a. Therefore A_4 is empty.

Next, let $v \in A_2$. If v is adjacent to b and c, then $G|bcdev \cong \overline{K_{2,3}}$. By symmetry, v is not adjacent to both of d and e. Hence, v is adjacent to exactly one of b and c and to exactly one of d and e. This proves (1).

(2) $G|(A_0 \cup S)$ *is semi-matched.*

First, we claim that at most one vertex $x \in A_0$ is adjacent to a, and if such a vertex x exists, then x is not adjacent to any other vertices in A_0. For suppose there are two vertices $x,y \in A_0$, both adjacent to a. If x and y are adjacent, then $G|abcxy \cong \overline{K_{2,3}}$, and if they are not adjacent, $G|abcdxy \cong F_7$. So there is at most one vertex in A_0 adjacent to a. Moreover if there is a vertex $x \in A_0$ adjacent to a, x is not adjacent to any other vertex $y \in A_0$ since otherwise $G|abcdxy \cong F_7$. This proves the claim.

To prove (2), it is enough to show that there do not exist vertices $u,v,w \in A_0 \cup \{a,b,c,d,e\}$ such that $G|uvw \cong C_3$ or $G|uvw \cong P_2$. If at least one of u,v,w is a member of S, then $G|uvw$ cannot be isomorphic to C_3 nor P_2 by the claim. So we may assume $u,v,w \in A_0$. But now if $G|uvw \cong C_3$, then $G|bcuvw \cong \overline{K_{2,3}}$ and if $G|uvw \cong P_2$, then $G|bcduvw \cong F_7$. This proves (2).

(3) Let $u, v \in A_2$ be non-adjacent. Then $N_{\{b,c,d,e\}}(u)$ is disjoint from $N_{\{b,c,d,e\}}(v)$. Moreover, exactly one of u and v is adjacent to a.

From (1) and by symmetry, we may assume that u is adjacent to b and d. Suppose that v is also adjacent to b and d. Then $G|bcdeuv \cong$ watch. Next, suppose that v is adjacent to b and e (or c and d). Then $G|bdeuv$ (or $G|bcduv$) is isomorphic to C_5. Consequently, v is adjacent to c and e.

Moreover, if u and v are both adjacent to a, then $G|abcuv \cong C_5$ and if u and v are both non-adjacent to a, then $G|abcdeuv \cong F_{17}$. Hence, exactly one of u and v is adjacent to a. This proves (3).

(4) $G|A_2$ is semi-antimatched.

It follows easily from (3) that there is no stable set of size 3 in A_2. Therefore, it is enough to show that there do not exist vertices $u, v, w \in A_2$ such that $G|uvw$ contains exactly one edge (say uv). For contradiction, suppose that such u, v, w exist. From (3) and by symmetry, we may assume that $\{u, v\}$ is complete to $\{b, d\}$, w is complete to $\{c, e\}$, and $N_{\{u,v,w\}}(a)$ is either $\{u, v\}$ or $\{w\}$. In the first case, $G|acuvw \cong \overline{K_{2,3}}$ and in the second case, $G|abuvw \cong \overline{K_{2,3}}$. Therefore, there do not exist $u, v, w \in A_2$ such that $G|uvw$ contains exactly one edge, and this proves (4).

It remains to show that $G|(A_0 \cup S)$ and $G|A_2$ are aligned. In (3), we have shown that for all non-adjacent $u, v \in A_2$ and all $w \in A_0 \cup S$, w is adjacent to exactly one of u and v. Hence, it suffices to show that for all $u \in A_2$ and all adjacent $v, w \in A_0 \cup S$, u is adjacent to exactly one of v, w. So suppose that for some u, v, w as above, u is adjacent to both of v, w. Let $x, y \in A_0 \cup S$ be adjacent such that $\{x, y\}$ is disjoint from $\{v, w\}$ (such x, y exist since $A_0 \cup S$ contain at least two edges). Then $G|uvwxy \cong \overline{K_{2,3}}$. Next, suppose that for some u, v, w as above, u is non-adjacent to both of v, w. Note that by (1), $\{v, w\}$ is disjoint from $\{b, c, d, e\}$. By (1) and without loss of generality, we may assume that u is adjacent to b and d. But then $G|bceuvw \cong F_7$. Therefore G is doubled and this proves 3.1. $\square$

3.2. *A graph containing P_5 but no graphs in $\mathcal{F}$ is doubled.*

Proof. Let G be a graph containing P_5 but no graphs in $\mathcal{F}$. Let $G|abcdef \cong P_5$ where ab, bc, cd, de, and ef are the five edges. By 3.1, we may assume that G or $\overline{G}$ does not contain $M_{2,1}$. Let $S = \{a, b, c, d, e, f\}$. For $0 \le i \le 4$, let $A_i \subseteq V(G) \setminus S$ denote the set of vertices that have i neighbors in $\{b, c, d, e\}$. Our goal is to show the following:

- $A_0 = A_2 = A_4 = \varnothing$, and
- $G|(A_1 \cup \{a, c, d, f\})$ is semi-matched, and
- $G|(A_3 \cup \{b, e\})$ is semi-antimatched, and
- $G|(A_1 \cup \{a, c, d, f\})$ and $G|(A_3 \cup \{b, e\})$ are aligned.

Together, these statements imply that G is doubled.

(1) $A_0 = A_2 = A_4 = \varnothing$.

First suppose $v \in A_0$. If v is non-adjacent to a, then $G|abdev \cong M_{2,1}$. So we may assume that v is adjacent to a and similarly v is adjacent to f; but then $G|abcdefv \cong F_{13}$. Therefore A_0 is empty.

Next, suppoose $v \in A_4$. If v is not adjacent to both of a and f, then $G|abcdefv \cong F_{14}$. If v is adjacent to exactly one of a and f, then $G|abcdefv \cong F_{15}$. So we may assume that v is adjacent to both a and f; but then $G|abcdefv \cong F_{16}$. Therefore A_4 is empty.

Finally suppose $v \in A_2$. Let $A_{bc} \subseteq A_2$ be those vertices that are adjacent to b and c, and define A_{bd}, A_{be}, A_{cd}, A_{ce}, A_{de} similarly. Suppose $v \in A_{bc}$. If v is adjacent to f, then $G|cdefv \cong C_5$; otherwise, $G|bcefv \cong \overline{K_{2,3}}$. Therefore A_{bc} is empty and by symmetry, A_{de} is empty. Next, suppose $v \in A_{bd}$. If v is not adjacent to a, then $G|abcdev \cong$ watch and if v is not adjacent to f, then $G|bcdefv \cong$ flag. So we may assume that v is adjacent to both of a and f; but then $G|abdefv \cong$ fish. Therefore A_{bd} is empty and by symmetry, A_{ce} is empty. Next, suppose $v \in A_{be}$. Then $G|bcdev \cong C_5$. Therefore A_{be} is empty. So we may assume that $v \in A_{cd}$. If v is adjacent to both of a and f, then $G|abcefv \cong$ flag. If v is adjacent to a and not to f, then $G|abcefv \cong$ TV, and if v is adjacent to f and not to a, then $G|abdefv \cong$ TV. So we may assume that v is not adjacent to either a or f; but then $G|abefv \cong M_{2,1}$. Therefore $A_2 = \varnothing$ and this proves (1).

(2) If $v \in A_1$, then v is adjacent to either b or e and is not adjacent to both a and f. Moreover, A_1 is a stable set.

Let $A_b \subseteq A_1$ be those vertices that are adjacent to b, and define A_c, A_d, A_e similarly. Suppose $v \in A_c$. If v is adjacent to a, then $G|abcdev \cong$ flag; otherwise, $G|abdev \cong M_{2,1}$. Therefore A_c is empty, and by symmetry, A_d is empty. Suppose $v \in A_b$. If v is adjacent to a, then $G|abdev \cong \overline{K_{2,3}}$ and if v is adjacent to f, then $G|acdfv \cong M_{2,1}$. Therefore v is anticomplete to $\{a, f\}$, and similarly, every vertex in A_e is anticomplete to $\{a, f\}$.

Next, suppose that $u, v \in A_b$ are adjacent. Then $G|bdeuv \cong \overline{K_{2,3}}$. Therefore A_b is a stable set and similarly, so is A_e. Finally, suppose that $u \in A_b$ and $v \in A_e$ are adjacent. Then $G|acduv \cong M_{2,1}$. Therefore $A_b \cup A_e = A_1$ is a stable set, and this proves (2).

(3) If $v \in A_3$, then $N_{\{b,c,d,e\}}(v)$ is either $\{b, c, e\}$ or $\{b, d, e\}$. Moreover, if $u, v \in A_3$ are not adjacent, then $N_{\{u,v\}}(c) \neq N_{\{u,v\}}(d)$ and $|N_{\{u,v\}}(a)| = |N_{\{u,v\}}(f)| = 1$.

Let $A_{bcd} \subseteq A_3$ be those vertices that are adjacent to b, c, and d, and define A_{bce}, A_{bde}, A_{cde} similarly. Suppose $v \in A_{cde}$. If v is not adjacent to a, then $G|abdev \cong \overline{K_{2,3}}$, and if v is adjacent to a and f, then $G|abcefv \cong$ fish. So we may assume that v is adjacent to a but not to f; but then $G|abcefv \cong$ flag. Therefore A_{cde} is empty and similarly, so is A_{bcd}.

Now suppose $u, v \in A_{bce}$ are not adjacent; then $G|cdeuv \cong K_{2,3}$. Therefore A_{bce} is a clique and similarly, so is A_{bde}. Suppose $u \in A_{bce}$ and $v \in A_{bde}$ are not adjacent. If u, v are both adjacent to a, then $G|abdeuv \cong \overline{\text{flag}}$, and if u, v are both non-adjacent to a, then $G|abcdeuv \cong \overline{F_{18}}$. Therefore exactly one of u and v is adjacent to a and similarly, exactly one of u and v is adjacent to f. This proves (3).

(4) $G|(A_3 \cup \{b, e\})$ is semi-antimatched.

It is enough to show that no set of three vertices $\{u, v, w\} \subseteq A_3 \cup \{b, e\}$ contains fewer than two edges. By (3), it is obvious that there are no stable sets of size 3 in $G|(A_3 \cup \{b, e\})$. Suppose $\{u, v, w\}$ contains exactly one edge uv. From (3) and symmetry, we may assume $u, v \in A_{bce}$ and $w \in A_{bde}$. But then $G|cdeuvw \cong \overline{\text{flag}}$.

8

This proves (4).

From (2) and (4), we have a candidate of a partition for G to be doubled. The subgraph $G|(A_1 \cup \{a, c, d, f\})$ contains only one edge (namely cd) and $G|(A_3 \cup \{b, e\})$ is semi-antimatched. Every $v \in A_3 \cup \{b, e\}$ has exactly one neighbor in $\{c, d\}$ and from (3), for every non adjacent pair $u, v \in A_3 \cup \{b, e\}$, $N_{\{u,v\}}(c) \neq N_{\{u,v\}}(d)$. Also by (3), if $u, v \in A_3 \cup \{b, e\}$ are nonadjacent, $|N_{\{u,v\}}(a)| = |N_{\{u,v\}}(f)| = 1$. Moreover, for $w \in A_1$, either $G|abcdew \cong P_5$ or $G|bcdefw \cong P_5$, and so $|N_{\{u,v\}}(w)| = 1$ for every $w \in A_1$, by an analogous argument to the one above. Therefore, $G|(A_3 \cup \{b, e\})$ and $G|(A_1 \cup \{a, c, d, f\})$ are aligned and so G is doubled; this proves 3.2. □

3.3. *A graph containing $\overline{C_6}$ but no graphs in $\mathcal{F}$ is doubled.*

Proof. Let G be a graph containing $\overline{C_6}$ but no graphs in $\mathcal{F}$. Let $G|abcdef \cong \overline{C_6}$ where $\{a, c, e\}$ and $\{b, d, f\}$ are the two triangles and the remaining edges are ad, be, and cf. Let $S = \{a, b, c, d, e, f\}$. By 3.1, we may assume G or $\overline{G}$ does not contain $M_{2,1}$. For $0 \leq i \leq 6$, let $A_i \subseteq V(G) \setminus S$ denote the set of vertices that have i neighbors in S. Our goal is to show that $A_i = \varnothing$ unless $i = 2$ and 4 vertices of S induce antimatching side and the rest of vertices (two in S together with vertices in A_2) induce matching side so that G is doubled.

(1) $A_i = \varnothing$ for $i = 0, 1, 3, 4, 5, 6$.

If $v \in A_0$, then $G|(S \cup \{v\}) \cong \overline{F_{19}}$, so A_0 is empty. Also, if $v \in A_6$, then $G|(S \cup \{v\}) \cong \overline{F_{17}}$, and so A_6 is empty.

Next, suppose $v \in A_1$. From symmetry, we may assume $N_S(v) = \{a\}$. Then $G|(S \cup \{v\}) \cong F_{18}$ and therefore A_1 is empty.

Next, suppose $v \in A_3$. From symmetry, we may assume $N_S(v)$ is one of $\{a, b, c\}, \{a, b, d\}, \{a, c, e\}$. If $N_S(v) = \{a, b, c\}$, then $G|abdev \cong K_{2,3}$ and if $N_S(v) = \{a, b, d\}$, then $G|abcfv \cong C_5$. So we may assume that $N_S(v) = \{a, c, e\}$; but then $G|abcdev \cong \overline{\text{watch}}$ and so A_3 is empty.

Next, suppose $v \in A_4$. From symmetry, we may assume $N_S(v)$ is one of $\{a, b, c, d\}, \{a, b, c, e\}$, and $\{a, b, d, e\}$. If $N_S(v) = \{a, b, c, d\}$, then $G|bcefv \cong K_{2,3}$ and if $N_S(v) = \{a, b, c, e\}$, then $G|acdefv \cong \overline{\text{watch}}$. So we may assume that $N_S(v) = \{a, b, d, e\}$; but then $G|abcfv \cong C_5$, and so A_4 is empty.

Finally, suppose $v \in A_5$. From symmetry, we may assume $N_S(v) = \{b, c, d, e, f\}$. Then $G|bcefv \cong \overline{M_{2,1}}$. Therefore A_5 is empty and this proves (1).

For $u, v \in S$, let $A_{uv} \subseteq A_2$ be those vertices that are adjacent to u and v.

(2) $A_{ab} = A_{bc} = A_{cd} = A_{de} = A_{ef} = A_{fa} = \varnothing$.

Suppose $v \in A_{ab}$. Then $G|abcfv \cong C_5$. Therefore A_{ab} is empty and similarly, so are $A_{bc}, A_{cd}, A_{de}, A_{ef}$,and A_{fa}.

(3) For every $x, y \in S$, A_{xy} is a stable set.

9

Suppose $u, v \in A_{ac}$ are adjacent. Then $G|acdfuv \cong \overline{\text{watch}}$. Therefore A_{ac} is a stable set and similarly so are A_{ae}, A_{ce}, A_{bd}, A_{bf}, and A_{df}. Suppose $u, v \in A_{ad}$ are adjacent. Then $G|acdfuv \cong \overline{\text{watch}}$. Therefore A_{ad} is a stable set and similarly so are A_{be} and A_{cf}. This proves (3).

(4) *If $A_{ac} \neq \varnothing$, then $A_{ae} = A_{ce} = A_{bd} = A_{bf} = \varnothing$.*

Suppose $u \in A_{ac}$, and $v \in A_{ae}$. If u and v are adjacent, then $G|aceuv \cong \overline{M_{2,1}}$, and otherwise $G|bcduv \cong M_{2,1}$. Therefore if A_{ac} is not empty, then $A_{ae} = \varnothing$ and similarly, $A_{ce} = \varnothing$.

Now suppose $v \in A_{bd}$. If u and v are adjacent, then $G|cdfuv \cong C_5$, and otherwise $G|bcefuv \cong \text{watch}$. Therefore if A_{ac} is not empty, then $A_{bd} = \varnothing$ and similarly, $A_{bf} = \varnothing$. This proves (4).

(5) *If $A_{ad} \neq \varnothing$, then $A_{ce} = A_{bf} = \varnothing$.*

Suppose $u \in A_{ad}$ and $v \in A_{ce}$. If u and v are adjacent, then $G|cdfuv \cong C_5$, and otherwise $G|cdeuv \cong \overline{K_{2,3}}$. Therefore if A_{ad} is not empty, then A_{ce} is empty and similarly, A_{bf} is empty as well. This proves (5).

(6) *If $u \in A_{ad}$, then $N_G(u) \setminus S \subseteq A_{be} \cup A_{cf}$.*

Suppose $u \in A_{ad}$. Then from (5), $A_{ce} = A_{bf} = \varnothing$, and from (3), u has no neighbors in A_{ad}. Now suppose $v \in A_{ac}$ is adjacent to u. Then $G|cdfuv \cong C_5$. Therefore u is anticomplete to A_{ac}, and similarly, u is anticomplete to A_{ae}, A_{bd}, and A_{df} as well. Therefore $N_G(u) \setminus S \subseteq A_{be} \cup A_{cf}$, and this proves (6).

(7) *If there are adjacent vertices $u \in A_{ad}$ and $v \in A_{be} \cup A_{cf}$, then $V(G) = S \cup \{u, v\}$ and G is doubled.*

From symmetry, we may assume $v \in A_{cf}$ is adjacent to $u \in A_{ad}$. We know that $A_{ae} \cup A_{ce} \cup A_{bd} \cup A_{bf} = \varnothing$ by (5). Suppose $w \in A_{ac} \cup A_{df}$. Then from (6), $\{u, v\}$ is anticomplete to w and so $G|beuvw \cong M_{2,1}$. Therefore $A_{ac} \cup A_{ae} \cup A_{ce} \cup A_{bd} \cup A_{bf} \cup A_{df} = \varnothing$.

Next, suppose $w(\neq v) \in A_{cf}$. From (3), w is not adjacent to v. If w is adjacent to u, then $G|abfuvw \cong \text{watch}$, and otherwise $G|beuvw \cong M_{2,1}$. Therefore $A_{cf} = \{v\}$ and similarly, $A_{ad} = \{u\}$.

Now suppose $w \in A_{be}$. If w is anticomplete to $\{u, v\}$, then $G|beuvw \cong \overline{K_{2,3}}$. Therefore w is adjacent to at least one of $\{u, v\}$ and by the same logic as above, $A_{be} = \{w\}$. If w is adjacent to exactly one of u and v (say u), then $G|abcuvw \cong \text{flag}$. So we may assume that w is adjacent to both u and v; but then $G|(S \cup \{u, v, w\}) \cong F_{23}$. Therefore $A_{be} = \varnothing$. But then $V(G) = S \cup \{u, v\}$. Since $G|uvbe$ is matched, $G|abcf$ is antimatched, and the two subgraphs are aligned, it follows that G is doubled. This proves (7).

(8) *If $v \in A_{ac}$, then $N_G(v) \setminus S \subseteq A_{df}$.*

Suppose $v \in A_{ac}$. From (3), v has no neighbors in A_{ac}. From (4), $A_{ae} = A_{ce} = A_{bd} = A_{bf} = \varnothing$ and from (5), $A_{be} = \varnothing$. Finally, from (6), v is anticomplete to $A_{ad} \cup A_{cf}$. Therefore $N_G(v) \setminus S \subseteq A_{df}$, and this proves (8).

(9) If there are adjacent vertices $u \in A_{ac}$ and $v \in A_{df}$, then $V(G) = S \cup \{u, v\}$ and G is doubled.

From (4) and (5), $A_{ae} = A_{ce} = A_{bd} = A_{bf} = A_{be} = \varnothing$. If $w \in A_{ad} \cup A_{cf}$, then from (8), w is anticomplete to $\{u, v\}$ and so $G|beuvw \cong M_{2,1}$. Therefore $A_{ad} \cup A_{cf}$ is empty and $V(G) \setminus S = A_{ac} \cup A_{df}$.

Now suppose $w(\neq v) \in A_{df}$. From (3), w is not adjacent to v. If w is adjacent to u, then $G|cdeuvw \cong$ flag, and otherwise $G|beuvw \cong M_{2,1}$. Therefore $A_{df} = \{v\}$ and similarly, $A_{ac} = \{u\}$. Hence, $V(G) = S \cup \{u, v\}$. Since $G|uvbe$ is matched, $G|abcf$ is antimatched, and the two subgraphs are aligned, it follows that G is doubled. This proves (9).

(10) If $G|(V(G) \setminus S)$ is a stable set, then G is doubled.

Suppose $G|(V(G) \setminus S)$ is a stable set. First, suppose $A_{ac} \cup A_{ce} \cup A_{ae} \cup A_{bd} \cup A_{bf} \cup A_{df} \neq \varnothing$. From symmetry, we may assume $A_{ac} \neq \varnothing$. Then from (4), $A_{ae} = A_{ce} = A_{bd} = A_{bf} = \varnothing$ and from (5), $A_{be} = \varnothing$. Therefore every vertex in $V(G) \setminus S$ has exactly one neighbor in $\{a, f\}$ and exactly one neighbor in $\{c, d\}$. Now it is easy to see that G is doubled with $G|acdf$ as the antimatched part.

Therefore we may assume $A_{ac} \cup A_{ce} \cup A_{ae} \cup A_{bd} \cup A_{bf} \cup A_{df} = \varnothing$. Suppose all three of the sets A_{ad}, A_{be}, and A_{cf} are not empty. Then for $u \in A_{ad}, v \in A_{be}$, and $w \in A_{cf}$, $G|afuvw \cong M_{2,1}$. Therefore from symmetry, we may assume A_{be} is empty. Now again, every vertex in $V(G) \setminus S$ has exactly one neighbor in $\{a, f\}$ and exactly one neighbor in $\{c, d\}$, so G is doubled with $G|acdf$ as the antimatched part. This proves (10).

By (10), we may assume that $G|V(G) \setminus S$ contains an edge uv. From symmetry, we may assume $u \in A_{ad}$ or $u \in A_{ac}$. If $u \in A_{ad}$, then by (6) and (7), $v \in A_{be} \cup A_{cf}$ and G is doubled. So we may assume that $u \in A_{ac}$; but then by (8) and (9), $v \in A_{df}$ and G is doubled. This proves 3.3. $\qquad\square$

3.4. *A graph containing $\overline{\text{domino}}$ but no graphs in $\mathcal{F}$ is doubled.*

Proof. Let G be a graph containing $\overline{\text{domino}}$ but no graphs in $\mathcal{F}$. By 3.1, 3.2, and 3.3, we may assume that G does not contain $M_{2,1}$, P_5, C_6, or their complements as induced subgraphs. Let $G|abcdef \cong \overline{\text{domino}}$, where ab, bc, ca, bd, ce, de, df, and ef are the edges; let $S = \{a, b, c, d, e, f\}$. For $0 \leq i \leq 4$, let $A_i \subseteq V(G) \setminus S$ denote the set of vertices that have i neighbors in $\{b, c, d, e\}$. Our goal is to show the following:

- $A_0 = A_1 = A_3 = A_4 = \varnothing$, and
- $G|(A_2 \cup \{a, f\})$ is a stable set, and
- $G|bcde$ is antimatched, and
- $A_2 \cup \{a, f\}$ and $\{b, c, d, e\}$ are aligned.

Together, these statements imply that G is doubled.

(1) $A_0 = A_1 = A_3 = A_4 = \varnothing$.

Suppose $v \in A_0$. If v is complete to $\{a, f\}$, then $G|abdfv \cong C_5$, and if v is anticomplete to $\{a, f\}$, then $G|acdfv \cong M_{2,1}$. So we may assume that v is adjacent to exactly one of a and f, say a; but then $G|adefv \cong \overline{K_{2,3}}$. Therefore $A_0 = \varnothing$.

Next, suppose $v \in A_1$. From symmetry, we may assume $N_{\{b,c,d,e\}}(v) = \{b\}$. If v is complete to $\{a, f\}$, then $G|acefv \cong C_5$, and if v is anticomplete to $\{a, f\}$, then $G|acdfv \cong M_{2,1}$. Furthermore, if v is adjacent

11

to a but not to f, then $G|abefv \cong \overline{K_{2,3}}$. So we may assume that v is adjacent to f but not to a; but then $G|bcefv \cong C_5$. Therefore $A_1 = \varnothing$.

Next, suppose $v \in A_3$. From symmetry, we may assume $N_{\{b,c,d,e\}}(v) = \{b, c, e\}$. If v is not adjacent to f, then $G|bcdefv \cong \overline{\text{flag}}$, and if v is complete to $\{a, f\}$, then $G|abcdfv \cong \text{watch}$. So we may assume that v is adjacent to f but not to a; but then $G|abdefv \cong \overline{\text{fish}}$. Therefore $A_3 = \varnothing$.

Finally, suppose $v \in A_4$. Then $G|bcdev \cong \overline{M_{2,1}}$. Therefore $A_4 = \varnothing$. This proves (1).

For $u, v \in \{b, c, d, e\}$, let $A_{uv} \subseteq A_2$ be those vertices that are adjacent to u and v.

(2) $A_{be} = A_{cd} = \varnothing$. Moreover, $A_2 \cup \{a, f\}$ is a stable set.

Suppose $v \in A_{be} \cup A_{cd}$; then $G|bcdev \cong K_{2,3}$. Therefore $A_{be} = A_{cd} = \varnothing$. Next, suppose $v \in A_{bc}$. If v is adjacent to a, then $G|abcdev \cong \overline{\text{watch}}$ and if v is adjacent to f, then $G|bcdefv \cong \overline{C_6}$. Therefore A_{bc} is anticomplete to $\{a, f\}$, and from symmetry, so is A_{de}.

Now suppose $v \in A_{bd}$. If v is adjacent to a, then $G|acdev \cong C_5$ and if v is adjacent to f, then $G|cdefv \cong C_5$. Therefore A_{bd} is anticomplete to $\{a, f\}$, and from symmetry, so is A_{ce}. It follows that A_2 is anticomplete to $\{a, f\}$. Note that for $v \in A_{bc} \cup A_{de}$, either $G|abcdev \cong \text{domino}$ or $G|bcdefv \cong \text{domino}$, and so by an argument analogous to the one above, we conclude that $A_{bc} \cup A_{de}$ is anticomplete to $A_{bd} \cup A_{ce}$ and that $A_{bc} \cup A_{de}$ is a stable set; hence $A_{bc} \cup A_{de} \cup \{a, f\}$ is a stable set.

It remains to show that $A_{bd} \cup A_{ce}$ is a stable set. For suppose $u, v \in A_{bd}$ are adjacent; then $G|bcdeuv \cong \overline{\text{watch}}$. Therefore A_{bd} is a stable set and from symmetry, so is A_{ce}. Next, suppose $u \in A_{bd}$ and $v \in A_{ce}$ are adjacent; then $G|bcdeuv \cong \overline{C_6}$. Therefore $A_2 \cup \{a, f\}$ is a stable set and this proves (2).

Now $\{b, c, d, e\}$ is anti-matched by definition and $A_2 \cup \{a, f\}$ is a stable set by (2). It remains to show that $A_2 \cup \{a, f\}$ and $\{b, c, d, e\}$ are aligned. Since $A_2 \cup \{a, f\}$ is a stable set, it suffices to show that for all $v \in A_2 \cup \{a, f\}$, v is adjacent to exactly one of b, e and exactly one of c, d. For $v \in \{a, f\}$ this is true by definition, and for $v \in A_2$ this follows from (2). Therefore G is doubled and this proves 3.4. $\square$

3.5. *A graph containing* tent_1 *but no graphs in* $\mathcal{F}$ *is doubled.*

Proof. Let G be a graph containing tent_1 but no graphs in $\mathcal{F}$. By 3.1, 3.2, 3.3, and 3.4, we may assume that G does not contain $M_{2,1}$, P_5, C_6, domino or their complements as induced subgraphs. Let $G|abcdef \cong \text{tent}_1$, where ab, bc, cd, de, fa, fb, fc, and fe are the edges; let $S = \{a, b, c, d, e, f\}$. For $0 \le i \le 4$, let $A_i \subseteq V(G) \backslash S$ denote the set of vertices that have i neighbors in $\{b, c, d, e\}$. Our goal is to show the following:

- $A_0 = A_2 = A_4 = \varnothing$, and
- $G|(A_1 \cup \{a, c, d\})$ is semi-matched, and
- $G|(A_3 \cup \{b, e, f\})$ is semi-antimatched, and
- $A_1 \cup \{a, c, d\}$ and $A_3 \cup \{b, e, f\}$ are aligned.

Together, these statements imply that G is doubled.

(1) $A_0 = A_2 = A_4 = \varnothing$.

Suppose $v \in A_0$. If v is adjacent to a, then $G|abcdev \cong P_5$, and if v is not adjacent to a, then $G|abdev \cong M_{2,1}$. Therefore $A_0 = \varnothing$.

Next, suppose $v \in A_4$. If v is adjacent to f, then $G|cdefv \cong \overline{M_{2,1}}$, and if v is not adjacent to f, then $G|bcdefv \cong \overline{P_5}$. Therefore $A_4 = \varnothing$.

Next, we show that $A_2 = \varnothing$. For $u, v \in \{b, c, d, e\}$, let $A_{uv} \subseteq A_2$ be those vertices that are adjacent to u and v. If $v \in A_{be}$, then $G|bcdev \cong C_5$, and so $A_{be} = \varnothing$. Now suppose $v \in A_{bc}$. If v is adjacent to a, then $G|abdev \cong \overline{K_{2,3}}$ and if v is adjacent to f, then $G|bcdefv \cong \overline{\text{watch}}$. So we may assume that v is not adjacent to either a or f; but then $G|S \cup \{v\} \cong F_{20}$. Therefore $A_{bc} = \varnothing$.

Next, suppose $v \in A_{bd}$. If v is not adjacent to f, then $G|bdefv \cong C_5$, and if v is not adjacent to a, then $G|abcdev \cong \text{watch}$. Hence, we may assume that v is adjacent to both a and f; but then $G|abdefv \cong \overline{\text{watch}}$. Therefore $A_{bd} = \varnothing$.

Next, suppose $v \in A_{cd}$. If v is adjacent to both a and f, then $G|abcfv \cong \overline{M_{2,1}}$. Next, if v is adjacent to a but not to f, then $G|abcdfv \cong \overline{\text{flag}}$, and if v is adjacent to f but not to a, then $G|abdefv \cong \text{fish}$. So we may assume that v is not adjacent to a or f; but then $G|abdfv \cong \overline{K_{2,3}}$. Therefore $A_{cd} = \varnothing$.

Next, suppose $v \in A_{ce}$. Then $G|abcdev \cong \text{domino}$ or flag depending on the adjacency between v and a. Therefore $A_{ce} = \varnothing$.

So we may assume that $v \in A_{de}$. If v is adjacent to a, then $G|abcdv \cong C_5$, and if v is not adjacent to a, then $G|abdev \cong \overline{K_{2,3}}$. Therefore $A_2 = \varnothing$ and this proves (1).

(2) A_1 is complete to b.

For $u \in \{b, c, d, e\}$, let $A_u \subseteq A_1$ be those vertices that are adjacent to u. We will show that $A_c = A_d = A_e = \varnothing$.

Suppose $v \in A_c$. If v is adjacent to a, then $G|abcdev \cong \text{flag}$, and if v is not adjacent to a, then $G|abdev \cong M_{2,1}$. Therefore $A_c = \varnothing$.

Next, suppose $v \in A_e$. Then $G|abcdev \cong P_5$ or C_6 depending on the adjacency between v and a. Therefore $A_e = \varnothing$.

Next, suppose $v \in A_d$. If v is adjacent to a, then $G|abcdv \cong C_5$, and if v is adjacent to f but not to a, then $G|abdefv \cong \text{fish}$. So we may assume that v is not adjacent to either a or f; but then $G|acdefv \cong \text{watch}$. Therefore $A_d = \varnothing$. This completes that proof of (2).

(3) $A_1 \cup \{a\}$ is a stable set.

Suppose $v \in A_1$ and a are adjacent; then $G|abdev \cong \overline{K_{2,3}}$. Therefore A_1 is anticomplete to a. Next, suppose $u, v \in A_1$ are adjacent. Then $G|bdeuv \cong \overline{K_{2,3}}$. Therefore $A_1 \cup \{a\}$ is a stable set and this proves (3).

(4) If $v \in A_3$, then $v \in A_{bce} \cup A_{bde}$.

For $u, v, w \in \{b, c, d, e\}$, let $A_{uvw} \subseteq A_3$ be those vertices that are adjacent to u, v and w.

Suppose $v \in A_{bcd}$. If v is not adjacent to f, then $G|bdefv \cong C_5$, and if v is adjacent to f but not to a, then $G|S \cup \{v\} \cong \overline{F_{21}}$. So we may assume that v is adjacent to both a and f; but then $G|abdefv \cong \overline{\text{watch}}$. Therefore $A_{bcd} = \varnothing$.

Next, suppose $v \in A_{cde}$. If v is adjacent to f, then $G|cdefv \cong \overline{M_{2,1}}$, and if v is not adjacent to f, then $G|bcdefv \cong \overline{\text{flag}}$. Therefore $A_{cde} = \varnothing$. This proves (4).

(5) $A_3 \cup \{f\}$ is a clique.

Suppose $v \in A_{bde}$ and v is not adjacent to f. Then $G|bcdefv \cong \overline{C_6}$. Next, suppose $v \in A_{bce}$ not adjacent to f. Then $G|cdefv \cong K_{2,3}$. Therefore A_3 is complete to f.

Next, suppose $u, v \in A_{bde} \cup A_{bce}$ are not adjacent. If $u, v \in A_{bde}$, then $G|bcduv \cong K_{2,3}$ and if $u, v \in A_{bce}$ then $G|cdeuv \cong K_{2,3}$. So we may assume that $u \in A_{bde}$ and $v \in A_{bce}$; but then $G|bcdeuv \cong \overline{C_6}$. Therefore $A_3 \cup \{f\}$ is a clique and this proves (5).

From (2) and (3), it follows that $A_1 \cup \{a, c, d\}$ is semi-matched with one edge (namely, cd). From (4) and (5), $A_3 \cup \{b, e, f\}$ is semi-antimatched with one nonedge (namely, be). Furthermore, it follows by definition and from (2) that for all $u \in A_1 \cup \{a, c, d\}$, u is adjacent to exactly one of b and e. It also follows by definition and from (4) that for all $v \in A_3 \cup \{b, e, f\}$, v is adjacent to exactly one of c and d. Therefore $A_1 \cup \{a, c, d\}$ and $A_3 \cup \{b, e, f\}$ are aligned and this proves 3.5. $\qquad\square$

3.6. *A graph containing* tent$_2$ *but no graphs in $\mathcal{F}$ is doubled.*

Proof. Let G be a graph containing tent$_2$ but no graphs in $\mathcal{F}$. By 3.1, 3.2, 3.3, 3.4, and 3.5, we may assume that G does not contain $M_{2,1}$, P_5, C_6, domino, tent$_1$ or their complements as induced subgraphs. Let $G|abcdef \cong$ tent$_2$, where ab, bc, cd, de, fa, fb, fd, and fe are the edges; let $S = \{a, b, c, d, e, f\}$. First, we show that if $v \in V(G) \setminus S$, then $N_S(v)$ is equal to $\{b, f\}$, $\{d, f\}$, or $\{a, b, d, e, f\}$.

Let A_{bf} be those vertices whose neighbor set in S is $\{b, f\}$ and define A_{df} and A_{abdef} similarly. We also prove that at least one of A_{bf}, A_{df} and A_{abdef} is empty. Then our goal is to show the following:

If $A_{abdef} = \varnothing$, then

- $G|(A_2 \cup \{a, e\})$ is semi-matched, and
- $G|\{b, c, d, f\}$ is antimatched, and
- $A_2 \cup \{a, e\}$ and $\{b, c, d, f\}$ are aligned.

If $A_{df} = \varnothing$, then

- $G|(A_2 \cup \{a, c, d\})$ is semi-matched, and
- $G|(A_5 \cup \{b, e, f\})$ is semi-antimatched, and
- $A_2 \cup \{a, c, d\}$ and $A_5 \cup \{b, e, f\}$ are aligned.

If $A_{bf} = \varnothing$, then

- $G|(A_2 \cup \{b, c, e\})$ is semi-matched, and
- $G|(A_5 \cup \{a, d, f\})$ is semi-antimatched, and
- $A_2 \cup \{b, c, e\}$ and $A_5 \cup \{a, d, f\}$ are aligned.

Together, these statements imply that G is doubled.

(1) For $v \in V(G) \setminus S$, $N_S(v)$ is equal to $\{b, f\}$, $\{d, f\}$, or $\{a, b, d, e, f\}$.

We show that $N_{\{b,c,d\}}(v)$ is equal to $\{b\}$, $\{d\}$, or $\{b, d\}$ and for each case, $N_S(v)$ is equal to $\{b, f\}$, $\{d, f\}$, or $\{a, b, d, e, f\}$, respectively.

First, suppose $N_{\{b,c,d\}}(v) = \varnothing$. If v is complete to $\{a,e\}$, then $G|abcdev \cong C_6$, and if v is adjacent to exactly one of a and e, then $G|abcdev \cong P_5$. So we may assume that v is anticomplete to $\{a,e\}$; but then $G|abdev \cong M_{2,1}$. Therefore $N_{\{b,c,d\}}(v)$ cannot be empty.

Next, suppose $N_{\{b,c,d\}}(v) = \{b\}$. If v is adjacent to e, then $G|bcdev \cong C_5$, and if v is adjacent to a, then $G|abdev \cong \overline{K_{2,3}}$. If v is not adjacent to f, then $G|bcdefv \cong \overline{\mathrm{tent}_1}$. Therefore $N_S(v) = \{b, f\}$. Similarly if $N_{\{b,c,d\}}(v) = \{d\}$, then $N_S(v) = \{d, f\}$.

Next, suppose $N_{\{b,c,d\}}(v) = \{c\}$. If v is complete to $\{a,e\}$, then $G|abcdev \cong$ domino, and if v is adjacent to exactly one of a and e, then $G|abcdev \cong$ flag. So we may assume that v is anticomplete to $\{a,e\}$; but then $G|abdev \cong M_{2,1}$. Therefore $N_{\{b,c,d\}}(v)$ cannot be equal to $\{c\}$.

Next, suppose $N_{\{b,c,d\}}(v) = \{b, c\}$. If v is complete to $\{e, f\}$, then $G|bcdefv \cong \overline{P_5}$. If v is adjacent to e but not to f, then $G|bcdefv \cong \overline{C_6}$, and if v is adjacent to f but not to e, then $G|bcdefv \cong \overline{\mathrm{flag}}$. So we may assume that v is anticomplete to $\{e, f\}$; but then $G|bcdefv \cong \overline{\mathrm{domino}}$. Therefore $N_{\{b,c,d\}}(v)$ cannot be $\{b, c\}$ and from symmetry, $N_{\{b,c,d\}}(v)$ cannot be $\{c, d\}$.

Next, suppose $N_{\{b,c,d\}}(v) = \{b, d\}$. If v is not adjacent to f, then $G|bcdfv \cong K_{2,3}$. If v is anticomplete to $\{a,e\}$, then $G|abcdev \cong$ watch. If v is adjacent to one of a and e, then $G|abcdev \cong \overline{\mathrm{tent}_1}$. Therefore $N_S(v) = \{a, b, d, e, f\}$.

Finally, suppose $N_{\{b,c,d\}}(v) = \{b, c, d\}$. If v is adjacent to f, then $G|bcdfv \cong \overline{M_{2,1}}$. If v is not adjacent to a, then $G|abcdfv \cong \overline{\mathrm{flag}}$, while if v is adjacent to a, then $G|abcdfv \cong \overline{P_5}$. Therefore v cannot be complete to $\{b, c, d\}$.

Together, these statements prove (1).

(2) $A_{bf} \cup A_{df}$ is a stable set, and A_{abdef} is a clique complete to $A_{bf} \cup A_{df}$.

Suppose $u, v \in A_{bf}$ are adjacent; then $G|bcdfuv \cong \overline{\mathrm{watch}}$. Therefore A_{bf} is a stable set and similarly, so is A_{df}. Now suppose $u \in A_{bf}$ and $v \in A_{df}$ are adjacent. Then $G|bcduv \cong C_5$. Therefore $A_{bf} \cup A_{df}$ is a stable set.

Next, suppose $u, v \in A_{abdef}$ are not adjacent; then $G|bcduv \cong K_{2,3}$. Therefore A_{abdef} is a clique.

Finally, suppose $u \in A_{bf}$ and $v \in A_{abdef}$ are not adjacent. Then $G|abcdev \cong \mathrm{tent}_2$ and u has only one neighbor in $\{a, b, c, d, e, v\}$, which is impossible by (1). Therefore A_{abdef} is complete to A_{bf} and similarly to A_{df}, and this proves (2).

(3) At least one of A_{bf}, A_{df}, and A_{abdef} is empty.

Suppose $u \in A_{bf}$, $v \in A_{df}$, and $w \in A_{abdef}$. From (2), w is complete to $\{u, v\}$ and u is not adjacent to v. It follows that $G|abcdefuvw \cong \overline{F_{22}}$. Therefore at least one of A_{bf}, A_{df}, and A_{abdef} is empty and this proves (3).

If $A_{abdef} = \varnothing$, then it follows from (2) that $G|(A_{bf} \cup A_{df} \cup \{a, e\})$ is a stable set. Also, $G|bcdf$ is antimatched by assumption and $A_{bf} \cup A_{df} \cup \{a, e\}$ and $\{b, c, d, f\}$ are aligned by assumption and definition. Hence, G is doubled.

So we may assume that $A_{abdef} \neq \varnothing$. Then by (3), one of A_{bf} and A_{df} is empty and from symmetry, we may assume A_{df} is empty. Then $G|(A_{bf} \cup \{a, c, d\})$ is semi-matched with an edge cd, and $G|(A_{abdef} \cup \{b, e, f\})$

is semi-antimatched with a non-edge be. It also follows from assumption and definition that for all $u \in A_{bf} \cup \{a, c, d\}$, u is adjacent to exactly one of b and e and for all $v \in A_{abdef} \cup \{b, e, f\}$, v is adjacent to exactly one of c and d. Hence, G is doubled. This proves 3.6. $\qquad\square$

We are now ready to prove the main result.

Proof of 1.2. The "only if" part is obvious since none of the graphs in $\mathcal{F}$ are doubled. For the "if" part, we may assume G is not almost-split and hence G or $\overline{G}$ contains one of $M_{2,1}$, P_5, C_6, domino, tent_1, and tent_2 as an induced subgraph. But then we are done by 3.1, 3.2, 3.3, 3.4, 3.5, or 3.6 applied to G or $\overline{G}$, keeping in mind that the complement of a doubled graph is doubled. $\qquad\square$

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