WEIGHTED KOPPELMAN FORMULAS AND THE $\bar\partial$-EQUATION ON AN ANALYTIC SPACE

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Abstract. Let X be an analytic space of pure dimension. We introduce a formalism to generate intrinsic weighted Koppelman formulas on X that provide solutions to the $\bar\partial$-equation. We obtain new existence results for the $\bar\partial$-equation, as well as new proofs of various known results.

1. Introduction

Let X be an analytic space of pure dimension n and let $\mathcal{O} = \mathcal{O}^X$ be the structure sheaf of (strongly) holomorphic functions. Locally X is a subvariety of a domain Ω in $\mathbb{C}^N$ and then $\mathcal{O}^X = \mathcal{O}^\Omega/\mathcal{J}$, where $\mathcal{J}$ is the sheaf in Ω of holomorphic functions that vanish on X. In the same way we say that ϕ is a smooth $(0,q)$-form on X, $\phi \in \mathcal{E}_{0,q}(X)$, if given a local embedding, there is a smooth form in a neighborhood in the ambient space such that ϕ is its pull-back to X_{reg}. It is well-known that this defines an intrinsic sheaf $\mathcal{E}^X_{0,q}$ on X. It was proved in [15] that if X is embedded as a reduced complete intersection in a pseudoconvex domain and ϕ is a $\bar\partial$-closed smooth form on X, then there is a solution ψ to $\bar\partial\psi = \phi$ on X_{reg}. It was an open question for long whether this holds more generally, and it was proved only in [6][1] that this is indeed true for any Stein space X.

In [6] we introduced fine (modules over the sheaf of smooth forms) sheaves $\mathcal{A}_k$ of $(0,k)$-currents on X, which coincide with the sheaves of smooth forms on X_{reg} and have rather "mild" singularities at X_{sing}. The main result in [6] is that

$$0 \to \mathcal{O}^X \to \mathcal{A}_0 \xrightarrow{\bar\partial} \mathcal{A}_1 \xrightarrow{\bar\partial} \tag{1.1}$$

is a (fine) resolution of $\mathcal{O}^X$. By the de Rham theorem it follows that the classical Dolbeault isomorphism for a smooth X extends to an arbitrary (reduced) singular space, but with the sheaves $\mathcal{A}_k$ instead of $\mathcal{E}_{0,k}$. In particular, if X is Stein, $\phi \in \mathcal{A}_{q+1}(X)$ and $\bar\partial\phi = 0$, then there is $u \in \mathcal{A}_q(X)$ such that $\bar\partial u = \phi$.

The results in [6] are based on semiglobal Koppelman formulas on X that we first describe for smooth forms.

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[1] The proof in [6] first appeared in [5].

Theorem 1.1. *Let X be an analytic subvariety of pure dimension n of a pseudoconvex domain $\Omega \subset \mathbb{C}^N$ and assume that $\Omega' \subset\subset \Omega$ and $X' := X \cap \Omega'$. There are linear operators $\mathcal{K}\colon \mathcal{E}_{0,q+1}(X) \to \mathcal{E}_{0,q}(X'_{reg})$ and $\mathcal{P}\colon \mathcal{E}_{0,0}(X) \to \mathcal{O}(\Omega')$ such that*

$$\phi(z) = \bar\partial \mathcal{K}\phi(z) + \mathcal{K}(\bar\partial\phi)(z), \quad z \in X'_{reg},\ \phi \in \mathcal{E}_{0,q}(X),\ q \geq 1, \tag{1.2}$$

and

$$\phi(z) = \mathcal{K}(\bar\partial\phi)(z) + \mathcal{P}\phi(z), \quad z \in X'_{reg},\ \phi \in \mathcal{E}_{0,0}(X). \tag{1.3}$$

Moreover, there is a number M such that

$$\mathcal{K}\phi(z) = \mathcal{O}(\delta(z)^{-M}), \tag{1.4}$$

where $\delta(z)$ is the distance to X'_{sing}.

The operators are given as

$$\mathcal{K}\phi(z) = \int_\zeta k(\zeta, z)\wedge\phi(\zeta), \quad \mathcal{P}\phi(z) = \int_\zeta p(\zeta, z)\wedge\phi(\zeta), \tag{1.5}$$

where k and p are intrinsic integral kernels on $X \times X'_{reg}$ and $X \times \Omega'$, respectively. They are locally integrable with respect to ζ on X_{reg} and the integrals in (1.5) are principal values at X_{sing}. If ϕ vanishes in a neighborhood of a point x, then $\mathcal{K}\phi$ is smooth at x. The distance $\delta(z)$ is the one induced from the ambient space; up to a constant it is independent of the particular embedding. The existence result in [15] for a reduced complete intersection is also obtained by an integral formula, which however does not give an intrinsic solution operator on X.

We cannot expect our solution $\mathcal{K}\phi$ to be smooth across X_{sing}, see, e.g., Example 1 in [6]. However, $\mathcal{K}$ and $\mathcal{P}$ extend to operators $\mathcal{K}\colon \mathcal{A}_{q+1}(X) \to \mathcal{A}_q(X')$ and $\mathcal{P}\colon \mathcal{A}_0(X) \to \mathcal{O}(\Omega')$, and the Koppelman formulas still hold, so in particular, $\bar\partial\mathcal{K}\phi = \phi$ if $\phi \in \mathcal{A}_{q+1}(X)$ and $\bar\partial\phi = 0$ (Theorem 4 in [6]).

There is an integer L, only depending on X, such that for each $k \geq L$, $\mathcal{K}\colon C^k_{0,q+1}(X) \to C^k_{0,q}(X'_{reg})$ and $\mathcal{P}\colon C^k_{0,0}(X) \to \mathcal{O}(\Omega')$. Here $\phi \in C^k_{0,q}(X)$ means that ϕ is the pullback to X_{reg} of a $(0,q)$-form of class C^k in a neighborhood of X in the ambient space. We have

Theorem 1.2. *Let X, X', Ω, Ω' be as in the previous theorem.*
(i) If $\phi \in C^k_{0,q+1}(X)$, $q \geq 0$, $k \geq L+1$, and $\bar\partial\phi = 0$, then there is $\psi \in C^k_{0,q}(X'_{reg})$ with $\psi(z) = \mathcal{O}(\delta(z)^{-M})$ and $\bar\partial\psi = \phi$.
(ii) If $\phi \in C^{L+1}_{0,0}(X)$ and $\bar\partial\phi = 0$ then ϕ is strongly holomorphic.

Part (ii) is well-known, [17] and [29], but $\mathcal{P}\phi$ provides an explicit holomorphic extension of ϕ to Ω'.

Our solution operator $\mathcal{K}$ behaves like a classical solution operator on X_{reg} and by introducing appropriate weight factors in the integral operators we get

Theorem 1.3. *Let X, X', Ω, Ω' be as in the previous theorem. Given $\mu \geq 0$ there is $\mu' \geq 0$ and a linear operator $\mathcal{K}$ such that if ϕ is a $\bar\partial$-closed $(0, q+1)$-form on X_{reg}, $q \geq 0$, with $\delta^{-\mu'}\phi \in L^p(X_{reg})$, $1 \leq p \leq \infty$, then $\bar\partial\mathcal{K}\phi = \phi$ and $\delta^{-\mu}\mathcal{K}\phi \in L^p(X'_{reg})$.*

The existence of such solutions was proved in [11] (even for (r,q)-forms) by resolutions of singularities and cohomological methods (for $p=2$, but the same method surely gives the more general results). By a standard technique this theorem implies global results for a Stein space X. In case X_{sing} is a single point more precise result are obtained in [21] and [10]. In particular, if ϕ has bidegree $(0,q)$, $q<\dim X$, then the image of $L^2(X_{reg})$ under $\bar\partial$ has finite codimension in $L^2(X_{reg})$. See also [19], and the references given there, for related results. In [9], Fornæss and Gavosto show that, for complex curves, a Hölder continuous solution exists if the right hand side is bounded. Special hypersurfaces and certain homogeneous varieties have been considered, e.g., in [24] and [25].

We can use our integral formulas to solve the $\bar\partial$-equation with compact support. As usual this leads to a Hartogs result in X, and a vanishing result in the complement of a Stein compact, for forms with not too high degree. The vanishing result is well-known but we can provide a description of the obstruction in the "limit" case. For a given analytic space X, let $\nu=\nu(X)$ be the minimal depth of the local rings $\mathcal{O}_x$ (the homological codimension). Since X has pure dimension, $\nu\geq 1$, and X is Cohen-Macaulay if and only if $\nu=n$.

Theorem 1.4. *Assume that X is a connected Stein space of pure dimension n with globally irreducible components X^ℓ and let K be a compact subset such that $X^\ell_{reg}\setminus K$ is connected for each ℓ.*

(i) If $\nu\geq 2$, then for each holomorphic function $\phi\in\mathcal{O}(X\setminus K)$ there is $\Phi\in\mathcal{O}(X)$ such that $\Phi=\phi$ in $X\setminus K$.

(ii) Assume that $\nu=1$ and let χ be a cutoff function that is identically 1 in a neighborhood of K and with support in a relatively compact Stein space $X'\subset\subset X$. There is an almost semi-meromorphic $\bar\partial$-closed $(n,n-1)$-current ω_{n-1} on X' that is smooth on X'_{reg} such that the function $\phi\in\mathcal{O}(X\setminus K)$ has a holomorphic extension Φ across K if and only if

$$\int_X \bar\partial\chi\wedge\omega_{n-1}\phi h=0,\quad h\in\mathcal{O}(X). \tag{1.6}$$

Part (i) is proved in [7, Ch. 1 Corollary 4.4]. If X is normal and $X\setminus K$ is connected, then the conditions of Theorem 1.4 (i) are fulfilled. If X is not normal it is necessary to assume that $X^\ell_{reg}\setminus K$ is connected; see Example 2 in Section 5 below. See [20] for a further discussion. For related results proved by other methods see, e.g., [18], [22], and [23].

The current ω_{n-1} is the top degree component of a *structure form* ω associated to X, see Section 2. Since ω_{n-1} is *almost semi-meromorphic*, see Section 2 and [6], the integrals (the action of ω_{n-1} on test forms) exist as principal values at X_{sing}. If the holomorphic extension Φ exists, then, since $\bar\partial\omega_{n-1}=0$, we have that

$$\int_X\bar\partial\chi\wedge\omega_{n-1}\phi h=\int_X\bar\partial\chi\wedge\omega_{n-1}\Phi h=-\int_X\chi\bar\partial(\omega_{n-1}\Phi h)=0,$$

and hence condition (1.6) is necessary; see, e.g., [6] for a discussion on currents on a singular space.

There is a similar result for $\bar\partial$-closed forms (currents) in $\mathcal{A}$:

Theorem 1.5. *Let X be a Stein space of pure dimension n and let $K \subset X$ be a Stein compact. Assume that $\phi \in \mathcal{A}_q(X \setminus K)$ and $\bar\partial\phi = 0$, and let $X' \subset\subset X$ be a Stein neighborhood of K.*
(i) If $q \le \nu - 2$, then there is $\Phi \in \mathcal{A}_q(X)$ such that $\bar\partial\Phi = 0$ and $\Phi = \phi$ outside X'.
(ii) If $q = \nu - 1$, then there is such a Φ if and only if

$$\int_X \bar\partial\chi\wedge\omega_{n-\nu}\wedge\phi h = 0, \quad h \in \mathcal{O}(X). \tag{1.7}$$

As usual this leads to a vanishing theorem for $\bar\partial$ in $X \setminus K$.

Corollary 1.6. *Assume that $\phi \in \mathcal{A}_q(X \setminus K)$ and $\bar\partial\phi = 0$.*
(i) If $1 \le q \le \nu - 2$, then there is $\psi \in \mathcal{A}_{q-1}(X \setminus K)$ such that $\bar\partial\psi = \phi$.
(ii) If $1 \le q = \nu - 1$, then there is $\psi \in \mathcal{A}_{q-1}(X \setminus K)$ such that $\bar\partial\psi = \phi$ if and only if (1.7) holds.

In view of the exactness of (1.1), part (i) is equivalent to that $H^q(X \setminus K, \mathcal{O}) = 0$ for $q \le \nu - 2$; this vanishing is well-known, see, e.g., [20, Section 2]. The novelty here is the proof with integral formulas. Part (ii) provides a representation of the cohomology for $q = \nu - 1$.

Remark 1. It follows from the proofs, and the semicontinuity of $x \mapsto \operatorname{depth} \mathcal{O}_x$ that these theorems hold with $\nu = \nu(K) := \min_{x \in K} \operatorname{depth} \mathcal{O}_x$. In Theorem 1.4 however, one must take the minimum over a Stein neighborhood of K, cf., [20, footnote on p. 2]. $\square$

In the same way we can obtain the existence of $\bar\partial$-closed extensions across $X \setminus A$ for any analytic, not necessarily pure dimensional, subset $A \subset X$, see Proposition 5.1 below. For instance A may be X_{sing}. This leads to vanishing results in $X \setminus A$.

Theorem 1.7. *Assume that X is a Stein space of pure dimension n, and let A be an analytic subset of dimension $d \ge 1$. Assume that $\phi \in \mathcal{A}_q(X \setminus A)$ and $\bar\partial\phi = 0$.*
(i) If $1 \le q \le \nu - 2 - d$, then there is a $\psi \in \mathcal{A}_{q-1}(X \setminus A)$ such that $\bar\partial\psi = \phi$.
(ii) If $1 \le q = \nu - 1 - d$, then the same conclusion holds if and only if

$$\int_X \bar\partial\chi\wedge\omega_{n-\nu}\wedge\phi\wedge h = 0 \tag{1.8}$$

for all smooth $\bar\partial$-closed $(0, d)$-forms h such that the $\operatorname{supp} h \cap \operatorname{supp} \bar\partial\chi$ is compact.

If $q = 0 \le \nu - 2 - d$ or $q = 0 = \nu - 1$ and (1.8) holds, then the conclusion is that ϕ is holomorphic and has a holomorphic extension across A.

Even in this case it is enough to take $\nu = \nu(A)$. Because of the exactness of (1.1), part (i) is equivalent to the vanishing of $H^q(X \setminus A, \mathcal{O})$ for $1 \le q \le \nu - 2 - d$, also this vanishing result is well-known, see [27], [31], and [28].

In [6] we introduced the sheaves $\mathcal{W}_{p,q}$ of pseudomeromorphic (p, q)-currents on X with the so-called *standard extension property* SEP. It is proved that the operators $\mathcal{K}$ and $\mathcal{P}$ in Theorem 1.1 extend to operators

$$\mathcal{W}_{0,q+1}(X) \to \mathcal{W}_{0,q}(X'), \quad \mathcal{W}_{0,0}(X) \to \mathcal{O}(\Omega').$$

Moreover, the Koppelman formulas hold if, in addition, ϕ is in the domain $\text{Dom}\,\bar\partial_X$ of the operator $\bar\partial_X$ introduced in [6]. The latter condition means that $\bar\partial\phi$ is in $\mathcal{W}_{0,*}$ and that ϕ satisfies a certain "boundary condition" at X_{sing}. If $\phi \in \mathcal{W}_{0,0}$, then ϕ is in $\text{Dom}\,\bar\partial_X$ and $\bar\partial_X\phi = 0$ if and only if ϕ is (strongly holomorphic), whereas $\bar\partial\phi = 0$ means that ϕ is weakly holomorphic in the sense of Barlet-Henkin-Passare, cf., [14].

We will mainly be interested here in the case when X is Cohen-Macaulay. Then we can always choose (at least semi-globally) a structure form ω that only has one component ω_0 that is a $\bar\partial$-closed $(n,0)$-form (current). The condition $\phi \in \text{Dom}\,\bar\partial_X$ then precisely means that there is a current ψ in $\mathcal{W}_{0,q+1}$ such that

$$\bar\partial(\phi\wedge\omega) = \psi\wedge\omega.$$

For other equivalent conditions, see Section 2 and [6].

Thus $\bar\partial\mathcal{K}\phi = \phi$ in X' if $\phi \in \mathcal{W}_{0,q}(X) \cap \text{Dom}\,\bar\partial_X$ and $\bar\partial_X\phi = 0$. Unfortunately we do not know whether $\mathcal{K}\phi$ is again in $\text{Dom}\,\bar\partial_X$; if it were, then $\mathcal{W}_{0,k} \cap \text{Dom}\,\bar\partial_X$ would provide a (fine) resolution of $\mathcal{O}$. It is however true, [6], that if $\phi \in \mathcal{W}_{0,0}$ and $\bar\partial_X\phi = 0$, then $\phi \in \mathcal{O}$. Moreover, the difference of two of our solutions is anyway $\bar\partial$-exact on X_{reg} if $q > 1$ and strongly holomorphic if $q = 1$. By an elaboration of these facts we can prove:

Theorem 1.8. *Assume that X is an analytic space of pure dimension n and that X is Cohen-Macaulay. Any $\bar\partial$-closed $\phi \in \mathcal{W}_{0,q}(X) \cap \text{Dom}\,\bar\partial_X$, $q \geq 1$, that is smooth on X_{reg} defines a canonical class in $H^q(X, \mathcal{O}^X)$; if this class vanishes then there is a global smooth form ψ on X_{reg} such that $\bar\partial\psi = \phi$. In particular, there is such a solution if X is a Stein space.*

Remark 2. If ϕ is not smooth, the conclusion is that there is a form $\psi \in \mathcal{W}_{q-1}(X)$ such that $\bar\partial\psi = \phi$ on X_{reg}.

A similar statement holds even if X is not Cohen-Macaulay. However, the proof then requires a hypothesis on ϕ that is (marginally) stronger than the $\text{Dom}\,\bar\partial_X$-condition, see Section 7. $\square$

The starting point is a certain residue current R, introduced in [3], that is associated to a subvariety $X \subset \Omega$, and the integral representation formulas from [2]. We discuss the current R, and its associated structure form ω on X, in Section 2, and in Section 3 we recall from [6] the construction of the Koppelman formulas.

In Section 6 we describe some concrete realizations of the "moment" condition (1.6) in Theorem 1.4. The remaining sections are devoted to the proofs.

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2. A residue current associated to X

Let X be a subvariety of pure dimension n of a pseudoconvex set $\Omega \subset \mathbb{C}^N$. The Lelong current $[X]$ is a classical analytic object that represents X. It is a d-closed (p,p)-current, $p = N - n$, such that

$$[X].\xi = \int_X \xi$$

for test forms ξ. If $\operatorname{codim} X = 1$, $X = \{f = 0\}$ and $df \neq 0$ on X_{reg}, then the Poincare-Lelong formula states that

$$\bar\partial \frac{1}{f} \wedge \frac{df}{2\pi i} = [X]. \tag{2.1}$$

To construct integral formulas we will use an analogue of the current $\bar\partial(1/f)$, introduced in [3], for a general variety X. It turns out that this current, contrary to $[X]$, also reflects certain subtleties of the variety at X_{sing} that are encoded by the algebraic description of X. Let $\mathcal{J}$ be the ideal sheaf over Ω generated by the variety X. In a slightly smaller set, still denoted Ω, one can find a free resolution

$$0 \to \mathcal{O}(E_M) \xrightarrow{f_M} \dots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \tag{2.2}$$

of the sheaf $\mathcal{O}/\mathcal{J}$. Here E_k are trivial vector bundles over Ω and $E_0 = \mathbb{C}$ is a trivial line bundle. This resolution induces a complex of trivial vector bundles

$$0 \to E_M \xrightarrow{f_M} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0 \tag{2.3}$$

that is pointwise exact outside X.

Let $\nu = \nu(X)$ be the minimal depth of the rings $\mathcal{O}_x^\Omega/\mathcal{J}_x = \mathcal{O}_x^X$. Then there is a resolution (2.2) with $M = N - \nu$. Since $\nu \geq 1$ we may thus assume that $M \leq N - 1$. If (and only if) X is Cohen-Macaulay, i.e., all the rings $\mathcal{O}_x^X$ are Cohen-Macaulay, there is a resolution (2.2) with $M = N - n$.

Given Hermitian metrics on E_k, in [3] was defined a current $U = U_1 + \cdots + U_M$, where U_k is a $(0, k-1)$-current that is smooth outside X and takes values in E_k, and a residue current with support on X,

$$R = R_p + R_{p+1} + \cdots + R_M, \tag{2.4}$$

where R_k is a $(0,k)$-current with values in E_k, satisfying

$$\nabla_f U = 1 - R, \tag{2.5}$$

and $\nabla_f = f - \bar\partial = \sum f_j - \bar\partial$.

Let $F = f_1$. The form-valued functions $\lambda \mapsto |F|^{2\lambda} u =: U^\lambda$ (here u is the restriction of U to $\Omega \setminus X$) and $1 - |F|^{2\lambda} + \bar\partial |F|^{2\lambda} \wedge u =: R^\lambda$, a priori defined for $\operatorname{Re}\lambda >> 0$, admit analytic continuations as current-valued functions to $\operatorname{Re}\lambda > -\epsilon$ and

$$U = U^\lambda|_{\lambda=0}, \quad R = R^\lambda|_{\lambda=0}. \tag{2.6}$$

Notice also that $\nabla_f U^\lambda = 1 - R^\lambda$.

It is proved in [6] that R has the *standard extension property*, SEP, with respect to X. This means that if h is a holomorphic function that does not vanish identically on any component of X (the most interesting case is when $\{h = 0\}$ contains X_{sing}), χ is a smooth approximand of the characteristic function for $[1, \infty)$, and $\chi_\delta = \chi(|h|/\delta)$, then

$$\lim_{\delta \to 0} \chi_\delta R = R. \tag{2.7}$$

The SEP can also be expressed as saying that R is equal to the value at $\lambda = 0$, $|h|^{2\lambda} R|_{\lambda=0}$, of (the analytic continuation of) $\lambda \mapsto |h|^{2\lambda} R|_{\lambda=0}$, see, e.g., [4].

It holds that $\nabla_f \circ \nabla_f = 0$, and in view of (2.5), thus $\nabla_f R = 0$, so in particular, $\bar\partial R_M = 0$.

We say that a current μ on X has the SEP on X if (with χ_δ as above) $\chi_\delta \mu \to \mu$ when $\delta \to 0$, for each holomorphic h that does not vanish identically on any irreducible component of X. We recall from [6] that a current μ on X is *almost semi-meromorphic* if it is the direct image of a semi-meromorphic current under a modification $\tilde X \to X$, see, [6]. Such a current μ is pseudomeromorphic and has the SEP on X, so in particular it is in $\mathcal{W}$.

It is proved in [6] that there is a (unique) almost semi-meromorphic current

$$\omega = \omega_0 + \omega_1 + \cdots + \omega_{n+M-N}$$

on X, where ω_r has bidegree (n, r) and takes values in $E^r := E_{N-n+r}|_X$, such that

$$i_*\omega = R \wedge dz_1 \wedge \cdots \wedge dz_N. \tag{2.8}$$

The current ω is smooth and nonvanishing ([6, Lemma 18]) on X_{reg} and

$$|\omega| = \mathcal{O}(\delta^{-M}) \tag{2.9}$$

for some $M \geq 0$, where δ is the distance to X_{sing}. We say that ω is a *structure form* for X, cf., Remark 3 below. The equality (2.8) means that

$$\int_\Omega R \wedge dz_1 \wedge \cdots \wedge dz_N \wedge \xi = \int_X \omega \wedge \xi$$

for each test form ξ in Ω. Here both integrals mean currents acting on the test form; the right hand side can also be interpreted as the principal value

$$\lim_{\delta \to 0} \int_X \chi_\delta \omega \wedge \xi.$$

In particular it follows that for a smooth form Φ, $R \wedge \Phi$ only depends on the pull-back of Φ to X_{reg}.

Remark 3. Let

$$E^r := E_{p+r}|_X, \quad f^r := f_{p+r}|_X$$

so that f^r becomes a holomorphic section of $\mathrm{Hom}\,(E^r, E^{r-1})$. Then $\nabla_f = f^\bullet - \bar\partial$ has a meaning on X. If ϕ is a meromorphic function, or even $\phi \in \mathcal{W}_{0,0}$ on X, then $\phi \wedge \omega$ is a well-defined current in $\mathcal{W}$ and ϕ is strongly holomorphic if and only if

$$\nabla_f(\phi \wedge \omega) = 0. \tag{2.10}$$

If X is Cohen-Macalay and $\omega = \omega_0$, then (2.10) precisely means that $\bar\partial(\phi \wedge \omega) = 0$ (which by definition means that ϕ is in $\mathrm{Dom}\,\bar\partial_X$ and $\bar\partial \phi = 0$). In this case $\bar\partial \omega_0 = 0$, i.e., ω_0 is a weakly holomorphic (in the Barlet-Henkin-Passare sense) $(n, 0)$-form; thus ω_0 is precisely so singular it possibly can be and still be $\bar\partial$-closed. $\square$

From the proof of Proposition 16 in [6] it follows that we can write $R = \gamma \lrcorner [X]$, where $\gamma = \gamma_0 + \cdots \gamma_{n-1}$ is smooth in $\Omega \setminus X_{sing}$, almost semi-meromorphic in Ω, and γ_r takes values in $E_{p+r} \otimes T^*_{0,r}(\Omega) \otimes \Lambda^p T_{1,0}(\Omega)$. In view of (2.8) it follows that

$$\int_X \omega \wedge \xi = \int R \wedge d\zeta \wedge \xi = \pm \int_X (\gamma \lrcorner d\zeta) \wedge \xi, \quad \xi \in \mathcal{D}_{0,*}(X), \tag{2.11}$$

so in particular, $\omega = \pm \gamma \lrcorner d\zeta$.

3. Construction of Koppelman formulas on X

Some of the material in this section overlap with [6] but it is included here for the reader's convenience and to make the proof of Theorem 1.8 more accessible. We first recall the construction of integral formulas in [1] on an open set Ω in $\mathbb{C}^N$. Let $(\eta_1, \ldots, \eta_N)$ be a holomorphic tuple in $\Omega_\zeta \times \Omega_z$ that span the ideal associated to the diagonal $\Delta \subset \Omega_\zeta \times \Omega_z$. For instance, one can take $\eta = \zeta - z$. Following the last section in [1] we consider forms in $\Omega_\zeta \times \Omega_z$ with values in the exterior algebra Λ_η spanned by $T^*_{0,1}(\Omega \times \Omega)$ and the $(1,0)$-forms $d\eta_1, \ldots, d\eta_N$. On such forms interior multiplication δ_η with

$$\eta = 2\pi i \sum_1^N \eta_j \frac{\partial}{\partial \eta_j}$$

has a meaning. We then introduce $\nabla_\eta = \delta_\eta - \bar\partial$, where $\bar\partial$ acts[2] on both ζ and z. Let $g = g_{0,0} + \cdots + g_{N,N}$ be a smooth form (in Λ_η) defined for z in $\Omega' \subset\subset \Omega$ and $\zeta \in \Omega$, such that $g_{0,0} = 1$ on the diagonal Δ in $\Omega' \times \Omega$ and $\nabla_\eta g = 0$. Here and in the sequel lower index (p,q) denotes bidegree. Since g takes values in Λ_η thus $g_{k,k}$ is the term that has degree k in $d\eta$. Such a form g will be called a *weight* with respect to Ω'. Notice that if g and g' are weights, then $g \wedge g'$ is again a weight.

Example 1. If Ω is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight with respect to some neighborhood Ω' of K, depending holomorphically on z, that has compact support (with respect to ζ) in Ω, see, e.g., [2, Example 2]. Here is an explicit choice when K is the closed ball $\overline{\mathbb{B}}$ and $\eta = \zeta - z$: If $\sigma = \bar\zeta \cdot d\eta / 2\pi i(|\zeta|^2 - \bar\zeta \cdot z)$, then $\delta_\eta \sigma = 1$ for $\zeta \neq z$ and

$$\sigma \wedge (\bar\partial\sigma)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\bar\zeta \cdot d\eta \wedge (d\bar\zeta \cdot d\eta)^{k-1}}{(|\zeta|^2 - \bar\zeta \cdot z)^k}.$$

If χ is a cutoff function that is 1 in a slightly larger ball, then we can take

$$g = \chi - \bar\partial\chi \wedge \frac{\sigma}{\nabla_\eta \sigma} = \chi - \bar\partial\chi \wedge [\sigma + \sigma \wedge \bar\partial\sigma + \sigma \wedge (\bar\partial\sigma)^2 + \cdots + \sigma \wedge (\bar\partial\sigma)^{N-1}].$$

Observe that $1/\nabla_\eta \sigma = 1/(1 - \bar\partial\sigma) = 1 + \bar\partial\sigma + (\bar\partial\sigma)^2 + \cdots$. One can find a g of the same form in the general case. $\square$

[2]For the time being, also $d\eta_j$ is supposed to include differentials with respect to both ζ and z; however, at the end only the $d_\zeta \eta_j$ come into play in this paper.

Let s be a smooth $(1,0)$-form in Λ_η such that $|s| \le C|\eta|$ and $|\delta_\eta s| \ge C|\eta|^2$; such an s is called *admissible*. Then $B = s/\nabla_\eta s$ is a locally integrable form and

$$\nabla_\eta B = 1 - [\Delta], \tag{3.1}$$

where $[\Delta]$ is the (N,N)-current of integration over the diagonal in $\Omega \times \Omega$. More concretely,

$$B_{k,k-1} = \frac{1}{(2\pi i)^k} \frac{s \wedge (\bar\partial s)^{k-1}}{(\delta_\eta s)^k}.$$

If $\eta = \zeta - z$, $s = \partial|\eta|^2$ will do, and we then refer to the resulting form B as the Bochner-Martinelli form. In this case

$$B_{k,k-1} = \frac{1}{(2\pi i)^k} \frac{\partial|\zeta - z|^2 \wedge (\bar\partial\partial|\zeta - z|^2)^{k-1}}{|\zeta - z|^{2k}}.$$

Assume now that Ω is pseudoconvex. Let us fix global frames for the bundles E_k in (2.3) over Ω. Then $E_k \simeq \mathbb{C}^{\operatorname{rank} E_k}$, and the morphisms f_k are just matrices of holomorphic functions. One can find (see [2] for explicit choices) $(k-\ell,0)$-form-valued Hefer morphisms, i.e., matrices, $H_k^\ell \colon E_k \to E_\ell$, depending holomorphically on z and ζ, such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$, and in general,

$$\delta_\eta H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}; \tag{3.2}$$

here f stands for $f(\zeta)$. Let

$$HU = \sum_k H_k^1 U_k, \quad HR = \sum_k H_k^0 R_k.$$

Thus HU takes a section Φ of E_0, i.e., a function, depending on ζ into a (current-valued) section $HU\Phi$ of E_1 depending on both ζ and z, and similarly, HR takes a section of E_0 into a section of E_0. We can have

$$g^\lambda = f(z) HU^\lambda + HR^\lambda$$

as smooth as we want by just taking $\operatorname{Re}\lambda$ large enough. If $\operatorname{Re}\lambda >> 0$, then, cf., [2, p. 235], g^λ is a weight, and in view of (3.1) thus

$$\nabla_\eta(g^\lambda \wedge g \wedge B) = g^\lambda \wedge g - [\Delta]$$

from which we get

$$\bar\partial (g^\lambda \wedge g \wedge B)_{N,N-1} = [\Delta] - (g^\lambda \wedge g)_{N,N}.$$

As in [2] we get the Koppelman formula

(3.3)

$$\Phi(z) = \int_\zeta (g^\lambda \wedge g \wedge B)_{N,N-1} \wedge \bar\partial\Phi + \bar\partial_z \int_\zeta (g^\lambda \wedge g \wedge B)_{N,N-1} \wedge \Phi + \int_\zeta (g^\lambda \wedge g)_{N,N} \wedge \Phi$$

for $z \in \Omega'$, and since $g^\lambda = HR^\lambda$ when $z \in X_{reg}$ we get

$$\begin{aligned} \Phi(z) = &\int_\zeta (HR^\lambda \wedge g \wedge B)_{N,N-1} \wedge \bar\partial\Phi + \\ &\bar\partial_z \int_\zeta (HR^\lambda \wedge g \wedge B)_{N,N-1} \wedge \Phi + \int_\zeta (HR^\lambda \wedge g)_{N,N} \wedge \Phi, \quad z \in X'_{reg}. \end{aligned} \tag{3.4}$$

It is proved in [6], see also [5] for a slightly different argument, that we can put $\lambda = 0$ in (3.4) and thus

$$\Phi(z) = \mathcal{K}\bar\partial\Phi + \bar\partial\mathcal{K}\Phi + \mathcal{P}\Phi, \quad z \in X'_{reg},$$

where

$$\mathcal{K}\Phi(z) = \int_\zeta (HR\wedge g\wedge B)_{N,N-1}\wedge\Phi, \quad z \in X'_{reg}, \tag{3.5}$$

and

$$\mathcal{P}\Phi(z) = \int_\zeta (HR\wedge g)_{N,N}\wedge\Phi, \quad z \in \Omega'. \tag{3.6}$$

If Φ is vanishing in a neighborhood of some given point x on X_{reg}, then $B\wedge\Phi$ is smooth in ζ for z close to x, and the integral in (3.5) is to be interpreted as the current R acting on a smooth form. It is clear that this integral depends smoothly on $z \in X'_{reg}$. Notice that

$$(HR\wedge g\wedge B)_{N,N-1} = \\ H^0_p R_p\wedge(g\wedge B)_{N-p,N-p-1} + H^0_{p+1}R_{p+1}\wedge(g\wedge B)_{N-p-1,N-p-2} + \cdots,$$

cf., (2.4), and that

$$(g\wedge B)_{N-k,N-k-1} = \mathcal{O}(1/|\eta|^{2N-2k-1}) \tag{3.7}$$

so it is integrable on X_{reg} for $k \geq N-n$. If Φ has support close to x, therefore (3.5) has a meaning as an approximative convolution and is again smooth in $z \in X_{reg}$ according to Lemma 3.2 below.

From Section 2 is is clear that these formulas only depend on the pullback ϕ of Φ to X_{reg}, and in view of (2.11) we have

Proposition 3.1. *Let g be any smooth weight in Ω with respect to Ω' and with compact support in Ω. For any smooth $(0,q)$-form ϕ on X, $\mathcal{K}\phi$ is a smooth $(0,q-1)$-form in X'_{reg}, $\mathcal{P}\phi$ is a smooth $(0,q)$-form in Ω', and we have the Koppelman formula*

$$\phi(z) = \bar\partial\mathcal{K}\phi(z) + \mathcal{K}(\bar\partial\phi)(z) + \mathcal{P}\phi(z), \quad z \in X'_{reg}. \tag{3.8}$$

where

$$\mathcal{K}\phi(z) = \int_\zeta k(\zeta,z)\wedge\phi(\zeta), \quad \mathcal{P}\phi(z) = \int_\zeta p(\zeta,z)\wedge\phi(\zeta), \tag{3.9}$$

and

$$k(\zeta,z) := \pm\gamma\lrcorner(H\wedge g\wedge B)_{N,N-1}, \quad p(\zeta,z) := \pm\gamma\lrcorner(H\wedge g)_{N,N}. \tag{3.10}$$

Since B has bidegree $(*,*-1)$, $\mathcal{K}\phi$ is a $(0,q-1)$-form and $\mathcal{P}\phi$ is $(0,q)$-form. It follows from (2.7) that the integrals in (3.9) exist as principal values at X_{sing}, i.e., $\mathcal{K}\phi = \lim\mathcal{K}(\chi_\delta\phi)$ and $\mathcal{P}\phi = \mathcal{P}(\chi_\delta\phi)$ if χ_δ is as in (2.7).

From (2.9) and (2.11) we find that

$$k(\zeta,z) = \omega(\zeta)\wedge\alpha(\zeta,z)/|\eta|^{2n}, \tag{3.11}$$

where α is a smooth form that is $\mathcal{O}(|\eta|)$.

Remark 4. Assume that ϕ is (smooth on X_{reg} and) in $\mathcal{W}_{0,q}(X)$. Then, see [6], $\mathcal{K}\phi$ and $\mathcal{P}\phi$ still define elements in $\mathcal{W}(X')$ that are smooth in X'_{reg}. Assume that ϕ in addition is in Dom $\bar\partial_X$. This means (implies) that $\bar\partial\chi_\delta\wedge\phi\wedge\omega\to 0$. Applying (3.9) to $\chi_\delta\phi$ for $z\in X'_{reg}$ and letting $\delta\to 0$, we conclude that (3.9) holds for ϕ as well. In particular, $\bar\partial\mathcal{K}\phi=\phi$ if $\bar\partial\phi=0$. □

Remark 5. In [6] we defined $\mathcal{A}$ as the smallest sheaf that is closed under multiplication with smooth forms and the action of any operator $\mathcal{K}$ as above with a weight g that is holomorphic in z. We can just as well admit any smooth weight g in the definition. The basic Theorem 2 in [6] holds also for this possibly slightly larger sheaf, that we still denote by $\mathcal{A}$. Basically the same proof works; the only difference is that in [6, (7.2)] we get an additional smooth term $\mathcal{P}\phi_{\ell-1}$, which however does not affect the conclusion. With this wider definition of $\mathcal{A}$ we have that $\mathcal{K}$ and $\mathcal{P}$ in (3.9) extend to operators $\mathcal{A}(X)\to\mathcal{A}(X')$ and $\mathcal{A}(X)\to\mathcal{E}_{0,*}(X')$, respectively. □

Lemma 3.2. *Suppose that $V\subset\Omega$ is smooth with codimension p and ξ has compact support and $\nu\le N-p$. If ξ is in $C^k(V)$, then*

$$h(z)=\int_{\zeta\in V}\frac{(\bar\zeta_i-\bar z_i)\xi(\zeta)}{|\zeta-z|^{2\nu}}$$

is in $C^k(V)$ as well for $i=1,\dots,N$.

4. Proofs of Theorems 1.1, 1.2, and 1.3

Proof of Theorem 1.1. If we choose g as the weight from Example 1 then $\mathcal{P}\phi$ will vanish for degree reasons unless ϕ has bidegree $(0,0)$, i.e., is a function, and in that case clearly $\mathcal{P}\phi$ will be holomorphic for all z in Ω'. Now Theorem 1.1 follows from (3.8) except for the asymptotic estimate (1.4).

After a slight regularization we may assume that $\delta(z)$ is smooth on X_{reg} or alternatively we can replace δ by $|h|$ where h is a tuple of functions in Ω such that $X_{sing}=\{h=0\}$, by virtue of Lojasiewicz' inequality, [16] and [17]. In fact, there is a number $r\ge 1$ such that

$$(1/C)\delta^r(\zeta)\le|h(\zeta)|\le C\delta(\zeta). \tag{4.1}$$

We have to estimate, cf., (3.11),

$$\int_\zeta\omega(\zeta)\wedge\frac{\alpha(\zeta,z)}{|\eta|^{2n}} \tag{4.2}$$

when $z\to X_{sing}$. To this end we take a smooth approximand χ of $\chi_{[1/4,\infty)}(t)$ and write (4.2) as

$$\int_\zeta\chi(\delta(\zeta)/\delta(z))\omega(\zeta)\wedge\frac{\alpha(\zeta,z)}{|\eta|^{2n}}\ +\ \int_\zeta\big(1\ -\ \chi(\delta(\zeta)/\delta(z))\big)\omega(\zeta)\wedge\frac{\alpha(\zeta,z)}{|\eta|^{2n}}.$$

In the first integral, $\delta(\zeta)\ge C\delta(z)$ and since the integrand is integrable we can use (2.9) and get the estimate $\lesssim\delta(z)^{-M}$ for some M. In the second integral we use instead that ω has some fixed finite order as a current so that its action can be estimates by a finite number of derivatives of $(1-\chi(\delta(\zeta)/\delta(z)))\alpha(\zeta,z)/|\eta|^{2n}$, which again is like $\delta(z)^{-M}$ for some M, since here $\delta(\zeta) \le \delta(z)/2$ and hence $C|\eta| \ge |\delta(z) - \delta(\zeta)| \ge \delta(z)/2$. Thus (1.4) holds. $\square$

Proof of Theorem 1.2. Suppose that ν is the order of the current R. Since $\mathcal{K}\Phi$ basically is the current R acting on Φ times a smooth form, it is clear that the Koppelman formula (3.8), but with Φ, remains true even if Φ is just of class $C^{\nu+1}$ in a neighborhood of X. For instance, for given Φ in $C^{\nu+1}$ this follows by approximating in $C^{\nu+1}$-norm by smooth forms.

It is a more delicate matter to check that $\mathcal{K}\Phi$ only depends on the pullback of Φ to X. The current R is (locally) the push-forward, under a suitable modification $\pi\colon Y \to \Omega$, of a finite sum $\tau = \sum \tau_j$ where each τ_j is a simple current of the form

$$\tau_j = \bar\partial \frac{1}{t_{j_1}^{a_{j_1}}} \wedge \frac{\alpha_j}{t_{j_2}^{a_{j_2}} \cdots t_{j_r}^{a_{j_r}}}, \tag{4.3}$$

with a smooth form α_j. Since R has the SEP with respect to X, arguing as in [4, Section 5], we can assume that the image of each of the divisors $t_{j_1} = 0$ is not fully contained in X_{sing}. Here is a sketch of a proof: Write $\tau = \tau' + \tau''$ where τ'' is the sum of all τ_j such that the image of $t_{1_j} = 0$ is contained in X_{sing}. Let $\chi_\delta = \chi(|h|/\delta)$, where h is a holomorphic tuple that cuts out X_{sing}. Then $\lim(\pi^*\chi_\delta)\tau'' = 0$ and $\lim(\pi^*\chi_\delta)\tau' = \tau'$. Since $R = \pi_*\tau$ and $\lim \chi_\delta R = R$, it follows that $R = \pi_*\tau'$.

Therefore, if $i\colon X \to \Omega$ and $i^*\Phi = 0$ on X_{reg}, then the pullback of $\pi^*\Phi$ to $t_{1_j} = 0$ must vanish. If Φ is in C^{L+1}, where L is the maximal sum of the powers in the denominators in (4.3), it follows that $\Phi\wedge R = \pi_*(\pi^*\Phi\wedge\tau) = 0$ and similarly $\bar\partial\Phi\wedge R = 0$. $\square$

Proof of Theorem 1.3. We will use an extra weight factor. In a slighly smaller domain $\Omega'' \subset\subset \Omega$ we can find a holomorphic tuple a such that $\{a = 0\} \cap X \cap \Omega'' = X_{sing} \cap \Omega''$. Let H^a be a holomorphic $(1,0)$-form in $\Omega'' \times \Omega''$ such that $\delta_\eta H^a = a(\zeta) - a(z)$. If ψ is a $(0,q)$-form that vanishes in a neighborhood of X_{sing} we can incorporate a suitable power of the weight

$$g_a = \frac{a(z)\cdot \bar a}{|a|^2} + H^a \cdot \bar\partial \frac{\bar a}{|a|^2} \tag{4.4}$$

in (3.8); we will use the weight $g_a^{\mu+n}\wedge g$ instead of just g, the usual weight with respect to $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ that has compact support and is holomorphic in z. For degree reasons, the second term on the right hand side of (4.4) can occur to the power at most n when pulled back to X, and hence the associated kernel

$$k^\mu(\zeta, z) = \gamma \lrcorner (H\wedge g_a^{\mu+n}\wedge g\wedge B)_{N,N-1}$$

is like, cf., (2.11),

$$\omega(\zeta)\wedge\Big(\frac{a(z)\cdot\overline{a(\zeta)}}{|a(\zeta)|^2}\Big)^\mu\wedge\mathcal{O}(1/|\eta|^{2n-1}).$$

The operators in Lemma 3.2 are bounded on L^p_{loc}, so we have that

$$\psi = \bar\partial \int_{X_{reg}} k^\mu(\zeta,z)\psi(\zeta) + \int_{X_{reg}} k^\mu(\zeta,z)\wedge\bar\partial\psi(\zeta) \tag{4.5}$$

for $(0,q)$-forms ψ, $q \geq 1$, in $L^p(X_{reg})$ that vanish in a neighborhood of X_{sing}. If ϕ is as in Theorem 1.3, thus (4.5) holds for $\psi = \chi_\epsilon \phi$, where $\chi_\epsilon = \chi(|a|^2/\epsilon)\phi$ and χ is a smooth approximand of the characteristic function for $[1,\infty)$.

If now $\mu' \geq M + r + \mu r$, where M is as in (2.9) and r as in (4.1), noting that $\bar\partial \chi_\epsilon \sim 1/|a|$, it follows that

$$\int \bar\partial \chi_\epsilon \wedge k^\mu \wedge \phi$$

tends to zero in L^p when $\epsilon \to 0$ if $\delta^{-\mu'}\phi \in L^p$. Therefore

$$u = \int_{X_{reg}} k^\mu(\zeta, z) \wedge \phi(\zeta)$$

is a solution such that $\delta^{-\mu} u \in L^p$. $\square$

5. Solutions with compact support

The proofs of Theorems 1.4, 1.5, and 1.7 relay on on the possibility to solve the $\bar\partial$-equation with compact support. To begin with, assume that X, X', Ω, Ω' are as in Theorem 1.1 and let $f \in \mathcal{A}_{q+1}(X)$ be $\bar\partial$-closed and with support in X'. Choose a resolution (2.2) of $\mathcal{O}^X = \mathcal{O}^\Omega/\mathcal{J}$ in (a slightly smaller set) Ω that ends at level $M = N - \nu$ where ν is the minimal depth of $\mathcal{O}^X_x$. Let $\tilde\chi$ be a cutoff function with support in Ω' that is identically 1 in a neighborhood of the support of f, and let g be the weight from Example 1 with this choice of $\tilde\chi$ but with z and ζ interchanged. This weight does not have compact support with respect to ζ, but since f has compact support itself we still have the Koppelman formula (3.8). (The one who is worried can include an extra weight factor with compact support that is identically 1 in a neighborhood of $\operatorname{supp} \bar\partial\tilde\chi$; we are then formally back to the situation in Proposition 3.1.) Clearly

$$v(z) = \int (HR \wedge g \wedge B)_{N,N-1} \wedge f$$

is in $\mathcal{A}_q(X')$ and has support in a neighborhood of the support of f, and it follows from (3.8) that it is indeed a solution if the associated integral $\mathcal{P}f$ vanishes. However, since now σ is holomorphic in ζ, for degree reasons we have that

$$\mathcal{P}f(z) = \pm \bar\partial\tilde\chi(z) \wedge \int HR_{N-q-1} \wedge \sigma \wedge (\bar\partial\sigma)^q \wedge f. \tag{5.1}$$

If $q \leq \nu - 2$, this integral vanishes since then $N - q - 1 \geq N - \nu + 1$ so that $R_{N-q-1} = 0$. If $q = \nu - 1$, then $\mathcal{P}f(z)$ vanishes if

$$\int R_{N-q-1} \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_N \wedge f h = \pm \int_X f \wedge h \omega_{n-\nu} = 0 \tag{5.2}$$

for all $h \in \mathcal{O}(X')$, and by approximation it is enough to assume that (5.2) holds for $h \in \mathcal{O}(X)$.

Remark 6. The condition (5.2) is necessary: Indeed if there is a solution $v \in \mathcal{A}_q(X')$ with compact support, then since $\bar\partial\omega_{n-\nu} = 0$ in X' we have that

$$\int_X f \wedge h\omega_{n-\nu} = \pm \int_X \bar\partial v \wedge h \omega_{n-\nu} = 0,$$

since $\bar\partial(v\omega_{n-\nu}) = \bar\partial v\wedge\omega_{n-\nu}$. This in turn holds, since $\nabla_f(v\wedge\omega) = -\bar\partial v\wedge\omega$, which directly follows from the definition of v being in $\mathcal{A} \subset \mathrm{Dom}\,\bar\partial_X$. $\square$

Proof of Theorem 1.4. Since X can be exhausted by holomorphically convex subsets each of which can be embedded in some affine space, we can assume from the beginning that $X \subset \Omega \subset \mathbb{C}^N$, where Ω is holomorphically convex (pseudoconvex). Let $\Omega' \subset\subset \Omega$ be a holomorphically convex open set in Ω that contains K. Let χ be a cutoff function with support in Ω' that is 1 in a neighborhood of K and let $f = \bar\partial\chi\wedge\phi$. Then $(1-\chi)\phi$ is a smooth function in X that coincides with ϕ outside a neighborhood of K. As we have seen above, one can find a $u \in \mathcal{A}_0(X)$ with support in X' such that $\bar\partial u = f$ if either $\nu \geq 2$ or (5.2), i.e., (1.6), holds.

Since X_{sing} is not contained in K, our solution u is, outside of K, only smooth on X_{reg}. Therefore $\Phi = (1-\chi)\phi + u$ is holomorphic in X_{reg}, in a neighborhood of K, and outside Ω'. Since $X^\ell_{reg}\backslash K$ is connected, $\Phi = \phi$ there. (It follows directly that Φ is in $\mathcal{O}(X)$, since it is in $\mathcal{A}_0(X)$ and $\bar\partial\Phi = 0$.) $\square$

Example 2. Let $X \subset \mathbb{C}^2$ be an irreducible curve with one transverse self intersection at $0 \in \mathbb{C}^2$. Close to 0, X has two irreducible components, A_1, A_2, each isomorphic to a disc in $\mathbb{C}$. Let $K \subset A_1$ be a closed annulus surrounding the intersection point $A_1 \cap A_2$. Then $X \setminus K$ is connected but $X_{reg} \setminus K$ is not. Denote the "bounded component" of $A_1 \setminus K$ by U_1 and put $U_2 = X\setminus(K\cup U_1)$. Let $\tilde\phi \in \mathcal{O}(X)$ satisfy $\tilde\phi(0) = 0$ and define ϕ to be 0 on U_1 and equal to $\tilde\phi$ on U_2. Then $\phi \in \mathcal{O}(X\setminus K)$ and a straight forward verification shows that ϕ satisfies the compatibility condition (1.6). However, it is clear that ϕ cannot be extended to a strongly holomorphic function on X. $\square$

Proof of Theorem 1.5 and Corollary 1.6. Theorem 1.5 is proved in pretty much the same way as Theorem 1.4. Again we can assume that $X \subset \Omega \subset \mathbb{C}^N$. Again take χ that is 1 in a neighborhood of K and with compact support in X'. There is then a solution in $\mathcal{A}_q(X')$ to $\bar\partial u = \bar\partial\chi\wedge\phi$ with support in X' if $q \leq \nu - 2$ or $q = \nu - 1$ and (5.2), i.e., (1.7) holds. Thus $\Phi = (1-\chi)\phi + u$ is in $\mathcal{A}_q(X)$, $\bar\partial\Phi = 0$, and $\Phi = \phi$ outside X'.

Let us now consider the corollary. We may assume that

$$K \subset \cdots X_{\ell+1} \subset\subset X_\ell \subset\subset \cdots X_0 \subset\subset X,$$

where all X_ℓ are Stein spaces. It follows from Theorem 1.5 that for each ℓ there is a $\bar\partial$-closed $\Phi_\ell \in \mathcal{A}_q(X)$ that coincides with ϕ outside X_ℓ, if $q \leq \nu - 2$ or $q = \nu - 1$ and (1.7) holds. From the exactness of (1.1) we have $u'_\ell \in \mathcal{A}_{q-1}(X)$ such that $\bar\partial u'_\ell = \Phi_\ell$. Since $\bar\partial(u'_\ell - u'_{\ell+1}) = 0$ outside X_ℓ, there is a $\bar\partial$-closed $w_\ell \in \mathcal{A}_{q-1}(X)$ such that $w_\ell = u'_\ell - u'_{\ell+1}$ outside X_ℓ (or at least outside $X_{\ell-1}$). If we let $u_k = u'_k - (w_1 + \cdots + w_{k-1})$ then $u = \lim u_k$ exists and solves $\bar\partial u = \phi$ in $X \setminus K$. $\square$

One can show directly that the conditions (1.6) and (1.7) are independent of the choice of metrics on $E_\bullet$: Let R' and R be the currents correspondning to two different metrics. With the notation in the proof of Theorem 4.1 in [3] we have

$$\nabla_f M = R - R', \tag{5.3}$$

where $M = \bar\partial |F|^{2\lambda} \wedge u' \wedge u|_{\lambda=0}$. It follows as in this proof that, outside X_{sing}, $M = \beta R_{N-n}$ where β is smooth. Following the proof of Proposition 16 in [6], we find that in fact $M\wedge dz = i_* m$, where $m = \beta\omega_0$ outside X_{sing}. However, β is a sum of terms like

$$(\bar\partial\sigma'_{n-\nu})\cdots(\bar\partial\sigma'_{r+1})\sigma'_r(\bar\partial\sigma_{r-1})\cdots(\bar\partial\sigma_{N-n+1}),$$

it is therefore almost semimeromorphic on X, and thus $m = \beta\omega_0$. Moreover, as in the proof of the main lemma [6, Lemma 27] it follows that $\bar\partial\chi_\delta\wedge\beta\wedge\omega_0\wedge\phi \to 0$ when $\delta\to 0$ if ϕ is in $\mathcal{A}$. Therefore,

$$\int_X \bar\partial m^{N-n}\wedge\phi\wedge h = \lim_{\delta\to 0}\int_X \chi_\delta\bar\partial m^{N-n}\wedge\phi\wedge h = \pm\lim_{\delta\to 0}\int_X m\wedge\bar\partial\chi_\delta\wedge\phi\; h = 0.$$

From (5.3) we have that $\bar\partial M_{N-\nu} = R'_{N-\nu} - R_{N-\nu}$ and hence $\bar\partial m_{n-\nu} = \omega'_{n-\nu} - \omega_{n-\nu}$. We thus have that (1.7) holds with $\omega_{n-\nu}$ if and only it holds with $\omega'_{n-\nu}$.

Remark 7. The proofs above for part (i) of the theorems can be seen as concrete realizations of abstract arguments. There is a long exact sequence

$$0\to H^0_K(X,\mathcal{O})\to H^0(X,\mathcal{O})\to H^0(X\setminus K,\mathcal{O})\to$$
$$\to H^1_K(X,\mathcal{O})\to H^1(X,\mathcal{O})\to H^1(X\setminus K,\mathcal{O})\to\cdots.$$

Since X is Stein, $H^k(X,\mathcal{O}) = 0$ for $k\geq 1$. Thus $H^0(X,\mathcal{O})\to H^0(X\setminus K,\mathcal{O})$ is surjective if $H^1_K(X,\mathcal{O}) = 0$, and in the same way, for $q\geq 1$, we have that $H^q(X\setminus K,\mathcal{O}) = 0$ if (and only if) $H^{q+1}_K(X,\mathcal{O}) = 0$. □

We now consider $X\setminus A$ where X is Stein and A is an analytic subset of positive codimension. For convenience we first consider the technical part concerning local solutions with compact support.

Proposition 5.1. *Let X be an analytic set defined in a neighborhood of the closed unit ball $\bar{\mathbb{B}}\subset\mathbb{C}^N$, A an analytic subset of X, and let $x\in A$, and let a be a holomorphic tuple such that $A = \{a = 0\}$ in a neighborhood of x and let $d = \dim A$. Assume that f is in $\mathcal{A}_{q+1}$ in a neighborhood of x, $\bar\partial f = 0$, and that f has support in $\{|a| < t\}$ for some small t. (We may assume that $f = 0$ close to A.)*

(i) If $0\leq q\leq \nu-d-2$, then one can find, in a neighborhood U of x, a $(0,q)$-form u in $\mathcal{A}_q$ with support in $\{|a|<t\}$ such that $\bar\partial u = f$ in $X\setminus A\cap U$.

(ii) If $0\leq q = \nu-d-1$, then one can find such a solution if and only if

$$\int_X f\wedge h\wedge\omega_{n-\nu} = 0 \tag{5.4}$$

for all smooth $\bar\partial$-closed $(0,d)$-forms h such that $\operatorname{supp} h\cap\{|a|\leq t\}$ is compact and contained in the set where $\bar\partial f = 0$.

Proof. Let χ_a be a cutoff function in $\mathbb{B}$, which in a neighborhood of x satisfies that $\chi_a = 1$ in a neighborhood of the support of f and $\chi_a = 0$ in a neighborhood of $\{|a|\geq t\}$. Close to x we can choose coordinates $z = (z', z'') = (z'_1,\dots,z'_d,z''_1,\dots,z''_{N-d})$ centered at x so that $A\subset\{|z''|\leq|z'|\}$.

Let H^a be a holomorphic $(1,0)$-form, as in the proof of Theorem 1.3, and define

$$g^a = \chi_a(z) - \bar\partial\chi_a(z) \wedge \frac{\sigma_a}{\nabla_\eta \sigma_a}, \quad \sigma_a = \frac{\overline{a(z)} \cdot H^a}{|a(z)|^2 - a(\zeta)\cdot \overline{a(z)}}.$$

Then g^a is a smooth weight for ζ on the support of f. Since f is supported close to A we can choose a function $\chi = \chi(\zeta')$, which is 1 close to x and such that $f\chi$ has compact support. Let $g = \chi - \bar\partial\chi \wedge \sigma/\nabla_\eta\sigma$ be the weight from Example 2 but built from z' and ζ'. Our Koppelman formula now gives that

$$u = \mathcal{K}f = \int (HR \wedge g^a \wedge g \wedge B)_{N,N-1} \wedge f$$

has the desired properties provided that the obstruction term

$$\mathcal{P}f = \int (HR \wedge g^a \wedge g)_{N,N} \wedge f$$

vanishes. Since g is built from ζ', g has at most degree d in $d\bar\zeta$. Moreover, HR has at most degree $N-\nu$ in $d\bar\zeta$ and g^a has no degree in $d\bar\zeta$. Thus, if $q \le \nu - d - 2$, then $(HR\wedge g^a \wedge g)_{N,N} \wedge f$ cannot have degree N in $d\bar\zeta$ and so $\mathcal{P}f = 0$ in that case. This proves (i). If $q = \nu - d - 1$, then

$$\mathcal{P}f = \bar\partial\chi_a(z) \wedge \int HR_{N-\nu} \wedge g_d \wedge \sigma_a \wedge (\bar\partial\sigma_a)^q f.$$

Now, H^a depends holomorphically on ζ and g_d is $\bar\partial$-closed since it is the top degree term of a weight. Also, g has compact support in the ζ'-direction, so $\mathrm{supp}(g) \cap \{|a| \le t\}$ is compact and thus $\mathcal{P}f = 0$ if (5.4) is fulfilled. On the other hand, it is clear that the existence of a solution with support in $\{|a| < t\}$ implies (5.4). □

Proof of Theorem 1.7. Arguing as in the proof of Corollary 1.6 above, we can conclude from Proposition 5.1: *Given a point x there is a neighborhood U such that if $\phi \in \mathcal{A}_q(U \cap X \setminus A)$ is $\bar\partial$-closed, $0 \le q \le \nu - d - 2$ or $0 \le q = \nu - d - 1$ and (1.8) holds, ϕ is strongly holomorphic if $q = 0$ and exact in $X \setminus A \cap U'$, for a possibly slightly smaller neighborhood U' of x, if $q \ge 1$.*

We define the analytic sheaves $\mathcal{F}_k$ on X by $\mathcal{F}_k(V) = \mathcal{A}_k(V\setminus A)$ for open sets $V \subset X$. Then $\mathcal{F}_k$ are fine sheaves and

$$0 \to \mathcal{O}_X \to \mathcal{F}_0 \xrightarrow{\bar\partial} \mathcal{F}_1 \xrightarrow{\bar\partial} \mathcal{F}_2 \xrightarrow{\bar\partial} \cdots \tag{5.5}$$

is exact for $k \le \nu - d - 2$. It follows that

$$H^k(X, \mathcal{O}_X) = \frac{\mathrm{Ker}_{\bar\partial} \mathcal{F}_k(X)}{\bar\partial \mathcal{F}_{k-1}(X)}$$

for $k \le \nu - d - 2$. Hence Theorem 1.7 follows for $q \le \nu - d - 2$. If $q = \nu - d - 1$ and (1.8) holds, then ϕ is in the image of $\mathcal{F}_{q-1} \to \mathcal{F}_q$, and then the result follows as well. □

6. Examples

We have already seen that if X is smooth, then ω_k is just a smooth (n,k)-form, and ω_0 is non-vanishing. At least semi-globally we can choose $\omega=\omega_0$, and then ω_0 is holomorphic.

Let now $X=\{h=0\}\subset\mathbb{B}\subset\mathbb{C}^{n+1}$, $h\in\mathcal{O}(\bar{\mathbb{B}})$, be a hypersurface in the unit ball in $\mathbb{C}^{n+1}$ and assume that $0\in X$. The depth (homological codimension) of $\mathcal{O}^X_x$ equals $\dim X=n$ for all $x\in X$. The residue current associated with X is simply $R=R_1=\bar\partial(1/h)$ and so by the Poincare-Lelong formula (2.1)

$$R\wedge d\zeta=\bar\partial\frac{1}{h}\wedge d\zeta=\bar\partial\frac{1}{h}\wedge\frac{dh}{2\pi i}\wedge\tilde\omega=\tilde\omega\wedge[X],$$

where, e.g.,

$$\tilde\omega=2\pi i\sum_{j=1}^{n+1}(-1)^{n-1}\frac{\overline{(\partial h/\partial\zeta_j)}}{|dh|^2}d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_{n+1}.$$

The structure form associated with X then is $\omega=i^*\tilde\omega$, where $i\colon X\hookrightarrow\mathbb{B}$. Alternatively, we can write $R=\gamma\lrcorner[X]$, and thus

$$\omega=\pm i^*(\gamma\lrcorner d\zeta_1\wedge\ldots\wedge d\zeta_{n+1}), \tag{6.1}$$

for

$$\gamma=-2\pi i\sum_{j=1}^{n+1}\frac{\overline{(\partial h/\partial\zeta_j)}}{|dh|^2}\frac{\partial}{\partial\zeta_j}. \tag{6.2}$$

Let $K=\{0\}\subset X$ and let $\phi\in\mathcal{A}_q(X\setminus K)$ be $\bar\partial$-closed. Since $\nu=n$ it follows from Theorem 1.5 and Corollary 1.6 that ϕ has a $\bar\partial$-closed extension in $\mathcal{A}_q(X)$ and is $\bar\partial$-exact in $X\setminus K$ if $q\le n-2$, or if $q=n-1$ and (1.7) holds. Let us consider (1.7) in our special case; assume therefore that $q=n-1$. The function χ in (1.7) may be any smooth function that is 1 in a neighborhood of K and has compact support in $\mathbb{B}$. Via Stokes' theorem, or a simple limit procedure, we can write the condition (1.7) as

$$0=\int_{X\cap\partial\mathbb{B}_\epsilon}\omega\wedge\phi\xi,\quad \xi\in\mathcal{O}(\mathbb{B}), \tag{6.3}$$

where ω is given by (6.1) and (6.2).

In case $X=\{\zeta_{n+1}=0\}$ we have $\omega=\pm2\pi i d\zeta_1\wedge\cdots\wedge d\zeta_n$ and (6.3) reduces to the usual condition for ϕ having a $\bar\partial$-closed extension across 0. Let instead $X=\{\zeta_1^r-\zeta_2^s=0\}\cap\mathbb{B}\subset\mathbb{C}^2$, where $2\le r<s$ are relatively prime integers. Then $\tau\mapsto(\tau^s,\tau^r)$ is the normalization of X. We have

$$\gamma=-2\pi i\frac{r\bar\zeta_1^{r-1}\partial/\partial\zeta_1-s\bar\zeta_2^{s-1}\partial/\partial\zeta_2}{r^2|\zeta_1|^{2(r-1)}+s^2|\zeta_2|^{2(s-1)}},$$

and it is straightforward to verify that $\omega=2\pi i d\tau/\tau^{(r-1)(s-1)}$. Let ϕ be holomorphic on $X\setminus\{0\}=X_{reg}$. Then, cf., (6.3), ϕ has a (strongly) holomorphic extension to X if and only if

$$\int_{|\tau|=\epsilon}\phi\xi\, d\tau/\tau^{(r-1)(s-1)}=0,\quad \xi\in\mathcal{O}(X).$$

7. Proof of Theorem 1.8

We now turn our attention to the proof of Theorem 1.8. We first assume that X is a subvariety of some domain Ω in $\mathbb{C}^N$. A basic problem with the globalization is that we cannot assume that there is one single resolution (2.2) of $\mathcal{O}/\mathcal{J}$ in the whole domain Ω. We therefore must patch together local solutions. To this end we will use Cech cohomology. Recall that if Ω_j is an open cover of Ω, then a k-cochain ξ is a formal sum

$$\xi = \sum_{|I|=k+1} \xi_I \wedge \epsilon_I$$

where I are multi-indices and ϵ_j is a nonsense basis, cf., e.g., [1, Section 8]. Moreover, in this language the coboundary operator ρ is defined as $\rho\xi = \epsilon\wedge\xi$, where $\epsilon = \sum_j \epsilon_j$.

If g is a weight as in Example 1 and $g' = (1-\chi)\sigma/\nabla_\eta\sigma$, then

$$\nabla_\eta g' = 1 - g. \tag{7.1}$$

Notice that the relations (3.2) for the Hefer morphism(s) can be written simply as

$$\delta_\eta H = Hf - f(z)H = Hf$$

if $z \in X$.

Proof of Theorem 1.8 in case $X \subset \Omega \subset \mathbb{C}^N$. Assume that ϕ is in $\mathcal{W}(X) \cap \operatorname{Dom}\bar\partial_X$, smooth on X_{reg}, and that $\bar\partial\phi = 0$. Let Ω_j be a locally finite open cover of Ω with convex polydomains (Cartesian products of convex domains in each variable), and for each j let g_j be a weight with support in a slightly larger convex polydomain $\tilde\Omega_j \supset\supset \Omega_j$ and holomorphic in z in a neighborhood of $\overline{\Omega}_j$. Moreover, for each j suppose that we have a given resolution (2.2) in $\tilde\Omega_j$, a choice of Hermitian metric, a choice of Hefer morphism, and let $(HR)_j$ be the resulting current. Then, cf., Remark 4 above,

$$u_j(z) = \int \big((HR)_j \wedge g_j \wedge B\big)_{N,N-1} \wedge \phi \tag{7.2}$$

is a solution in $\Omega_j \cap X_{reg}$ to $\bar\partial u_j = \phi$. We will prove that $u_j - u_k$ is (strongly) holomorphic on $\Omega_{jk} \cap X$ if $q = 1$ and $u_j - u_k = \bar\partial u_{jk}$ on $\Omega_{jk} \cap X_{reg}$ if $q > 1$, and more generally:

Claim I *Let u^0 be the 0-cochain $u^0 = \sum u_j \wedge \epsilon_j$. For each $k \le q-1$ there is a k-cochain of $(0, q-k-1)$-forms on X_{reg} such that $\rho u^k = \bar\partial u^{k+1}$ if $k < q-1$ and ρu^{q-1} is a (strongly) holomorphic q-cocycle.*

The holomorphic q-cocycle ρu^{q-1} defines a class in $H^q(\Omega, \mathcal{O}/\mathcal{J})$ and if Ω is pseudoconvex this class must vanish, i.e., there is a holomorphic $q-1$-cochain h such that $\rho h = \rho u^{q-1}$. By standard arguments this yields a global solution to $\bar\partial\psi = \phi$. For instance, if $q = 1$ this means that we have holomorphic functions h_j in Ω_j such that $u_j - u_k = h_j - h_k$ in $\Omega_{jk} \cap X$. It follows that $u_j - h_j$ is a global solution in X_{reg}.

We thus have to prove Claim I. To begin with we assume that we have a fixed resolution with a fixed metric and Hefer morphism; thus a fixed choice of current HR. Notice that if

$$g_{jk} = g_j \wedge g'_k - g_k \wedge g'_j,$$

cf., (7.1), then

$$\nabla_\eta g_{jk} = g_j - g_k$$

in $\tilde{\Omega}_{jk}$. With g^λ as in Section 3, and in view of (3.1), we have

$$\nabla_\eta(g^\lambda \wedge g_{jk} \wedge B) = g^\lambda \wedge g_j \wedge B - g^\lambda \wedge g_k \wedge B - g^\lambda \wedge g_{jk} + g^\lambda \wedge g_{jk} \wedge [\Delta].$$

However, the last term must vanish since $[\Delta]$ has full degree in $d\eta$ and g_{jk} has at least degree 1. Therefore

$$-\bar\partial(g^\lambda \wedge g_{jk} \wedge B)_{N,N-2} = (g^\lambda \wedge g_j \wedge B)_{N,N-1} - (g^\lambda \wedge g_k \wedge B)_{N,N-1} - (g^\lambda \wedge g_{jk})_{N,N-1}$$

and as in Section 3 we can take $\lambda = 0$ and get, assuming that $\bar\partial\phi = 0$ and arguing as in Remark 4,

$$u_j - u_k = \int (HR \wedge g_{jk})_{N,N-1} \wedge \phi + \bar\partial_z \int (HR \wedge g_{jk} \wedge B)_{N,N-2} \wedge \phi. \tag{7.3}$$

Since g_{jk} is holomorphic in z in Ω_{jk} it follows that $u_j - u_k$ is (strongly) holomorphic in $\Omega_{jk} \cap X$ if $q = 1$ and $\bar\partial$-exact on $\Omega_{jk} \cap X_{reg}$ if $q > 1$.

Claim II *Assume that we have a fixed resolution but different choices of Hefer forms and metrics and thus different $a_j = (HR)_j$ in $\tilde{\Omega}_j$. Let ϵ'_j be a nonsense basis. If $A^0 = \sum a_j \wedge \epsilon'_j$, then for each $k > 0$ there is a k-cochain*

$$A^k = \sum_{|I|=k+1} A_I \wedge \epsilon'_I,$$

where A_I are currents on $\tilde{\Omega}_I$ with support on $\tilde{\Omega}_I \cap X$ and holomorphic in z in Ω_I, such that

$$\rho' A^k = \epsilon' \wedge A^k = \nabla_\eta A^{k+1}. \tag{7.4}$$

Moreover,

$$\bar\partial \chi_\delta \wedge \phi \wedge A^k \to 0, \quad \delta \to 0. \tag{7.5}$$

For the last statement we use that X is Cohen-Macaulay.

In particular we have currents a_{jk} with support on X and such that $\nabla_\eta a_{jk} = a_j - a_k$ in $\tilde{\Omega}_{jk}$. If

$$w_{jk} = a_{jk} \wedge g_j \wedge g_k + a_j \wedge g_j \wedge g'_k - a_k \wedge g_k \wedge g'_j,$$

then

$$\nabla_\eta w_{jk} = a_j \wedge g_j - a_k \wedge g_k.$$

Notice that w_{jk} is a globally defined current. By a similar argument as above (and via a suitable limit process), cf., Remark 4 and (7.5), one gets that

$$u_j - u_k = \int (w_{jk})_{N,N-1} \wedge \phi + \bar\partial_z \int (w_{jk} \wedge B)_{N,N-2} \wedge \phi$$

in $\Omega_{jk} \cap X_{reg}$ as before. In general we put

$$\epsilon' = g = \sum g_j \wedge \epsilon_j.$$

If, cf., (7.1),

$$g' = \sum g'_j \wedge \epsilon_j$$

then

$$\nabla_\eta g' = \epsilon - g = \epsilon - \epsilon'.$$

If a_I is a form on $\tilde{\Omega}_I$, then $a_I \wedge \epsilon'_I$ is a well-defined global form. Therefore A, and hence also

$$W = A\wedge e^{g'},$$

i.e., $W^k = \sum_j A^{k-j}(g')^j/j!$, has globally defined coefficients and

$$\rho W = \nabla_\eta W.$$

In fact, since A and g' have even degree,

$$\nabla_\eta (A\wedge e^{g'}) = \epsilon' \wedge A\wedge e^{g'} + A\wedge e^{g'}\wedge(\epsilon - \epsilon') = \epsilon\wedge A\wedge e^{g'}.$$

By the yoga above the k-cochain

$$u^k = \int (W^k \wedge B)_{N,N-k-1}\wedge\phi$$

satisfies

$$\rho u^k = \bar\partial_z \int (W^{k+1}\wedge B)_{N,N-k-2}\phi + \int (W^{k+1})_{N,N-k-1}\wedge\phi.$$

Thus $\rho u^k = \bar\partial u^{k+1}$ for $k < q-1$ whereas $\rho\wedge u^{q-1}$ is a holomorphic q-cocycle as desired.

It remains to consider the case when we have different resolutions in Ω_j. For each pair j,k choose a weight $g_{s_{jk}}$ with support in $\tilde\Omega_{jk}$ that is holomorphic in z in $\Omega_{s_{jk}} = \Omega_{jk}$. By [12, Theorem 3 Ch. 6 Section F] we can choose a resolution in $\tilde\Omega_{s_{jk}} = \tilde\Omega_{jk}$ in which both of the resolutions in $\tilde\Omega_j$ and $\tilde\Omega_k$ restricted to $\Omega_{s_{jk}}$ are direct summands. Let us fix metric and Hefer form and thus a current $a_{s_{jk}} = (HR)_{s_{jk}}$ in $\Omega_{s_{jk}}$ and thus a solution $u_{s_{jk}}$ corresponding to $(HR)_{s_{jk}}\wedge g_{s_{jk}}$. If we extend the metric and Hefer form from $\tilde\Omega_j$ in a way that respects the direct sum, then $(HR)_j$ with these extended choices will be unaffected, cf., [3, Section 4]. On $\tilde\Omega_{js_{jk}}$ we therefore practically speaking have just one single resolution and as before thus $u_j - u_s$ is holomorphic (if $q=1$) and $\bar\partial u_{js_{jk}}$ if $q>1$. It follows that $u_j - u_k = u_j - u_s + u_s - u_k$ is holomorphic on Ω_{jk} if $q=1$ and equal to $\bar\partial$ of

$$u_{jk} = u_{js_{jk}} + u_{s_{jk}k}$$

if $q>1$. We now claim that each 1-cocycle

$$u_{jk} + u_{kl} + u_{lj} \tag{7.6}$$

is holomorphic on Ω_{jkl} if $q=2$ and $\bar\partial$-exact on $\Omega_{jkl}\cap X_{reg}$ if $q>2$. On $\tilde\Omega_{s_{jkl}} = \tilde\Omega_{jkl}$ we can choose a resolution in which each of the resolutions associated with the indices s_{jk}, s_{kl} and s_{kj} are direct summands. It follows that $u_{js_{jk}} + u_{s_{jk}s_{jkl}} + u_{s_{jkl}j}$ is holomorphic if $q=2$ and $\bar\partial u_{js_{jk}s_{jkl}}$ if $q>2$. Summing up, the statement about (7.6) follows. If we continue in this way Claim I follows.

It remains to prove Claim II. It is not too hard to check by an appropriate induction procedure, cf., the very construction of Hefer morphisms in [2], that if we have two choices of (systems of) Hefer forms H_j and H_k for the same resolution f, then there is a form H_{jk} such that

$$\delta_\eta H_{jk} = H_j - H_k + f(z)H_{jk} - H_{jk}f. \tag{7.7}$$

More generally, if

$$H^0 = \sum H_j \wedge \epsilon_j$$

then for each k there is a (holomorphic) k-cochain H^k such that (assuming $f(z) = 0$ for simplicity)

$$\delta_\eta H^k = \epsilon \wedge H^{k-1} - H^k f \tag{7.8}$$

(the difference in sign between (7.7) and (7.8) is because in the latter one f is to the right of the basis elements).

Elaborating the construction in [3, Section 4], cf., [1, Section 8], one finds, given $R^0 = \sum R_j \wedge \epsilon_j$, k-cochains of currents R^k such that

$$\nabla_f R^{k+1} = \epsilon \wedge R^k. \tag{7.9}$$

(With the notation in [3], if $R_j = \bar\partial |F|^{2\lambda} \wedge u^j|_{\lambda=0}$, then the coefficient for $\epsilon_j \wedge \epsilon_k \wedge \epsilon_\ell$ is $\bar\partial |F|^{2\lambda} \wedge u^j u^k u^\ell|_{\lambda=0}$, etc.)

We define a product of forms in the following way. If the multiindices I, J have no index in common, then $(\epsilon_I, \epsilon_J) = 0$, whereas

$$(\epsilon_I \wedge \epsilon_\ell, \epsilon_\ell \wedge \epsilon_J) = \frac{|I|!|J|!}{(|I| + |J| + 1)!} \epsilon_I \wedge \epsilon_J.$$

We then extend it to any forms bilinearly in the natural way. It is easy to check that

$$(H^k f, R^\ell) = -(H^k, f R^\ell).$$

Using (7.8) and (7.9) (and keeping in mind that H^k and R^ℓ have odd order) one can verify that

$$\nabla_\eta (H^k, R^\ell) = (\epsilon \wedge H^{k-1}, R^\ell) + (H^k, \epsilon \wedge R^\ell).$$

By a similar argument one can finally check that

$$A^k = \sum_{j=0}^{k} (H^j, R^{k-j})$$

will satisfy (7.4).

Since X is Cohen-Macaulay, each R^k will be a smooth form times the principal term $(R_j)_{N-n}$ for R_j corresponding to some choice of metric. The case with two different metrics is described in [3, Section 4] and the general case is similar; compare also to the discussion preceding Remark 7. Thus (7.5) holds, and thus Claim II holds, and so Theorem 1.8 is proved in case X is a subvariety of $\Omega \subset \mathbb{C}^N$. $\square$

Remark 8. If X is not Cohen-Macaulay, then we must assume explicitly that $\bar\partial \chi_\delta \wedge \phi \wedge R^k \to 0$ for all R^k. $\square$

The extension to a general analytic space X is done in pretty much the same way and we just sketch the idea. First assume that we have a fixed η as before but two different choices s and $\tilde s$ of admissible form, and let B

and $\tilde{B}$ be the corresponding locally integrable forms. Then, one can check, arguing as in [6, Section 5], that

$$\nabla_\eta(B\wedge\tilde{B}) = \tilde{B} - B \tag{7.10}$$

in the current sense, and by a minor modification of Lemma 3.2 one can check that

$$\int (HR\wedge g\wedge B\wedge\tilde{B})_{N,N-2}\wedge\phi$$

is smooth on $X_{reg}\cap\Omega'$; for degree reasons it vanishes if $q=1$. It follows from (7.10) that $\nabla_\eta(HR^\lambda\wedge g\wedge B\wedge\tilde{B}) = HR^\lambda\wedge g\wedge\tilde{B} - HR^\lambda\wedge g\wedge B$ from which we can conclude that

$$\begin{aligned}\bar\partial_z \int (HR\wedge g\wedge B\wedge\tilde{B})_{N,N-2}\wedge\phi = \\ \int (HR\wedge g\wedge B)_{N,N-1}\wedge\phi - \int (HR\wedge g\wedge\tilde{B})_{N,N-1}\wedge\phi, \quad z\in\Omega'\cap X_{reg}.\end{aligned} \tag{7.11}$$

Now let us assume that we have two local solutions, in say Ω and Ω', obtained from two different embeddings of slightly larger sets $\tilde\Omega$ and $\tilde\Omega'$ in subsets of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We want to compare these solutions on $\Omega\cap\Omega'$. Localizing further, as before, we may assume that the weights both have support in $\tilde\Omega\cap\tilde\Omega'$. After adding nonsense variables we may assume that both embeddings are into the same $\mathbb{C}^N$, and after further localization there is a local biholomorphism in $\mathbb{C}^N$ that maps one embedding onto the other one, see [12]. (Notice that a solution obtained via an embedding in $\mathbb{C}^{N_1}$ also can be obtained via an embedding into a larger $\mathbb{C}^N$, by just adding dummy variables in the first formula.) In other words, we may assume that we have the same embedding in some open set $\Omega\subset\mathbb{C}^N$ but two solutions obtained from different η and η'. (Arguing as before, however, we may assume that we have the same resolution and the same residue current R.) Locally there is an invertible matrix h_{jk} such that

$$\eta'_j = \sum h_{jk}\eta_k. \tag{7.12}$$

We define a vector bundle mapping $\alpha^*\colon \Lambda_{\eta'}\to\Lambda_\eta$ as the identity on $T^*_{0,*}(\Omega\times\Omega)$ and so that

$$\alpha^* d\eta'_j = \sum h_{jk}d\eta_k.$$

It is readily checked that

$$\nabla_\eta\alpha^* = \alpha^*\nabla_{\eta'}.$$

Therefore, α^*g' is an η-weight if g' is an η'-weight. Moreover, if H is an η'-Hefer morphism, then α^*H is an η-Hefer morphism, cf., (3.2). If B' is obtained from an η' admissible form s', then α^*s' is an η-admissible form and α^*B' is the corresponding locally integrable form. We claim that the η'-solution

$$v' = \int (H'R\wedge g'\wedge B')_{N,N-1}\wedge\phi \tag{7.13}$$

is comparable to the η-solution

$$v = \int \alpha^*(H'R)\wedge\alpha^*g'\wedge\alpha^*B'\wedge\phi. \tag{7.14}$$

Notice that we are only interested in the $d\zeta$-component of the kernels. We have that ($d\eta = d\eta_1\wedge\ldots\wedge d\eta_N$ etc)

$$(H'R\wedge g'\wedge B')_{N,N-1} = A\wedge d\eta' \sim A\wedge \det(\partial\eta'/\partial\zeta)d\zeta$$

and

$$\alpha^*(H'R\wedge g'\wedge B')_{N,N-1} = A\wedge \det h\wedge d\eta \sim A\wedge \det h \det(\partial\eta/\partial\zeta)d\zeta.$$

Thus

$$\alpha^*(H'R\wedge g'\wedge B')_{N,N-1} \sim \gamma(\zeta,z)(H'R\wedge g'\wedge B')_{N,N-1}$$

with

$$\gamma = \det h \det \frac{\partial\eta}{\partial\zeta}\Big(\det\frac{\partial\eta'}{\partial\zeta}\Big)^{-1}.$$

From (7.12) we have that $\partial\eta'_j/\partial\zeta_\ell = \sum_k h_{jk}\partial\eta_k/\partial\zeta_\ell + \mathcal{O}(|\eta|)$ which implies that γ is 1 on the diagonal. Thus γ is a smooth (holomorphic) weight and therefore (7.13) and (7.14) are comparable, and thus the claim is proved. This proves Theorem 1.8 in the case $q=1$, and elaborating the idea as in the previous proof we obtain the general case.

Remark 9. In case X is a Stein space and X_{sing} is discrete there is a much simpler proof of Theorem 1.8. To begin with we can solve $\bar\partial v = \phi$ locally, and modifying by such local solutions we may assume that ϕ is vanishing identically in a neighborhood of X_{sing}. There exists a sequence of holomorphically convex open subsets X_j such that X_j is relatively compact in X_{j+1} and X_j can be embedded as a subvariety of some pseudoconvex set Ω_j in $\mathbb{C}^{N_j}$. Let K_ℓ be the closure of X_ℓ. By Theorem 1.1 we can solve $\bar\partial u_\ell = \phi$ in a neighborhood of K_ℓ and u_ℓ will be smooth. If $q>1$ we can thus solve $\bar\partial w_\ell = u_{\ell+1} - u_\ell$ in a neighborhood of K_ℓ, and since X_{sing} is discrete we can assume that $\bar\partial w_\ell$ is smooth in X. Then $v_\ell = u_\ell - \sum_1^{\ell-1}\bar\partial w_k$ defines a global solution. If $q=1$, then one obtains a global solution in a similar way by a Mittag-Leffler type argument. □

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