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CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS III

ZHI-Hong Sun

School of the Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, PR China
Email: zhihongsun@yahoo.com
Homepage: http://www.hytc.edu.cn/xsjl/szh

Abstract. Let $p>3$ be a prime, and let m be an integer with $p\nmid m$. In the paper we solve some conjectures of Z.W. Sun concerning $\sum_{k=0}^{p-1}\frac{(6k)!}{m^k(3k)!k!^3} \pmod p$, and show that for integers m,n with $p\nmid m$,

$$\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\frac{(6k)!}{(3k)!k!^3}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k \pmod p,$$

where $(\frac{a}{p})$ is the Legendre symbol and $[x]$ is the greatest integer function. Let $\{P_n(x)\}$ be the Legendre polynomials. We also prove congruences for $P_{[\frac p6]}(t)$ and $P_{[\frac p3]}(t) \pmod p$ by using character sums and Morton's work.

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1. Introduction.

In 2003, Rodriguez-Villegas[RV] conjectured that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{(6k)!}{1728^k(3k)!k!^3}\equiv\delta(p)C_{q^p}\ q\prod_{n=1}^{\infty}(1-q^{4n})^6 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv C_{q^p}\ q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3 \pmod{p^2},$$

where $C_{q^n}f(q)$ denotes the coefficient of q^n in the power series expansion of $f(q)$ and $\delta(p)=-1$ or 1 according as $p\equiv 5 \pmod{12}$ or not. The two conjectures were partially solved by Mortenson[M]. For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly say that $n=ax^2+by^2$. In 1892, Klein and Fricke[KF] showed that

$$C_{q^p}\ q\prod_{n=1}^{\infty}(1-q^{4n})^6=\begin{cases}0 & \text{if } p\equiv 3 \pmod 4,\\ 4x^2-2p & \text{if } p=x^2+4y^2\equiv 1 \pmod 4.\end{cases}$$

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In 1985, Stienstra and Beukers[SB] proved that

$$C_{q^p}\ q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3=\begin{cases}0 & \text{if } p\equiv 2 \pmod 3,\\ 4x^2-2p & \text{if } p=x^2+3y^2\equiv 1 \pmod 3.\end{cases}$$

Let $\mathbb{Z}$ be the set of integers, and let $[x]$ be the greatest integer function. For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is coprime to p, and let $(\frac{a}{m})$ be the Jacobi symbol. Recently the author's brother Zhi-Wei Sun[Su] posed many conjectures involving $\sum_{k=0}^{p-1}\frac{(6k)!}{m^k(3k)!k!^3}$ and $\sum_{k=0}^{p-1}\frac{1}{m^k}\binom{2k}{k}^2\binom{3k}{k}$ modulo p^2, where $p>3$ is a prime and $m\in\mathbb{Z}$ with $p\nmid m$. For example, Zhi-Wei Sun conjectured ([Su, Conjectures A8 and A9]) that for any prime $p>3$,

(1.1)

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3,\\ L^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } 4p=L^2+27M^2,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{(6k)!}{(-96)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod{p^2} & \text{if } (\frac{p}{19})=-1,\\ (\frac{-6}{p})(x^2-2p) \pmod{p^2} & \text{if } (\frac{p}{19})=1 \text{ and so } 4p=x^2+19y^2.\end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_0(x)=1,\ P_1(x)=x,\ (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)\ (n\ge 1).$$

It is well known that (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{k}(-1)^k\binom{2n-2k}{n}x^{n-2k}=\frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n. \tag{1.2}$$

From (1.2) we see that

$$P_n(-x)=(-1)^nP_n(x),\quad P_{2m+1}(0)=0\quad\text{and}\quad P_{2m}(0)=\frac{(-1)^m}{2^{2m}}\binom{2m}{m}. \tag{1.3}$$

We also have the following formula due to Murphy ([G, (3.135)]):

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{x-1}{2}\Big)^k=\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}\Big(\frac{x-1}{2}\Big)^k. \tag{1.4}$$

We remark that $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$.

Let $p>3$ be a prime. Based on the work of Brillhart and Morton ([BM],[Mo1]), in the paper we study $P_{[\frac{p}{6}]}(t)$ and $P_{[\frac{p}{3}]}(t) \pmod p$. For $t\in\mathbb{Z}_p$ we prove that

$$P_{[\frac{p}{6}]}(t)\equiv-\Big(\frac{3}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x+2t}{p}\Big) \pmod p, \tag{1.5}$$

$$P_{[\frac{p}{3}]}(t)\equiv-\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p. \tag{1.6}$$

By using the Legendre polynomials, character sums and some results developed by the author in [S5], in the paper we determine $\sum_{k=0}^{[\frac p6]}\frac{(6k)!}{m^k(3k)!k!^3}\pmod p$ for $m=-640320^3$, $-5280^3,-960^3,-96^3,-32^3-15^3,20^3,66^3,255^3,54000$. We also determine $\sum_{k=0}^{[p/3]}\frac{(3k)!}{(-216)^k\cdot k!^3}$ and $\sum_{k=0}^{[p/3]}\frac{\binom{2k}k^2\binom{3k}k}{(-192)^k}\pmod p$. Thus we solve some conjectures in [Su] and [S4]. For example, we confirm (1.1) when the modulus is p. We also show that for any prime $p>3$ and $m,n\in\mathbb Z_p$ with $m\not\equiv 0\pmod p$,

$$(1.7)\qquad \Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\frac{(6k)!}{(3k)!k!^3}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k\pmod p.$$

2. Congruences for $P_{[\frac p6]}(x)\pmod p$.

Lemma 2.1. *Let p be a prime greater than 3, and let x be a variable. Then*

$$P_{[\frac p6]}(x)\equiv\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{1-x}{864}\Big)^k\pmod p.$$

Proof. Suppose that $r\in\{1,5\}$ is given by $p\equiv r\pmod 6$. Then clearly

$$\begin{aligned}\binom{[\frac p6]+k}{2k}&=\frac{(\frac{p-r}6+k)(\frac{p-r}6+k-1)\cdots(\frac{p-r}6-k+1)}{(2k)!}\\&=\frac{(p+6k-r)(p+6k-r-6)\cdots(p-(6k+r-6))}{6^{2k}\cdot(2k)!}\\&\equiv(-1)^k\frac{(6k-r)(6k-r-6)\cdots(6-r)\cdot r(r+6)\cdots(6k+r-6)}{6^{2k}\cdot(2k)!}\\&=\frac{(-1)^k\cdot(6k)!}{(2\cdot 4\cdots 6k)(3\cdot 9\cdot 15\cdots(6k-3))\cdot 6^{2k}\cdot(2k)!}\\&=\frac{(-1)^k\cdot(6k)!}{2^{3k}(3k)!\cdot 3^k\frac{(2k)!}{2\cdot 4\cdot 6\cdots 2k}\cdot 36^k(2k)!}\equiv\frac{(6k)!k!}{(-432)^k(3k)!(2k)!^2}\pmod p.\end{aligned}$$

Hence

$$(2.1)\qquad \binom{[\frac p6]}{k}\binom{[\frac p6]+k}{k}=\binom{[\frac p6]+k}{2k}\binom{2k}{k}\equiv\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}\pmod p.$$

This together with (1.4) yields the result.

Lemma 2.2. *Let p be a prime of the form $12k+5$ and $p=a^2+b^2(a,b\in\mathbb Z)$ with $a\equiv 1\pmod 4$ and $b\equiv a\pmod 3$. Then*

$$\binom{\frac{p-1}2}{\frac{p-5}{12}}\equiv 2b\pmod p.$$

Proof. From [Mo1, p.246] we have $\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 12(-432)^{\frac{p-5}{12}}\binom{\frac{p-5}{6}}{\frac{p-5}{12}} \pmod p$. Since $\binom{(p-1)/2}{k} \equiv \binom{-\frac12}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$ (see [S4, Lemma 2.4]), by the above and Gauss' congruence $\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p$ we have

$$\binom{\frac{p-1}{2}}{\frac{p-5}{12}} \equiv \frac{1}{(-4)^{\frac{p-5}{12}}}\binom{\frac{p-5}{6}}{\frac{p-5}{12}} \equiv \frac{2a}{(-4)^{\frac{p-5}{12}}\cdot 12(-432)^{\frac{p-5}{12}}} = \frac{2a}{2^{\frac{p-1}{2}}3^{\frac{p-1}{4}}} \pmod p.$$

By [S1, Theorem 2.2] we have $2^{\frac{p-1}{2}}3^{\frac{p-1}{4}} \equiv (-3)^{\frac{p-1}{4}} \equiv -\frac{b}{a} \equiv \frac{a}{b} \pmod p$. Thus $\binom{\frac{p-1}{2}}{\frac{p-5}{12}} \equiv \frac{2a}{a/b} = 2b \pmod p$. This proves the lemma.

We remark that R.J. Evans informed the author Lemma 2.2 can also be proved by using octic Eisenstein sums.

Theorem 2.1. *Let p be a prime greater than 3. Then*

$$\sum_{k=0}^{[p/6]}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv \begin{cases} 0 \pmod p & \text{if } p\equiv 7,11 \pmod{12},\\ 2a \pmod p & \text{if } 12\mid p-1,\ p=a^2+b^2,\ 4\mid a-1 \text{ and } 3\nmid a,\\ -2a \pmod p & \text{if } 12\mid p-1,\ p=a^2+b^2,\ 4\mid a-1 \text{ and } 3\mid a,\\ 2b \pmod p & \text{if } 12\mid p-5,\ p=a^2+b^2,\ 4\mid a-1 \text{ and } 3\mid b-a.\end{cases}$$

Proof. From Lemma 2.1 and (1.3) we have

$$\sum_{k=0}^{[p/6]}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv P_{[\frac p6]}(0) = \begin{cases} 0 \pmod p & \text{if } p\equiv 7,11 \pmod{12},\\ \frac{(-1)^{[\frac{p}{12}]}}{2^{[\frac p6]}}\binom{[\frac p6]}{[\frac{p}{12}]} \pmod p & \text{if } p\equiv 1,5 \pmod{12}.\end{cases}$$

By [S4, Lemma 2.4], $\binom{(p-1)/2}{k} \equiv \binom{-\frac12}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$. Thus, for $p\equiv 1,5 \pmod{12}$,

$$\sum_{k=0}^{[p/6]}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]} \pmod p.$$

If $p\equiv 1 \pmod{12}$ and $p=a^2+b^2$ with $a\equiv 1 \pmod 4$, by [HW, Corollary 4.2.2] and Gauss' congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2a \pmod p$ we have

$$\binom{\frac{p-1}{2}}{\frac{p-1}{12}} \equiv \begin{cases} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p & \text{if } 3\nmid a,\\ -\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv -2a \pmod p & \text{if } 3\mid a.\end{cases}$$

Now combining all the above with Lemma 2.2 we deduce the result.

Remark 2.1 Let $p>3$ be a prime. In [Su] Zhi-Wei Sun conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p\equiv 7,11 \pmod{12},\\ (-1)^{[\frac a6]}(2a-\frac{p}{2a}) \pmod{p^2} & \text{if } 12\mid p-1,\ p=a^2+b^2 \text{ and } 4\mid a-1,\\ (\frac{ab}{3})(2b-\frac{p}{2b}) \pmod{p^2} & \text{if } 12\mid p-5,\ p=a^2+b^2 \text{ and } 4\mid a-1.\end{cases}$$

Let $W_n(x)$ be the Deuring polynomial given by

$$W_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k. \tag{2.2}$$

It is known that ([G,(3.134)],[BM],[Mo2])

$$W_n(x)=(1-x)^nP_n\Big(\frac{1+x}{1-x}\Big). \tag{2.3}$$

Theorem 2.2. *Let $p>3$ be a prime and $m,n\in\mathbb{Z}_p$ with $m\not\equiv 0\pmod p$. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\equiv\begin{cases}-(-3m)^{\frac{p-1}{4}}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big)\pmod p & \text{if } p\equiv 1\pmod 4,\\ -\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big)\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. We first assume $4m^3+27n^2\equiv 0\pmod p$. It is easily seen that $x^3+mx+n\equiv(x-\frac{3n}{m})(x+\frac{3n}{2m})^2\pmod p$. Thus

$$\begin{aligned}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)&=\sum_{x=0}^{p-1}\Big(\frac{(x-\frac{3n}{m})(x+\frac{3n}{2m})^2}{p}\Big)=\sum_{\substack{x=0\\ x\not\equiv-\frac{3n}{2m}\pmod p}}^{p-1}\Big(\frac{x-\frac{3n}{m}}{p}\Big)\\&=\sum_{t=0}^{p-1}\Big(\frac tp\Big)-\Big(\frac{-\frac{3n}{2m}-\frac{3n}{m}}{p}\Big)=-\Big(\frac{-2mn}{p}\Big).\end{aligned}$$

Since $4m^3+27n^2\equiv 0\pmod p$ we see that $2m^2\equiv\pm 3n\sqrt{-3m}\pmod p$. Thus, if $p\equiv 1\pmod 4$, then $[\frac p6]$ is even and so

$$\begin{aligned}&(-3m)^{\frac{p-1}{4}}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\&\equiv(-3m^3)^{\frac{p-1}{4}}\Big(\frac mp\Big)P_{[\frac p6]}(\pm1)\equiv\Big(\frac{9n}{2}\Big)^{\frac{p-1}{2}}\Big(\frac mp\Big)\equiv\Big(\frac{-2mn}{p}\Big)\pmod p;\end{aligned}$$

if $p\equiv 3\pmod 4$, then $[\frac p6]$ is odd and so

$$\begin{aligned}&\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\&\equiv(-3m)^{\frac{p-3}{4}}\frac{2m^2}{3n}\cdot\frac{3n\sqrt{-3m}}{2m^2}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\equiv(-3m)^{\frac{p-3}{4}}\frac{2m^2}{3n}(\pm1)P_{[\frac p6]}(\pm1)\\&=-\frac{2(-3m^3)^{\frac{p+1}{4}}}{9nm^{\frac{p-1}{2}}}\equiv-\frac{2}{9n}\Big(\frac mp\Big)\Big(\frac{9n}{2}\Big)^{\frac{p+1}{2}}\equiv-\Big(\frac{2mn}{p}\Big)=\Big(\frac{-2mn}{p}\Big)\pmod p,\end{aligned}$$

Therefore the result is true in the case $4m^3+27n^2\equiv 0\pmod p$.

From now on we assume $4m^3+27n^2\not\equiv 0 \pmod p$. Set

$$u=\frac{-4m^3+27n^2+12mn\sqrt{-3m}}{4m^3+27n^2}.$$

It is easy to check that

$$\frac{1+u}{1-u}=\frac{3n\sqrt{-3m}}{2m^2}\quad\text{and}\quad u+\frac{1}{u}=\frac{-2(4m^3-27n^2)}{4m^3+27n^2}.$$

From [Mo1, Theorem 3.3] we have

$$\begin{aligned}(2.4)\quad &\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\ &\equiv -(-48m)^{\frac{1-(\frac{p}{3})}{2}}(864n)^{\frac{1-(\frac{-1}{p})}{2}}(-16(4m^3+27n^2))^{[\frac{p}{12}]}J_p\Big(\frac{2^8\cdot 3^3m^3}{4m^3+27n^2}\Big) \pmod p,\end{aligned}$$

where $J_p(t)$ is a certain Jacobi polynomial given by

$$J_p(t)=1728^{[\frac{p}{12}]}P_{[\frac{p}{12}]}^{(-\frac13(\frac p3),-\frac12(\frac{-1}{p}))}\Big(1-\frac{t}{864}\Big)$$

and

$$P_k^{(\alpha,\beta)}(x)=\frac{1}{2^k}\sum_{r=0}^{k}\binom{k+\alpha}{r}\binom{k+\beta}{k-r}(x-1)^{k-r}(x+1)^r.$$

Let $A_k(x)$ and $B_k(x)$ be monic integral polynomials of degree k such that

$$W_{2k}(x)=x^kA_k\Big(x+\frac1x\Big)\quad\text{and}\quad W_{2k+1}(x)=(x+1)x^kB_k\Big(x+\frac1x\Big).$$

Then Morton ([Mo1, Theorem 1.1]) proved that

$$J_p(t)\equiv\begin{cases}(-432)^{[\frac{p}{12}]}A_{[\frac{p}{12}]}(2-\frac{t}{432}) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (-432)^{[\frac{p}{12}]}B_{[\frac{p}{12}]}(2-\frac{t}{432}) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Hence

$$\begin{aligned}&J_p\Big(\frac{2^8\cdot 3^3m^3}{4m^3+27n^2}\Big)\\ &\equiv\begin{cases}(-432)^{[\frac{p}{12}]}A_{[\frac{p}{12}]}(\frac{-2(4m^3-27n^2)}{4m^3+27n^2})=(-432)^{[\frac{p}{12}]}A_{[\frac{p}{12}]}(u+\frac1u) \pmod p & \text{if } 4\mid p-1,\\ (-432)^{[\frac{p}{12}]}B_{[\frac{p}{12}]}(\frac{-2(4m^3-27n^2)}{4m^3+27n^2})=(-432)^{[\frac{p}{12}]}B_{[\frac{p}{12}]}(u+\frac1u) \pmod p & \text{if } 4\mid p-3.\end{cases}\end{aligned}$$

For $p\equiv 1\pmod 4$, from (2.3) and the above we deduce

$$\begin{aligned}A_{[\frac{p}{12}]}\Big(u+\frac1u\Big)&=u^{-[\frac{p}{12}]}W_{2[\frac{p}{12}]}(u)=\Big(u+\frac1u-2\Big)^{[\frac{p}{12}]}P_{[\frac p6]}\Big(\frac{1+u}{1-u}\Big)\\ &=\Big(\frac{-2(4m^3-27n^2)}{4m^3+27n^2}-2\Big)^{[\frac{p}{12}]}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\ &=\Big(\frac{-16m^3}{4m^3+27n^2}\Big)^{[\frac{p}{12}]}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&\equiv -(-48m)^{\frac{1-(\frac p3)}{2}}(-16(4m^3+27n^2))^{[\frac p{12}]}(-432)^{[\frac p{12}]}A_{[\frac p{12}]}\Big(u+\frac 1u\Big)\\&= -(-48m)^{\frac{1-(\frac p3)}{2}}(432\cdot 16(4m^3+27n^2))^{[\frac p{12}]}\Big(\frac{-16m^3}{4m^3+27n^2}\Big)^{[\frac p{12}]}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\&\equiv -(-3m)^{\frac{p-1}{4}}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big) \pmod p.\end{aligned}$$

Similarly, for $p\equiv 3\pmod 4$ we have

$$B_{[\frac p{12}]}\Big(u+\frac 1u\Big)=\frac{2m^2}{3n\sqrt{-3m}}\Big(\frac{-16m^3}{4m^3+27n^2}\Big)^{[\frac p{12}]}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big).$$

Hence, from the above we deduce

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&\equiv -(-48m)^{\frac{1-(\frac p3)}{2}}864n(-16(4m^3+27n^2))^{[\frac p{12}]}(-432)^{[\frac p{12}]}B_{[\frac p{12}]}\Big(u+\frac 1u\Big)\\&= -(-48m)^{\frac{1-(\frac p3)}{2}}864n(432\cdot 16(4m^3+27n^2))^{[\frac p{12}]}\\&\quad\times\frac{2m^2}{3n\sqrt{-3m}}\Big(\frac{-16m^3}{4m^3+27n^2}\Big)^{[\frac p{12}]}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\&\equiv -\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big) \pmod p.\end{aligned}$$

This completes the proof.

Corollary 2.1. *Let $p>5$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{11\sqrt 5}{25}\Big)\equiv\begin{cases}5^{\frac{p-1}{4}}(\frac 5p)2A \pmod p & \text{if } 12\mid p-1,\ p=A^2+3B^2 \text{ and } 3\mid A-1,\\ -5^{\frac{p-3}{4}}(\frac 5p)2A\sqrt 5 \pmod p & \text{if } 12\mid p-7,\ p=A^2+3B^2 \text{ and } 3\mid A-1,\\ 0 \pmod p & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. By [S2, Lemma 2.3] or [S5, Corollary 2.1 and (2.4)] we have

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-15x+22}{p}\Big)\\&=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+8}{p}\Big)=\begin{cases}-2A & \text{if } 3\mid p-1,\ p=A^2+3B^2 \text{ and } 3\mid A-1,\\ 0 & \text{if } p\equiv 2 \pmod 3.\end{cases}\end{aligned}\tag{2.5}$$

Thus, taking $m=-15$ and $n=22$ in Theorem 2.2 we obtain the result.

Corollary 2.2. *Let $p\ne 2,3,11$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{21\sqrt{33}}{121}\Big)\equiv\begin{cases}(\frac{33}{p})(-33)^{\frac{p-1}4}2a\pmod p&\text{if } 4\mid p-1,\ p=a^2+b^2 \text{ and } 4\mid a-1,\\ 0\pmod p&\text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. By [S5, Corollary 2.1 and (2.3)] we have

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-11x+14}{p}\Big)\\ &=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-4x}{p}\Big)=\begin{cases}(-1)^{\frac{p+3}4}2a&\text{if } 4\mid p-1,\ p=a^2+b^2\text{ and } 4\mid a-1,\\ 0&\text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}\tag{2.6}$$

Thus, taking $m=-11$ and $n=14$ in Theorem 2.2 we obtain the result.

Remark 2.2 Let p be a prime of the form $4k+1$, and $p=a^2+b^2$ with $2\mid b$ and $a+b\equiv 1\pmod 4$. By [S1, Theorem 2.2] we have

$$(-3)^{\frac{p-1}4}\equiv\begin{cases}1\pmod p&\text{if } 3\mid b,\\ -1\pmod p&\text{if } 3\mid a,\\ \pm\frac ba\pmod p&\text{if } 3\mid a\pm b,\end{cases}$$

and

$$(-11)^{\frac{p-1}4}\equiv\begin{cases}1\pmod p&\text{if } 11\mid b\text{ or } a\equiv\pm 2b\pmod{11},\\ -1\pmod p&\text{if } 11\mid a\text{ or } a\equiv\pm 5b\pmod{11},\\ \pm\frac ba\pmod p&\text{if } a\equiv\pm b,\pm 3b,\pm 4b\pmod{11}.\end{cases}$$

Thus we may obtain the congruence for $33^{\frac{p-1}4}\pmod p$.

Corollary 2.3. *Let $p>5$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{7\sqrt{10}}{25}\Big)\equiv\begin{cases}(-1)^{\frac d2}(\frac 5p)5^{\frac{p-1}4}2c\pmod p&\text{if } 8\mid p-1,\ p=c^2+2d^2\text{ and } 4\mid c-1,\\ (\frac 5p)5^{\frac{p-3}4}2d\sqrt{10}\pmod p&\text{if } 8\mid p-3,\ p=c^2+2d^2\text{ and } 4\mid d-1,\\ 0\pmod p&\text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Proof. From [BE, Theorems 5.12 and 5.17] we know that

$$\sum_{k=0}^{p-1}\Big(\frac{x^3-4x^2+2x}{p}\Big)=\begin{cases}(-1)^{[\frac p8]+1}2c&\text{if } p=c^2+2d^2\equiv 1,3\pmod 8\text{ with } 4\mid c-1,\\ 0&\text{otherwise.}\end{cases}$$

As $27(x^3-4x^2+2x)=(3x-4)^3-30(3x-4)-56$, we see that

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-4x^2+2x}{p}\Big)\\ &=\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{(3x-4)^3-30(3x-4)-56}{p}\Big)=\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-30x-56}{p}\Big)\\ &=\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{(-x)^3-30(-x)-56}{p}\Big)=\Big(\frac{-3}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-30x+56}{p}\Big).\end{aligned}$$

Thus, from the above we deduce

(2.7)
$$\sum_{x=0}^{p-1}\Big(\frac{x^3-30x+56}{p}\Big)=\begin{cases}(-1)^{\frac{p+7}{8}}(\frac{3}{p})2c & \text{if } p\equiv 1 \pmod 8,\ p=c^2+2d^2 \text{ and } 4\mid c-1,\\ (-1)^{\frac{p-3}{8}}(\frac{3}{p})2c & \text{if } p\equiv 3 \pmod 8,\ p=c^2+2d^2 \text{ and } 4\mid c-1,\\ 0 & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

By [S3, p.1317] we have

$$2^{[\frac{p}{4}]}\equiv\begin{cases}(-1)^{\frac{c^2-1}{8}}=(-1)^{\frac{p-1}{8}+\frac{d}{2}} \pmod p & \text{if } p=c^2+2d^2\equiv 1 \pmod 8,\\ (-1)^{\frac{c^2-1}{8}}\frac{d}{c}=(-1)^{\frac{p-3}{8}}\frac{d}{c} \pmod p & \text{if } p=c^2+2d^2\equiv 3 \pmod 8 \text{ with } 4\mid c-d.\end{cases}$$

Now taking $m=-30$ and $n=56$ in Theorem 2.2 and applying the above we deduce the result.

Corollary 2.4. *Let $p>7$ be a prime. Then*

$$P_{[\frac{p}{6}]}\Big(\frac{3\sqrt{105}}{25}\Big)\equiv\begin{cases}2(\frac{p}{15})15^{\frac{p-1}{4}}C \pmod p\\ \qquad \text{if } p\equiv 1,9,25 \pmod{28},\ p=C^2+7D^2 \text{ and } 4\mid C-1,\\ 2(\frac{p}{15})15^{\frac{p-3}{4}}D\sqrt{105} \pmod p\\ \qquad \text{if } p\equiv 11,15,23 \pmod{28},\ p=C^2+7D^2 \text{ and } 4\mid D-1,\\ 0 \pmod p \quad \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Since $(-x-7)^3-35(-x-7)+98=-(x^3+21x^2+112x)$, applying [R1,R2] we see that

(2.8)
$$\begin{aligned}\sum_{x=0}^{p-1}\Big(\frac{x^3-35x+98}{p}\Big)&=(-1)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3+21x^2+112x}{p}\Big)\\ &=\begin{cases}(-1)^{\frac{p+1}{2}}2(\frac{C}{7})C & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and } p=C^2+7D^2,\\ 0 & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}\end{aligned}$$

Suppose $p\equiv 1,2,4 \pmod 7$ and so $p=C^2+7D^2$. By [S3, p.1317] we have

(2.9)
$$7^{[\frac{p}{4}]}\equiv\begin{cases}(\frac{C}{7}) \pmod p & \text{if } p\equiv 1,9,25 \pmod{28} \text{ and } C\equiv 1 \pmod 4,\\ -(\frac{C}{7})\frac{D}{C} \pmod p & \text{if } p\equiv 11,15,23 \pmod{28} \text{ and } D\equiv 1 \pmod 4.\end{cases}$$

Now taking $m=-35$ and $n=98$ in Theorem 2.2 and applying all the above we deduce the result.

Corollary 2.5. *Let $p\ne 2,3,11$ be a prime.*

(i) *If $p\equiv 2,6,7,8,10 \pmod{11}$, then $P_{[\frac{p}{6}]}(\frac{7}{32}\sqrt{22})\equiv 0 \pmod p$.*

(ii) *If* $p\equiv 1,3,4,5,9 \pmod{11}$ *and hence* $4p=u^2+11v^2$ *for some* $u,v\in\mathbb{Z}$, *then*

$$P_{[\frac p6]}\Big(\frac{7}{32}\sqrt{22}\Big)\equiv\begin{cases}-(\frac p3)(-2)^{\frac{p-1}4}u \pmod p&\text{if } 4\mid p-1 \text{ and } 4\mid u-1,\\ (\frac p3)2^{\frac{p-1}4}u \pmod p&\text{if } 4\mid p-1 \text{ and } 8\mid u-2,\\ -(\frac p3)(-2)^{\frac{p-3}4}v\sqrt{22} \pmod p&\text{if } 4\mid p-3 \text{ and } 4\mid v-1,\\ (\frac p3)2^{\frac{p-3}4}v\sqrt{22} \pmod p&\text{if } 4\mid p-3 \text{ and } 8\mid v-2.\end{cases}$$

Proof. It is known that (see [PR] and [JM])

$$(2.10)\quad \sum_{x=0}^{p-1}\Big(\frac{x^3-96\cdot 11x+112\cdot 11^2}{p}\Big)=\begin{cases}(\frac 2p)(\frac u{11})u&\text{if } (\frac p{11})=1 \text{ and } 4p=u^2+11v^2,\\ 0&\text{if } (\frac p{11})=-1.\end{cases}$$

Thus applying Theorem 2.2 we deduce

$$P_{[\frac p6]}(\frac{7}{32}\sqrt{22})\equiv\begin{cases}-(\frac p3)22^{\frac{p-1}4}(\frac u{11})u \pmod p&\text{if } (\frac p{11})=1,\ 4\mid p-1 \text{ and } 4p=u^2+11v^2,\\ -(\frac p3)22^{\frac{p-3}4}(\frac u{11})u \pmod p&\text{if } (\frac p{11})=1,\ 4\mid p-3 \text{ and } 4p=u^2+11v^2,\\ 0 \pmod p&\text{if } (\frac p{11})=-1.\end{cases}$$

Now assume $(\frac p{11})=1$ and so $4p=u^2+11v^2$. If $u\equiv v\equiv 1 \pmod 4$, by [S3, Theorem 4.3] we have

$$(-11)^{[\frac p4]}\equiv\begin{cases}(\frac u{11}) \pmod p&\text{if } p\equiv 1 \pmod 4,\\ (\frac u{11})\frac vu \pmod p&\text{if } p\equiv 3 \pmod 4.\end{cases}$$

If $u\equiv v\equiv 0 \pmod 2$, by [S3, Corollary 4.6] we have

$$11^{[\frac p4]}\equiv\begin{cases}-(\frac u{11}) \pmod p&\text{if } p\equiv 1 \pmod 4 \text{ and } 8\mid u-2,\\ -(\frac u{11})\frac vu \pmod p&\text{if } p\equiv 3 \pmod 4 \text{ and } 8\mid v-2.\end{cases}$$

Now combining all the above we derive the result.

From [RPR], [JM] and [PV] we know that for any prime $p>3$,
(2.11)

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-8\cdot 19x+2\cdot 19^2}{p}\Big)=\begin{cases}(\frac 2p)(\frac u{19})u&\text{if } (\frac p{19})=1 \text{ and } 4p=u^2+19v^2,\\ 0&\text{if } (\frac p{19})=-1,\end{cases}$$

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-80\cdot 43x+42\cdot 43^2}{p}\Big)=\begin{cases}(\frac 2p)(\frac u{43})u&\text{if } (\frac p{43})=1 \text{ and } 4p=u^2+43v^2,\\ 0&\text{if } (\frac p{43})=-1,\end{cases}$$

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-440\cdot 67x+434\cdot 67^2}{p}\Big)=\begin{cases}(\frac 2p)(\frac u{67})u&\text{if } (\frac p{67})=1 \text{ and } 4p=u^2+67v^2,\\ 0&\text{if } (\frac p{67})=-1,\end{cases}$$

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-80\cdot 23\cdot 29\cdot 163x+14\cdot 11\cdot 19\cdot 127\cdot 163^2}{p}\Big)$$
$$=\begin{cases}(\frac 2p)(\frac u{163})u&\text{if } (\frac p{163})=1 \text{ and } 4p=u^2+163v^2,\\ 0&\text{if } (\frac p{163})=-1.\end{cases}$$

Thus, using the method in the proof of Corollary 2.5 one can similarly prove the following results.

Corollary 2.6. *Let $p \neq 2,3,19$ be a prime.*

(i) *If* $(\frac{p}{19}) = -1$, *then* $P_{[\frac{p}{6}]}(\frac{3}{32}\sqrt{114}) \equiv 0 \pmod p$.

(ii) *If* $(\frac{p}{19}) = 1$ *and hence* $4p = u^2 + 19v^2$ *for some* $u, v \in \mathbb{Z}$, *then*

$$P_{[\frac{p}{6}]}\Big(\frac{3}{32}\sqrt{114}\Big) \equiv \begin{cases} -(\frac{p}{3})6^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 4 \mid u-1, \\ 6^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 8 \mid u-2, \\ (\frac{p}{3})6^{\frac{p-3}{4}}v\sqrt{114} \pmod p & \text{if } 4 \mid p-3 \text{ and } 4 \mid v-1, \\ (\frac{-6}{p})6^{\frac{p-3}{4}}v\sqrt{114} \pmod p & \text{if } 4 \mid p-3 \text{ and } 8 \mid v-2. \end{cases}$$

Corollary 2.7. *Let $p \neq 2,3,5,43$ be a prime.*

(i) *If* $(\frac{p}{43}) = -1$, *then* $P_{[\frac{p}{6}]}(\frac{63\sqrt{645}}{1600}) \equiv 0 \pmod p$.

(ii) *If* $(\frac{p}{43}) = 1$ *and hence* $4p = u^2 + 43v^2$ *for some* $u, v \in \mathbb{Z}$, *then*

$$P_{[\frac{p}{6}]}\Big(\frac{63\sqrt{645}}{1600}\Big) \equiv \begin{cases} -(\frac{p}{15})15^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 4 \mid u-1, \\ (\frac{p}{15})(-15)^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 8 \mid u-2, \\ (\frac{p}{15})15^{\frac{p-3}{4}}v\sqrt{645} \pmod p & \text{if } 4 \mid p-3 \text{ and } v \equiv 1,2,5 \pmod 8. \end{cases}$$

Corollary 2.8. *Let p be a prime such that $p \neq 2,3,5,11,67$.*

(i) *If* $(\frac{p}{67}) = -1$, *then* $P_{[\frac{p}{6}]}(\frac{651}{96800}\sqrt{22110}) \equiv 0 \pmod p$.

(ii) *If* $(\frac{p}{67}) = 1$ *and hence* $4p = u^2 + 67v^2$ *for some* $u, v \in \mathbb{Z}$, *then*

$$P_{[\frac{p}{6}]}\Big(\frac{651}{96800}\sqrt{22110}\Big) \equiv \begin{cases} -(\frac{165}{p})330^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 4 \mid u-1, \\ (\frac{330}{p})330^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 8 \mid u-2, \\ -(\frac{165}{p})330^{\frac{p-3}{4}}v\sqrt{22110} \pmod p & \text{if } 4 \mid p-3 \text{ and } 4 \mid v-1, \\ -(\frac{330}{p})330^{\frac{p-3}{4}}v\sqrt{22110} \pmod p & \text{if } 4 \mid p-3 \text{ and } 8 \mid v-2. \end{cases}$$

Corollary 2.9. *Let p be a prime such that $p \neq 2,3,5,23,29,163$.*

(i) *If* $(\frac{p}{163}) = -1$, *then* $P_{[\frac{p}{6}]}(\frac{557403}{26680^2}\sqrt{1630815}) \equiv 0 \pmod p$.

(ii) *If* $(\frac{p}{163}) = 1$ *and hence* $4p = u^2 + 163v^2$ *for some* $u, v \in \mathbb{Z}$, *then*

$$P_{[\frac{p}{6}]}\Big(\frac{557403}{26680^2}\sqrt{1630815}\Big)$$
$$\equiv \begin{cases} -(\frac{10005}{p})(-10005)^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 4 \mid u-1, \\ (\frac{10005}{p})(-10005)^{\frac{p-1}{4}}u \pmod p & \text{if } 4 \mid p-1 \text{ and } 8 \mid u-2, \\ -(\frac{10005}{p})10005^{\frac{p-3}{4}}v\sqrt{1630815} \pmod p & \text{if } 4 \mid p-3 \text{ and } v \equiv 1,2,5 \pmod 8. \end{cases}$$

Corollary 2.10. *Let p be a prime such that $p\ne 2,3,5,7,17$.*

(i) *If $p\equiv 3,5,6 \pmod 7$, then $P_{[\frac p6]}(\frac{171\sqrt{1785}}{85^2})\equiv 0 \pmod p$.*

(ii) *If $p\equiv 1,2,4 \pmod 7$ and so $p=C^2+7D^2$ for some $C,D\in\mathbb{Z}$, then*

$$P_{[\frac p6]}\Big(\frac{171\sqrt{1785}}{85^2}\Big)\equiv\begin{cases}(\frac{255}{p})255^{\frac{p-1}{4}}\cdot 2C \pmod p & \text{if } 4\mid p-1 \text{ and } 4\mid C-1,\\ (\frac{255}{p})255^{\frac{p-3}{4}}\cdot 2D\sqrt{1785} \pmod p & \text{if } 4\mid p-3 \text{ and } 4\mid D-1.\end{cases}$$

Proof. From [W, p.296] we know that

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{(x^2+6x+2)(3x^2+16x)}{p}\Big)\\&=\begin{cases}-2(\frac{-2}{p})(\frac C7)C-(\frac 3p) & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and } p=C^2+7D^2,\\ -(\frac 3p) & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}\end{aligned}$$

As $(x^2+6x+2)(3x^2+16x)=x^4(3+34/x+102/x^2+32/x^3)$, we see that

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{(x^2+6x+2)(3x^2+16x)}{p}\Big)\\&=\sum_{x=1}^{p-1}\Big(\frac{3+34/x+102/x^2+32/x^3}{p}\Big)=\sum_{x=1}^{p-1}\Big(\frac{3+34x+102x^2+32x^3}{p}\Big)\\&=\Big(\frac 2p\Big)\sum_{x=1}^{p-1}\Big(\frac{6+68x+204x^2+64x^3}{p}\Big)=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+\frac{51}{4}x^2+17x+6}{p}\Big)-\Big(\frac{12}{p}\Big)\end{aligned}$$

and

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+\frac{51}{4}x^2+17x+6}{p}\Big)\\&=\sum_{x=0}^{p-1}\Big(\frac{(x-\frac{17}{4})^3+\frac{51}{4}(x-\frac{17}{4})^2+17(x-\frac{17}{4})+6}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{595}{16}x+\frac{5586}{64}}{p}\Big)\\&=\sum_{x=0}^{p-1}\Big(\frac{(\frac x4)^3-\frac{595}{16}\cdot\frac x4+\frac{5586}{64}}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x^3-595x+5586}{p}\Big).\end{aligned}$$

Now combining all the above we deduce

(2.12)
$$\sum_{x=0}^{p-1}\Big(\frac{x^3-595x+5586}{p}\Big)=\begin{cases}(-1)^{\frac{p+1}{2}}2C(\frac C7) & \text{if } p=C^2+7D^2\equiv 1,2,4 \pmod 7,\\ 0 & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Now taking $m=-595$ and $n=5586$ in Theorem 2.2 and then applying (2.12) and (2.9) we deduce the result.

Conjecture 2.1. *Let $p>3$ be a prime. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-120x+506}{p}\Big)=\begin{cases}(\frac 2p)L & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ 0 & \text{if } p\equiv 2\pmod 3).\end{cases}$$

If Conjecture 2.1 is true, using Theorem 2.2 we deduce that for any prime $p>5$,

$$(2.13)\qquad P_{[\frac p6]}\Big(\frac{253\sqrt{10}}{800}\Big)\equiv\begin{cases}-(\frac{10}p)10^{\frac{p-1}4}L \pmod p\\ \qquad\qquad \text{if } 12\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ (\frac{10}p)10^{\frac{p-3}4}L\sqrt{10}\pmod p\\ \qquad\qquad \text{if } 12\mid p-7,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1,\\ 0\pmod p\quad \text{if } p\equiv 2\pmod 3).\end{cases}$$

Theorem 2.3. *Let $p>3$ be a prime and $m,n\in\mathbb Z_p$ with $m\not\equiv 0\pmod p$. Then*

$$P_{[\frac p6]}\Big(\frac{n}{2m^3}\Big)\equiv\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{2m^3-n}{12^3m^3}\Big)^k\equiv-\Big(\frac{3m}p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3m^2x+n}{p}\Big)\pmod p.$$

Proof. Replacing m by $-3m^2$ in Theorem 2.2 and then applying Lemma 2.1 we deduce the result.

Theorem 2.4. *Let $p>3$ be a prime. Then*

$$(2.14)\qquad P_{[\frac p6]}(t)\equiv-\Big(\frac 3p\Big)\sum_{x=0}^{p-1}(x^3-3x+2t)^{\frac{p-1}2}\pmod p.$$

Proof. Taking $m=1$ and $n=2t$ in Theorem 2.3 we see that (2.14) is true for $t=0,1,\ldots,p-1$. Since both sides of (2.14) are polynomials of t with degree less than $(p-1)/2$, applying Lagrange's theorem we see that (2.14) holds when t is a variable.

Theorem 2.5. *Let $p>3$ be a prime and $t\in\mathbb Z_p$.*

(i) *If $t^2+3\not\equiv 0\pmod p$, then*

$$P_{\frac{p-1}2}(t)\equiv\begin{cases}(-t^2-3)^{\frac{p-1}4}P_{[\frac p6]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2})\pmod p & \text{if } p\equiv 1\pmod 4),\\ -\frac{(-t^2-3)^{\frac{p+1}4}}{\sqrt{t^2+3}}P_{[\frac p6]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2})\pmod p & \text{if } p\equiv 3\pmod 4).\end{cases}$$

(ii) *If $3t+5\not\equiv 0\pmod p$, then*

$$P_{[\frac p4]}(t)\equiv\begin{cases}(6t+10)^{\frac{p-1}4}P_{[\frac p6]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2})\pmod p & \text{if } p\equiv 1\pmod 4),\\ \frac{(6t+10)^{\frac{p+1}4}}{\sqrt{6t+10}}P_{[\frac p6]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2})\pmod p & \text{if } p\equiv 3\pmod 4).\end{cases}$$

Proof. Part (i) follows from [S4, Theorem 2.11] and Theorem 2.2, and part (ii) follows from [S5, Theorem 2.3(ii)] and Theorem 2.2.

Corollary 2.11. *Let $p>3$ be a prime and $m\in\mathbb{Z}_p$ with $m\not\equiv 0 \pmod p$. Then*

$$P_{[\frac p4]}\Big(\frac{2m^2-5}{3}\Big)\equiv\Big(\frac{2m}{p}\Big)P_{[\frac p6]}\Big(\frac{3m^2-4}{m^3}\Big) \pmod p.$$

Proof. Taking $t=(2m^2-5)/3$ in Theorem 2.5(ii) we deduce the result.

Corollary 2.12. *Let $p>3$ be a prime and $m\in\mathbb{Z}_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\binom{2k}{k}\Big(\frac{4-m^2}{192}\Big)^k\equiv\Big(\frac{2m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{(m+1)(m-2)^2}{864m^3}\Big)^k \pmod p.$$

Proof. This is immediate from [S5, Theorem 2.3(ii)], Lemma 2.1 and Corollary 2.11.

3. Congruences for $\sum_{k=0}^{[p/6]}\frac{(6k)!}{m^k\cdot(3k)!k!^3} \pmod p$.

For any nonnegative integer n, following [S5] we define

$$(3.1)\qquad S_n(x)=\sum_{k=0}^n\binom nk\binom{n+k}{k}\binom{2k}{k}x^k=\sum_{k=0}^n\binom{2k}{k}^2\binom{n+k}{2k}x^k.$$

By [S5, Theorem 3.1] we have

$$(3.2)\qquad S_n(x)=P_n(\sqrt{1+4x})^2\quad\text{and}\quad P_n(x)^2=S_n\Big(\frac{x^2-1}{4}\Big).$$

Theorem 3.1. *Let p be an odd prime and $m\in\mathbb{Z}_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{m^k\cdot(3k)!k!^3}\equiv P_{[\frac p6]}(\sqrt{1-12^3/m})^2\equiv\Big(\sum_{x=0}^{p-1}(x^3-3x+2\sqrt{1-12^3/m})^{\frac{p-1}{2}}\Big)^2 \pmod p.$$

Proof. By (2.1) and (3.1) we have

$$S_{[\frac p6]}\Big(-\frac{432}{m}\Big)=\sum_{k=0}^{[\frac p6]}\binom{2k}{k}^2\binom{[\frac p6]+k}{2k}\Big(-\frac{432}{m}\Big)^k\equiv\sum_{k=0}^{[p/6]}\frac{(6k)!}{m^k(3k)!k!^3} \pmod p.$$

On the other hand, by (3.2) and Theorem 2.4 we have

$$S_{[\frac p6]}\Big(-\frac{432}{m}\Big)=P_{[\frac p6]}(\sqrt{1-12^3/m})^2\equiv\Big(\sum_{x=0}^{p-1}(x^3-3x+2\sqrt{1-12^3/m})^{\frac{p-1}{2}}\Big)^2 \pmod p.$$

Thus the result follows.

Theorem 3.2. *Let $p>3$ be a prime and $m,n\in\mathbb{Z}_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\frac{(6k)!}{(3k)!k!^3}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k \pmod p.$$

Proof. From Theorem 2.2 we have

$$\begin{aligned}\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2&\equiv(-3m)^{\frac{p-1}{2}}P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)^2\\&\equiv\Big(\frac{-3m}{p}\Big)P_{[\frac p6]}\Big(\sqrt{\frac{-27n^2}{4m^3}}\Big)^2 \pmod p.\end{aligned}$$

By (3.1), (3.2) and (2.1),

$$\begin{aligned}P_{[\frac p6]}\Big(\sqrt{\frac{-27n^2}{4m^3}}\Big)^2&=S_{[\frac p6]}\Big(\frac{\frac{-27n^2}{4m^3}-1}{4}\Big)=\sum_{k=0}^{[\frac p6]}\binom{2k}{k}^2\binom{[\frac p6]+k}{2k}\Big(-\frac{4m^3+27n^2}{16m^3}\Big)^k\\&\equiv\sum_{k=0}^{[p/6]}\frac{(6k)!}{(3k)!k!^3}\Big(\frac{4m^3+27n^2}{432\cdot 16m^3}\Big)^k \pmod p.\end{aligned}$$

Thus the result follows.

Lemma 3.1. *Let p be an odd prime, n be a positive integer and $t\in\mathbb{Z}_p$ with $t\not\equiv 0,-\frac14 \pmod p$.*

(i) *If $P_n(\sqrt{1+4t})\equiv 0 \pmod p$, then*

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}t^k\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}kt^k\equiv 0 \pmod p.$$

(ii) *If*

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}t^k\equiv 0 \pmod p,$$

then

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}kt^k\equiv 0 \pmod p.$$

Proof. By (3.2) we have $S_n(x)=P_n(\sqrt{1+4x})^2$ and so $S_n'(x)=2P_n(\sqrt{1+4x})\cdot\frac{d}{dx}P_n(\sqrt{1+4x})$. Thus, (i) is true. Also, $S_n(t)\equiv 0 \pmod p$ implies $P_n(\sqrt{1+4t})\equiv 0 \pmod p$ and so $S_n'(t)\equiv 0 \pmod p$. This together with (3.1) proves the lemma.

Theorem 3.3. *Let p be an odd prime, and $m\in\mathbb{Z}_p$ with $m\not\equiv 0,1728 \pmod p$. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{m^k(3k)!k!^3}\equiv 0 \pmod p \quad \textit{implies} \quad \sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{m^k(3k)!k!^3}\equiv 0 \pmod p.$$

Proof. By (2.1) we have $\binom{2k}{k}^2\binom{[\frac p6]+k}{2k}(-\frac{432}{m})^k \equiv \frac{(6k)!}{m^k(3k)!k!^3} \pmod p$. Thus taking $n=[\frac p6]$ and $t=-\frac{432}{m}$ in Lemma 3.1(ii) we deduce the result.

Theorem 3.4 ([Su, Conjecture A26]). *Let $p\ne 2,11$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-32)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{p}{11})=-1,\\ (\frac{-2}{p})x^2 \pmod p & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2.\end{cases}$$

(ii) *If $(\frac{p}{11})=-1$, then $\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-32)^{3k}(3k)!k!^3}\equiv 0 \pmod p$.*

Proof. Taking $m=-96\cdot 11$ and $n=112\cdot 11^2$ in Theorem 3.2 and then applying (2.10) and Theorem 3.3 we deduce the result.

Theorem 3.5 ([Su, Conjecture A9]). *Let $p\ne 2,3,19$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-96)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{p}{19})=-1,\\ (\frac{-6}{p})x^2 \pmod p & \text{if } (\frac{p}{19})=1 \text{ and so } 4p=x^2+19y^2.\end{cases}$$

(ii) *If $(\frac{p}{19})=-1$, then $\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-96)^{3k}(3k)!k!^3}\equiv 0 \pmod p$.*

Proof. Taking $m=-8\cdot 19$ and $n=2\cdot 19^2$ in Theorem 3.2 and then applying (2.11) and Theorem 3.3 we deduce the result.

Theorem 3.6 ([Su, Conjecture A10]). *Let $p\ne 2,3,5,43$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-960)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{p}{43})=-1,\\ (\frac{p}{15})x^2 \pmod p & \text{if } (\frac{p}{43})=1 \text{ and so } 4p=x^2+43y^2.\end{cases}$$

(ii) *If $(\frac{p}{43})=-1$, then $\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-960)^{3k}(3k)!k!^3}\equiv 0 \pmod p$.*

Proof. Taking $m=-80\cdot 43$ and $n=42\cdot 43^2$ in Theorem 3.2 and then applying (2.11) and Theorem 3.3 we deduce the result.

Theorem 3.7 ([Su, Conjecture A11]). *Let p be a prime such that $p\ne 2,3,5,11,67$.*

(i) *We have*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-5280)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{p}{67})=-1,\\ (\frac{-330}{p})x^2 \pmod p & \text{if } (\frac{p}{67})=1 \text{ and so } 4p=x^2+67y^2.\end{cases}$$

(ii) *If $(\frac{p}{67})=-1$, then $\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-5280)^{3k}(3k)!k!^3}\equiv 0 \pmod p$.*

Proof. Taking $m=-440\cdot 67$ and $n=434\cdot 67^2$ in Theorem 3.2 and then applying (2.11) and Theorem 3.3 we deduce the result.

Theorem 3.8 ([Su, Conjecture A12]). *Let p be a prime with $p\ne 2,3,5,23,29,163$.*
(i) *We have*

$$\begin{aligned}&\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-640320)^{3k}(3k)!k!^3}\\&\equiv\begin{cases}0 \pmod p & \text{if } (\frac{p}{163})=-1,\\ (\frac{-10005}{p})x^2 \pmod p & \text{if } (\frac{p}{163})=1 \text{ and so } 4p=x^2+163y^2.\end{cases}\end{aligned}$$

(ii) *If $(\frac{p}{163})=-1$, then* $\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-640320)^{3k}(3k)!k!^3}\equiv 0 \pmod p$.

Proof. Taking $m=-80\cdot 23\cdot 29\cdot 163$ and $n=14\cdot 11\cdot 19\cdot 127\cdot 163^2$ in Theorem 3.2 and then applying (2.11) and Theorem 3.3 we deduce the result.

Theorem 3.9 ([S4, Conjecture 2.8]). *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-15)^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7,\\ (\frac{p}{15})4C^2 \pmod p & \text{if } p=C^2+7D^2\equiv 1,2,4 \pmod 7\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{(-15)^{3k}(3k)!k!^3}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 3,5,6 \pmod 7.$$

Proof. Taking $m=-35$ and $n=98$ in Theorem 3.2 and then applying (2.8) and Theorem 3.3 we deduce the result.

Theorem 3.10 ([S4, Conjecture 2.9]). *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{255^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } p\equiv 3,5,6 \pmod 7,\\ (\frac{p}{255})4C^2 \pmod p & \text{if } p=C^2+7D^2\equiv 1,2,4 \pmod 7\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{255^{3k}(3k)!k!^3}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 3,5,6 \pmod 7.$$

Proof. Taking $m=-595$ and $n=5586$ in Theorem 3.2 and then applying (2.12) and Theorem 3.3 we deduce the result.

Theorem 3.11 ([S4, Conjecture 2.4]). *Let p be a prime such that $p\ne 2,3,11$. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{66^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } p\equiv 3 \pmod 4,\\ (\frac{p}{33})4a^2 \pmod p & \text{if } p=a^2+b^2\equiv 1 \pmod 4 \text{ and } 2\nmid a\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{66^{3k}(3k)!k!^3}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 3 \pmod 4.$$

Proof. Taking $m=-11$ and $n=14$ in Theorem 3.2 and then applying (2.6) and Theorem 3.3 we deduce the result.

Theorem 3.12 ([S4, Conjecture 2.5]). *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{20^{3k}(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } p\equiv 5,7 \pmod 8,\\ (\frac{-5}{p})4c^2 \pmod p & \text{if } p=c^2+2d^2\equiv 1,3 \pmod 8\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{20^{3k}(3k)!k!^3}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 5,7 \pmod 8.$$

Proof. Taking $m=-30$ and $n=56$ in Theorem 3.2 and then applying (2.7) and Theorem 3.3 we deduce the result.

Theorem 3.13 ([S4, Conjecture 2.6]). *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{54000^k(3k)!k!^3}\equiv\begin{cases}0 \pmod p & \text{if } p\equiv 2 \pmod 3,\\ (\frac{p}{5})4A^2 \pmod p & \text{if } p=A^2+3B^2\equiv 1 \pmod 3\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{54000^k(3k)!k!^3}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 2 \pmod 3.$$

Proof. Taking $m=-15$ and $n=22$ in Theorem 3.2 and then applying (2.5) and Theorem 3.3 we deduce the result.

Remark 3.1 Let $p>5$ be a prime. If Conjecture 2.1 is true, by Theorem 3.2 we get

$$\sum_{k=0}^{[p/6]}\frac{(6k)!}{(-12288000)^k(3k)!k!^3}\equiv\begin{cases}(\frac{10}{p})L^2 \pmod p & \text{if } 3\mid p-1 \text{ and so } 4p=L^2+27M^2,\\ 0 \pmod p & \text{if } 3\mid p-2.\end{cases}$$

This is a special case of [S4, Conjecture 2.7].

Lemma 3.2 ([S5, Lemma 3.2]). *For any positive integer n we have the following identities:*

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k}{(-4)^k}=\begin{cases}-\frac{n^2}{2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2\nmid n,\\ \frac{n(n+1)}{2^{2n}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2\mid n,\end{cases}$$

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k^2}{(-4)^k}=\begin{cases}-\frac{n^2(2n^2+2n-1)}{3\cdot 2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2\nmid n,\\ \frac{n^2(n+1)^2}{3\cdot 2^{2n-1}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2\mid n,\end{cases}$$

$$\sum_{k=1}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\frac{k^3}{(-4)^k}=\begin{cases}-\frac{n^2(4n^2(n+1)^2-n(n+1)+1)}{15\cdot 2^{2n-2}}\binom{n-1}{\frac{n-1}{2}}^2 & \text{if } 2\nmid n,\\ \frac{n^2(n+1)^2(2n+1)^2}{15\cdot 2^{2n}}\binom{n}{\frac{n}{2}}^2 & \text{if } 2\mid n.\end{cases}$$

Theorem 3.14. *Let $p\equiv 1 \pmod 4$ be a prime and $p=a^2+b^2$ with $a,b\in\mathbb{Z}$ and $2\nmid a$. Then*

$$\begin{aligned}\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv -\frac{5}{9}\Big(\frac{p}{3}\Big)a^2 \pmod p,\\ \sum_{k=0}^{[p/6]}\frac{k^2\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv \frac{25}{486}\Big(\frac{p}{3}\Big)a^2 \pmod p,\\ \sum_{k=0}^{[p/6]}\frac{k^3\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv \frac{5}{2187}\Big(\frac{p}{3}\Big)a^2 \pmod p.\end{aligned}$$

Proof. For $p=5$ clearly the result is true. Now suppose $p>5$. Since $p\equiv 1\pmod 4$ we see that $[\frac p6]$ is even. Taking $n=[\frac p6]$ in Lemma 3.2 and applying (2.1) and the congruence $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac12}{k}=\frac{1}{(-4)^k}\binom{2k}{k}\pmod p$ (see [S4, Lemma 2.4]) we see that

$$\begin{aligned}\sum_{k=0}^{[p/6]}\frac{k\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv\frac{[\frac p6]([\frac p6]+1)}{2^{2[\frac p6]}}\binom{[\frac p6]}{[\frac p{12}]}^2\equiv -\frac{5}{36}\binom{\frac{p-1}{2}}{[\frac p{12}]}^2 \pmod p,\\ \sum_{k=0}^{[p/6]}\frac{k^2\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv\frac{[\frac p6]^2([\frac p6]+1)^2}{3\cdot 2^{2[\frac p6]-1}}\binom{[\frac p6]}{[\frac p{12}]}^2\equiv \frac{25}{1944}\binom{\frac{p-1}{2}}{[\frac p{12}]}^2 \pmod p,\\ \sum_{k=0}^{[p/6]}\frac{k^3\cdot(6k)!}{12^{3k}(3k)!k!^3}&\equiv\frac{[\frac p6]^2([\frac p6]+1)^2(2[\frac p6]+1)^2}{15\cdot 2^{2[\frac p6]}}\binom{[\frac p6]}{[\frac p{12}]}^2\equiv \frac{5}{8748}\binom{\frac{p-1}{2}}{[\frac p{12}]}^2 \pmod p.\end{aligned}$$

From Lemma 2.2 and the proof of Theorem 2.1 we know that

$$\binom{\frac{p-1}{2}}{[\frac p{12}]}^2\equiv\begin{cases}4a^2 \pmod p & \text{if } p\equiv 1 \pmod{12},\\ 4b^2\equiv -4a^2 \pmod p & \text{if } p\equiv 5 \pmod{12}.\end{cases}$$

Thus the result follows.

We remark that the first congruence in Theorem 3.14 was conjectured by Zhi-Wei Sun in [Su, Conjecture A29].

4. Congruences for $P_{[p/3]}(t) \pmod p$.

Theorem 4.1. *Let $p>3$ be a prime and $t\in\mathbb{Z}_p$. Then*

$$P_{[\frac p3]}(t)\equiv -\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p.$$

Proof. By (1.4) we have $P_n(1)=1$. Since $P_{[\frac p3]}(1)=1$ and

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-3x-2}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(x+1)^2(x-2)}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x-2}{p}\Big)-\Big(\frac{-1-2}{p}\Big)=-\Big(\frac p3\Big),$$

we see that the result is true for $t\equiv 1\pmod p$. Since $P_{[\frac p3]}(-1)=(-1)^{[\frac p3]}P_{[\frac p3]}(1)=(\frac p3)$ and

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-27x+54}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(-3x)^3-27(-3x)+54}{p}\Big)=\Big(\frac{-3}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x-2}{p}\Big)=-1,$$

we see that the result is also true for $t\equiv -1\pmod p$.

Now we assume $t\not\equiv\pm 1\pmod p$. Set $W_n(x)=\sum_{k=0}^n\binom nk^2x^k$. From [BM, Theorem 6] we know that

$$W_{[\frac p3]}\Big(1-\frac{x}{27}\Big)\equiv u_p(x)(x-27)^{[\frac p{12}]}J_p\Big(\frac{x(x-24)^3}{x-27}\Big)\pmod p,$$

where $J_p(x)$ is a certain Jacobi polynomial given in the proof of Theorem 2.2 and

$$u_p(x)=\begin{cases}1&\text{if } p\equiv 1\pmod{12},\\ -3(x-24)&\text{if } p\equiv 5\pmod{12},\\ x^2-36x+216&\text{if } p\equiv 7\pmod{12},\\ -3(x-24)(x^2-36x+216)&\text{if } p\equiv 11\pmod{12}.\end{cases}$$

Set $x=54/(t+1)$. We then have

(4.1)

$$W_{[\frac p3]}((t-1)/(t+1))$$
$$\equiv\begin{cases}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p&\text{if } p\equiv 1\pmod{12},\\ \frac{18(4t-5)}{t+1}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p&\text{if } p\equiv 5\pmod{12},\\ \frac{108(2t^2-14t+11)}{(t+1)^2}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p&\text{if } p\equiv 7\pmod{12},\\ \frac{1944(4t-5)(2t^2-14t+11)}{(t+1)^3}(\frac{27(1-t)}{1+t})^{[\frac p{12}]}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\pmod p&\text{if } p\equiv 11\pmod{12}.\end{cases}$$

By (2.3) we have

$$W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)=\Big(1-\frac{t-1}{t+1}\Big)^{[\frac p3]}P_{[\frac p3]}\Big(\frac{1+(t-1)/(t+1)}{1-(t-1)/(t+1)}\Big)=\Big(\frac{2}{t+1}\Big)^{[\frac p3]}P_{[\frac p3]}(t).\tag{4.2}$$

If $p\equiv 2\pmod 3$ and $t\equiv\frac54\pmod p$, from the above we get

$$P_{[\frac p3]}\Big(\frac54\Big)=\Big(\frac{\frac54+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{\frac54-1}{\frac54+1}\Big)\equiv 0\pmod p.$$

On the other hand,

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)=\sum_{y=0}^{p-1}\Big(\frac{y-27/4}{p}\Big)=0.$$

Thus the result is true when $p\equiv 2 \pmod 3$ and $t\equiv \frac{5}{4} \pmod p$. Now assume $p\equiv 1 \pmod 3$ or $t\not\equiv \frac{5}{4} \pmod p$. If $p\equiv 3 \pmod 4$ and $2t^2-14t+11\equiv 0 \pmod p$, from the above we deduce

$$P_{[\frac p3]}(t)=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\equiv 0 \pmod p.$$

On the other hand,

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\\&=\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(-x)^3+3(4t-5)(-x)}{p}\Big)=-\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x}{p}\Big)=0.\end{aligned}$$

Thus the result is true when $p\equiv 3 \pmod 4$ and $2t^2-14t+11\equiv 0 \pmod p$. From now on we assume $p\equiv 1 \pmod 4$ or $2t^2-14t+11\not\equiv 0 \pmod p$. Set $m=3(4t-5)$ and $n=2(2t^2-14t+11)$. Then

$$4m^3+27n^2=-432(1-t)(1+t)^3\quad\text{and so}\quad \frac{2^8\cdot 3^3m^3}{4m^3+27n^2}=\frac{432(5-4t)^3}{(1-t)(1+t)^3}.$$

By (2.4) we have

$$\begin{aligned}&J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big)=J_p\Big(\frac{2^8\cdot 3^3m^3}{4m^3+27n^2}\Big)\\&\equiv -(-48m)^{\frac{(\frac p3)-1}{2}}(864n)^{\frac{(\frac{-1}{p})-1}{2}}(-16(4m^3+27n^2))^{-[\frac p{12}]}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big) \pmod p.\end{aligned}$$

If $p\equiv 1 \pmod{12}$, from all the above we deduce

$$\begin{aligned}&P_{[\frac p3]}(t)\\&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-1}{3}}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-1}{12}}J_p\Big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\Big)\\&\equiv -2^{-\frac{p-1}{3}}(3(t+1))^{\frac{p-1}{4}}(1-t)^{\frac{p-1}{12}}(16\cdot 432(1-t)(1+t)^3)^{-\frac{p-1}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&\equiv -\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big) \pmod p.\end{aligned}$$

If $p\equiv 5 \pmod{12}$, from all the above we deduce

$$\begin{aligned}
&P_{[\frac p3]}(t)\\
&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-2}{3}}\frac{18(4t-5)}{t+1}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-5}{12}}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\\
&\equiv 2^{-\frac{p-5}{3}}3^{\frac{p+3}{4}}(4t-5)(1+t)^{\frac{p-5}{4}}(1-t)^{\frac{p-5}{12}}(144(4t-5))^{-1}\\
&\qquad\times(16\cdot 432(1-t)(1+t)^3)^{-\frac{p-5}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

If $p\equiv 7 \pmod{12}$, from all the above we deduce

$$\begin{aligned}
P_{[\frac p3]}(t)&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\\
&\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-1}{3}}\frac{108(2t^2-14t+11)}{(t+1)^2}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-7}{12}}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\\
&\equiv -2^{-\frac{p-7}{3}}3^{\frac{p+5}{4}}(2t^2-14t+11)(1+t)^{\frac{p-7}{4}}(1-t)^{\frac{p-7}{12}}(1728(2t^2-14t+11))^{-1}\\
&\qquad\times(16\cdot 432(1-t)(1+t)^3)^{-\frac{p-7}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

If $p\equiv 11 \pmod{12}$, from all the above we deduce

$$\begin{aligned}
P_{[\frac p3]}(t)&=\Big(\frac{t+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{t-1}{t+1}\Big)\\
&\equiv\Big(\frac{t+1}{2}\Big)^{\frac{p-2}{3}}\frac{1944(4t-5)(2t^2-14t+11)}{(t+1)^3}\Big(\frac{27(1-t)}{1+t}\Big)^{\frac{p-11}{12}}J_p\big(\frac{432(5-4t)^3}{(1-t)(1+t)^3}\big)\\
&\equiv 2^{-\frac{p-11}{3}}3^{\frac{p-11}{4}+5}(4t-5)(2t^2-14t+11)(1+t)^{\frac{p-11}{4}}(1-t)^{\frac{p-11}{12}}(48m)^{-1}(864n)^{-1}\\
&\qquad\times(16\cdot 432(1-t)(1+t)^3)^{-\frac{p-11}{12}}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\
&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\pmod p.
\end{aligned}$$

This proves the theorem.

Corollary 4.1. *Let $p > 3$ be a prime and let t be a variable. Then*

$$\begin{aligned}&\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}\Big(\frac{1-t}{54}\Big)^k\\ &\equiv P_{[\frac p3]}(t)\equiv -\Big(\frac p3\Big)\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}{2}} \pmod p.\end{aligned}$$

Proof. From [S4, Lemma 2.3] we have $P_{[\frac p3]}(t)\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}(\frac{1-t}{54})^k \pmod p$ By Theorem 4.1 and Euler's criterion, the result is true for $t = 0, 1, \dots, p-1$. Since both sides are polynomials of t with degree at most $p-1$. Using Lagrange's theorem we obtain the result.

Corollary 4.2. *Let $p \ge 17$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\\ &=\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t+5)x+2(2t^2+14t+11)}{p}\Big).\end{aligned}$$

Proof. Since $P_{[\frac p3]}(-t)=(-1)^{[\frac p3]}P_{[\frac p3]}(t)=\big(\frac p3\big)P_{[\frac p3]}(t)$, by Theorem 4.1 we have

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\\ &\equiv\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t+5)x+2(2t^2+14t+11)}{p}\Big) \pmod p.\end{aligned}$$

By Weil's estimate ([BEW, p.183]) we have

$$\begin{aligned}&\Big|\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}{p}\Big)\Big|\le 2\sqrt p,\\ &\Big|\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t+5)x+2(2t^2+14t+11)}{p}\Big)\Big|\le 2\sqrt p.\end{aligned}$$

Since $4\sqrt p < p$ for $p \ge 17$, from the above we deduce the result.

Theorem 4.2. *Let $p > 3$ be a prime and let t be a variable. Then*

$$P_{[\frac p3]}(t)\equiv\begin{cases}(5-4t)^{\frac{p-1}{4}}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p & \text{if } p\equiv 1\pmod 4,\\ -\frac{(5-4t)^{\frac{p+1}{4}}}{\sqrt{5-4t}}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Proof. Since both sides are polynomials of t with degree at most $p-2$. It suffices to show that the congruence is true for all $t\in\mathbb{Z}_p$ with $t\not\equiv\frac54\pmod p$. Set $m=3(4t-5)$ and $n=2(2t^2-14t+11)$. Then

$$\frac{3n\sqrt{-3m}}{2m^2}=\frac{(2t^2-14t+11)\sqrt{5-4t}}{(5-4t)^2}.$$

Thus, by Theorems 4.1 and 2.2 we have

$$\begin{aligned}P_{[\frac p3]}(t)&\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&\equiv\begin{cases}\big(\frac p3\big)(9(5t-4))^{\frac{p-1}4}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p&\text{if }4\mid p-1,\\ \big(\frac p3\big)\frac{(9(5t-4))^{\frac{p+1}4}}{\sqrt{9(5-4t)}}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p&\text{if }4\mid p-3.\end{cases}\end{aligned}$$

For $p\equiv1\pmod 4$ we have $9^{\frac{p-1}4}\big(\frac p3\big)\equiv\big(\frac3p\big)\big(\frac p3\big)=1\pmod p$, For $p\equiv3\pmod 4$ we have $9^{\frac{p+1}4}\frac13\big(\frac p3\big)\equiv\big(\frac3p\big)\big(\frac p3\big)=-1\pmod p$. Thus the result follows.

Corollary 4.3. *Let $p>3$ be a prime and $m\in\mathbb{Z}_p$ with $m\not\equiv0\pmod p$. Then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}\Big(\frac{m^2-1}{216}\Big)^k\equiv\Big(\frac{-m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{(m+1)(3-m)^3}{2^8\cdot3^3m^3}\Big)^k\pmod p.$$

Proof. Taking $t=\frac{5-m^2}{4}$ in Theorem 4.2 and then applying Corollary 4.1 and Lemma 2.1 we deduce the result.

Theorem 4.3. *Let $p>3$ be a prime. Then*

(i) *If $p\equiv2\pmod 3$, then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{(-216)^k\cdot k!^3}\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{24^k\cdot k!^3}\equiv P_{[\frac p3]}\Big(\frac54\Big)\equiv0\pmod p.$$

(ii) *If $p\equiv1\pmod 3$ and so $4p=L^2+27M^2$ with $L,M\in\mathbb{Z}$ and $L\equiv1\pmod 3$, then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{(-216)^k\cdot k!^3}\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{24^k\cdot k!^3}\equiv P_{[\frac p3]}\Big(\frac54\Big)\equiv-L\equiv\Big(\frac{-2}{p}\Big)\binom{\frac{2(p-1)}{3}}{[\frac p{12}]}\pmod p.$$

Proof. Putting $t=\pm\frac54$ in Corollary 4.1 we get

$$P_{[\frac p3]}\Big(\frac54\Big)\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{(-216)^k\cdot k!^3}\pmod p\quad\text{and}\quad P_{[\frac p3]}\Big(-\frac54\Big)\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{24^k\cdot k!^3}\pmod p.$$

This together with (1.3) yields

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{(-216)^k\cdot k!^3}\equiv\Big(\frac p3\Big)\sum_{k=0}^{[p/3]}\frac{(3k)!}{24^k\cdot k!^3}\pmod p.$$

From the above and the proof of Theorem 4.1 we obtain (i).

Now assume $p\equiv 1\pmod 3$, $p=A^2+3B^2$, $4p=L^2+27M^2$ and $A\equiv L\equiv 1\pmod 3$. It is known that $2^{\frac{p-1}3}\equiv 1\pmod p$ if and only if $3\mid B$. When $3\nmid B$ we choose the sign of B so that $B\equiv 1\pmod 3$. By [S2, (2.12)] we have $2^{(p-1)/3}\equiv \frac12(-1-\frac AB)\pmod p$. From Theorem 4.1 and [S2, (2.9)-(2.11)] we deduce that

$$\begin{aligned}-P_{[\frac p3]}\Big(\frac54\Big)&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)=1+\sum_{x=0}^{p-1}\Big(\frac{x^3-27/4}{p}\Big)\\&=\begin{cases}-2A=L\pmod p&\text{if } 2^{\frac{p-1}3}\equiv 1\pmod p,\\ A+3B=L\pmod p&\text{if } 2^{\frac{p-1}3}\not\equiv 1\pmod p \text{ and } B\equiv 1\pmod 3.\end{cases}\end{aligned}$$

On the other hand, by the proof of Theorem 4.1,

$$\begin{aligned}P_{[\frac p3]}\Big(\frac54\Big)&=\Big(\frac{\frac54+1}{2}\Big)^{[\frac p3]}W_{[\frac p3]}\Big(\frac{\frac54-1}{\frac54+1}\Big)\\&\equiv\begin{cases}\big(\frac98\big)^{\frac{p-1}3}\big(\frac{27(1-\frac54)}{1+\frac54}\big)^{\frac{p-1}{12}}J_p(0)\equiv(-1)^{\frac{p-1}{12}}3^{-\frac{p-1}4}J_p(0)\pmod p&\text{if } p\equiv 1\pmod{12},\\ \big(\frac98\big)^{\frac{p-1}3}\frac{108(2(\frac54)^2-14\cdot\frac54+11)}{(\frac54+1)^2}\big(\frac{27(1-\frac54)}{1+\frac54}\big)^{\frac{p-7}{12}}J_p(0)\\ \quad\equiv -8(-1)^{\frac{p-7}{12}}3^{-\frac{p-7}4}J_p(0)\pmod p&\text{if } p\equiv 7\pmod{12}.\end{cases}\end{aligned}$$

By the definition of $J_p(x)$, we have

$$\begin{aligned}J_p(0)&=1728^{[\frac p{12}]}\cdot 2^{-[\frac p{12}]}\sum_{r=0}^{[\frac p{12}]}\binom{[\frac p{12}]-\frac13(\frac p3)}{r}\binom{[\frac p{12}]-\frac12(\frac{-1}p)}{[\frac p{12}]-r}0^{[\frac p{12}]-r}2^r\\&=1728^{[\frac p{12}]}\binom{[\frac p{12}]-\frac13(\frac p3)}{[\frac p{12}]}=(-1728)^{[\frac p{12}]}\binom{\frac13(\frac p3)-1}{[\frac p{12}]}.\end{aligned}$$

Hence

$$J_p(0)\equiv(-1728)^{[\frac p{12}]}\binom{\frac{2(p-1)}3}{[\frac p{12}]}\pmod p$$

and therefore

$$P_{[\frac p3]}\Big(\frac54\Big)\equiv\begin{cases}(-1)^{\frac{p-1}{12}}3^{-\frac{p-1}4}(-1728)^{\frac{p-1}{12}}\binom{\frac{2(p-1)}3}{\frac{p-1}{12}}\equiv\big(\frac2p\big)\binom{\frac{2(p-1)}3}{\frac{p-1}{12}}\pmod p&\text{if } 12\mid p-1,\\ -8(-1)^{\frac{p-7}{12}}3^{-\frac{p-7}4}(-1728)^{\frac{p-7}{12}}\binom{\frac{2(p-1)}3}{\frac{p-7}{12}}\equiv-\big(\frac2p\big)\binom{\frac{2(p-1)}3}{\frac{p-7}{12}}\pmod p&\text{if } 12\mid p-7.\end{cases}$$

Now putting all the above together we deduce the result.

Remark 4.1 For any prime $p>3$, Zhi-Wei Sun conjectured ([Su, Conjecture A46])

$$\sum_{k=0}^{p-1}\frac{(3k)!}{24^k\cdot k!^3}\equiv\Big(\frac p3\Big)\sum_{k=0}^{p-1}\frac{(3k)!}{(-216)^k\cdot k!^3}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2}&\text{if } p\equiv 1\pmod 3,\\ 0\pmod p&\text{if } p\equiv 2\pmod 3.\end{cases}$$

—

Theorem 4.4. *Let $p>3$ be a prime such that $p\equiv 3\pmod 4$. Then*

$$P_{[\frac p3]}\Big(\frac{7\pm 3\sqrt 3}{2}\Big)\equiv\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}\Big(\frac{3\pm\sqrt 3}{36}\Big)^k\equiv 0\pmod p.$$

Proof. Set $t=(7\pm 3\sqrt 3)/2$. Then $2t^2-14t+11=0$. By Corollary 4.1 we have

$$P_{[\frac p3]}(t)\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}(x^3+3(4t-5)x)^{\frac{p-1}2}\pmod p.$$

By Corollary 4.1 and (1.3) we also have

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{k!^3}\Big(\frac{3\pm\sqrt 3}{36}\Big)^k\equiv P_{[\frac p3]}(-t)=\Big(\frac p3\Big)P_{[\frac p3]}(t)\pmod p.$$

Since $p\equiv 3\pmod 4$ we see that

$$\sum_{x=0}^{p-1}(x^3+3(4t-5)x)^{\frac{p-1}2}\equiv\sum_{x=0}^{p-1}((-x)^3+3(4t-5)(-x))^{\frac{p-1}2}=-\sum_{x=0}^{p-1}(x^3+3(4t-5)x)^{\frac{p-1}2}\pmod p.$$

Therefore $\sum_{x=0}^{p-1}(x^3+3(4t-5)x)^{\frac{p-1}2}\equiv 0\pmod p$ and so $P_{[\frac p3]}(t)\equiv 0\pmod p$. This completes the proof.

5. Congruences for $\sum_{k=0}^{[p/3]}\frac{\binom{2k}k^2\binom{3k}k}{m^k}\pmod p$.

Let $p>3$ be a prime and $m\in\mathbb Z$ with $p\nmid m$. In [Su], Zhi-Wei Sun posed some conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}k^2\binom{3k}k}{m^k}\pmod{p^2}$. For example, he conjectured (see [Su, Conjecture A13])

$$(5.1)\qquad\sum_{k=0}^{p-1}\frac{\binom{2k}k^2\binom{3k}k}{(-27)^k}\equiv\begin{cases}0\pmod{p^2}&\text{if } p\equiv 7,11,13,14\pmod{15},\\ 4x^2-2p\pmod{p^2}&\text{if } p=x^2+15y^2\equiv 1,4\pmod{15},\\ 20x^2-2p\pmod{p^2}&\text{if } p=5x^2+3y^2\equiv 2,8\pmod{15}.\end{cases}$$

Theorem 5.1. *Let $p>3$ be a prime, $m\in\mathbb Z_p$, $m\not\equiv 0\pmod p$ and $t=\sqrt{1-108/m}$. Then*

$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}k^2\binom{3k}k}{m^k}\equiv P_{[\frac p3]}(t)^2\equiv\Big(\sum_{x=0}^{p-1}(x^3+3(4t-5)x+2(2t^2-14t+11))^{\frac{p-1}2}\Big)^2\pmod p.$$

Proof. By the proof of [S4, Lemma 2.3] we have

$$(5.2)\qquad\binom{[\frac p3]+k}{2k}\equiv\frac{1}{(-27)^k}\binom{3k}k\pmod p.$$

Thus, using (3.1) and the above we have

$$S_{[\frac p3]}\Big(-\frac{27}m\Big)=\sum_{k=0}^{[\frac p3]}\binom{2k}k^2\binom{[\frac p3]+k}{2k}\Big(-\frac{27}m\Big)^k\equiv\sum_{k=0}^{[p/3]}\frac{\binom{2k}k^2\binom{3k}k}{m^k}\pmod p.$$

On the other hand, by (3.2) we have $S_{[\frac p3]}(-\frac{27}m)=P_{[\frac p3]}(\sqrt{1-\frac{108}m})^2$. Thus the result follows from the above and Corollary 4.1.

Theorem 5.2. *Let p be an odd prime, and $m\in\mathbb{Z}_p$ with $m\not\equiv 0,108\pmod p$. Then*
$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\equiv 0\pmod p\quad\text{implies}\quad\sum_{k=0}^{[p/3]}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{m^k}\equiv 0\pmod p.$$

Proof. By (5.2) we have $\binom{[\frac p3]+k}{2k}(-\frac{27}{m})^k\equiv\frac{1}{m^k}\binom{3k}{k}\pmod p$. Thus taking $n=[\frac p3]$ and $t=-\frac{27}{m}$ in Lemma 3.1(ii) we deduce the result.

Theorem 5.3. *Let $p>3$ be a prime. Then*
$$\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\begin{cases}L^2\pmod p & \text{if } p\equiv 1\pmod 3 \text{ and so } 4p=L^2+27M^2,\\ 0\pmod p & \text{if } p\equiv 2\pmod 3\end{cases}$$
and
$$\sum_{k=0}^{[p/3]}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv 0\pmod p\quad\text{for}\quad p\equiv 2\pmod 3.$$

Proof. Taking $m=-192$ in Theorem 5.1 we obtain $\sum_{k=0}^{[p/3]}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv P_{[\frac p3]}(\frac54)^2\pmod p$. Now applying Theorems 4.3 and 5.2 we obtain the result.

Remark 5.1 Theorem 5.3 was conjectured by Zhi-Wei Sun. See [Su, A8].

Theorem 5.4. *Let $p\equiv 1\pmod 3$ be a prime and hence $p=A^2+3B^2$ with $A,B\in\mathbb{Z}$. Then*
$$\sum_{k=0}^{(p-1)/3}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv-\frac89A^2\pmod p,$$
$$\sum_{k=0}^{(p-1)/3}\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\frac{32}{243}A^2\pmod p,$$
$$\sum_{k=0}^{(p-1)/3}\frac{k^3\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\frac{16}{10935}A^2\pmod p.$$

Proof. Taking $n=\frac{p-1}{3}$ in Lemma 3.2 and then applying (5.2) and the congruence $\binom{\frac{p-1}{2}}{k}\equiv\binom{-\frac12}{k}=\frac{1}{(-4)^k}\binom{2k}{k}\pmod p$ we derive
$$\sum_{k=0}^{(p-1)/3}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\frac{\frac{p-1}{3}(\frac{p-1}{3}+1)}{2^{\frac{2(p-1)}{3}}}\binom{\frac{p-1}{3}}{\frac{p-1}{6}}^2\equiv-\frac29\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\pmod p,$$
$$\sum_{k=0}^{(p-1)/3}\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\frac{(\frac{p-1}{3})^2(\frac{p-1}{3}+1)^2}{3\cdot 2^{\frac{2(p-1)}{3}-1}}\binom{\frac{p-1}{3}}{\frac{p-1}{6}}^2\equiv\frac{8}{243}\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\pmod p,$$
$$\sum_{k=0}^{(p-1)/3}\frac{k^3\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\frac{(\frac{p-1}{3})^2(\frac{p-1}{3}+1)^2(\frac{2(p-1)}{3}+1)^2}{15\cdot 2^{\frac{2(p-1)}{3}}}\binom{\frac{p-1}{3}}{\frac{p-1}{6}}^2$$
$$\equiv\frac{4}{15\cdot 3^6}\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2\pmod p.$$

From [BEW, Theorem 9.4.4] we know that $\binom{\frac{p-1}{2}}{\frac{p-1}{6}}^2 \equiv 4A^2 \pmod p$. Thus the result follows.

We mention that the first congruence in Theorem 5.4 was conjectured by Zhi-Wei Sun in [Su, Conjecture A55]. By Theorem 5.1, for any prime $p>3$,

$$\sum_{k=0}^{[p/3]} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \equiv P_{[\frac p3]}(\sqrt5)^2 \pmod p.$$

This together with (5.1) entices us to make the following conjecture.

Conjecture 5.1. *let $p>5$ be a prime. Then*

$$P_{[\frac p3]}(\sqrt5) \equiv \begin{cases} 0 \pmod p & \text{if } p\equiv 7,11,13,14 \pmod{15},\\ 2x(\frac x3) \pmod p & \text{if } p=x^2+15y^2\equiv 1,4 \pmod{15},\\ 2x(\frac x3)\sqrt5 \pmod p & \text{if } p=5x^2+3y^2\equiv 2,8 \pmod{15}.\end{cases}$$

From Zhi-Wei Sun's Conjecture A14 on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}$ and Theorem 5.1 we form the following conjecture.

Conjecture 5.2. *Let $p>3$ be a prime. Then*

$$P_{[\frac p3]}\Big(\frac{1}{\sqrt2}\Big) \equiv \begin{cases} 0 \pmod p & \text{if } p\equiv 13,17,19,23 \pmod{24},\\ 2x(\frac x3) \pmod p & \text{if } p=x^2+6y^2\equiv 1,7 \pmod{24},\\ -2x(\frac x3)\sqrt2 \pmod p & \text{if } p=2x^2+3y^2\equiv 5,11 \pmod{24}.\end{cases}$$

From Zhi-Wei Sun's Conjecture A4 on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$ and Theorem 5.1 we form the following conjecture.

Conjecture 5.3. *Let $p\ne 2,3,11$ be a prime. Then*

$$P_{[\frac p3]}\Big(\frac{\sqrt{-11}}{4}\Big) \equiv \begin{cases} 0 \pmod p & \text{if } (\frac{p}{11})=-1,\\ 2x(\frac x3) \pmod p & \text{if } 3\mid p-1,\ (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2,\\ -y(\frac y3)\sqrt{-11} \pmod p & \text{if } 3\mid p-2,\ (\frac{p}{11})=1 \text{ and so } 4p=x^2+11y^2.\end{cases}$$

Using Theorem 5.1 one may produce many similar conjectures from Zhi-Wei Sun's conjectures in [Su].

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