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Lie Supergroups: An operator viewpoint

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Abstract

Representation-theoretical Lie supergroups are introduced as supermanifolds with underlying base manifold being the Lie group associated to the even part of a given Lie superalgebra. The Lie superalgebra is required to be represented by invariant derivations on the space of superfunctions. This approach does not involve the morphisms for multiplication, inverse and identity as in the category-theoretical approach to Lie supergroups. The constructed category is not isomorphic to the category of Harish-Chandra superpairs. Variation of structure occurs. First examples where positive-dimensional parameter spaces arise are computed.

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Lie supergroups are defined as group objects in the category of supermanifolds. From this point of view there is a unique Lie supergroup $\mathcal{G}$ associated to a Harish-Chandra superpair $(G, \mathfrak{g})$ (see e.g. [Kost77] for the real and [Vis09] for the complex case). In contrast to this equivalence of categories there is a representation-theoretical approach to Lie supergroups which does not lead to uniqueness of the Lie supergroup structure but still inherits the identification of morphisms of supergroup-objects and morphisms of Lie superalgebras. In numerous applications it is only important to have the operators stemming from the Lie superalgebra, e.g., the radial operators, and the group-theoretic morphisms are extraneous. Therefore we have initiated this investigation.

Associated to a Harish-Chandra superpair we will discuss supermanifolds with underlying Lie group and two representations of the Lie superalgebra by superderivations on superfunctions replacing the notion of left- and right-invariant vector fields. Furthermore a splitting of

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the supermanifold will be fixed and morphisms are assumed to preserve this splitting. This restriction is perhaps unsuitable for superanalysis, nevertheless it is the first step in the solution a larger problem, because omitting the condition of a fixed splitting the morphisms between the obtained objects are no longer identified with morphisms of Lie superalgebras. A second step after our analysis, one that is well beyond the final goal of this article, is to parameterize the possible splittings and to decompose a morphism which does not preserve the splitting into one which does followed by a change of the splitting. This describes a category closer to supersymmetry.

Let us now outline the contents of the article. In the first section we give the definition of a representation-theoretical Lie supergroup. The isomorphy classes of these objects for a fixed Harish-Chandra superpair turn out to be special classes of homogeneous spaces with respect to the category-theoretical Lie supergroup $\mathcal{G}$. Furthermore we observe that the constructions of Lie supergroups by Kostant (see [Kost77]) and Berezin (see [Ber87]) apply naturally to the construction of representation-theoretical Lie supergroups.

In the remaining sections we analyze in detail the structures of special representation- theoretical Lie supergroups for the Lie superalgebra $\mathfrak{gl}_{\mathbb{K}}(1/1)$ with underlying Lie group $(\mathbb{K}^{\times})^2$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ determining the parameter space of such structures. In this space we find the points defining Kostant's and Berezin's constructions and prove that they are not isomorphic as representation-theoretical Lie supergroups. Since larger matrix Lie superalgebras contain $\mathfrak{gl}_{\mathbb{K}}(1/1)$ as a Lie subsuperalgebra, non-isomorphy holds in a more general context. It should be noted that in [Kal10] we prove that Kostant's construction of real Lie supergroups is also valid in the complex case. So referring to Kostant's construction in this article we already allow $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. That article as well as this one is a part of the author's dissertation [Kal11].

Let us conclude this introduction by fixing some notation: We denote a supermanifold by the symbol $\mathcal{M} = (M, \mathcal{A}_{\mathcal{M}})$ where M designates the underlying manifold and $\mathcal{A}_{\mathcal{M}}$ the sheaf of superfunctions. In order to simplify notation we denote the sheaf of smooth, respectively holomorphic, functions on M by $\mathcal{F}_{\mathbb{R},G} := \mathcal{C}^{\infty}_{G}$, respectively $\mathcal{F}_{\mathbb{C},G} := \mathcal{O}_G$. A morphism of supermanifolds is denoted by symbols of the form $\mathbf{\Psi} = (\psi, \psi^{\#})$ with underlying morphism of manifolds ψ and pull-back of superfunctions $\psi^{\#}$. A supermanifold with fixed splitting (M, E) is a manifold M together with a vector bundle $E \to M$, inducing via the full exterior product ΛE a sheaf $\mathcal{A}_E$ of superalgebras on M. This sheaf defines the structure of a globally split supermanifold $(M, \mathcal{A}_E)$. Due to the theorem of Batchelor (see [Bat80]) all real supermanifolds can be (non-canonically) generated in this way. A morphism of supermanifolds with fixed splittings is a morphism of supermanifolds such that the pull-back of superfunctions comes from a morphism of vector bundles.

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1 Definition of a Lie Supergroup by invariant Operators

After defining representation-theoretical Lie supergroups and their morphisms we point out the equivalence to special actions of a Lie supergroup on its own underlying supermanifold. Finally we analyze the relation to the constructions by Kostant and Berezin and restrict our view to planed representation-theoretical Lie supergroups.

1.1 Representation-theoretical Lie Supergroups

Starting from Lie groups we motivate a definition of Lie supergroups. Let G be a Lie group of dimension n over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the sheaf of smooth, respectively holomorphic, functions $\mathcal{F}_{\mathbb{R},G}$, respectively $\mathcal{F}_{\mathbb{C},G}$. The group structure defines left-, respectively right-invariant derivations yielding a frame for the tangent bundle. This frame can be identified with a basis of the tangent space T_eG, respectively $T^{(1,0)}_{\mathbb{C},e}G$ at the neutral element $e \in G$, the Lie algebra $\mathfrak{g}_0$ of G. A vector $X_0 \in \mathfrak{g}_0$ is represented on a function by the left-invariant derivation

$$(X_0.f)(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g \cdot \exp(tX_0)) \tag{1}$$

for $g \in G$. Note that the right-invariant derivations are automatically associated to left-invariant derivations via $S \circ \chi_L \circ S$ for a left-invariant derivation χ_L and

$$S : \mathcal{F}_{\mathbb{K},G} \longrightarrow \mathcal{F}_{\mathbb{K},G}, \quad f \longmapsto \big(g \mapsto S(f)(g) := f(g^{-1})\big) \ .$$

Left- and right-invariant derivations commute.

Generalizing this notion we fix a suitable definition of a Lie supergroup associated to a Harish-Chandra superpair $(G, \mathfrak{g})$. Recall that this is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a Lie group G with $Lie(G) = \mathfrak{g}_0$ such that the adjoint action $Ad : G \to GL(\mathfrak{g})$ integrating the adjoint representation exists. The foundation of a Lie supergroup is a supermanifold $(G, \mathcal{A})$ with underlying manifold G and sheaf of superfunctions $\mathcal{A}$. Since $\mathfrak{g}$ should again take the place of the tangent space at the neutral element $e \in G$ the odd dimension of $(G, \mathcal{A})$ is fixed to be $dim(\mathfrak{g}_1)$.

We also demand parallel to the classical case that for a basis $\{X_i\} \subset \mathfrak{g}$ the X_i should uniquely define nowhere vanishing global superderivations in independent directions on the sheaf of superfunctions replacing the notion of left-, respectively right-invariant vector fields. This yields two global frames for the tangent bundle, which motivates the definition of a Lie supergroup by the supermanifold $(G, \mathcal{A})$ with fixed splitting where $\mathcal{A}$ is the sheaf of sections in the trivial bundle $pr_G : G \times \Lambda\mathfrak{g}_1^* \to G$. Weaving the classical image into the definition of a graded object we additionally postulate that even derivations act on numerical functions classically. More exactly:

Definition 1. *Associate to a Harish-Chandra superpair the supermanifold with fixed splitting $(G, \mathcal{F}_{\mathbb{K},G} \otimes \Lambda\mathfrak{g}_1^*)$ and projection onto numerical functions*

$$pr_{\mathcal{F}_{\mathbb{K},G}} : \mathcal{F}_{\mathbb{K},G} \otimes \Lambda\mathfrak{g}_1^* \to \mathcal{F}_{\mathbb{K},G}, \quad \sum f_I \theta^I \mapsto f_0 \quad \textit{for a basis } \{\theta_i\} \textit{ of } \mathfrak{g}_1^*$$

together with a representation ρ of $\mathfrak{g}$ by superderivations on the sheaf $\mathcal{F}_{\mathbb{K},G} \otimes \Lambda\mathfrak{g}_1^*$ such that

- *(nativeness) even elements $X_0 \in \mathfrak{g}_0$ are represented on functions $f \in \mathcal{F}_{\mathbb{K},G} \otimes \mathbf{1}$ by equation (1) and*
- *(definiteness) the sheaf $U \mapsto \{f \in \mathcal{F}_{\mathbb{K},G}(U) \otimes \Lambda\mathfrak{g}_1^* \mid X.f \equiv 0 \ \ \forall\ X \in \mathfrak{g}\}$ is isomorphic to the sheaf of $\mathbb{K}$-valued constant functions on G.*

Such an object $(G, \mathfrak{g}, \rho)$ is called a left-representation-theoretical Lie supergroup, abbreviated LRT Lie supergroup. Via the classical representation by right-invariant vector fields, an RRT Lie supergroup can be defined analogously. A representation-theoretical Lie supergroup or RT Lie supergroup is an LRT Lie supergroup $(G, \mathfrak{g}, \rho_L)$ which carries at the same time the structure of an RRT Lie supergroup $(G, \mathfrak{g}, \rho_R)$ according to the same splitting with the additional condition

- *(bilaterality)* $\rho_R(X) \circ \rho_L(Y) = (-1)^{|X||Y|}\rho_L(Y) \circ \rho_R(X)$ *for all homogeneous* $X, Y \in \mathfrak{g}$.

In the classical setting a morphism of Lie groups ψ induces a unique morphism of invariant vector fields ψ_*. Analogously a morphism of RT Lie supergroups shall be determined by a morphism of Lie superalgebras. To ensure this correspondence the preservation of the splitting is necessary. This motivates:

Definition 2. *A morphism of LRT Lie supergroups* $\mathbf{\Psi} : (G_a, \mathfrak{g}_a, \rho_a) \to (G_b, \mathfrak{g}_b, \rho_b)$ *is a morphism of supermanifolds with fixed splitting*

$$(\psi, \psi^{\#}) : (G_a, \mathcal{F}_{\mathbb{K},G_a} \otimes \Lambda\mathfrak{g}^*_{a,1}) \to (G_b, \mathcal{F}_{\mathbb{K},G_b} \otimes \Lambda\mathfrak{g}^*_{b,1})$$

together with an (even) morphism of Lie superalgebras $\psi^\diamond : \mathfrak{g}_a \to \mathfrak{g}_b$ *such that*

- *(compatibility)* $\psi : G_a \to G_b$ *is a morphism of Lie groups and*

$$\psi^{\#}(\sum f_I\theta^I) = \sum \psi^*(f_I)\psi^{\diamond *}(\theta^I) \quad \textit{for a basis } \{\theta_i\} \textit{ of } \mathfrak{g}^*_{b,1} \quad \textit{and}$$

- *(equivariance)* $\rho_a(X) \circ \psi^{\#} = \psi^{\#} \circ \rho_b(\psi^\diamond(X)) : \mathcal{F}_{\mathbb{K},G_b} \otimes \Lambda\mathfrak{g}^*_{b,1} \to \psi_* \left(\mathcal{F}_{\mathbb{K},G_a} \otimes \Lambda\mathfrak{g}^*_{a,1}\right)$.

Analogously, morphisms of RRT and RT Lie supergroups are defined, the equivariance condition appears twice in the later case. A morphism of LRT, RRT, respectively RT, Lie supergroups is called an isomorphism if $(\psi, \psi^{\#})$ *is an isomorphism of supermanifolds. Then* $\psi^\diamond$ *is an isomorphism of Lie superalgebras.*

1.2 RT Lie Supergroups and Actions of Lie Supergroups

Before constructing LRT Lie supergroups we briefly describe the objects defined above by actions of Lie supergroups. For this we first recall the definition of an action of a Lie supergroup from Kostant's point of view. Secondly we give the connection between RT Lie supergroups and such actions.

Let $\mathcal{G}$ be a Lie supergroup with Lie superalgebra $\mathfrak{g}$ and morphisms

$$\mathbf{m}_\mathcal{G} = (m_\mathcal{G}, m^{\#}_\mathcal{G}), \qquad \mathbf{s}_\mathcal{G} = (s_\mathcal{G}, s^{\#}_\mathcal{G}) \quad \text{and} \quad \mathbf{u}_\mathcal{G} = (u_\mathcal{G}, u^{\#}_\mathcal{G})$$

for multiplication, inverse and identity and let $\mathcal{M}$ be a supermanifold. A **Lie supergroup (right-)action** of $\mathcal{G}$ on $\mathcal{M}$ is a morphism of supermanifolds of the form $\mathbf{\Psi} : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ with the properties

$$\mathbf{\Psi} \circ (\mathbf{Id}_\mathcal{M} \otimes \mathbf{m}_\mathcal{G}) = \mathbf{\Psi} \circ (\mathbf{\Psi} \otimes \mathbf{Id}_\mathcal{G}) \ \text{ and } \ \mathbf{\Psi} \circ (\mathbf{Id}_\mathcal{M} \otimes \mathbf{u}_\mathcal{G}) = \mathbf{Id}_\mathcal{M} \ ,$$

where we make the identification $\mathcal{M} \times pt \cong \mathcal{M}$. The action $\mathbf{\Psi}$ is called **transitive** if the underlying action is transitive and the sheaf of invariant superfunctions on the supermanifold is isomorphic to the sheaf of constant numerical functions on it. It is called **free** if the underlying action is free and the induced local representation of $\mathfrak{g}$ on $\mathcal{A}_{\mathcal{M}}$ is injective.

We briefly recall Kostant's formalism (see [Kost77, §2.11, 3.5] or [Kal10]) for supermanifolds and Lie supergroups.[1] For a supermanifold $\mathcal{M} = (M, \mathcal{A}_{\mathcal{M}})$ define the sheaf

$$\mathcal{A}^*_{\mathcal{M}}(U) := \{\varphi \in Hom_{\mathbb{R}-vect}(\mathcal{A}_{\mathcal{M}}(U), \mathbb{R}) | \exists \text{ ideal } I \subset \mathcal{A}_{\mathcal{M}}(U) \text{ with } codim(I) < \infty, \\ I \subset ker(\varphi)\}$$

for open $U \subset M$. Taking the direct sum over the stalks of this sheaf,

$$\mathbf{A}^*_{\mathcal{M}} = \bigoplus\nolimits_{p \in M} \mathcal{A}^*_{\mathcal{M},p} ,$$

we obtain the super co-commutative supercoalgebra of local differential operators. The supercoalgebra structure is induced by the superalgebra structure of superfunctions. For a Lie supergroup $\mathcal{G}$ the group morphisms additionally induce the structure of a superalgebra on $\mathbf{A}^*_{\mathcal{G}}$. Using the notion of the smash product (see e.g. [Swe69, chap.VII]) we have $\mathbf{A}^*_{\mathcal{G}} \cong \mathbb{K}(G) \# E(\mathfrak{g})$.

Following the presentation of Kostant (see [Kost77, §3.9]) a Lie supergroup action of the form $\mathbf{\Psi} : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ is uniquely described by the structure of a right-$\mathbf{A}^*_{\mathcal{G}}$-supermodule on the super co-commutative supercoalgebra $\mathbf{A}^*_{\mathcal{M}}(U)$ such that the multiplication map

$$\mathbf{A}^*_{\mathcal{M}} \otimes \mathbf{A}^*_{\mathcal{G}} \to \mathbf{A}^*_{\mathcal{M}}, \quad w \otimes u \mapsto w \cdot u$$

is a morphism of super co-commutative supercoalgebras. These right-$\mathbf{A}^*_{\mathcal{G}}$-supermodule structures on $\mathbf{A}^*_{\mathcal{M}}$ are uniquely determined by the representation $\pi_{\mathbf{\Psi}} : \mathbf{A}^*_{\mathcal{G}} \to End(\mathcal{A}_{\mathcal{M}})$ obtained from $\mathbf{\Psi}$. Regarding the action on the supermanifold $\mathcal{M} = \mathcal{G}$, we obtain the following result.

Proposition 1. *The structures of LRT Lie supergroups $(G, \mathfrak{g}, \rho)$ for a Harish-Chandra superpair $(G, \mathfrak{g})$ correspond bijectively up to isomorphisms to the transitive and free right-actions $\mathbf{\Psi} = (\psi, \psi^{\#}) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ with $\psi = m_G$ of the associated Lie supergroup $\mathcal{G}$ onto the supermanifold $\mathcal{G}$ with splitting fixed via the construction by Koszul.*[2] *An analogous statement holds for RRT Lie supergroups and left-actions. For RT Lie supergroups left- and right-actions $\mathbf{\Psi}_L$, respectively $\mathbf{\Psi}_R$, are obtained with*

$$\mathbf{\Psi}_R \circ (\mathbf{\Psi}_L \otimes Id_{\mathcal{G}}) = \mathbf{\Psi}_L \circ (Id_{\mathcal{G}} \otimes \mathbf{\Psi}_R) \ . \tag{2}$$

Proof. We fix the splitting $\mathcal{A}_{\mathcal{G}} := \mathcal{F}_{\mathbb{K},G} \otimes \Lambda \mathfrak{g}_1^*$ on $\mathcal{G}$ by Koszul's construction (see [Kosz82]). The representation ρ given by the LRT Lie supergroup structure can be continued to a representation $\rho : E(\mathfrak{g}) \to End(\mathcal{A}_{\mathcal{G}})$ inducing

$$\pi_{\Phi} : \mathbf{A}^*_{\mathcal{G}} \longrightarrow End(\mathcal{A}_{\mathcal{G}}), \quad g\#X \longmapsto \big(\Phi \mapsto (\rho(X) \circ \Phi)(\bullet \cdot g\#\mathbf{1})\big) \ .$$

[1] We allow $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The complex case is handled in [Kal10].

[2] Note that we do not request any compatibility of action and splitting. The splitting is only fixed to guarantee uniqueness up to isomorphisms.

Here we used the identification $\mathbf{A}_{\mathcal{G}}^* \cong \mathbb{K}(G)\#E(\mathfrak{g})$ and

$$\mathcal{A}_{\mathcal{G}}(U) = \Big\{ \Phi \in Hom_{\mathbb{K}-vect}(\mathbb{K}(U)\#E(\mathfrak{g}),\mathbb{K}) \mid \Big(U \to \mathbb{K},\ g \mapsto \Phi(g\#Z)\Big) \in \mathcal{C}_G^{\infty}(U)\ \forall Z \in E(\mathfrak{g})\Big\} \tag{3}$$

in [Kost77, §3.7]. The underlying map of the associated supermodule structure is the map $m_{\mathcal{G}}$. Freedom and transitivity follow from the definition of LRT Lie supergroups. For the opposite direction, starting with a supermodule structure π_Φ with underlying multiplication we obtain by restriction the representation ρ for an LRT Lie supergroup. Since the splitting is fixed, the correspondence is unique up to isomorphism. This yields the result for LRT and RRT Lie supergroups while (2) for RT Lie supergroups follows from bilaterality. ☐

Hence LRT, respectively RRT, Lie supergroup structures for a Harish-Chandra superpair with associated Lie supergroup $\mathcal{G}$ correspond to the $\mathcal{G}$-homogeneous spaces with fixed splitting and transitive and free action. RT Lie supergroups define $\mathcal{G}$-homogeneous spaces with fixed splitting and both actions supercommuting in the sense of (2). As mentioned in the introduction, omitting the fixed splitting in the definition of RT Lie supergroups would lead to a characterization of the isomorphy classes of $\mathcal{G}$-homogeneous spaces with transitive and free action. This can not be approached within the bounds of this article.

From now on we will restrict to LRT Lie supergroup structures. RRT Lie supergroup structures can be dealt with in an analogous way. RT Lie supergroup structures will also be discussed in the context of the constructions by Kostant and Koszul, respectively Berezin. In particular it should be noted that the final result in Theorem 3 holds for RT Lie supergroup structures.

1.3 Relation to the Construction by Kostant and Koszul

The construction of Lie supergroups by Kostant and Koszul in [Kost77, §3.7] and [Kosz82], respectively the construction by Berezin in [Ber87, chap.II.2.2], presented also in [Kal10], respectively [HK10], can be used to define LRT Lie supergroup structures associated to Harish-Chandra superpairs. The representation of the Lie superalgebra on superfunctions will be in both cases the representation by left-invariant operators. Let us now analyze this in detail.

Let $(G,\mathfrak{g})$ be a Harish-Chandra superpair over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Using the construction by Koszul we associate the structure of an LRT Lie supergroup to $(G,\mathfrak{g})$ via the isomorphism of sheaves $\mathcal{A}_{\mathcal{G}} \to \mathcal{F}_{\mathbb{K},G}\otimes\Lambda\mathfrak{g}_1^*$ in [Kosz82, §1]. Fixing a basis $\{X_i\}$ of $\mathfrak{g}_0$ and $\{Y_j\}$ of $\mathfrak{g}_1$, the preimage in Kostant's formalism of a function $\alpha \in \mathbf{1}\otimes\mathfrak{g}_1^*$ is described as follows:

$$\Phi_\alpha : \mathbb{K}(G)\#E(\mathfrak{g}) \to \mathbb{K}, \qquad \Phi_\alpha(g\#X^I\gamma(Y^J)) = \begin{cases} \alpha(Y^J) & \text{if } I=0, |J|=1 \\ 0 & \text{otherwise} \end{cases}$$

where $I \in \mathbb{N}^{dim(\mathfrak{g}_0)}$ and $J \in \mathbb{Z}_2^{dim(\mathfrak{g}_i)}$ denote multi-indexes. Applying a left-invariant superderivation $X \in \mathfrak{g}_0$ to Φ_α we obtain

$$(X.\Phi_\alpha)(g\#X^I\gamma(Y^J)) = \Phi_\alpha((g\#X^I\gamma(Y^J))\cdot(e\#X)) = \Phi_\alpha(g\#(X^I\gamma(Y^J)X))$$

and use

$$\gamma(Y_{j_1}\wedge\ldots\wedge Y_{j_k})X = X\gamma(Y_{j_1}\wedge\ldots\wedge Y_{j_k}) + \gamma([Y_{j_1},X]\wedge Y_{j_2}\wedge\ldots\wedge Y_{j_k}) + \cdots + \gamma(Y_{j_1}\wedge\ldots\wedge Y_{j_{k-1}}\wedge [Y_{j_k},X])\ .$$

This yields $(X.\Phi_\alpha)(g\#X^I\gamma(Y^J)) = 0$ for $|I|\neq 0$ or $|J|\neq 1$. Therefore, $(X.\Phi_\alpha) = \Phi_{\alpha([\cdot,X])}$.

Definition 3. *A planed LRT Lie supergroup is an LRT Lie supergroup* $(G,\mathfrak{g},\rho)$ *satisfying* $X_0.(\alpha) = \alpha([\cdot,X_0])$ *for all* $X_0\in\mathfrak{g}_0$ *and* $\alpha\in\mathbf{1}\otimes\mathfrak{g}_1^*\subset\mathcal{F}_{\mathbb{K},G}\otimes\Lambda\mathfrak{g}_1^*$. *A planed RT Lie supergroup is an RT Lie supergroup with planed LRT Lie supergroup structure.*

Note that, due to compatibility and equivariance, the property "planed" is preserved by morphisms of LRT Lie supergroups. The LRT Lie supergroups analyzed here carry the natural structure of an RT Lie supergroup by the representation by right-invariant superderivations given in Kostant's construction. Regarding isomorphisms of LRT, respectively RT Lie supergroups, we have the desired functoriality.

Proposition 2. *Isomorphic Harish-Chandra superpairs induce by the construction by Kostant and Koszul isomorphic planed LRT, respectively planed RT Lie supergroups.*

Proof. An isomorphism of Harish-Chandra superpairs is fixed by an isomorphism L of the underlying Lie groups and a compatible isomorphism ℓ of the Lie superalgebras yielding an isomorphism $L\#\ell$ of Lie-Hopf superalgebras such that the induced morphism of supermanifolds by Koszul's construction satisfies the equivariance condition in Definition 2. Denote the associated isomorphism of Lie supergroups by $(\psi,\psi^\#)$. Since $\psi^\#(\Phi) = \Phi\circ(L\#\ell)$ in the notion of (3), compatibility follows from Koszul's construction. This defines an isomorphism of planed LRT, respectively RT Lie supergroups. ☐

1.4 Relation to the Construction by Berezin

In Berezin's construction of a Lie supergroup $\mathcal{G} = (G,\mathcal{A}_\mathcal{G})$ with Lie superalgebra $\mathfrak{g}$ the sheaf of holomorphic, respectively real-analytic, superfunctions $\mathcal{A}_\mathcal{G}$ is embedded into $\mathcal{F}_{\mathbb{K},\tilde{G}}\otimes\Lambda\mathfrak{g}_1^*$ by Grassmann analytical continuation (see [Ber87] or [HK10]). Here $\tilde{G}$ denotes the Lie group associated to the Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g}_0\otimes(\Lambda\mathfrak{g}_1^*)_0\oplus\mathfrak{g}_1\otimes(\Lambda\mathfrak{g}_1^*)_1$ which contains G as a Lie subgroup. By this procedure a superfunction $\mu: G\to\Lambda\mathfrak{g}_1^*$ with constant value in $\mathfrak{g}_1^*$ is continued to

$$\tilde{\mu}(\exp(\sum a_i\xi_i)\exp(\sum\sigma_j\Xi_j)) = \sum\sigma_j\mu(\Xi_j)$$

for a basis $\{\xi_i\}$ of $\mathfrak{g}_0$ and $\{\Xi_j\}$ of $\mathfrak{g}_1$ and coefficients $a_i\in(\Lambda\mathfrak{g}_1^*)_0$ and $\sigma_i\in(\Lambda\mathfrak{g}_1^*)_1$.

Deriving $\tilde{\mu}$ by the left-invariant superderivation defined by $Y\in\mathfrak{g}_0$ we obtain

$$\begin{aligned}
&(Y.\tilde{\mu})(\exp(\sum a_i\xi_i)\exp(\sum\sigma_j\Xi_j)) = \frac{d}{dt}\Big|_{t=0}\tilde{\mu}(\exp(\sum a_i\xi_i)\exp(\sum\sigma_j\Xi_j)\exp(tY))\\
&= \frac{d}{dt}\Big|_{t=0}\tilde{\mu}(\exp(\sum a_i\xi_i)\exp(tY)\exp(-tY)\exp(\sum\sigma_j\Xi_j)\exp(tY))\\
&\overset{(*_1)}{=} \frac{d}{dt}\Big|_{t=0}\tilde{\mu}(\exp(-tY)\exp(\sum\sigma_j\Xi_j)\exp(tY)) \overset{(*_2)}{=} \frac{d}{dt}\Big|_{t=0}\sum\sigma_j\mu(\exp(-tY)\Xi_j\exp(tY))\\
&= \sum\sigma_j\mu([\Xi_j,Y]) = \widetilde{\mu\circ[\cdot,Y]}(\exp(\sum a_i\xi_i)\exp(\sum\sigma_j\Xi_j))\ ,
\end{aligned}$$

where we have used in $(*_1)$ that $\exp(-tY)\exp(\sum \sigma_j \Xi_j)\exp(tY)$ is of the form $\exp(\sum \sigma_j' \Xi_j)$ and $\exp(\sum a_i \xi_i)\exp(tY)$ is again of the form $\exp(\sum a_i' \xi_i)$. In $(*_2)$ we have used the equality for matrix representations $V^{-1}\exp(W)V = \exp(V^{-1}WV)$.

Hence, the construction by Berezin associates to a Harish-Chandra superpair $(G, \mathfrak{g})$ a planed LRT Lie supergroup $(G, \mathfrak{g}, \rho)$, where $\rho : \mathfrak{g} \to Der(\mathcal{O}_G \otimes \Lambda \mathfrak{g}_1^*)$ is given by the representation of $\mathfrak{g}$ on Grassmann analytically continued functions. An analogous version of Proposition 2 holds for Berezin's construction by arguments which are similar to those used in that proposition. Recall that the construction by Berezin also yields a representation by right-invariant superderivations inducing structures of RT Lie supergroups.

Proposition 3. *Isomorphic Harish-Chandra superpairs induce by the construction of Berezin isomorphic planed LRT, respectively planed RT, Lie supergroups.*

Proof. Note that equivariance follows immediately from the definition of a morphism of Harish-Chandra superpairs. The isomorphism $\ell : \mathfrak{g}_a \to \mathfrak{g}_b$ in the proof of Proposition 2 can be continued Grassmann-linearly to an isomorphism $\widetilde{\ell} : \widetilde{\mathfrak{g}}_a \to \widetilde{\mathfrak{g}}_b$ inducing a continuation of $L : G_a \to G_b$ to $\widetilde{L}_0 : \widetilde{G}_{a,0} \to \widetilde{G}_{b,0}$. Altogether this yields an isomorphism of Lie groups

$$\widetilde{L} : \widetilde{G}_a \longrightarrow \widetilde{G}_b, \quad \tilde{g}_{a,0} \cdot \exp(\Xi) \longmapsto \widetilde{L}_0(\tilde{g}_{a,0}) \cdot \exp(\widetilde{\ell}(\Xi))$$

for $\tilde{g}_{a,0} \in \widetilde{G}_{a,0}$ and $\Xi \in \widetilde{\mathfrak{g}}_{a,1}$. Now the pull-back of a superfunction $f \otimes \alpha \in \mathcal{F}_{\mathbb{K},G_b} \otimes \Lambda \mathfrak{g}_{b,1}^*$ by the induced isomorphism of Lie supergroups $(\psi, \psi^\#) : \mathcal{G}_a \to \mathcal{G}_b$ is given by the restriction of $(\hat{f} \circ \widetilde{L}_0) \otimes (\alpha \circ \tilde{\ell})$ to G_a. Hence compatibility is satisfied. □

An interesting question is whether the constructions by Kostant and Koszul, respectively Berezin, yield isomorphic LRT, respectively RT Lie supergroups. We will discuss this in the case of the example $\mathfrak{gl}_{\mathbb{K}}(1/1)$ in the following section and prove that they are indeed not isomorphic.

2 Planed LRT Lie Supergroup Structures for $\mathfrak{gl}_{\mathbb{K}}(1/1)$

We classify all planed LRT Lie supergroup structures for the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}_{\mathbb{K}}(1/1)$ with underlying Lie group $G = (\mathbb{K}^\times)^2$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and compare the constructions by Kostant and Berezin in this example. We conclude with a remark concerning the uniqueness of RT Lie supergroup structures for Harish-Chandra superpairs of higher dimension.

2.1 Obstructions for planed LRT Lie Supergroup Structures

Here the Lie superalgebra $\mathfrak{gl}_{\mathbb{K}}(1/1)$ is introduced and a suitable set of generating functions for the superalgebra of superfunctions is determined.

2.1.1 The Lie Superalgebra $\mathfrak{gl}_{\mathbb{K}}(1/1)$

The Lie superalgebra $\mathfrak{gl}_{\mathbb{K}}(1/1)$ can be represented by 2×2-matrices with entries in $\mathbb{K}$. As a basis of the even part $\mathfrak{g}_0$ we choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and for the odd part $\mathfrak{g}_1$

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here $\mathfrak{g}_0$ is the Cartan subalgebra of $\mathfrak{gl}_{\mathbb{K}}(1/1)$ and the basis of $\mathfrak{g}_1$ determines the root spaces. The superbracket is given by $[X, Y] = XY - (-1)^{|X||Y|}YX$ for homogeneous elements. Of course $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$ and otherwise

$$\begin{aligned} &[A, C] = -[B, C] = C \quad \text{and} \quad -[A, D] = [B, D] = D, \\ &[C, D] = A + B \quad \text{and} \quad [C, C] = [D, D] = 0. \end{aligned} \tag{4}$$

A Lie group associated to $\mathfrak{g}_0$ is $G = (\mathbb{K}^\times)^2$ where we choose the standard coordinates (z, w) on $\mathbb{K}^2$. The following identities are required by the definition a planed LRT Lie supergroup.

$$\begin{aligned} &-A.C^* = B.C^* = C^*, \qquad A.D^* = -B.D^* = D^*, \\ &(A.f)(z, w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\begin{smallmatrix} e^t \cdot z & 0 \\ 0 & w \end{smallmatrix}\right) = \frac{\partial f}{\partial z}(z, w) \cdot z \\ &(B.f)(z, w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\begin{smallmatrix} z & 0 \\ 0 & e^t \cdot w \end{smallmatrix}\right) = \frac{\partial f}{\partial w}(z, w) \cdot w \end{aligned} \tag{5}$$

for $f \in \mathcal{F}_{\mathbb{K},G} \cong \mathcal{F}_{\mathbb{K},G} \otimes \mathbf{1}$. From (5) it follows that

$$A.(C^* \wedge D^*) = B.(C^* \wedge D^*) = 0 \; .$$

For further calculations it is very convenient to chose an appropriate set of generators for the algebra $\mathcal{F}_{\mathbb{K},G}$. For this We fix

$$f_{n,m} \in \mathcal{F}_{\mathbb{K},G}, \qquad f_{n,m}(z, w) = z^n w^m \text{ for } n, m \in \mathbb{Z}$$

and obtain

$$A.f_{n,m} = n \cdot f_{n,m} \quad \text{and} \quad B.f_{n,m} = m \cdot f_{n,m}. \tag{6}$$

2.1.2 The 16 structural Constants

Using (4) and (6) we compute

$$\begin{aligned} C.f_{n,m} &= [A, C].f_{n,m} = (AC - CA).f_{n,m} = A.(C.f_{n,m}) - nC.f_{n,m} \\ &\Rightarrow A.(C.f_{n,m}) = (n+1) \cdot C.f_{n,m} \text{ and} \\ C.f_{n,m} &= -[B, C].f_{n,m} = -(BC - CB).f_{n,m} = -B.(C.f_{n,m}) + mC.f_{n,m} \\ &\Rightarrow B.(C.f_{n,m}) = (m-1) \cdot C.f_{n,m} \; . \end{aligned}$$

Thus $C.f_{n,m}$ lies in the eigenspace of A of the eigenvalue $n+1$ and of B for the eigenvalue $m-1$. Since $C.f_{n,m}$ is of the form $g_1 C^* + g_2 D^*$ for $g_1, g_2 \in \mathcal{F}_{\mathbb{K},G}$, it follows that g_1 is a scalar multiple of $f_{n+2,m-2}$ and g_2 of $f_{n,m}$.

A similar calculation yields that $D.f_{n,m}$ is a linear combination of $f_{n,m}C^*$ and $f_{n-2,m+2}D^*$. For eight scalar constants $c^z_{C^*}, c^w_{C^*}, c^z_{D^*}, c^w_{D^*}, d^z_{C^*}, d^w_{C^*}, d^z_{D^*}$ and $d^w_{D^*}$ we note

$$C.f_{1,0} = c^z_{C^*} f_{3,-2}C^* + c^z_{D^*} f_{1,0}D^* \quad \text{and} \quad C.f_{0,1} = c^w_{C^*} f_{2,-1}C^* + c^w_{D^*} f_{0,1}D^* \ ,$$
$$D.f_{1,0} = d^z_{C^*} f_{1,0}C^* + d^z_{D^*} f_{-1,2}D^* \quad \text{and} \quad D.f_{0,1} = d^w_{C^*} f_{0,1}C^* + d^w_{D^*} f_{-2,3}D^* \ .$$

All other odd derivatives of functions in $\mathcal{F}_{\mathbb{K},G}$ can be obtained using the properties of a superderivation and the fact $f_{n,m} = f^n_{1,0} \cdot f^m_{0,1}$. More explicitly

$$\begin{aligned} C.f_{n,m} &= n(C.f_{1,0})f_{n-1,m} + m(C.f_{0,1})f_{n,m-1} \\ &= (nc^z_{C^*} + mc^w_{C^*})f_{n+2,m-2}C^* + (nc^z_{D^*} + mc^w_{D^*})f_{n,m}D^* \ , \\ D.f_{n,m} &= n(D.f_{1,0})f_{n-1,m} + m(D.f_{0,1})f_{n,m-1} \\ &= (nd^z_{C^*} + md^w_{C^*})f_{n,m}C^* + (nd^z_{D^*} + md^w_{D^*})f_{n-2,m+2}D^* \ . \end{aligned}$$

Now using (4) and (5) we obtain

$$\begin{aligned} C.C^* &= [A,C].C^* = (AC - CA).C^* = A.(C.C^*) + C.C^*, \\ C.C^* &= -[B,C].C^* = -(BC - CB).C^* = -B.(C.C^*) + C.C^*, \\ C.D^* &= [A,C].D^* = (AC - CA).D^* = A.(C.D^*) - C.D^* \qquad \text{and} \\ C.D^* &= -[B,C].D^* = -(BC - CB).D^* = -B.(C.D^*) - C.D^* \end{aligned}$$

leading to

$$A.(C.C^*) = 0, \quad B.(C.C^*) = 0, \quad A.(C.D^*) = 2C.D^* \ \text{ and } \ B.(C.D^*) = -2C.D^* \ .$$

An analogous calculation yields

$$A.(D.C^*) = -2D.C^*, \quad B.(D.C^*) = 2D.C^*, \quad A.(D.D^*) = 0 \ \text{ and } \ B.(D.D^*) = 0 \ .$$

This way we obtain again eight constants $c^{C^*}_1$,$c^{D^*}_1$, $c^{C^*}_\wedge$,$c^{D^*}_\wedge$,$d^{C^*}_1$,$d^{D^*}_1$,$d^{C^*}_\wedge$ and $d^{D^*}_\wedge$ with

$$C.C^* = c^{C^*}_1 + c^{C^*}_\wedge C^* \wedge D^* \quad \text{and} \quad C.D^* = f_{2,-2}(c^{D^*}_1 + c^{D^*}_\wedge C^* \wedge D^*) \ ,$$
$$D.C^* = f_{-2,2}(d^{C^*}_1 + d^{C^*}_\wedge C^* \wedge D^*) \quad \text{and} \quad D.D^* = d^{D^*}_1 + d^{D^*}_\wedge C^* \wedge D^* \ .$$

Note that in addition

$$C.(C^* \wedge D^*) = c^{C^*}_1 D^* - f_{2,-2}c^{D^*}_1 C^* \quad \text{and} \quad D.(C^* \wedge D^*) = f_{-2,2}d^{C^*}_1 D^* - d^{D^*}_1 C^* \ .$$

A given representation of $\mathfrak{g}$ on $\mathcal{F}_{\mathbb{K},G} \otimes \Lambda\mathfrak{g}^*_1$ by superderivations is uniquely determined by these 16 constants, because all superfunctions can be approximated by linear combinations of products of the $f_{1,0}$,$f_{0,1}$,C^* and D^*. Now we discuss which sets of constants are allowed.

2.1.3 The 25 structural Conditions

In terms of the above parameters we now determine the conditions which guarantee that a $\mathfrak{g}$-representation is defined. From $[C,C] = 0$ we obtain

$$0 = [C,C].f_{1,0} = 2C.C.f_{1,0} \ \Leftrightarrow \ \begin{cases} 2c^z_{C^*}c^w_{D^*} - 2c^z_{C^*}c^z_{D^*} + c^z_{C^*}c^{C^*}_\wedge + c^z_{D^*}c^{D^*}_\wedge = 0 & (i) \\ c^z_{C^*}c^{C^*}_1 + c^z_{D^*}c^{D^*}_1 = 0 & (ii) \end{cases}$$
$$0 = [C,C].f_{0,1} = 2C.C.f_{0,1} \ \Leftrightarrow \ \begin{cases} 2c^w_{C^*}c^w_{D^*} - 2c^w_{C^*}c^z_{D^*} + c^w_{C^*}c^{C^*}_\wedge + c^w_{D^*}c^{D^*}_\wedge = 0 & (iii) \\ c^w_{C^*}c^{C^*}_1 + c^w_{D^*}c^{D^*}_1 = 0 & (iv) \end{cases}$$

Furthermore,

$$0=[C,C].C^*=2C.C.C^* \;\Leftrightarrow\; \begin{cases} c_\wedge^{C^*}c_1^{C^*}=0 & (v)\\ c_\wedge^{C^*}c_1^{D^*}=0 & (vi)\end{cases}$$

$$0=[C,C].D^*=2C.C.D^* \;\Leftrightarrow\; \begin{cases} 2c^z_{C^*}c_1^{D^*}-2c^w_{C^*}c_1^{D^*}-c_\wedge^{D^*}c_1^{D^*}=0 & (vii)\\ 2c^z_{D^*}c_1^{D^*}-2c^w_{D^*}c_1^{D^*}+c_\wedge^{D^*}c_1^{C^*}=0 & (viii)\end{cases}$$

In addition, from $[D,D]=0$ we obtain

$$0=[D,D].f_{1,0}=2D.D.f_{1,0} \;\Leftrightarrow\; \begin{cases} 2d^z_{D^*}d^w_{C^*}-2d^z_{D^*}d^z_{C^*}+d^z_{C^*}d_\wedge^{C^*}+d^z_{D^*}d_\wedge^{D^*}=0 & (ix)\\ d^z_{C^*}d_1^{C^*}+d^z_{D^*}d_1^{D^*}=0 & (x)\end{cases}$$

$$0=[D,D].f_{0,1}=2D.D.f_{0,1} \;\Leftrightarrow\; \begin{cases} 2d^w_{D^*}d^w_{C^*}-2d^w_{D^*}d^z_{C^*}+d^w_{C^*}d_\wedge^{C^*}+d^w_{D^*}d_\wedge^{D^*}=0 & (xi)\\ d^w_{C^*}d_1^{C^*}+d^w_{D^*}d_1^{D^*}=0 & (xii)\end{cases}$$

and

$$0=[D,D].D^*=2D.D.C^* \;\Leftrightarrow\; \begin{cases} 2d^w_{C^*}d_1^{C^*}-2d^z_{C^*}d_1^{C^*}-d_\wedge^{C^*}d_1^{D^*}=0 & (xiii)\\ 2d^w_{D^*}d_1^{C^*}-2d^z_{D^*}d_1^{C^*}+d_\wedge^{C^*}d_1^{C^*}=0 & (xiv)\end{cases}$$

$$0=[D,D].D^*=2D.D.D^* \;\Leftrightarrow\; \begin{cases} d_\wedge^{D^*}d_1^{C^*}=0 & (xv)\\ d_\wedge^{D^*}d_1^{D^*}=0 & (xvi)\end{cases}$$

Finally from $[C,D]=A+B$ applied to $f_{1,0}$ and $f_{0,1}$ it follows that

$$\begin{pmatrix} c_\wedge^{C^*} & c_\wedge^{D^*}+2c^w_{C^*}-4c^z_{C^*} & 0 & 2c^z_{C^*} & 0 & 0 & c^z_{C^*} & c^z_{D^*}\\ 0 & -2c^w_{C^*} & c_\wedge^{C^*} & c_\wedge^{D^*}-2c^z_{C^*}+4c^w_{C^*} & 0 & 0 & c^w_{C^*} & c^w_{D^*}\\ c_1^{C^*} & c_1^{D^*} & 0 & 0 & c^z_{C^*} & c^z_{D^*} & 0 & 0\\ 0 & 0 & c_1^{C^*} & c_1^{D^*} & c^w_{C^*} & c^w_{D^*} & 0 & 0 \end{pmatrix}$$

$$\cdot\begin{pmatrix} d^z_{C^*} & d^z_{D^*} & d^w_{C^*} & d^w_{D^*} & d_1^{C^*} & d_1^{D^*} & d_\wedge^{C^*} & d_\wedge^{D^*}\end{pmatrix}^T=\begin{pmatrix}0\\0\\1\\1\end{pmatrix} \qquad \begin{matrix}(xvii)\\(xviii)\\(xix)\\(xx)\end{matrix}$$

Applied to C^* and D^* we obtain

$$\begin{pmatrix} 2c^w_{C^*}-2c^z_{C^*} & -c_\wedge^{C^*} & -c_1^{D^*} & 0\\ 2c^w_{D^*}-2c^z_{D^*}+c_\wedge^{C^*} & 0 & c_1^{C^*} & 0\end{pmatrix}\cdot\begin{pmatrix} d_1^{C^*} & d_1^{D^*} & d_\wedge^{C^*} & d_\wedge^{D^*}\end{pmatrix}^T=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{matrix}(xxi)\\(xxii)\end{matrix}$$

$$\begin{pmatrix} 0 & 2d^z_{C^*}-2d^w_{C^*}-d_\wedge^{D^*} & 0 & -d_1^{D^*}\\ d_\wedge^{D^*} & 2d^z_{D^*}-2d^w_{D^*} & 0 & d_1^{C^*}\end{pmatrix}\cdot\begin{pmatrix} c_1^{C^*} & c_1^{D^*} & c_\wedge^{C^*} & c_\wedge^{D^*}\end{pmatrix}^T=\begin{pmatrix}0\\0\end{pmatrix} \qquad \begin{matrix}(xxiii)\\(xxiv)\end{matrix}$$

These 24 equations are satisfied if and only if we obtain a representation of $\mathfrak{g}$ by superderivations on superfunctions.

Now we have to find a condition on the constants yielding the definiteness in the definition of LRT Lie supergroups. Therefore we must describe the sheaf $\mathcal{F}_{\mathfrak{g}_0}$ of superfunctions which are in the kernel of all even derivations. We observe from (5) and (6) that

$$\mathcal{F}_{\mathfrak{g}_0}=\left\{a\cdot f_{1,-1}C^*+b\cdot f_{-1,1}D^* \mid a,b\in\mathbb{K}\right\}\,.$$

The definiteness can be reformulated as

$$\begin{pmatrix} C.(f_{1,-1}C^*) & C.(f_{-1,1}D^*) \\ D.(f_{1,-1}C^*) & D.(f_{-1,1}D^*) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \quad a = b = 0$$

which is by direct calculation equivalent to:

$$det \begin{pmatrix} c^w_{D^*} - c^z_{D^*} + c^{C^*}_\wedge & c^w_{C^*} - c^z_{C^*} + c^{D^*}_\wedge \\ d^w_{D^*} - d^z_{D^*} + d^{C^*}_\wedge & d^w_{C^*} - d^z_{C^*} + d^{D^*}_\wedge \end{pmatrix} \neq 0 \quad \text{or} \quad det \begin{pmatrix} c^{C^*}_1 & c^{D^*}_1 \\ d^{C^*}_1 & d^{D^*}_1 \end{pmatrix} \neq 0 \qquad (xxv)$$

2.2 Classification of planed LRT Lie Supergroup Structures

Here we classify all planed LRT Lie supergroup structures for $\mathfrak{gl}_{\mathbb{K}}(1/1)$ up to isomorphy.

2.2.1 The possible Structures

First we determine the sets of allowed parameters. In some cases we will point out that there are several structures satisfying all equations but inequalities (xxv). We call these objects, which are not of further interest here, non-definite planed LRT Lie supergroups.

Denote

$$M_C := \begin{pmatrix} c^z_{C^*} & c^z_{D^*} \\ c^w_{C^*} & c^w_{D^*} \end{pmatrix} , \qquad M_D := \begin{pmatrix} d^z_{C^*} & d^z_{D^*} \\ d^w_{C^*} & d^w_{D^*} \end{pmatrix} \quad \text{and} \quad M_1 := \begin{pmatrix} c^{C^*}_1 & c^{D^*}_1 \\ d^{C^*}_1 & d^{D^*}_1 \end{pmatrix} .$$

From $det(M_C), det(M_D) \neq 0$ we obtain with (ii),(iv),(x) and (xii) $M_1 = 0$ contrary to the equations (xix) and (xx) which also excludes $M_C = M_D = 0$.

Lemma 1. $det(M_C) = det(M_D) = 0$.

Proof. We will find a contradiction to $det(M_C) \neq 0$ and $det(M_D) = 0$ follows analogously. For this note that (ii) and (iv) yield $c^{C^*}_1 = c^{D^*}_1 = 0$ satisfying (v) to $(viii)$. Furthermore, $d^{C^*}_1 = d^{D^*}_1 = 0$ which contradicts (xix) and (xx).

Case 1: Assume $det(M_C) \neq 0$ and $d^{C^*}_1, d^{D^*}_1 \neq 0$ and set $\alpha \in \mathbb{K}^\times$ with $d^{C^*}_1 = \alpha d^{D^*}_1$.
(xv) and (xvi) yield $d^{D^*}_\wedge = 0$ and (xix) and (xx) yield $\alpha c^z_{C^*} + c^z_{D^*} = \alpha c^w_{C^*} + c^w_{D^*} = (d^{D^*}_1)^{-1}$. From (x) and (xii) we have $\alpha d^z_{C^*} = -d^z_{D^*}$ and $\alpha d^w_{C^*} = -d^w_{D^*}$. The conditions (xxi) and $(xxii)$ imply $c^{C^*}_\wedge = 2(c^z_{D^*} - c^w_{D^*})$ and $(xxiii)$ and $(xxiv)$ lead to $c^{D^*}_\wedge = 0$ satisfying (i) and (iii). Now (ix) and (xi) lead to

$$d^z_{D^*}(2(d^z_{D^*} - d^w_{D^*}) - d^{C^*}_\wedge) = 0 \qquad \text{and} \qquad d^w_{D^*}(2(d^z_{D^*} - d^w_{D^*}) - d^{C^*}_\wedge) = 0 .$$

Case 1a: Assume additionally to case 1 $d^z_{D^*} = d^w_{D^*} = 0$.
This includes $d^z_{C^*} = d^w_{C^*} = 0$. Here $(xvii)$ and $(xviii)$ lead to $d^{C^*}_\wedge = 0$ and $(xiii)$ and (xiv) are satisfied. Thus all equations are fullfilled, but not the inequalities (xxv). Thus we obtain equivalence classes of non-definite planed LRT Lie supergroup.

Case 1b: Assume additionally to case 1 that it is not the case $d^z_{D^*} = d^w_{D^*} = 0$.
This includes $d^{C^*}_\wedge = 2(d^z_{D^*} - d^w_{D^*})$. The equations $(xiii)$ and (xiv) are satisfied as well as $(xvii)$ and $(xviii)$. In (xxv) both determinants vanish so we obtain only non-definite planed LRT Lie supergroups

Case 2: Assume $det(M_C) \neq 0$ and $d_1^{C^*} \neq 0$, $d_1^{D^*} = 0$.
We obtain from (x) and (xii) $d^z_{C^*} = d^w_{C^*} = 0$ and from (xv) and $(xxiv)$ $d_\wedge^{D^*} = c_\wedge^{D^*} = 0$. (xix) and (xx) yield the restriction $c^z_{C^*} = c^w_{C^*} = (d_1^{C^*})^{-1}$, (xiv) yields $d_\wedge^{C^*} = 2(d^z_{D^*} - d^w_{D^*})$ and $(xxii)$ yields $c_\wedge^{C^*} = 2(c^z_{D^*} - c^w_{D^*})$. Such constants satisfy all other equations but inequalities (xxv). So we only obtain non-definite planed LRT Lie supergroup structures.

Case 3: Assume $det(M_C) \neq 0$ and $d_1^{C^*} = 0$, $d_1^{D^*} \neq 0$.
With arguments parallel to case 2 we obtain $d^z_{D^*} = d^w_{D^*} = c_\wedge^{C^*} = c_\wedge^{D^*} = d_\wedge^{C^*} = d_\wedge^{D^*} = 0$ and $c^z_{D^*} = c^w_{D^*} = (d_1^{D^*})^{-1}$. Again (xxv) is the only condition which is not satisfied. □

Lemma 2. $c_\wedge^{C^*} = d_\wedge^{D^*} = 0$.

Proof. Assuming $c_\wedge^{C^*} \neq 0$ we find a contradiction. That $d_\wedge^{D^*} = 0$ follows analogously. From $c_\wedge^{C^*} \neq 0$ it follows from equations (v) and (vi) that $c_1^{C^*} = c_1^{D^*} = 0$. Further, (xix) and (xx) imply that $(d_1^{C^*}, d_1^{D^*}) \neq (0,0)$. Then from $(xxiii)$ and $(xxiv)$ we see that $c_\wedge^{D^*} = 0$ and (xv) and (xvi) imply $d_\wedge^{D^*} = 0$.

Case 1: Assume $c_\wedge^{C^*} \neq 2(c^z_{D^*} - c^w_{D^*})$.
Then (i) and (iii) yield $c^z_{C^*} = c^w_{C^*} = 0$ and (xxi) and $(xxii)$ imply $d_1^{C^*} = d_1^{D^*} = 0$, contradicting (xix) and (xx).

Case 2: Assume that $c_\wedge^{C^*} = 2(c^z_{D^*} - c^w_{D^*})$ and $d_1^{C^*} = 0$.
We conclude from (x) and (xii) that $d^z_{D^*} = d^w_{D^*} = 0$ and from $(xiii)$ that $d_\wedge^{C^*} = 0$. Then $(xvii)$ and $(xviii)$ are equivalent to $(c^z_{D^*} - c^w_{D^*})d^z_{C^*} = (c^z_{D^*} - c^w_{D^*})d^w_{C^*} = 0$. In contrast (xxv) requires $d^z_{C^*} \neq d^w_{C^*}$, but $c^z_{D^*} = c^w_{D^*}$ contradicts $c_\wedge^{C^*} \neq 0$.

Case 3: Assume $c_\wedge^{C^*} = 2(c^z_{D^*} - c^w_{D^*})$ and $d_1^{C^*} \neq 0$.
Then (xiv) yields $d_\wedge^{C^*} = 2(d^z_{D^*} - d^w_{D^*})$. Plugging this into $(xvii)$ and $(xviii)$ and substracting both equations we obtain the first determinant in (xxv) which thus has to be zero. □

Lemma 3. $c_\wedge^{D^*} = 2(c^z_{C^*} - c^w_{C^*})$ *and* $d_\wedge^{C^*} = 2(d^z_{D^*} - d^w_{D^*})$.

Proof. We assume that $c_\wedge^{D^*} \neq 2(c^z_{C^*} - c^w_{C^*})$ and derive a contradition. The parallel equation $d_\wedge^{C^*} = 2(d^z_{D^*} - d^w_{D^*})$ follows analogously. From (vii) it follows that $c_1^{D^*} = 0$ and from (ii), (iv) and $(viii)$ we derive $c^z_{C^*}c_1^{C^*} = c^w_{C^*}c_1^{C^*} = c_\wedge^{D^*}c_1^{C^*} = 0$. Thus $c_1^{C^*} \neq 0$ contradicts the assumption. Hence $c_1^{C^*} = 0$. By (xix) and (xx) we have $(d_1^{C^*}, d_1^{D^*}) \neq (0,0)$. Hence, by $(xxiii)$ and $(xxiv)$ we see that $c_\wedge^{D^*} = 0$. Furthermore, (xxi) yields $d_1^{C^*} = 0$, since otherwise $c^z_{C^*} = c^w_{C^*}$, contradicting the assumption. Now (xix) and (xx) yield $d_1^{D^*} \neq 0$ and $c^z_{D^*} = c^w_{D^*}$ while (x) and (xii) yield $d^z_{D^*} = d^w_{D^*} = 0$. Finally (xxv) includes $d_\wedge^{C^*} \neq 0$ which yields together with $(xvii)$ and $(xviii)$ the fact that $c^z_{C^*} = c^w_{C^*} = 0$, contrary to the assumption. □

As a consequence of Lemmas 2 and 3 there remain the 12 parameters in M_C, M_D and M_1 to determine a planed LRT Lie supergroup structure. From equations (i) to (xvi) remain

$$M_C \cdot \begin{pmatrix} c_1^{C^*} \\ c_1^{D^*} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad M_D \cdot \begin{pmatrix} d_1^{C^*} \\ d_1^{D^*} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

and from equations $(xvii)$ to $(xxiv)$ remain

$$\begin{aligned}
&M_D \cdot \begin{pmatrix} c_1^{C^*} \\ c_1^{D^*} \end{pmatrix} + M_C \cdot \begin{pmatrix} d_1^{C^*} \\ d_1^{D^*} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&(c^z_{C^*} - c^w_{C^*})d_1^{C^*} + (d^z_{D^*} - d^w_{D^*})c_1^{D^*} = 0 \\
&(c^z_{D^*} - c^w_{D^*})d_1^{C^*} - (d^z_{D^*} - d^w_{D^*})c_1^{C^*} = 0 \\
&(d^z_{C^*} - d^w_{C^*})c_1^{D^*} - (c^z_{C^*} - c^w_{C^*})d_1^{D^*} = 0 \\
&(d^z_{D^*} - d^w_{D^*})c_1^{D^*} + (c^z_{C^*} - c^w_{C^*})d_1^{C^*} = 0
\end{aligned} \tag{8}$$

while (xxv) becomes

$$det \begin{pmatrix} c^w_{D^*} - c^z_{D^*} & c^w_{C^*} - c^z_{C^*} \\ d^w_{D^*} - d^z_{D^*} & d^w_{C^*} - d^z_{C^*} \end{pmatrix} \neq 0 \quad \text{or} \quad det \begin{pmatrix} c_1^{C^*} & c_1^{D^*} \\ d_1^{C^*} & d_1^{D^*} \end{pmatrix} \neq 0 \ .$$

Since $det(M_C) = det(M_D) = 0$, by Lemma 1 we may introduce new parameters μ_C,μ_D, ν_C, ν_D, c^z, c^w, d^z, d^w, c_1 and d_1 with

$$M_C = \begin{pmatrix} \mu_C c^z & \nu_C c^z \\ \mu_C c^w & \nu_C c^w \end{pmatrix} \quad M_D = \begin{pmatrix} \mu_D d^z & \nu_D d^z \\ \mu_D d^w & \nu_D d^w \end{pmatrix} \quad M_1 = \begin{pmatrix} \nu_C c_1 & -\mu_C c_1 \\ \nu_D d_1 & -\mu_D d_1 \end{pmatrix} \tag{9}$$

satisfying the equations (7) while the equations (8) are transfered to

$$det \left(\begin{smallmatrix} \nu_C & \mu_C \\ \nu_D & \mu_D \end{smallmatrix} \right) \cdot (d^z c_1 - c^z d_1) = 1 \qquad det \left(\begin{smallmatrix} \nu_C & \mu_C \\ \nu_D & \mu_D \end{smallmatrix} \right) \cdot (d^w c_1 - c^w d_1) = 1 \ . \tag{10}$$

Since this implies $det \left(\begin{smallmatrix} \nu_C & \mu_C \\ \nu_D & \mu_D \end{smallmatrix} \right) \neq 0$, the inequalities (xxv) become

$$(c^z - c^w) \cdot (d^z - d^w) \neq 0 \quad \text{or} \quad c_1 \cdot d_1 \neq 0 \ . \tag{11}$$

Note that we gain with the new parameters in (9) one accessory artificial degree of freedom in both M_C and M_D.

This yields a first result for the classification of planed LRT Lie supergroups for $\mathfrak{gl}_{\mathbb{K}}(1/1)$.

Theorem 1. *The possible structures of planed LRT Lie supergroups for $\mathfrak{gl}_{\mathbb{K}}(1/1)$ and $(\mathbb{K}^\times)^2$ are parameterized by a 6-dimensional algebraic subvariety of $\mathbb{K}^{10}$ intersected with the complements of two 9-dimensional algebraic subvarieties. Explicitly this set is parameterized by M_C, M_D and M_1 in (9) satisfying (10) and (11) and it is generically 6-dimensional.*

2.2.2 The Isomorphisms

In order to determine isomorphy classes of planed LRT Lie supergroups we first have to determine the Lie superalgebra automorphism $\psi^\diamond : \mathfrak{gl}_{\mathbb{K}}(1/1) \to \mathfrak{gl}_{\mathbb{K}}(1/1)$. These are given by

$$\psi^{+,\#}_{x,y,u,v,*} : \mathfrak{g} \to \mathfrak{g}, \qquad \begin{cases} A & \mapsto & uA + (u-1)B \\ B & \mapsto & vA + (v+1)B \\ C & \mapsto & xC \\ D & \mapsto & yD \end{cases}$$

$$\psi^{-,\#}_{x,y,u,v,*} : \mathfrak{g} \to \mathfrak{g}, \qquad \begin{cases} A & \mapsto & uA + (u+1)B \\ B & \mapsto & vA + (v-1)B \\ C & \mapsto & xD \\ D & \mapsto & yC \end{cases}$$

for $x, y \in \mathbb{K}^\times$ and $u, v \in \mathbb{K}$ with $x \cdot y = u + v$. Note that we have the additional restriction $u, v \in \mathbb{Z}$ in the case $\mathbb{K} = \mathbb{C}$ in order for $\psi^\diamond|_{\mathfrak{gl}_{\mathbb{K},0}(1/1)}$ to induce a Lie group morphism.

Hence the automorphisms of a planed LRT Lie supergroup structure for $\mathfrak{gl}_{\mathbb{K}}(1/1)$ and $(\mathbb{K}^\times)^2$ depend on 3 free real parameters in the case $\mathbb{K} = \mathbb{R}$ and 1 free complex and two free integer parameters in the case $\mathbb{K} = \mathbb{C}$.

Let us now analyze the isomorphisms of planed LRT Lie supergroups

$$\mathbf{\Psi}^+_{x,y,u,v} = (\psi^+_{x,y,u,v}, \psi^{+,\#}_{x,y,u,v})$$

induced by $\psi^{+,\#}_{x,y,u,v,*}$ in the case $\mathbb{K} = \mathbb{R}$. Here we will denote the images $\psi^{+,\#}_{x,y,u,v,*}(X)$ by $\hat{X}$ and the dual basis to $\{\hat{C}, \hat{D}\} \subset \mathfrak{g}_1$ by $\{\hat{C}^*, \hat{D}^*\}$. Note that $C^* = x\hat{C}^*$ and $D^* = y\hat{D}^*$. Furthermore, a superfunction f with pull-back $\psi^{+,\#}_{x,y,u,v}(f) = f_{n,m}$ will be denoted by $\hat{f}_{n,m}$. Since

$$\begin{pmatrix} u & u-1 \\ v & v+1 \end{pmatrix}^{-1} = \frac{1}{xy} \begin{pmatrix} 1+v & 1-u \\ -v & u \end{pmatrix}$$

we conclude that $\hat{f}_{1,0} = f_{\frac{1+v}{xy}, \frac{-v}{xy}}$, $\hat{f}_{0,1} = f_{\frac{1-u}{xy}, \frac{u}{xy}}$ and

$$\hat{f}_{n,m} = \hat{f}^n_{1,0} \cdot \hat{f}^m_{0,1} = f_{n\frac{1+v}{xy} + m\frac{1-u}{xy}, n\frac{-v}{xy} + m\frac{u}{xy}} = f_{(n+m)\frac{1+v}{xy} - m, (n+m)\frac{-v}{xy} + m} \tag{12}$$

which implies that

$$\begin{aligned}
\hat{C}.\hat{f}_{1,0} &= \frac{x}{y}((1+v)c^z_{C^*} - vc^w_{C^*})\hat{f}_{3,-2}\hat{C}^* + ((1+v)c^z_{D^*} - vc^w_{D^*})\hat{f}_{1,0}\hat{D}^* \ , \\
\hat{C}.\hat{f}_{0,1} &= \frac{x}{y}((1-u)c^z_{C^*} + uc^w_{C^*})\hat{f}_{2,-1}\hat{C}^* + ((1-u)c^z_{D^*} + uc^w_{D^*})\hat{f}_{0,1}\hat{D}^* \ , \\
\hat{D}.\hat{f}_{1,0} &= ((1+v)d^z_{C^*} - vd^w_{C^*})\hat{f}_{1,0}\hat{C}^* + \frac{y}{x}((1+v)d^z_{D^*} - vd^w_{D^*})\hat{f}_{-1,2}\hat{D}^* \ , \\
\hat{D}.\hat{f}_{0,1} &= ((1-u)d^z_{C^*} + ud^w_{C^*})\hat{f}_{0,1}\hat{C}^* + \frac{y}{x}((1-u)d^z_{D^*} + ud^w_{D^*})\hat{f}_{-2,3}\hat{D}^* \ ,
\end{aligned}$$

and

$$\begin{aligned}
\hat{C}.\hat{C}^* &= c_1^{C^*} + xyc_\wedge^{C^*}\hat{C}^* \wedge \hat{D}^* \ , & \hat{C}.\hat{D}^* &= \hat{f}_{2,-2}(\frac{x}{y}c_1^{D^*} + y^2 c_\wedge^{D^*}\hat{C}^* \wedge \hat{D}^*) \ , \\
\hat{D}.\hat{C}^* &= \hat{f}_{-2,2}(\frac{y}{x}d_1^{C^*} + y^2 d_\wedge^{C^*}\hat{C}^* \wedge \hat{D}^*) \quad \text{and} & \hat{D}.\hat{D}^* &= d_1^{D^*} + xyd_\wedge^{D^*}\hat{C}^* \wedge \hat{D}^* \ .
\end{aligned}$$

Together with an analogous calculation for $\mathbf{\Psi}^-_{x,y,u,v}$, we have the following summary.

Proposition 4. *For $\mathbb{K} = \mathbb{R}$ the isomorphisms of planed LRT Lie supergroups $\mathbf{\Psi}^+_{x,y,u,v}$ and $\mathbf{\Psi}^-_{x,y,u,v}$ induce the following transformations of parameters:*

$$\left\{ \begin{matrix} \begin{pmatrix} c^z_{C^*} & c^z_{D^*} \\ c^w_{C^*} & c^w_{D^*} \end{pmatrix} \\ \begin{pmatrix} d^z_{C^*} & d^z_{D^*} \\ d^w_{C^*} & d^w_{D^*} \end{pmatrix} \\ \begin{pmatrix} c_1^{C^*} & c_1^{D^*} \\ d_1^{C^*} & d_1^{D^*} \end{pmatrix} \end{matrix} \right\} \xrightarrow{\mathbf{\Psi}^+_{x,y,u,v}} \left\{ \begin{matrix} \begin{pmatrix} \frac{x}{y}((1+v)c^z_{C^*} - vc^w_{C^*}) & (1+v)c^z_{D^*} - vc^w_{D^*} \\ \frac{x}{y}((1-u)c^z_{C^*} + uc^w_{C^*}) & (1-u)c^z_{D^*} + uc^w_{D^*} \end{pmatrix} \\ \begin{pmatrix} (1+v)d^z_{C^*} - vd^w_{C^*} & \frac{y}{x}((1+v)d^z_{D^*} - vd^w_{D^*}) \\ (1-u)d^z_{C^*} + ud^w_{C^*} & \frac{y}{x}((1-u)d^z_{D^*} + ud^w_{D^*}) \end{pmatrix} \\ \begin{pmatrix} c_1^{C^*} & \frac{x}{y}c_1^{D^*} \\ \frac{y}{x}d_1^{C^*} & d_1^{D^*} \end{pmatrix} \end{matrix} \right\}$$

and

$$\left\{\begin{matrix}\begin{pmatrix} c^z_{C^*} & c^z_{D^*} \\ c^w_{C^*} & c^w_{D^*}\end{pmatrix} \\ \begin{pmatrix} d^z_{C^*} & d^z_{D^*} \\ d^w_{C^*} & d^w_{D^*}\end{pmatrix} \\ \begin{pmatrix} c_1^{C^*} & c_1^{D^*} \\ d_1^{C^*} & d_1^{D^*}\end{pmatrix}\end{matrix}\right\} \xrightarrow{\mathbf{\Psi}^-_{x,y,u,v}} \left\{\begin{matrix}\begin{pmatrix} \frac{x}{y}((1-v)d^z_{D^*}+vd^w_{D^*}) & (1-v)d^z_{C^*}+vd^w_{C^*} \\ \frac{x}{y}((1+u)d^z_{D^*}-ud^w_{D^*}) & (1+u)d^z_{C^*}-ud^w_{C^*}\end{pmatrix} \\ \begin{pmatrix} (1-v)c^z_{D^*}+vc^w_{D^*} & \frac{y}{x}((1-v)c^z_{C^*}+vc^w_{C^*}) \\ (1+u)c^z_{D^*}-uc^w_{D^*} & \frac{y}{x}((1+u)c^z_{C^*}-uc^w_{C^*})\end{pmatrix} \\ \begin{pmatrix} d_1^{D^*} & \frac{x}{y}d_1^{C^*} \\ \frac{y}{x}c_1^{D^*} & c_1^{C^*}\end{pmatrix}\end{matrix}\right\}.$$

Using this description we obtain from (7) and (8) the following results.

Theorem 2. *The subset of* $\mathbb{R}^{10}$ *of generic dimension six which parameterizes planed LRT-Lie supergroups associated to the Lie superalgebra* $\mathfrak{gl}_{\mathbb{R}}(1/1)$ *in Theorem 1 decomposes into isomorphy classes of generic dimension three.*

Therefore, without being precise about the quotient by isomorphisms, we have shown that there is a three-dimensional parameter space of mutually non-isomorphic planed LRT Lie supergroups. Note that this description of isomorphy fails in the complex case since $\hat{f}_{n,m}$ in (12) is only well-defined if $x \cdot y$ divides v and $1+v$ and hence equals ± 1. Since in any case $x \cdot y \in \mathbb{Z}$, it is nevertheless of certain interest to analyze this set C_R of isomorphisms of the form $\mathbf{\Psi}^+_{x,x^{-1},u,\pm 1-u}$ and $\mathbf{\Psi}^-_{x,x^{-1},u,\pm 1-u}$ with $x \in \mathbb{C}^\times$ and $u \in \mathbb{Z}$. Using considerations analogous to those in the real case, we obtain the following result.

Proposition 5. *The subset of* $\mathbb{C}^{10}$ *of generic dimension six which parameterizes planed LRT-Lie supergroups associated to* $\mathfrak{gl}_{\mathbb{C}}(1/1)$ *in Theorem 1 decomposes into isomorphy classes which are generically* 1*-dimensional.*

Therefore, in the rough sense indicated above, the space of mutually non-isomorphic planed structures is complex 5-dimensional.

2.3 Comparison of the Constructions by Berezin and by Kostant

Berezin's analytic construction of a Lie supergroup induces naturally a planed LRT Lie supergroup as does the construction by Kostant and Koszul. Here we determine the 12 essential structural constants for both constructions for the example of $\mathfrak{gl}_{\mathbb{K}}(1/1)$ and $(\mathbb{K}^\times)^2$ and show that the results are not isomorphic in the category of planed LRT, respectively planed RT, Lie supergroups.

2.3.1 The 12 essential Constants for Kostant and Koszul

We start with Kostant and Koszul where $f_{n,m}$, $f_{n,m}C^*$, $f_{n,m}D^*$ and $f_{n,m}C^*\wedge D^*$ are identified with the functions

$$\Phi_{f_{n,m}}, \Phi_{f_{n,m}C^*}, \Phi_{f_{n,m}D^*}, \Phi_{f_{n,m}C^*\wedge D^*} : \mathbb{K}(G)\#E(\mathfrak{g}) \to \mathbb{K}$$

defined by

$$\begin{aligned}
\Phi_{f_{n,m}}(g\#X_0\gamma(a_1+a_{C^*}C+a_{D^*}D+a_\wedge C\wedge D)) &= a_1(X_0.f_{n,m})(g)\\
\Phi_{f_{n,m}C^*}(g\#X_0\gamma(a_1+a_{C^*}C+a_{D^*}D+a_\wedge C\wedge D)) &= a_{C^*}(X_0.f_{n,m})(g)\\
\Phi_{f_{n,m}D^*}(g\#X_0\gamma(a_1+a_{C^*}C+a_{D^*}D+a_\wedge C\wedge D)) &= a_{D^*}(X_0.f_{n,m})(g)\\
\Phi_{f_{n,m}C^*\wedge D^*}(g\#X_0\gamma(a_1+a_{C^*}C+a_{D^*}D+a_\wedge C\wedge D)) &= a_\wedge(X_0.f_{n,m})(g)
\end{aligned}$$

for $g \in G$, $X_0 \in E(\mathfrak{g}_0)$ and $a_1, a_{C^*}, a_{D^*}, a_\wedge \in \mathbb{K}$. Using

$$\gamma(C\wedge D) = \frac{1}{2}(CD - DC),\ CD + DC = A + B \ \Rightarrow \ \begin{array}{c} CD = \frac{1}{2}(A+B) + \gamma(C\wedge D) \\ DC = \frac{1}{2}(A+B) - \gamma(C\wedge D) \end{array}$$

it follows that

$$\begin{aligned} & C.\Phi_{f_{n,m}}(g\#X_0\gamma(a_1 + a_{C^*}C + a_{D^*}D + a_\wedge C\wedge D)) \\ & \quad = -\Phi_{f_{n,m}}\left((g\#X_0(a_1 + a_{C^*}C + a_{D^*}D + a_\wedge\gamma(C\wedge D))\cdot(e\#C)\right) \\ & \quad = -\Phi_{f_{n,m}}\left(g\#X_0(a_1C + a_{D^*}D\cdot C + a_\wedge\frac{1}{2}(A+B)C)\right) \\ & \quad = -\Phi_{f_{n,m}}\Big(g\#a_1X_0\gamma(C) + g\#\frac{a_{D^*}}{2}X_0(A+B) - g\#a_{D^*}X_0\gamma(C\wedge D) \\ & \qquad\qquad +g\#\frac{a_\wedge}{2}X_0(A+B)\gamma(C)\Big) \\ & \quad = -\frac{a_{D^*}}{2}(X_0(A+B)).(f_{n,m}) = -\frac{a_{D^*}}{2}(n+m)(X_0.f_{n,m}) \\ & \quad = -\frac{1}{2}(n+m)\Phi_{f_{n,m}D^*}(g\#X_0\gamma(a_1 + a_{C^*}C + a_{D^*}D + a_\wedge C\wedge D)). \end{aligned}$$

Analogous calculations lead to the list

$$\begin{array}{ll} C.\Phi_{f_{n,m}} = -\frac{1}{2}(n+m)\Phi_{f_{n,m}D^*} & D.\Phi_{f_{n,m}} = -\frac{1}{2}(n+m)\Phi_{f_{n,m}C^*} \\ C.\Phi_{f_{n,m}C^*} = -\Phi_{f_{n,m}} - \frac{1}{2}(n+m)\Phi_{f_{n,m}C^*\wedge D^*} & C.\Phi_{f_{n,m}D^*} = 0 \\ D.\Phi_{f_{n,m}C^*} = 0 & D.\Phi_{f_{n,m}D^*} = -\Phi_{f_{n,m}} - \frac{1}{2}(n+m)\Phi_{f_{n,m}C^*\wedge D^*} \end{array}$$

generating the set of parameters:

$$\left\{ \begin{pmatrix} c^z_{C^*} & c^z_{D^*} \\ c^w_{C^*} & c^w_{D^*} \end{pmatrix} \begin{pmatrix} d^z_{C^*} & d^z_{D^*} \\ d^w_{C^*} & d^w_{D^*} \end{pmatrix} \begin{pmatrix} c_1^{C^*} & c_1^{D^*} \\ d_1^{C^*} & d_1^{D^*} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{\textbf{KK}}$$

2.3.2 The 12 essential Constants for Berezin

In the case of Berezin's construction, associated to the superfunctions $f_{n,m}$,$f_{n,m}C^*$,$f_{n,m}D^*$ and $f_{n,m}C^*\wedge D^*$ we obtain the maps

$$F_{n,m,\mathbf{1}}, F_{n,m,C^*}, F_{n,m,D^*}, F_{n,m,C^*\wedge D^*} : \widetilde{G} \to \Lambda\mathfrak{g}_1^*$$

given by

$$\begin{aligned}
&F_{n,m,\mathbf{1}}\left(\exp\left(\begin{smallmatrix} a_1+a_\wedge C^*\wedge D^* & 0 \\ 0 & b_1+b_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & c_{C^*}C^*+c_{D^*}D^* \\ d_{C^*}C^*+d_{D^*}D^* & 0\end{smallmatrix}\right)\right)\\
&\quad = e^{na_1+mb_1}\cdot(1+(na_\wedge+mb_\wedge)C^*\wedge D^*)\ ,\\
&F_{n,m,C^*}\left(\exp\left(\begin{smallmatrix} a_1+a_\wedge C^*\wedge D^* & 0 \\ 0 & b_1+b_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & c_{C^*}C^*+c_{D^*}D^* \\ d_{C^*}C^*+d_{D^*}D^* & 0\end{smallmatrix}\right)\right)\\
&\quad = e^{na_1+mb_1}\cdot(c_{C^*}C^*+c_{D^*}D^*)\ ,\\
&F_{n,m,D^*}\left(\exp\left(\begin{smallmatrix} a_1+a_\wedge C^*\wedge D^* & 0 \\ 0 & b_1+b_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & c_{C^*}C^*+c_{D^*}D^* \\ d_{C^*}C^*+d_{D^*}D^* & 0\end{smallmatrix}\right)\right)\\
&\quad = e^{na_1+mb_1}\cdot(d_{C^*}C^*+d_{D^*}D^*)\ \text{ and}\\
&F_{n,m,C^*\wedge D^*}\left(\exp\left(\begin{smallmatrix} a_1+a_\wedge C^*\wedge D^* & 0 \\ 0 & b_1+b_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & c_{C^*}C^*+c_{D^*}D^* \\ d_{C^*}C^*+d_{D^*}D^* & 0\end{smallmatrix}\right)\right)\\
&\quad = e^{na_1+mb_1}\cdot(c_{C^*}d_{D^*}-c_{D^*}d_{C^*})C^*\wedge D^*
\end{aligned}$$

for $a_1, b_1 \in \mathbb{C}^\times$ and $a_\wedge, b_\wedge, c_{C^*}, c_{D^*}, d_{C^*}, d_{D^*} \in \mathbb{C}$. Note that we used

$$\exp(x+yC^*\wedge D^*) = e^x(1+yC^*\wedge D^*)$$

for $x, y \in \mathbb{C}$.

Before calculating the derivatives of these functions we must identify a matrix in $\widetilde{G}$ with a pair of matrices as they appear in the argument of the functions above. For this we set $\tilde{R} := \tilde{c}_{C^*}\tilde{d}_{D^*} - \tilde{c}_{D^*}\tilde{d}_{C^*}$ and obtain

$$\begin{aligned}
&\exp\left(\begin{smallmatrix} \tilde{a}_1+\tilde{a}_\wedge C^*\wedge D^* & 0 \\ 0 & \tilde{b}_1+\tilde{b}_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & \tilde{c}_{C^*}C^*+\tilde{c}_{D^*}D^* \\ \tilde{d}_{C^*}C^*+\tilde{d}_{D^*}D^* & 0\end{smallmatrix}\right)\\
&= \left(\begin{smallmatrix} e^{\tilde{a}_1}(1+\tilde{a}_\wedge C^*\wedge D^*) & 0 \\ 0 & e^{\tilde{a}_1}(1+\tilde{b}_\wedge C^*\wedge D^*)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 1+\tilde{R}C^*\wedge D^* & \tilde{c}_{C^*}C^*+\tilde{c}_{D^*}D^* \\ \tilde{d}_{C^*}C^*+\tilde{d}_{D^*}D^* & 1-\tilde{R}C^*\wedge D^*\end{smallmatrix}\right)\\
&= \left(\begin{smallmatrix} e^{\tilde{a}_1}(1+(\tilde{a}_\wedge+\tilde{R})C^*\wedge D^*) & e^{\tilde{a}_1}(\tilde{c}_{C^*}C^*+\tilde{c}_{D^*}D^*) \\ e^{\tilde{b}_1}(\tilde{d}_{C^*}C^*+\tilde{d}_{D^*}D^*) & e^{\tilde{b}_1}(1+(\tilde{b}_\wedge-\tilde{R})C^*\wedge D^*)\end{smallmatrix}\right)\ .
\end{aligned}$$

Hence, in order to determine a decomposition of an element

$$A = \left(\begin{smallmatrix} \hat{a}_1+\hat{a}_\wedge C^*\wedge D^* & \hat{c}_{C^*}C^*+\hat{c}_{D^*}D^* \\ \hat{d}_{C^*}C^*+\hat{d}_{D^*}D^* & \hat{b}_1+\hat{b}_\wedge C^*\wedge D^*\end{smallmatrix}\right) \in \widetilde{G} \tag{13}$$

into $A = \exp(X_0)\cdot\exp(X_1)$ for $X_i \in \tilde{\mathfrak{g}}_i$ we have to solve the equations

$$\begin{aligned}
&\hat{a}_1 = e^{\tilde{a}_1} && \hat{a}_\wedge = e^{\tilde{a}_1}(\tilde{a}_\wedge+\tilde{R}) && \hat{b}_1 = e^{\tilde{b}_1} && \hat{b}_\wedge = e^{\tilde{b}_1}(\tilde{b}_\wedge-\tilde{R}) \\
&\hat{c}_{C^*} = e^{\tilde{a}_1}\tilde{c}_{C^*} && \hat{c}_{D^*} = e^{\tilde{a}_1}\tilde{c}_{D^*} && \hat{d}_{C^*} = e^{\tilde{b}_1}\tilde{d}_{C^*} && \hat{d}_{D^*} = e^{\tilde{b}_1}\tilde{d}_{D^*} && \tilde{R} = \tilde{c}_{C^*}\tilde{d}_{D^*}-\tilde{c}_{D^*}\tilde{d}_{C^*}\ .
\end{aligned} \tag{14}$$

For odd superderivations it is necessary to understand the argument

$$\exp\left(\begin{smallmatrix} a_1+a_\wedge C^*\wedge D^* & 0 \\ 0 & b_1+b_\wedge C^*\wedge D^*\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & c_{C^*}C^*+c_{D^*}D^* \\ d_{C^*}C^*+d_{D^*}D^* & 0\end{smallmatrix}\right)\cdot\exp\left(\begin{smallmatrix} 0 & t\alpha \\ 0 & 0\end{smallmatrix}\right)$$

with $\alpha = \alpha_{C^*}C^* + \alpha_{D^*}D^*$. In the notation of (13) the product is determined by

$$\begin{aligned}
&\hat{a}_1 = e^{a_1} && \hat{a}_\wedge = e^{a_1}(a_\wedge+R) && \hat{b}_1 = e^{b_1} && \hat{b}_\wedge = e^{b_1}\big(b_\wedge - R \\
& && && && \qquad + t(d_{C^*}\alpha_{D^*} - d_{D^*}\alpha_{C^*})\big) \\
&\hat{c}_{C^*} = e^{a_1}(c_{C^*}+t\alpha_{C^*}) && \hat{c}_{D^*} = e^{a_1}(c_{D^*}+t\alpha_{D^*}) && \hat{d}_{C^*} = e^{b_1}d_{C^*} && \hat{d}_{D^*} = e^{b_1}d_{D^*}\ .
\end{aligned}$$

Solving the nine equations in (14) we obtain

$$\begin{array}{llll} \tilde{a}_1 = a_1 & \tilde{a}_\wedge = a_\wedge + t(\alpha_{D^*}d_{C^*} - \alpha_{C^*}d_{D^*}) & \tilde{b}_1 = b_1 & \tilde{b}_\wedge = b_\wedge \\ \tilde{c}_{C^*} = (c_{C^*} + t\alpha_{C^*}) & \tilde{c}_{D^*} = (c_{D^*} + t\alpha_{D^*}) & \tilde{d}_{C^*} = d_{C^*} & \tilde{d}_{D^*} = d_{D^*} \end{array}$$

yielding

$$\begin{array}{ll} C.F_{n,m,\mathbf{1}} = nF_{n,m,D^*} & C.F_{n,m,C^*} = F_{n,m,\mathbf{1}} \\ C.F_{n,m,D^*} = 0 & C.F_{n,m,C^*\wedge D^*} = -F_{n,m,D^*} \ . \end{array}$$

A parallel calculation for D leads to

$$\begin{array}{ll} D.F_{n,m,\mathbf{1}} = mF_{n,m,C^*} & D.F_{n,m,C^*} = 0 \\ D.F_{n,m,D^*} = F_{n,m,\mathbf{1}} & D.F_{n,m,C^*\wedge D^*} = F_{n,m,C^*} \end{array}$$

and finally to the parameters

$$\left\{ \begin{pmatrix} c^z_{C^*} & c^z_{D^*} \\ c^w_{C^*} & c^w_{D^*} \end{pmatrix} \begin{pmatrix} d^z_{C^*} & d^z_{D^*} \\ d^w_{C^*} & d^w_{D^*} \end{pmatrix} \begin{pmatrix} c^{C^*}_1 & c^{D^*}_1 \\ d^{C^*}_1 & d^{D^*}_1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (\mathbf{Ber})$$

Hence, we are finally in a position where we can compare the structures.

Theorem 3. *In the case $\mathfrak{gl}_{\mathbb{R}}(1/1)$ the constructions by Kostant-Koszul and Berezin define non-isomorphic RT Lie supergroup structures.*

Proof. For the example of $\mathfrak{gl}_{\mathbb{R}}(1/1)$ we obtain as conditions for the existence of an isomorphism of type $\mathbf{\Psi}^+_{x,y,u,v}$ between the parameter sets (**Ber**) and (**KK**)

$$\begin{array}{l} c^z_{D^*,Ber} = (1+v)c^z_{D^*,KK} - vc^w_{D^*,KK} \ \Rightarrow \ (1+v) - v = -2 \\ d^z_{C^*,Ber} = (1+v)d^z_{C^*,KK} - vd^w_{C^*,KK} \ \Rightarrow \ (1+v) - v = 0 \ . \end{array}$$

The type $\mathbf{\Psi}^-_{x,y,u,v}$ yields an analogous contradiction. Here we have discussed planed LRT Lie supergroups, but since isomorphisms of LRT Lie supergroups preserve the property "planed", non-isomorphy also holds in the category of general LRT Lie supergroup structures in the case of $\mathfrak{gl}_{\mathbb{R}}(1/1)$. The constructions of Kostant and Berezin yield RT Lie supergroups, so non-isomorphy also holds in this case. □

It should be remarked that, since $\mathfrak{gl}_{\mathbb{R}}(1/1)$ can be found as a subsuperalgebra in many higher dimensional Lie superalgebras, it is therefore a generally occurring phenomenon that the Kostant-Koszul and Berezin constructions produce non-isomorphic RT-structures.

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