

**TORSION UNITS IN INTEGRAL GROUP RINGS
OF CONWAY SIMPLE GROUPS**

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V.A. BOVDI, A.B. KONOVALOV, AND S. LINTON

The paper is dedicated to Professor Said Sidki on his 70th birthday.

ABSTRACT. Using the Luthar–Passi method, we investigate the possible orders and partial augmentations of torsion units of the normalized unit group of integral group rings of Conway simple groups Co_1 , Co_2 and Co_3 .

Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring $\mathbb{Z}G$ of a finite group G , and $V(\mathbb{Z}G)$ be its normalized unit group

$$V(\mathbb{Z}G) = \left\{ \sum_{g \in G} \alpha_g g \in U(\mathbb{Z}G) \mid \sum_{g \in G} \alpha_g = 1 \right\}.$$

The structure of $U(\mathbb{Z}G)$ is completely determined by its normalized unit group since $U(\mathbb{Z}G) = U(\mathbb{Z}) \times V(\mathbb{Z}G)$. Throughout the paper (unless stated otherwise) any unit of $\mathbb{Z}G$ is always *normalized* and not equal to the identity element of G .

The following longstanding conjecture is due to H. Zassenhaus (see [24]):

(ZC) every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G .

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [23] for the alternating group A_5 (for its further applications, see also [2] and [21]).

The conjecture **(ZC)** is still open for all sporadic simple groups, and for several of them results are available that either prove **(ZC)** for some orders or restrict possible partial augmentations of torsion units. For some recent results on Mathieu, Janko, Higman-Sims, McLaughlin, Held, Rudvalis, Suzuki and O’Nan simple groups we refer to [3, 4, 5, 6, 7, 8, 9, 10, 11, 13]. Here we continue these investigations for the Conway simple groups Co_1 , Co_2 and Co_3 .

Let G be a finite group. Denote by $\mathcal{C} = \{C_1, C_{k_1 t_1}, \dots, C_{k_s t_r}\}$ the collection of all conjugacy classes of G , where $C_1 = \{1\}$, and C_{kt} denote the conjugacy class with representatives of order k , labelled by the distinguishing letter t (throughout the paper we use the ordering of conjugacy classes as used in the GAP Character Table Library). Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ is a non-trivial torsion unit. The partial augmentation of u with respect to C_{nt} is defined as $\nu_{nt} = \nu_{nt}(u) = \sum_{g \in C_{nt}} \alpha_g$.

The criterion for **ZC** can be formulated in terms of the vanishing of partial augmentations of torsion units (see Proposition 3 below). Therefore, it is useful

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to know for each possible order of a torsion unit in $V(\mathbb{Z}G)$, which combinations of partial augmentations may arise. Such an answer is provided by our main results.

Theorem 1. *Let G denote the Conway simple group Co_3 . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ with the tuple ν of length 42 containing partial augmentations for all conjugacy classes of G . The following properties hold.*

- (i) *There are no elements of order 33, 46, 55, 69, 77, 115, 161, 253 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{28, 35, 36, 40, 42, 44, 45, 56, 60, 63, 70, 72, 84, 88, 90, 105, 120, 126, 140, 168, 180, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260, 2520\}$, then $|u|$ is the order of some element $g \in G$.*
- (ii) *If $|u| = 7$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 2$, then $\nu_{kx} = 0$ for $kx \notin \{2a, 2b\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(-2, 3), (-1, 2), (0, 1), (1, 0), (2, -1), (3, -2)\}.$$
- (iv) *If $|u| = 3$, all partial augmentations of u are zero except possibly $\nu_{3a}, \nu_{3b}, \nu_{3c}$ and the triple $(\nu_{3a}, \nu_{3b}, \nu_{3c})$ is one of those given in Appendix A.*
- (v) *If $|u| = 5$, then $\nu_{kx} = 0$ for $kx \notin \{5a, 5b\}$ and*

$$(\nu_{5a}, \nu_{5b}) \in \{(-4, 5), (-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0)\}.$$
- (vi) *If $|u| = 11$, then all partial augmentations of u are zero except possibly ν_{11a}, ν_{11b} , and the pair (ν_{11a}, ν_{11b}) is one of*

$$\{(\nu_{11a}, \nu_{11b}) \mid -11 \leq \nu_{11a} \leq 12, \nu_{11a} + \nu_{11b} = 1\}.$$
- (vii) *If $|u| = 23$, then all partial augmentations of u are zero except possibly ν_{23a}, ν_{23b} , and the pair (ν_{23a}, ν_{23b}) is one of*

$$\{(\nu_{23a}, \nu_{23b}) \mid -5 \leq \nu_{23a} \leq 6, \nu_{23a} + \nu_{23b} = 1\}.$$
- (viii) *If $|u| = 35$, then all partial augmentations of u are zero except possibly $\nu_{5a}, \nu_{5b}, \nu_{7a}$, and the triple $(\nu_{5a}, \nu_{5b}, \nu_{7a})$ is one of $\{(3, 12, -14), (4, 11, -14)\}$.*

Theorem 2. *Let G denote the Conway simple group Co_2 . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ with the tuple ν of length 60 containing partial augmentations for all conjugacy classes of G . The following properties hold.*

- (i) *There are no elements of order 21, 22, 33, 46, 55, 69, 77, 115, 161, 253 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{36, 40, 45, 48, 56, 60, 70, 72, 80, 90, 112, 120, 140, 144, 180, 240, 280, 360, 560, 720\}$, then $|u|$ is the order of some $g \in G$.*
- (ii) *If $|u| \in \{7, 11\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 2$, all partial augmentations of u are zero except possibly $\nu_{2a}, \nu_{2b}, \nu_{2c}$, and the triple $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ is one of those given in Appendix B.*
- (iv) *If $|u| = 3$, then $\nu_{kx} = 0$ for $kx \notin \{3a, 3b\}$ and*

$$(\nu_{3a}, \nu_{3b}) \in \{(-2, 3), (-1, 2), (0, 1), (1, 0)\}.$$
- (v) *If $|u| = 5$, then $\nu_{kx} = 0$ for $kx \notin \{5a, 5b\}$ and*

$$(\nu_{5a}, \nu_{5b}) \in \{(-4, 5), (-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0)\}.$$
- (vi) *If $|u| = 23$, then all partial augmentations of u are zero except possibly ν_{23a}, ν_{23b} , and the pair (ν_{23a}, ν_{23b}) is one of*

$$\{(\nu_{23a}, \nu_{23b}) \mid -32 \leq \nu_{23a} \leq 33, \nu_{23a} + \nu_{23b} = 1\}.$$
- (vii) *If $|u| = 35$, then all partial augmentations of u are zero except possibly $\nu_{5a}, \nu_{5b}, \nu_{7a}$, and the triple $(\nu_{5a}, \nu_{5b}, \nu_{7a})$ is one of $\{(3, 12, -14), (4, 11, -14)\}$.*

Theorem 3. *Let G denote the Conway simple group Co_1 . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ with the tuple ν of length 101 containing partial augmentations for all conjugacy classes of G . The following properties hold.*

- (i) *There are no elements of order 46, 69, 77, 91, 115, 143, 161, 253 and 299 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{55, 65, 110, 130, 165, 195, 220, 260, 330, 390, 440, 495, 520, 585, 660, 780, 880, 990, 1040, 1170, 1320, 1560, 1980, 2340, 2640, 3120, 3960, 4680, 7920, 9360\}$, then $|u|$ is the order of some $g \in G$.*
- (ii) *If $|u| \in \{11, 13\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 7$, then all partial augmentations of u are zero except possibly ν_{7a} and ν_{7b} , and the pair (ν_{7a}, ν_{7b}) is one of*

$$\{ (\nu_{7a}, \nu_{7b}) \mid \nu_{7a} + \nu_{7b} = 1, \quad -7 \leq \nu_{7a} \leq 39 \}.$$

- (iv) *If $|u| = 23$, then all partial augmentations of u are zero except possibly ν_{23a} and ν_{23b} , and the pair (ν_{23a}, ν_{23b}) is one of*

$$\{ (\nu_{23a}, \nu_{23b}) \mid \nu_{7a} + \nu_{7b} = 1, \quad -29293 \leq \nu_{23a} \leq 29294 \}.$$

- (v) *If $|u| = 55$, then all partial augmentations of u are zero except possibly ν_{5a} , ν_{5b} , ν_{5c} and ν_{11a} , and the tuple $(\nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{11a})$ is one of*

$$\begin{aligned} &(-2, 2, -10, 11), (-2, 3, -11, 11), (-1, -2, -7, 11), (-1, -1, -8, 11), \\ &(-1, 0, -9, 11), (-1, 1, -10, 11), (0, -6, -4, 11), (0, -5, -5, 11), \\ &(0, -4, -6, 11), (0, -3, -7, 11), (0, -2, -8, 11), (0, -1, -9, 11), \\ &(0, 0, -10, 11), (1, -8, -3, 11), (1, -7, -4, 11), (1, -6, -5, 11), \\ &(1, -5, -6, 11), (1, -4, -7, 11), (1, -3, -8, 11), (1, -2, -9, 11), \\ &(2, -9, -3, 11), (2, -8, -4, 11), (2, -7, -5, 11), (2, -6, -6, 11), \\ &(2, -5, -7, 11), (2, -4, -8, 11), (2, -3, -9, 11), (3, -11, -2, 11), \\ &(3, -10, -3, 11), (3, -9, -4, 11), (3, -8, -5, 11), (3, -7, -6, 11), \\ &(3, -6, -7, 11), (4, -12, -2, 11), (4, -11, -3, 11), (4, -10, -4, 11). \end{aligned}$$

- (vi) *If $|u| = 65$, then all partial augmentations of u are zero except possibly ν_{5a} , ν_{5b} , ν_{5c} and ν_{13a} , and the tuple $(\nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{13a})$ is one of*

$$\begin{aligned} &(-3, 2, -24, 26), (-2, -2, -21, 26), (-2, -1, -22, 26), (-2, 0, -23, 26), \\ &(-1, -3, -21, 26), (-1, -2, -22, 26), (-1, -1, -23, 26), (5, -4, 39, -39), \\ &(5, -3, 38, -39), (6, -7, 41, -39), (6, -6, 40, -39), (6, -5, 39, -39), \\ &(7, -8, 41, -39), (7, -7, 40, -39). \end{aligned}$$

For the determination of possible orders of torsion units in $V(\mathbb{Z}G)$ first of all we start with the following well-known bound.

Proposition 1 ([14]). *The order of a torsion element $u \in V(\mathbb{Z}G)$ divides $\exp(G)$.*

Moreover, the partial augmentations of torsion units are also bounded.

Proposition 2 (see [20]). *Let C_1, \dots, C_n be conjugacy classes of a finite group G . Let u be a torsion unit in $V(\mathbb{Z}G)$ and $\nu_i(u)$ denote the partial augmentation of u with respect to the conjugacy class C_i . Then $\nu_i(u)^2 \leq |C_i|$ and, moreover,*

$$\sum_{i=1}^n \frac{\nu_i(u)^2}{|C_i|} \leq 1.$$

The following result allows a reformulation of the Zassenhaus conjecture in terms of the vanishing of partial augmentations of torsion units.

Proposition 3 (see [23]). *Let $u \in V(\mathbb{Z}G)$ be of order k . Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

The next result is a reformulation of the Proposition 3.1 [21] (which was originally proved for a group ring over an arbitrary Dedekind ring of characteristic zero) for the case of integral group rings. It restricts possible values of some partial augmentations of torsion units.

Proposition 4 (see [21], Proposition 3.1). *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$ of order k . If x is an element of G whose order does not divide k , then $\varepsilon_x(u) = 0$.*

The basis of the Luthar–Passi method which produces further restrictions on possible orders of torsion units and their partial augmentations is the following.

Proposition 5 (see [21, 23]). *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k -th root of unity and χ is either a ordinary character or a p -Brauer character of G , then for every integer l we define*

$$(1) \quad \mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}(\chi(u^d)z^{-dl}).$$

Then $\mu_l(u, \chi, p)$ is a non-negative integer not greater than $\text{deg}(\chi)$.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Let u be a normalized unit of order k , where k divides $\text{exp}(G)$ by Proposition 1. From the Berman–Higman Theorem (see [1]) one knows that $\text{tr}(u) = \nu_1 = 0$, so

$$(2) \quad \sum_{C_{nt} \in \mathcal{C} \setminus C_1} \nu_{nt} = 1.$$

On the next step we apply Proposition 4 for every appropriate prime p , such that $k = p^m t$, where $(p, t) = 1$, to eliminate partial augmentations of conjugacy classes of elements of G with representatives of order $p^n s$, where $(p, s) = 1$ and $n > m$. If after this step for torsion units of some order k we have only one non-zero partial augmentation, then **ZC** holds for this order by Proposition 3.

Otherwise, we have to produce and solve a system of constraints. For the unit u of order k we denote by $\nu_1^{(k)}, \dots, \nu_n^{(k)}$ its non-vanishing partial augmentations for conjugacy classes C_{q_1}, \dots, C_{q_n} (we will also omit the upper index and denote them by ν_1, \dots, ν_n for the clarity of notation). Let d_1, \dots, d_s be the set of all non-negative integers dividing k , where $d_i > 1$ and $d_s = k$. Furthermore, let $k_i = k/d_i$ and let $\nu_1^{(k_i)}, \dots, \nu_{n_i}^{(k_i)}$ be the non-vanishing partial augmentations for elements of order k_i , corresponding to conjugacy classes $C_{q_1}^{(k_i)}, \dots, C_{q_{n_i}}^{(k_i)}$. Then the right-hand side in (1) from Proposition 5 formula may be written as

$$\begin{aligned} \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}(\chi(u^d)z^{-dl}) &= \frac{1}{k} (\text{Tr}_{\mathbb{Q}(z)/\mathbb{Q}}(\chi(u)z^{-l}) + \\ &\quad \text{Tr}_{\mathbb{Q}(z^{d_1})/\mathbb{Q}}(\chi(u^{d_1})z^{-d_1 l}) + \dots + \\ &\quad \text{Tr}_{\mathbb{Q}(z^{d_i})/\mathbb{Q}}(\chi(u^{d_i})z^{-d_i l}) + \dots + \\ &\quad \text{Tr}_{\mathbb{Q}(z^{d_{s-1}})/\mathbb{Q}}(\chi(u^{d_{s-1}})z^{-d_{s-1} l}) + \chi(1)), \end{aligned}$$

where the summand $\chi(1)$ comes from $d_s = k$.

Clearly, $\chi(u) = \sum_{j=1}^n \chi(h_j) \nu_j$ and $\chi(u^{d_i}) = \sum_{j=1}^{n_i} \chi(h_j) \nu_j^{(k_i)}$ for any character χ , where h_j is a representative of the conjugacy class C_j , and $\nu_j^{(k_i)}$ is the partial augmentation for the conjugacy class C_j for an element u^{d_i} of order $k_i = d/d_i$.

Since the trace is a linear mapping, this gives us $\mu_l(u, \chi, p)$ as a linear combination of corresponding partial augmentations:

$$\begin{aligned} \mu_l(u, \chi, p) = & \frac{1}{k} (c_1 \nu_1 + \cdots + c_n \nu_n + \\ & c_1^{(k_1)} \nu_1^{(k_1)} + \cdots + c_n^{(k_1)} \nu_n^{(k_1)} + \cdots + \\ & c_1^{(k_i)} \nu_1^{(k_i)} + \cdots + c_n^{(k_i)} \nu_n^{(k_i)} + \cdots + \\ & c_1^{(k_{s-1})} \nu_1^{(k_{s-1})} + \cdots + c_n^{(k_{s-1})} \nu_n^{(k_{s-1})} + \chi(1)) \geq 0. \end{aligned}$$

Since all the trace values must lie in \mathbb{Q} , we may be able to deduce at this stage that some more partial augmentations must be zero, when the corresponding character values are irrational.

Now to form the constraint satisfaction problem (CSP) for units of order k we put together: all inequalities for $\mu_l(u, \chi, p_i)$ for units of order k for all possible $0 \leq l < k$, characters χ and p_i ; similarly produced on earlier steps systems of inequalities with indeterminates $\nu_1^{(k_i)}, \dots, \nu_n^{(k_i)}$ for units of order k_i ; equation $\nu_1 + \cdots + \nu_n = 1$ and equations $\nu_1^{(k_i)} + \cdots + \nu_n^{(k_i)} = 1$ for every order k_i . Now, if this CSP has no solutions, this can be seen immediately, and this approach is much more efficient than enumerating all cases determined by possible partial augmentations for units of orders k_i , used, for example, in [5, 13].

Proposition 5 may be reformulated for elements of order st . Let s and t be two primes such that G contains no element of order st , and let u be a normalized torsion unit of order st . We denote by ν_k the sum of partial augmentations of u with respect all conjugacy classes of elements of order k in G , i.e. $\nu_2 = \nu_{2a} + \nu_{2b}$, etc. Then by (2) and Proposition 4 we obtain that $\nu_s + \nu_t = 1$ and $\nu_k = 0$ for $k \notin \{s, t\}$. For each character χ of G (an ordinary character or a Brauer character in characteristic not dividing st) that is constant on the elements of order s and constant on the elements of order t , we have $\chi(u) = \nu_s \chi(C_s) + \nu_t \chi(C_t)$, where $\chi(C_r)$ denote the value of the character χ on any element of order r of G .

Let s and t be two primes dividing $|G|$, and let χ be an ordinary or p -Brauer character of G for p not dividing st . Then χ is called a (s, t) -constant character, if χ is constant on all elements of order s and constant on all elements of order t .

From Proposition 5 we obtain that the values

$$(3) \quad \begin{aligned} \mu_l(u, \chi) = & \frac{1}{st} \left(\chi(1) + Tr_{\mathbb{Q}(z^s)/\mathbb{Q}}(\chi(u^s)z^{-sl}) \right. \\ & \left. + Tr_{\mathbb{Q}(z^t)/\mathbb{Q}}(\chi(u^t)z^{-tl}) + Tr_{\mathbb{Q}(z)/\mathbb{Q}}(\chi(u)z^{-l}) \right) \end{aligned}$$

are nonnegative integers, and if χ is (s, t) -constant character then we get

$$(4) \quad \mu_l(u, \chi) = \frac{1}{st} (m_1 + \nu_s m_s + \nu_t m_t),$$

where

$$(5) \quad \begin{aligned} m_1 &= \chi(1) + \chi(C_t) Tr_{\mathbb{Q}(z^s)/\mathbb{Q}}(z^{-sl}) + \chi(C_s) Tr_{\mathbb{Q}(z^t)/\mathbb{Q}}(z^{-tl}), \\ m_s &= \chi(C_s) Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}), \quad m_t = \chi(C_t) Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}). \end{aligned}$$

Since Proposition 5 and its reformulation are valid for any character (not necessarily irreducible), we are interested in a systematic search for (s, t) -constant characters that are capable of producing new constraints on partial augmentations. For example, if we have only two conjugacy classes of elements of order k , namely C_{ka} and C_{kb} , then if there are two characters χ_1 and χ_2 such that

$$\chi_1(ka) - \chi_1(kb) = \chi_2(kb) - \chi_2(ka),$$

then for the character $\chi = \chi_1 + \chi_2$ we will have that $\chi(ka) = \chi(kb)$.

If we have two (s, t) -constant characters χ_1 and χ_2 , then $\chi_1 + \chi_2$ can not give us any further restrictions on partial augmentations, as it is shown by the following.

Proposition 6. *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k -th root of unity and χ_1, χ_2 are both either classical characters or p -Brauer characters of G , then $\mu_l(u, \chi_1 + \chi_2, p)$ is a non-negative integer whenever both $\mu_l(u, \chi_1, p)$ and $\mu_l(u, \chi_2, p)$ are non-negative integers.*

Indeed, put $\xi = \chi_1 + \chi_2$. It is easy to check that

$$\begin{aligned} \mu_l(u, \xi, p) &= \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}\{(\chi_1 + \chi_2)(u^d)z^{-dl}\} \\ &= \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi_1(u^d)z^{-dl}\} + \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi_2(u^d)z^{-dl}\} \\ &= \mu_l(u, \chi_1, p) + \mu_l(u, \chi_2, p). \end{aligned}$$

Thus, the task is to find all (s, t) -constant characters that can not be represented as a sum of other (s, t) -constant characters. We will call such characters (s, t) -irreducible characters. The search can be performed by analyzing relative differences between values of irreducible characters on all conjugacy classes of the given order (see example in the proof for units of order 35 from Theorem 1).

1. PROOF OF THEOREM 1

Let $G \cong \text{Co}_3$. It is well known [15, 18] that $|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ and $\text{exp}(G) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. The ordinary and p -Brauer character tables of G for $p \in \{2, 3, 5, 7, 11, 23\}$ can be found using the computational algebra system GAP [18], which derives its data from [16, 22]. For characters and conjugacy classes we will use throughout the paper the same notation, including indexation, as used in the GAP Character Table Library.

The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 20, 21, 22, 23, 24 and 30. We begin with units of prime orders: 2, 3, 5, 7, 11, 23. We do not give here our results for the remaining cases of torsion units of orders 4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24 and 30 because they are rather complex. For example, using our implementation of the Luthar–Passi method, which we intend to make available in the GAP package LAGUNA [12], together with constraint solvers MINION [19] and ECLiPSe [17], we can compute 510 possible cases for partial augmentations $(\nu_{2a}, \nu_{2b}, \nu_{4a}, \nu_{4b})$ for torsion units of order 4 and five possible cases for partial augmentations $(\nu_{2a}, \nu_{2b}, \nu_{7a}, \nu_{14a})$ for torsion units of order 14. To complete the proof, we will investigate units of orders which do not appear in G .

- Let $|u| = 7$. Using Proposition 4 we obtain that all partial augmentations except one are zero. Thus by Proposition 3 part (ii) of Theorem 1 is proved.

• Let $|u| = 2$. By (2) and Proposition 4 we get $\nu_{2a} + \nu_{2b} = 1$. Put $t_1 = 7\nu_{2a} - \nu_{2b}$. Now using Proposition 5 we obtain the following system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{2}(t_1 + 23) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{2}(-t_1 + 23) \geq 0,$$

which has only six solutions listed in part (iii) of Theorem 1.

• Let u be a unit of order 3. By (2) and Proposition 4 we get $\nu_{3a} + \nu_{3b} + \nu_{3c} = 1$. Put $t_1 = 4\nu_{3a} - 5\nu_{3b} + \nu_{3c}$, $t_2 = 10\nu_{3a} + 10\nu_{3b} + \nu_{3c}$ and $t_3 = 14\nu_{3a} + 5\nu_{3b} + 2\nu_{3c}$. Using Proposition 5 we obtain the following system of inequalities

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{3}(-2t_1 + 23) \geq 0; & \mu_1(u, \chi_2, *) &= \frac{1}{3}(t_1 + 23) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{3}(2t_2 + 253) \geq 0; & \mu_1(u, \chi_3, *) &= \frac{1}{3}(-t_2 + 253) \geq 0; \\ \mu_0(u, \chi_3, 2) &= \frac{1}{3}(2t_3 + 230) \geq 0; & \mu_1(u, \chi_3, 2) &= \frac{1}{3}(-t_3 + 230) \geq 0, \end{aligned}$$

which has only 155 solutions listed in the Appendix A such that all $\mu_i(u, \chi_j)$ are non-negative integers.

• Let u be a unit of order 5. By (2) and Proposition 4 we get $\nu_{5a} + \nu_{5b} = 1$. Put $t_1 = 2\nu_{5a} - 3\nu_{5b}$. Using Proposition 5 we obtain the system of two inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{5}(-4t_1 + 23) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{5}(t_1 + 23) \geq 0,$$

which has only six solutions listed in part (v) of Theorem 1 such that all $\mu_i(u, \chi_2, *)$ are non-negative integers.

• Let u be a unit of order 11. By (2) and Proposition 4 we get $\nu_{11a} + \nu_{11b} = 1$. Put $t_1 = 6\nu_{11a} - 5\nu_{11b}$ and $t_2 = -5\nu_{11a} + 6\nu_{11b}$ (observe that $t_2 = 1 - t_1$). Now

$$\mu_1(u, \chi_3, 3) = \frac{1}{11}(t_1 + 126) \geq 0; \quad \mu_2(u, \chi_3, 3) = \frac{1}{11}(-t_1 + 127) \geq 0,$$

and this system has only 24 solutions listed in part (vi) of Theorem 1 such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 23. By (2) and Proposition 4 we get $\nu_{23a} + \nu_{23b} = 1$. Put $t_1 = 12\nu_{23a} - 11\nu_{23b}$. Now using Proposition 5 we obtain the system of inequalities

$$\mu_1(u, \chi_3, 3) = \frac{1}{23}(t_1 + 126) \geq 0; \quad \mu_5(u, \chi_3, 3) = \frac{1}{23}(-t_1 + 127) \geq 0,$$

which has only 12 solutions listed in part (vii) of Theorem 1 such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

Now we consider orders which do not appear in G .

• Let $|u| = 33$. For these units we consider partial augmentations $\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{11a}$ and ν_{11b} . Since $|u^3| = 11$ and $|u^{11}| = 3$, by Proposition 5 we obtain the system of inequalities with 5 more variables $\nu_{11a}^{(3)}, \nu_{11b}^{(3)}, \nu_{3a}^{(11)}, \nu_{3b}^{(11)}$ and $\nu_{3c}^{(11)}$. Replacing $\nu_{11a}^{(3)}$ and $\nu_{11b}^{(3)}$ by their numerical values from part (vi) of Theorem 1, we get the system

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{33}(-2t_1 + 33) \geq 0; & \mu_{11}(u, \chi_2, *) &= \frac{1}{33}(t_1 + 33) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{33}(-t_2 + 22) \geq 0; & \mu_3(u, \chi_2, *) &= \frac{1}{33}(2t_2 + 22) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{33}(2t_3 + 253) \geq 0; & \mu_{11}(u, \chi_3, *) &= \frac{1}{33}(-t_3 + 253) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{33}(t_4 + 253) \geq 0; & \mu_3(u, \chi_3, *) &= \frac{1}{33}(-2t_4 + 253) \geq 0; \\ \mu_0(u, \chi_6, *) &= \frac{1}{33}(2t_5 + 891) \geq 0; & \mu_{11}(u, \chi_6, *) &= \frac{1}{33}(-t_5 + 891) \geq 0; \\ \mu_1(u, \chi_6, *) &= \frac{1}{33}(t_6 + \alpha) \geq 0; & \mu_3(u, \chi_6, *) &= \frac{1}{33}(-2t_6 + \alpha) \geq 0, \end{aligned}$$

where the values of α are given in the following table

(ν_{11a}, ν_{11b})	(12, -11)	(11,-10)	(10, -9)	(9, -8)	(8,-7)	(7,-6)	(6,-5)	(5,-4)
α	1023	1012	1001	990	979	968	957	946
(ν_{11a}, ν_{11b})	(4,-3)	(3,-2)	(2, -1)	(1,0)	(0,1)	(-1,2)	(-2,3)	(-3,4)
α	935	924	913	902	891	880	869	858
(ν_{11a}, ν_{11b})	(-4,5)	(-5,6)	(-6,7)	(-7,8)	(-8,9)	(-9,10)	(-10,11)	(-11,12)
α	847	836	825	814	803	792	781	770

and

$$\begin{aligned}
t_1 &= 40\nu_{3a} - 50\nu_{3b} + 10\nu_{3c} - 10\nu_{11a} - 10\nu_{11b} + 4\nu_{3a}^{(11)} - 5\nu_{3b}^{(11)} + \nu_{3c}^{(11)}; \\
t_2 &= 4\nu_{3a} - 5\nu_{3b} + \nu_{3c} - \nu_{11a} - \nu_{11b} - 4\nu_{3a}^{(11)} + 5\nu_{3b}^{(11)} - \nu_{3c}^{(11)}; \\
t_3 &= 100\nu_{3a} + 100\nu_{3b} + 10\nu_{3c} + 10\nu_{3a}^{(11)} + 10\nu_{3b}^{(11)} + \nu_{3c}^{(11)}; \\
t_4 &= 10\nu_{3a} + 10\nu_{3b} + \nu_{3c} - 10\nu_{3a}^{(11)} - 10\nu_{3b}^{(11)} - \nu_{3c}^{(11)}; \\
t_5 &= 320\nu_{3a} - 40\nu_{3b} - 70\nu_{3c} - 5\nu_{11a} - 5\nu_{11b} + 32\nu_{3a}^{(11)} - 4\nu_{3b}^{(11)} - 7\nu_{3c}^{(11)}; \\
t_6 &= 32\nu_{3a} - 4\nu_{3b} - 7\nu_{3c} - 6\nu_{11a} + 5\nu_{11b} - 32\nu_{3a}^{(11)} + 4\nu_{3b}^{(11)} + 7\nu_{3c}^{(11)}.
\end{aligned}$$

It is not difficult to check that for any value of α listed in the table this system has no integral solutions such that each $\mu_i(u, \chi_j, *)$ is a non-negative integer.

• Let u be a unit of order 35. Using Proposition 5 for the 3-Brauer character χ_3 , for which $\chi(C_5) = 1$ and $\chi(C_7) = 0$, we obtain the system

$$\mu_0(u, \chi_3, 3) = \frac{1}{35}(24\nu_5 + 130) \geq 0; \quad \mu_7(u, \chi_3, 3) = \frac{1}{35}(-6\nu_5 + 125) \geq 0,$$

from which $(\nu_5, \nu_7) = (15, -14)$.

Any ordinary or Brauer (5, 7)-constant character can not eliminate this pair. Indeed, since we have only one conjugacy class of elements of order 7, to enumerate (5, 7)-irreducible characters we need to look only on character values on elements of order 5. First, for ordinary characters and Brauer characters for $p \in \{11, 23\}$ the set of differences $\chi(5a) - \chi(5b)$ is $\{\pm 5, \pm 10, \pm 15\}$.

Thus, besides irreducible (5, 7)-constant characters, all other (5, 7)-irreducible characters can be parametrized by the tuples from the following set:

$$\begin{aligned}
&\{ (5, 5), (10, -10), (15, -15), (10, -5, -5), (5, 5, -10), (15, -5, -5, -5), \\
&\quad (5, 5, 5, -15), (15, -5, -10), (5, 10, -15), (10, 10, -15, -5), \\
&\quad (15, 5, -10, -10), (15, 15, -10, -10, -10), (10, 10, 10, -15, -15) \},
\end{aligned}$$

where, for example, the tuple $(5, 5, -10)$ mean that the character is the sum of three irreducible characters, for two of them $\chi(5a) - \chi(5b) = 5$ and for the last one $\chi(5a) - \chi(5b) = -10$. Enumerating all possible combinations of characters for each tuple from the list above, and any other (5, 7)-constant character would be a sum of some already known (5, 7)-constant characters. We need also to repeat the same procedure for 3-Brauer characters, where $\chi(5a) - \chi(5b) \in \{\pm 5, \pm 10, 15, 20\}$, and for 2-Brauer characters, where $\chi(5a) - \chi(5b) \in \{\pm 5, \pm 10, -20\}$. Using the GAP system [18], we verified that $\mu_i(u, \chi, p)$ are non-negative integers when $(\nu_5, \nu_7) = (15, -14)$ for any (5, 7)-constant character χ obtained by the procedure described above, so it is not possible to prove non-existence of torsion units of order 35 using this method.

For further detalisation, we consider six cases from part (v) of Theorem 1. If

$$\chi(u^7) \in \{ \chi(5a), \chi(5b), -\chi(5a) + 2\chi(5b), -2\chi(5a) + 3\chi(5b) \},$$

then we combine the condition $\mu_0(u, \chi_2, *) = \frac{1}{35}(-24t + \alpha) \geq 0$, where $t = 2\nu_{5a} - 3\nu_{5b} - 2\nu_{7a}$ and α is equal to 27, 47, 67 and 87 respectively, with the condition $\mu_5(u, \chi_2, *) = \frac{1}{35}(4t + 13) \geq 0$, when $\chi(u^7) = \chi(5a)$ and with $\mu_7(u, \chi_2, *) = \frac{1}{35}(6t + \beta) \geq 0$, where β is equal to 32, 27 and 22 respectively in the other three cases to show that there are no solutions.

In the remaining two cases we obtain the system of inequalities

$$\begin{aligned} \mu_1(u, \chi_2, *) &= \frac{1}{35}(t_1 + \gamma) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{35}(-6t_1 + \delta) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{35}(24t_2 + 271) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{35}(-6t_2 + 256) \geq 0, \end{aligned}$$

where $t_1 = -2\nu_{5a} + 3\nu_{5b} + 2\nu_{7a}$, $t_2 = 3\nu_{5a} + 3\nu_{5b} + \nu_{7a}$ and $(\gamma, \delta) = (3, 17)$ for $\chi(u^7) = -3\chi(5a) + 4\chi(5b)$ and $(\gamma, \delta) = (-2, 12)$ for $\chi(u^7) = -4\chi(5a) + 5\chi(5b)$. Each of these cases has one solution $(\nu_{5a}, \nu_{5b}, \nu_{7a}) = (3, 12, -14)$ and $(3, 12, -14)$ respectively. Note that in both cases $(\nu_5, \nu_7) = (15, -14)$ as it was concluded before.

• It remains to prove that $V(\mathbb{Z}G)$ has no elements of orders 46, 55, 69, 77, 115, 161 and 253. We give a detailed proof for the order 46. Other cases can be derived similarly from the table below containing the data for the constraints on partial augmentations ν_p and ν_q for possible orders pq (including the order 46 as well) accordingly to (3)–(5).

If $|u| = 46$, using Proposition 5 for the ordinary character χ_{23} with $\chi(C_2) = -55$ and $\chi(C_{23}) = 0$, we obtain the system

$$\begin{aligned} \mu_0(u, \chi_{23}, *) &= \frac{110}{46}(-11\nu_2 + 287) \geq 0; & \mu_1(u, \chi_{23}, *) &= \frac{55}{46}(-\nu_2 + 576) \geq 0; \\ \mu_{23}(u, \chi_{23}, *) &= \frac{110}{46}(11\nu_2 + 288) \geq 0, \end{aligned}$$

which yields $\nu_2 \in \{-22, 24\}$. Put $t = 462\nu_2 - 22\nu_{23}$. Using the ordinary character $\chi = \chi_2 + \chi_5 + \chi_8$ such that $\chi(C_2) = 21$, $\chi(C_{23}) = -1$ we eliminate the case $\nu_2 = -22$ from the condition $\mu_0(u, \chi, *) = \frac{1}{46}(t + 2068) \geq 0$ and the case $\nu_2 = 24$ from $\mu_{23}(u, \chi, *) = \frac{1}{46}(-t + 2026) \geq 0$.

The data for the remaining orders are given in the table below.

$ u $	p	q	ξ, τ	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
46	2	23	$\xi = (23)_{[*]}$	-55	0	1	31570	-1210	0
			$\xi = (23)_{[*]}$	-55	0	0	31680	-55	0
			$\xi = (23)_{[*]}$	-55	0	23	31680	1210	0
			$\tau = (2, 5, 8)_{[*]}$	21	-1	0	2068	462	-22
			$\tau = (2, 5, 8)_{[*]}$	21	-1	23	2026	-462	22
55	5	11	$\xi = (3)_{[*]}$	3	0	0	265	120	0
				1	0	1	250	3	0
				11	0	11	250	-30	0
69	3	23	$\xi = (3, 5, 8, 15, 19)_{[5]}$ $\xi = (3, 5, 8, 15, 19)_{[5]}$ $\tau = (3, 3, 6)_{[2]}$	25	0	0	48189	1100	0
				25	0	23	48114	-550	0
				12	1	23	1966	-264	-22
77	7	11	$\xi = (3)_{[*]}$	1	0	0	259	60	0
				11	0	11	252	-10	0
115	5	23	$\xi = (3)_{[*]}$	3	0	0	265	264	0
				23	0	23	250	-66	0
161	7	23	$\xi = (2)_{[*]}$	2	0	0	35	264	0
				23	0	23	21	-44	0
253	11	23	$\xi = (2)_{[*]}$	1	0	0	33	220	0
				23	0	23	33	-10	0
							22	-22	0

In the table above the notation $\xi = (i)_{[p]}$ is used for p -Brauer characters.

2. PROOF OF THEOREM 2

Let $G \cong \text{Co}_2$. Then $|G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ and $\exp(G) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ (see [15, 18]). The ordinary and p -Brauer character tables of G for $p \in \{2, 3, 5, 7, 11, 23\}$ can be found using the GAP system [18], and the same remarks as in the case of Co_3 regarding the notation for characters and conjugacy classes applies.

The group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 23, 24, 28 and 30. As before, first we consider units of prime orders: 2, 3, 5, 7, 11 and 23. Then we investigate products of two different primes from this list which are not orders of elements of G .

- Let $|u| \in \{7, 11\}$. Using Proposition 4 we obtain that all partial augmentations except one are zero. Thus by Proposition 3 part (ii) of Theorem 2 is proved.

- Let $|u| = 2$. By (2) and Proposition 4 we get $\nu_{2a} + \nu_{2b} + \nu_{2c} = 1$. Put $t_1 = 9\nu_{2a} - 7\nu_{2b} + \nu_{2c}$ and $t_2 = 29\nu_{2a} + 13\nu_{2b} - 11\nu_{2c}$. Now using Proposition 5 we get

$$\begin{aligned}\mu_1(u, \chi_2, *) &= \frac{1}{2}(t_1 + 23) \geq 0; & \mu_0(u, \chi_2, *) &= \frac{1}{2}(-t_1 + 23) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{2}(t_2 + 253) \geq 0; & \mu_1(u, \chi_3, *) &= \frac{1}{2}(-t_2 + 253) \geq 0,\end{aligned}$$

which has only 48 solutions listed in the Appendix B such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

- Let u be a unit of order 3. By (2) and Proposition 4 we get $\nu_{3a} + \nu_{3b} = 1$. Put $t_1 = 4\nu_{3a} - 5\nu_{3b}$. Using Proposition 5 we obtain the following system

$$\mu_0(u, \chi_2, *) = \frac{1}{3}(-2t_1 + 23) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{3}(t_1 + 23) \geq 0,$$

which has only four solutions listed in part (iv) of Theorem 2 such that all $\mu_i(u, \chi_2, *)$ are non-negative integers.

- Let u be a unit of order 5. By (2) and Proposition 4 we get $\nu_{5a} + \nu_{5b} = 1$. If put $t_1 = 2\nu_{5a} - 3\nu_{5b}$, then using Proposition 5 we obtain

$$\mu_0(u, \chi_2, *) = \frac{1}{5}(-4t_1 + 23) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{5}(t_1 + 23) \geq 0,$$

which has only six solutions listed in part (v) of Theorem 2 such that all $\mu_i(u, \chi_2, *)$ are non-negative integers.

- Let u be a unit of order 23. By (2) and Proposition 4 we get $\nu_{23a} + \nu_{23b} = 1$. Put $t_1 = 12\nu_{23a} - 11\nu_{23b}$. Using Proposition 5 we obtain the system

$$\begin{aligned}\mu_1(u, \chi_4, 2) &= \frac{1}{23}(-t_1 + 748) \geq 0; & \mu_1(u, \chi_9, 3) &= \frac{1}{23}(t_1 + 9372) \geq 0; \\ \mu_5(u, \chi_4, 2) &= \frac{1}{23}(11\nu_{23a} - 12\nu_{23b} + 748) \geq 0,\end{aligned}$$

which has only 66 solutions listed in part (vi) of Theorem 2 such that all $\mu_i(u, \chi_j)$ are non-negative integers.

Now we will deal with torsion units of orders which do not appear in G .

- Let $|u| = 21$. We need to consider four cases defined by part (iv) of Theorem 2.

Case 1. $\chi(u^7) = \chi(3a)$. Put $t_1 = 4\nu_{3a} - 5\nu_{3b} - 2\nu_{7a}$. Then we obtain the system

$$\mu_3(u, \chi_2, *) = \frac{1}{21}(2t_1 + 13) \geq 0; \quad \mu_0(u, \chi_2, *) = \frac{1}{21}(-12t_1 + 27) \geq 0,$$

which has no solution.

Case 2. $\chi(u^7) = \chi(3b)$. Put $t_1 = -4\nu_{3a} + 5\nu_{3b} + 2\nu_{7a}$, $t_2 = 10\nu_{3a} + 10\nu_{3b} + \nu_{7a}$. Then we obtain the following incompatible system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{21}(12t_1 + 45) \geq 0; & \mu_1(u, \chi_2, *) &= \frac{1}{21}(t_1 + 16) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{21}(t_2 + 242) \geq 0; & \mu_7(u, \chi_3, *) &= \frac{1}{21}(-6t_2 + 249) \geq 0; \\ \mu_7(u, \chi_2, *) &= \frac{1}{21}(-6t_1 + 30) \geq 0; & \mu_0(u, \chi_3, *) &= \frac{1}{21}(12t_2 + 279) \geq 0; \\ \mu_1(u, \chi_5, *) &= \frac{1}{21}(-11\nu_{3a} + 16\nu_{3b} + 1755) \geq 0.\end{aligned}$$

Case 3. $\chi(u^7) = -\chi(3a) + 2\chi(3b)$. Put $t_1 = 4\nu_{3a} - 5\nu_{3b} - 2\nu_{7a}$. The system

$$\begin{aligned}\mu_7(u, \chi_2, *) &= \frac{1}{21}(6t_1 + 21) \geq 0; & \mu_0(u, \chi_2, *) &= \frac{1}{21}(-12t_1 + 63) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{21}(-t_1 + 7) \geq 0,\end{aligned}$$

has no integral solutions.

Case 4. $\chi(u^7) = -2\chi(3a) + 3\chi(3b)$. Put $t_1 = -4\nu_{3a} + 5\nu_{3b} + 2\nu_{7a}$ and $t_2 = 10\nu_{3a} + 10\nu_{3b} + \nu_{7a}$. Then the following system is incompatible:

$$\begin{aligned}\mu_1(u, \chi_2, *) &= \frac{1}{21}(t_1 - 2) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{21}(-6t_1 + 12) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{21}(12t_2 + 279) \geq 0; & \mu_1(u, \chi_3, *) &= \frac{1}{21}(t_2 + 242) \geq 0; \\ \mu_7(u, \chi_3, *) &= \frac{1}{21}(-6t_2 + 249) \geq 0.\end{aligned}$$

• Let u be a unit of order 22. We will use the same approach as in the case of units of order 33 in the proof of Theorem 1. Taking into account part (ii) of the Theorem 2 with respect to torsion units of order 11, we obtain the system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{22}(-t_1 + 33) \geq 0; & \mu_{11}(u, \chi_2, *) &= \frac{1}{22}(t_1 + 33) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{22}(-t_2 + 22) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{22}(t_2 + 22) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{22}(t_3 + 253) \geq 0; & \mu_{11}(u, \chi_3, *) &= \frac{1}{22}(-t_3 + 253) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{22}(t_4 + 253) \geq 0; & \mu_2(u, \chi_3, *) &= \frac{1}{22}(-t_4 + 253) \geq 0; \\ \mu_0(u, \chi_4, *) &= \frac{1}{22}(t_5 + 275) \geq 0; & \mu_{11}(u, \chi_4, *) &= \frac{1}{22}(-t_5 + 275) \geq 0,\end{aligned}$$

where

$$\begin{aligned}t_1 &= 90\nu_{2a} - 70\nu_{2b} + 10\nu_{2c} - 10\nu_{11a} + 9\nu_{2a}^{(11)} - 7\nu_{2b}^{(11)} + \nu_{2c}^{(11)}; \\ t_2 &= 9\nu_{2a} - 7\nu_{2b} + \nu_{2c} - \nu_{11a} - 9\nu_{2a}^{(11)} + 7\nu_{2b}^{(11)} - \nu_{2c}^{(11)}; \\ t_3 &= 290\nu_{2a} + 130\nu_{2b} - 110\nu_{2c} + 29\nu_{2a}^{(11)} + 13\nu_{2b}^{(11)} - 11\nu_{2c}^{(11)}; \\ t_4 &= 29\nu_{2a} + 13\nu_{2b} - 11\nu_{2c} - 29\nu_{2a}^{(11)} - 13\nu_{2b}^{(11)} + 11\nu_{2c}^{(11)}; \\ t_5 &= 510\nu_{2a} + 350\nu_{2b} + 110\nu_{2c} + 51\nu_{2a}^{(11)} + 35\nu_{2b}^{(11)} + 11\nu_{2c}^{(11)},\end{aligned}$$

which has no integral solutions such that each $\mu_i(u, \chi_j, *)$ is a non-negative integer.

• Let u be a unit of order 33. Using Proposition 5 for the ordinary character χ_3 , for which $\chi(C_3) = 10$ and $\chi(C_{11}) = 0$, we obtain the system

$$\mu_0(u, \chi_3, *) = \frac{1}{33}(200\nu_3 + 273) \geq 0; \quad \mu_{11}(u, \chi_3, *) = \frac{1}{33}(-100\nu_3 + 243) \geq 0,$$

which has no solutions such that all $\mu_i(u, \chi_3, *)$ are non-negative integers.

• Let u be a unit of order 35. For this order we have situation similar to the 3rd Conway group. It is possible to show that the best restriction that we can get

is the system obtained by Proposition 5 for the 3-Brauer character χ_7 , for which $\chi(C_5) = 4$ and $\chi(C_7) = 0$:

$$\mu_0(u, \chi_7, 3) = \frac{1}{35}(96\nu_5 + 2270) \geq 0; \quad \mu_7(u, \chi_7, 3) = \frac{1}{35}(-24\nu_5 + 2250) \geq 0,$$

from which $\nu_5 \in \{-20, 15, 50, 85\}$. Using further analysis we are able to eliminate three of these opportunities, but not all. Now considering six cases defined by part (v) of Theorem 1, we obtain exactly the same systems of inequalities that either has no solutions or lead to solutions listed in part (vii) of Theorem 2.

• To complete the proof of part (i), it remains to show that there are no elements of orders 46, 55, 69, 77, 115, 161 and 253 in $V(\mathbb{Z}G)$. As in the proof of Theorem 1, below we give the table containing the data describing the constraints on partial augmentations ν_p and ν_q accordingly to (3)–(5) for all these orders. From this table part (i) of Theorem 2 is derived in the same way as in the proof of Theorem 1.

$ u $	p	q	ξ, τ	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
46	2	23	$\xi = (2, 3, 5)_{[*]}$	-1	0	1	2046	-22	0
			$\xi = (2, 3, 5)_{[*]}$	-1	0	2	2046	1	0
			$\xi = (2, 3, 5)_{[*]}$	-1	0	23	2048	22	0
			$\tau = (2, 4, 5)_{[*]}$	21	-1	0	2068	462	-22
			$\tau = (2, 4, 5)_{[*]}$	21	-1	23	2026	-462	22
55	5	11	$\xi = (3)_{[*]}$	3	0	0	265	120	0
						5	265	-12	0
						11	250	-30	0
69	2	23	$\xi = (3)_{[*]}$	10	0	0	273	440	0
						23	243	-220	0
77	7	11	$\xi = (4)_{[*]}$	3	0	0	287	120	0
						11	273	-20	0
115	5	23	$\xi = (3)_{[*]}$	3	0	0	265	264	0
						23	250	-66	0
161	7	23	$\xi = (2)_{[*]}$	2	0	0	35	264	0
						23	21	-44	0
253	11	23	$\xi = (2)_{[*]}$	1	0	0	33	220	0
						11	33	-10	0
						23	22	-22	0

3. PROOF OF THEOREM 3

Let $G \cong \text{Co}_1$. Then (see [15, 18]) we know that $|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$ and $\exp(G) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23$. The ordinary and p -Brauer character tables of G for $p \in \{7, 11, 23\}$ can be found using the GAP system [18], and we use the same approach to the notation for characters and conjugacy classes.

- Let $|u| \in \{11, 12\}$. Using Proposition 4 we obtain that all partial augmentations except one are zero. Thus by Proposition 3 part (ii) of Theorem 3 is proved.
- Let $|u| = 7$. By (2) and Proposition 4 we get $\nu_{7a} + \nu_{7b} = 1$. Put $t = 10\nu_{7a} + 3\nu_{7b}$. Then using Proposition 5 we obtain the system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{7}(6t + 276) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{7}(-t + 276) \geq 0,$$

which yields only 47 solutions listed in part (iii) of Theorem 3.

- Let $|u| = 23$. Similarly to the above, $\nu_{23a} + \nu_{23b} = 1$ and we get the system

$$\begin{aligned} \mu_1(u, \chi_{17}, *) &= \frac{1}{23}(12\nu_{23a} - 11\nu_{23b} + 673750) \geq 0; \\ \mu_5(u, \chi_{17}, *) &= \frac{1}{23}(-11\nu_{23a} + 12\nu_{23b} + 673750) \geq 0, \end{aligned}$$

which has only 58588 solutions listed in part (iv) of Theorem 3.

Now we consider torsion units of orders which do not appear in G . First we will show that $V(\mathbb{Z}G)$ has no units of order 46, 69, 77, 91, 115, 143, 161, 253 and 299.

As in the proof of Theorem 1, below we give the table containing the data describing the constraints on partial augmentations ν_p and ν_q accordingly to (3)–(5) for orders 91, 143, 161, 253 and 299.

$ u $	p	q	ξ, τ	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
91	7	13	$\xi = (3)_{[*]}$	5	0	0	329	360	0
						13	294	-60	0
143	2	0	$\xi = (3)_{[*]}$	2	0	0	319	240	0
						13	297	-24	0
161	7	23	$\xi = (3)_{[*]}$	5	0	0	329	660	0
						23	294	-110	0
253	11	23	$\xi = (3)_{[*]}$	2	0	0	319	440	0
						11	319	-20	0
299	13	23	$\xi = (2)_{[*]}$	3	0	0	312	792	0
						23	273	-66	0

- Let $|u| = 77$. Similarly to the case of order 33 in Theorem 1 we have that

$$\begin{aligned} \mu_{11}(u, \chi_2, *) &= \frac{1}{77}(-t_1 - 30\nu_{7b} - 10\nu_{11a} + 10\nu_{11a}^{(7)} - 3\nu_{7b}^{(11)} + 276) \geq 0; \\ \mu_0(u, \chi_3, *) &= \frac{1}{77}(3t_1 + 300\nu_{7b} + 120\nu_{11a} + 20\nu_{11a}^{(7)} + 30\nu_{7b}^{(11)} + 299) \geq 0; \\ \mu_0(u, \chi_4, *) &= \frac{1}{77}(6t_2 + 1771) \geq 0; \quad \mu_{11}(u, \chi_4, *) = \frac{1}{77}(-t_2 + 1771) \geq 0; \\ \mu_1(u, \chi_4, *) &= \frac{1}{77}(t_3 + 1771) \geq 0; \quad \mu_7(u, \chi_4, *) = \frac{1}{77}(-6t_3 + 1771) \geq 0; \\ \mu_0(u, \chi_7, *) &= \frac{1}{77}(2t_4 + 27300) \geq 0; \quad \mu_0(u, \chi_{15}, 13) = \frac{1}{77}(-t_4 + 474145) \geq 0, \end{aligned}$$

where $t_1 = 100\nu_{7a} + 10\nu_{7a}^{(11)}$, $t_2 = 140\nu_{7a} + 14\nu_{7a}^{(11)}$, $t_3 = 14\nu_{7a} - 14\nu_{7a}^{(11)}$ and $t_4 = 420\nu_{7a} - 60\nu_{11a} - 10\nu_{11a}^{(7)} + 42\nu_{7a}^{(11)}$. This system has no integral solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

- Let $|u| \in \{46, 69, 115\}$. Using the same method as in the case of order 33 in Theorem 1, we constructed systems of constraints and verified with the constraint solver ECLiPSe [17] that they have no solutions, using the lower and upper bounds on partial augmentations given by Proposition 2. Remarkably, we immediately check that there are no units of order 69, while the enumeration of all possible partial augmentations for $|u| \in \{3, 23\}$ requires $15239 \cdot 58588 = 892822532$ cases.

- Let $|u| \in \{55, 65\}$. Clearly, $|u^{11}| = |u^{13}| = 5$. Using the LAGUNA package [12] together with ECLiPSe [17], we produce 1041 possible tuples of partial augmentations for units of order 5. From corresponding systems we computed solutions listed in parts (v) and (vi). Note that p -Brauer character tables for G are not known for $p \in \{2, 3, 5\}$ (see <http://www.math.rwth-aachen.de/~MOC/work.html>), and hopefully, further progress could be made when p -Brauer character tables for missing values of p will become available.

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Appendix A.

Partial augmentations $(\nu_{3a}, \nu_{3b}, \nu_{3c})$ for units of order 3 in $\mathbb{Z}\mathbb{C}o_3$:

$(-9, -3, 13)$	$(-9, -2, 12)$	$(-9, -1, 11)$	$(-8, -5, 14)$	$(-8, -4, 13)$	$(-8, -3, 12)$
$(-8, -2, 11)$	$(-8, -1, 10)$	$(-8, 0, 9)$	$(-7, -5, 13)$	$(-7, -4, 12)$	$(-7, -3, 11)$
$(-7, -2, 10)$	$(-7, -1, 9)$	$(-7, 0, 8)$	$(-6, -4, 11)$	$(-6, -3, 10)$	$(-6, -2, 9)$
$(-6, -1, 8)$	$(-6, 0, 7)$	$(-6, 1, 6)$	$(-5, -4, 10)$	$(-5, -3, 9)$	$(-5, -2, 8)$
$(-5, -1, 7)$	$(-5, 0, 6)$	$(-5, 1, 5)$	$(-4, -3, 8)$	$(-4, -2, 7)$	$(-4, -1, 6)$
$(-4, 0, 5)$	$(-4, 1, 4)$	$(-4, 2, 3)$	$(-3, -3, 7)$	$(-3, -2, 6)$	$(-3, -1, 5)$
$(-3, 0, 4)$	$(-3, 1, 3)$	$(-3, 2, 2)$	$(-2, -2, 5)$	$(-2, -1, 4)$	$(-2, 0, 3)$
$(-2, 1, 2)$	$(-2, 2, 1)$	$(-2, 3, 0)$	$(-1, -2, 4)$	$(-1, -1, 3)$	$(-1, 0, 2)$
$(-1, 1, 1)$	$(-1, 2, 0)$	$(-1, 3, -1)$	$(0, -1, 2)$	$(0, 0, 1)$	$(0, 1, 0)$
$(0, 2, -1)$	$(0, 3, -2)$	$(0, 4, -3)$	$(1, -1, 1)$	$(1, 0, 0)$	$(1, 1, -1)$
$(1, 2, -2)$	$(1, 3, -3)$	$(1, 4, -4)$	$(2, 0, -1)$	$(2, 1, -2)$	$(2, 2, -3)$
$(2, 3, -4)$	$(2, 4, -5)$	$(2, 5, -6)$	$(3, 0, -2)$	$(3, 1, -3)$	$(3, 2, -4)$
$(3, 3, -5)$	$(3, 4, -6)$	$(3, 5, -7)$	$(4, 1, -4)$	$(4, 2, -5)$	$(4, 3, -6)$
$(4, 4, -7)$	$(4, 5, -8)$	$(4, 6, -9)$	$(5, 1, -5)$	$(5, 2, -6)$	$(5, 3, -7)$
$(5, 4, -8)$	$(5, 5, -9)$	$(5, 6, -10)$	$(6, 2, -7)$	$(6, 3, -8)$	$(6, 4, -9)$
$(6, 5, -10)$	$(6, 6, -11)$	$(6, 7, -12)$	$(7, 2, -8)$	$(7, 3, -9)$	$(7, 4, -10)$
$(7, 5, -11)$	$(7, 6, -12)$	$(7, 7, -13)$	$(8, 3, -10)$	$(8, 4, -11)$	$(8, 5, -12)$
$(8, 6, -13)$	$(8, 7, -14)$	$(8, 8, -15)$	$(9, 3, -11)$	$(9, 4, -12)$	$(9, 5, -13)$
$(9, 6, -14)$	$(9, 7, -15)$	$(9, 8, -16)$	$(10, 4, -13)$	$(10, 5, -14)$	$(10, 6, -15)$
$(10, 7, -16)$	$(10, 8, -17)$	$(10, 9, -18)$	$(11, 4, -14)$	$(11, 5, -15)$	$(11, 6, -16)$
$(11, 7, -17)$	$(11, 8, -18)$	$(11, 9, -19)$	$(12, 5, -16)$	$(12, 6, -17)$	$(12, 7, -18)$
$(12, 8, -19)$	$(12, 9, -20)$	$(12, 10, -21)$	$(13, 5, -17)$	$(13, 6, -18)$	$(13, 7, -19)$
$(13, 8, -20)$	$(13, 9, -21)$	$(13, 10, -22)$	$(14, 6, -19)$	$(14, 7, -20)$	$(14, 8, -21)$
$(14, 9, -22)$	$(14, 10, -23)$	$(14, 11, -24)$	$(15, 6, -20)$	$(15, 7, -21)$	$(15, 8, -22)$
$(15, 9, -23)$	$(15, 10, -24)$	$(15, 11, -25)$	$(16, 7, -22)$	$(16, 8, -23)$	$(16, 9, -24)$
$(16, 10, -25)$	$(16, 11, -26)$	$(16, 12, -27)$	$(17, 7, -23)$	$(17, 8, -24)$	

Appendix B.

Partial augmentations $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ for units of order 2 in $\mathbb{Z}\mathbb{C}o_2$:

$(-4, -3, 8)$	$(-4, -2, 7)$	$(-4, -1, 6)$	$(-3, -5, 9)$	$(-3, -4, 8)$	$(-3, -3, 7)$	$(-3, -2, 6)$
$(-3, -1, 5)$	$(-3, 0, 4)$	$(-2, -4, 7)$	$(-2, -3, 6)$	$(-2, -2, 5)$	$(-2, -1, 4)$	$(-2, 0, 3)$
$(-2, 1, 2)$	$(-1, -3, 5)$	$(-1, -2, 4)$	$(-1, -1, 3)$	$(-1, 0, 2)$	$(-1, 1, 1)$	$(-1, 2, 0)$
$(0, -2, 3)$	$(0, -1, 2)$	$(0, 0, 1)$	$(0, 1, 0)$	$(0, 2, -1)$	$(0, 3, -2)$	$(1, -1, 1)$
$(1, 0, 0)$	$(1, 1, -1)$	$(1, 2, -2)$	$(1, 3, -3)$	$(1, 4, -4)$	$(2, 0, -1)$	$(2, 1, -2)$
$(2, 2, -3)$	$(2, 3, -4)$	$(2, 4, -5)$	$(2, 5, -6)$	$(3, 1, -3)$	$(3, 2, -4)$	$(3, 3, -5)$
$(3, 4, -6)$	$(3, 5, -7)$	$(3, 6, -8)$	$(4, 2, -5)$	$(4, 3, -6)$	$(4, 4, -7)$	

INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, P.O. BOX 12, H-4010 DEBRECEN,
HUNGARY

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF ST ANDREWS, NORTH HAUGH, ST ANDREWS,
FIFE, KY16 9SX, SCOTLAND

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF ST ANDREWS, NORTH HAUGH, ST ANDREWS,
FIFE, KY16 9SX, SCOTLAND