On union ultrafilters

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We present some new results on union ultrafilters. We characterize stability for union ultrafilters and, as the main result, we construct a new kind of unordered union ultrafilter.

Introduction

The equivalent notions of union and strongly summable ultrafilters have been important examples of idempotent ultrafilters ever since they were first conceived in [Hin72], [Bla87]. Their unique properties have been a useful tool in set theory, algebra in the Stone-Čech compactification and set theoretic topology. For example, strongly summable ultrafilters were, in a manner of speaking, the first idempotent ultrafilters known, cf. [Hin72] and [HS98, notes to Chapter 5]; they were the first strongly right maximal idempotents known and, even stronger, they are the only known idempotent with a maximal group isomorphic to $\mathbb{Z}$; their existence is independent of *ZFC*, since it implies the existence of (rapid) P-points, cf. [BH87]; since a strongly summable is strongly right maximal, its orbit is a van Douwen space, cf. [HS02].

This article will focus on union ultrafilters, studying the various kinds of union ultrafilters and as the main result constructing of a new kind of union ultrafilter answering a question of Andreas Blass.

The presentation of the proofs is inspired by [Ler83] and [Lam95] splitting the proofs into different levels, at times adding [[in the elevator]] comments in between. The typesetting incorporates ideas from [Tuf05] highlighting details in the proofs and structural remarks in the margin. Online discussion is possible through the author's website at `http://peter.krautzberger.info/paper`

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1 Preliminaries

Let us begin by giving a non-exhaustive selection of standard terminology in which we follow N. Hindman and D. Strauss [HS98]; for standard set theoretic notation we refer to T. Jech [Jec03], e.g., natural numbers are considered as ordinals, i.e., $n = \{0, \dots, n-1\}$. We work in *ZFC* throughout. The main objects of this paper are *(ultra)filters* on an infinite set S, i.e., (maximal) proper subsets of the power set $\mathfrak{P}(S)$ closed under taking finite intersections and supersets. S carries the discrete topology in which case the set of ultrafilters is βS, its Stone-Čech compactification. The Stone topology on βS is generated by basic clopen sets induced by subsets $A \subseteq S$ in the form $\overline{A} := \{p \in \beta S \mid A \in p\}$. Filters are usually denoted by upper case Roman letters, mostly F, G, H, ultrafilters by lower case Roman letters, mostly p, q, r, u.

The set S is always assumed to be the domain of a *(partial) semigroup* $(S, \cdot)$, i.e., the (partial) operation $\cdot$ fulfills the associativity law $s \cdot (t \cdot v) = (s \cdot t) \cdot v$ (in the sense that if one side is defined, then so is the other and they are equal). For a partial semigroup S and $s \in S$ the set of elements compatible with s is denoted by $\sigma(s) := \{t \in S \mid s \cdot t \text{ is defined}\}$. A partial semigroup is also assumed to be *adequate*, i.e., $\{\sigma(s) \mid s \in S\}$ has the finite intersection property. We denote the generated filter by $\sigma(S)$ and the corresponding closed subset of βS by δS. For partial semigroups S, T a map $\varphi : S \to T$ is a *partial semigroup homomorphism* if $\varphi[\sigma(s)] \subseteq \sigma(\varphi(s))$ (for $s \in S$) and

(Partial) Semigroup

$$(\forall s \in S)(\forall s' \in \sigma(s))\ \varphi(s \cdot s') = \varphi(s) \cdot \varphi(s').$$

To simplify notation in a partial semigroup, $s \cdot t$ is always meant to imply $t \in \sigma(s)$. For $s \in S$, the restricted multiplication to s from the left (right) is denoted by λ_s (ρ_s).

It is easy to see that the operation of a partial semigroup can always be extended to a full semigroup operation by adjoining a (multiplicative) zero which takes the value of all undefined products. One key advantage of partial semigroups is that partial subsemigroups are usually much more diverse than subsemigroups. Nevertheless, it is convenient to think about most theoretical aspects (such as extension to βS) with a full operation in mind.

The semigroups considered in this paper are $(\mathbb{N}, +)$ (with $\mathbb{N} := \omega \setminus \{0\}$) and the most important adequate partial semigroup $\mathbb{F}$.

Definition 1.1 On $\mathbb{F} := \{s \subseteq \omega \mid \emptyset \neq s \text{ finite}\}$ we define a partial semigroup structure by

The partial semigroup $\mathbb{F}$

$$s \cdot t := s \cup t \text{ if and only if } s \cap t = \emptyset.$$

The theory of the Stone-Čech compactification allows for the (somewhat unique) extension of any operation on S to its compactification, in particular a semigroup operation.

Definition 1.2 For a semigroup $(S, \cdot)$, $s \in S$ and $A \subseteq S$, $p, q \in \beta S$ we define the following. The semigroup βS

- $s^{-1}A := \{t \in S \mid st \in A\}$.
- $A^{-q} := \{s \in S \mid s^{-1}A \in q\}$.
- $p \cdot q := \{A \subseteq S \mid A^{-q} \in p\}$.
  Equivalently, $p \cdot q$ is generated by sets $\bigcup_{v \in V} v \cdot W_v$ for $V \in p$ and each $W_v \in q$.
- $A^\star := A^{-q} \cap A$.
  This notation will only be used when there is no confusion regarding the chosen ultrafilter.

As is well known, this multiplication on βS is well defined and extends the operation on S. It is associative and right topological, i.e., the operation with fixed right hand side, ρ_q, is continuous. For these and all other theoretical background we refer to [HS98].

In the case of a partial semigroup, ultrafilters in δS in a way multiply as if the partial operation was total. With the arguments from the following proposition it is a simple but useful exercise to check that if $(S, \cdot)$ is partial the above definitions still work just as well in the sense that $s^{-1}A := \{t \in \sigma(s) \mid st \in A\}$ and $p \cdot q$ is only defined if it is an ultrafilter.

Proposition 1.3
Let S be a partial subsemigroup of a semigroup T. Then δS is a subsemigroup of βT. *The semigroup δS*

Proof. (1.) Simply observe that, by strong associativity, for all $a \in S$

$$\bigcup_{b \in \sigma(a)} b \cdot (\sigma(ab) \cap \sigma(b)) \subseteq \sigma(a).$$

(2.) Therefore $\sigma(S) \subseteq p \cdot q$ whenever $p, q \in \delta S$. □

It is easy to similarly check that partial semigroup homomorphisms extend to full semigroup homomorphisms on δS.

Since A^{-q} is not an established notation, the following useful observations present a good opportunity to test it.

Proposition 1.4
Let $p, q \in \beta S$, $A \subseteq S$ and $s, t \in S$. *Tricks with A^{-q}*

- $t^{-1}s^{-1}A = (st)^{-1}A$.
- $s^{-1}A^{-q} = (s^{-1}A)^{-q}$.
- $(A \cap B)^{-q} = A^{-q} \cap B^{-q}$.
- $(s^{-1}A)^\star = s^{-1}A^\star$ *(with respect to the same ultrafilter).*

- $(A^{-q})^{-p} = A^{-(p \cdot q)}$.

  Proof. This is straightforward to check. □

The proverbial big bang for the theory of ultrafilters on semigroups is the following theorem.

Theorem 1.5 (Ellis-Numakura Lemma)
If $(S, \cdot)$ is a compact, right topological semigroup then there exists an idempotent *element in S, i.e., an element $p \in S$ such that $p \cdot p = p$.*

Proof. See, e.g., [HS98, notes to Chapter 2]. □

Therefore the following classical fact is meaningful.

Lemma 1.6 (Galvin Fixpoint Lemma)
For idempotent $p \in \beta S$, $A \in p$ implies $A^\star \in p$ and $(A^\star)^\star = A^\star$.

Proof. $(A^\star)^\star = A^\star \cap (A^\star)^{-p} = A^\star \cap (A \cap A^{-p})^{-p} = A^\star \cap A^{-p} \cap A^{-p \cdot p} = A^\star \cap A^{-p} = A^\star$. □

The following definitions are central in what follows. Even though we mostly work in $\mathbb{N}$ and $\mathbb{F}$ we formulate them for a general setting.

FP-sets, **x**-support and condensations

Definition 1.7 Let $\mathbf{x} = (x_n)_{n<N}$ (with $N \leq \omega$) be a sequence in a partial semigroup $(S, \cdot)$ and let $K \leq \omega$.

- The set of finite products (the *FP-set*) is defined as

$$FP(\mathbf{x}) := \{\prod_{i \in v} x_i \mid v \in \mathbb{F}\},$$

  where products are in increasing order of the indices. In this case, all products are assumed to be defined.[1]

- $\mathbf{x}$ has *unique representations* if for $v, w \in \mathbb{F}$ the fact $\prod_{i \in v} x_i = \prod_{j \in w} x_j$ implies $v = w$.

- If $\mathbf{x}$ has unique representations and $z \in FP(\mathbf{x})$ we can define the $\mathbf{x}$-*support of* z, short $\mathbf{x}\text{-supp}(z)$, by the equation $z = \prod_{j \in \mathbf{x}\text{-supp}(z)} x_j$. We can then also define $\mathbf{x}\text{-min} := \min \circ \mathbf{x}\text{-supp}$, $\mathbf{x}\text{-max} := \max \circ \mathbf{x}\text{-supp}$.

- A sequence $\mathbf{y} = (y_j)_{j<K}$ is called a *condensation* of $\mathbf{x}$, in short $\mathbf{y} \sqsubseteq \mathbf{x}$, if

$$FP(\mathbf{y}) \subseteq FP(\mathbf{x}).$$

  In particular, $\{y_i \mid i < K\} \subseteq FP(\mathbf{x})$. For convenience, $\mathbf{x}\text{-supp}(\mathbf{y}) := \mathbf{x}\text{-supp}[\{y_i \mid i \in \omega\}]$.

- Define $FP_k(\mathbf{x}) := FP(\mathbf{x}')$ where $x'_n = x_{n+k}$ for all n.

[1]Note that we will mostly deal with commutative semigroups so the order of indices is not too important in what follows.

- FP-sets have a natural partial subsemigroup structure induced by $\mathbb{F}$, i.e., $(\prod_{i\in s} x_i)\cdot(\prod_{i\in t} x_i)$ is defined as in S but only if $\max(s) < \min(t)$. With respect to this restricted operation define $FP^\infty(\mathbf{x}) := \delta FP(\mathbf{x}) = \bigcap_{k\in\omega} \overline{FP_k(\mathbf{x})}$.
- If the semigroup is written additively, we write $FS(\mathbf{x})$ etc. accordingly (for finite sums); for $\mathbb{F}$ we write $FU(\mathbf{x})$ etc. (for finite unions).

Instead of saying that a sequence has certain properties it is often convenient to say that the generated FP-set does.

The following classical result is the starting point for most applications of algebra in the Stone-Čech compactification. We formulate it for partial semigroups.

Theorem 1.8 (Galvin-Glazer Theorem)
Let $(S,\cdot)$ be a partial semigroup, $p\in\delta S$ idempotent and $A\in p$. Then there exists $\mathbf{x}=(x_i)_{i\in\omega}$ in A such that

$$FP(\mathbf{x})\subseteq A.$$

Proof. This can be proved essentially just like the the original theorem, cf. [HS98, Theorem 5.8], using the fact that $\sigma(S)\subseteq p$ to guarantee all products are defined. □

An immediate corollary is, of course, the following classical theorem, originally proved combinatorially for $\mathbb{N}$ in [Hin74].

Theorem 1.9 (Hindman's Theorem)
Let $S = A_0 \cup A_1$. Then there exists $i\in\{0,1\}$ and a sequence $\mathbf{x}$ such that $FP(\mathbf{x})\subseteq A_i$.

2 Union Ultrafilters

This paper deals primarily with ultrafilters on the partial semigroup $\mathbb{F}$. The following definitions enable us to speak about the relevant properties of condensations in $\mathbb{F}$ smoothly.

Definition 2.1 (Condensation, ordered, meshed) Let $s,t\in\mathbf{F}$ and $\mathbf{s}=(s_i)_{i<N}$ be a disjoint sequence in $\mathbb{F}$ with $N\le\omega$.

Condensation, ordered, meshed

- We say that the pair (s,t) is *ordered*, in short $s<t$, if $\max(s)<\min(t)$.
- $\mathbf{s}$ is called *ordered* if $s_i<s_j$ for all $i<j<N$.
- $v,w\in\mathbb{F}$ are said to *mesh*, in short $v\sqcap w$, if neither $v<w$ nor $w<v$.

The following three kinds of ultrafilters were first described in [Bla87].

Definition 2.2 (union ultrafilters) An ultrafilter u on $\mathbb{F}$ is called

Union ultrafilters

- *union* if it has a base of FU-sets.

- *ordered union* if it has a base of FU-sets from ordered sequences.
- *stable union* if it is union and for any sequence $(FU(\mathbf{s}^\alpha))_{\alpha<\omega}$ in u there exists $FU(\mathbf{t}) \in u$ such that
$$(\forall \alpha < \omega)\ \mathbf{t} \sqsubseteq^* \mathbf{s}^\alpha,$$
i.e., $\mathbf{t}$ almost condenses all the sequences $\mathbf{s}^\alpha$ at once.

It is obvious yet important to note that FU-sets always have unique products and all products are defined. At this point it is also good to check the following. Union ultrafilters are elements of $\delta\mathbb{F}$ and it is not difficult to check that they are idempotent, cf. [Bla87, Proposition 3.3], [HS98, Theorem 12.19]. Even though the partial operation on $\mathbb{F}$ was defined only for disjoint elements it could just as well have been restricted to ordered elements. Of course, this would significantly change the operation on $\mathbb{F}$ (for example it would not be commutative anymore), but $\sigma(\mathbb{F})$ would still be the same and therefore $\delta\mathbb{F}$. Additionally, the operation on $\delta\mathbb{F}$ is not changed – it is after all still the extension of $\cup$ (or Δ) to $\beta\mathbb{F}$.

The following notion was introduced in [BH87] to help differentiate union ultrafilters.

Definition 2.3 (Additive isomorphism) Given partial semigroups S, T, call two ultrafilters $p \in \beta S, q \in \beta T$ *additively isomorphic* if there exist $FP(\mathbf{x}) \in p, FP(\mathbf{y}) \in q$ both with unique products such that the following map maps p to q

Additive isomorphism

$$\varphi : FP(\mathbf{x}) \to FP(\mathbf{y}), \prod_{i\in s} x_i \mapsto \prod_{i \in s} y_i.$$

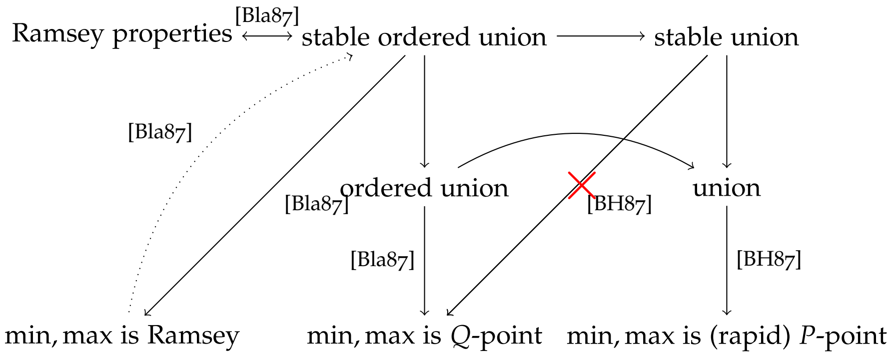
We call this map *the natural (partial semigroup) isomorphism* for FP-sets. As mentioned, it extends to a homomorphism (in fact, isomorphism) between $FP^\infty(\mathbf{s})$ and $FP^\infty(\mathbf{t})$.

This notion is a special case of equivalence in the Rudin-Keisler order, but arguably the natural notion for union ultrafilters since every idempotent ultrafilter is isomorphic to an ultrafilter that is not idempotent. For an example, consider the map $\mathbb{F} \to \mathbb{F}, s \mapsto \{\max(s)\}$; its extension to $\delta\mathbb{F}$ does have a product of ultrafilters in its range since the set of singletons does not contain any non-trivial products, i.e., union of two disjoint elements.

Figure 1 recapitulates the known implications between the types of union ultrafilters with references; the Ramsey properties will be described later. The dotted arrow represents the following: under CH, given two non-isomorphic Ramsey ultrafilters, there exists a stable ordered union ultrafilter that maps to them via min and max.

The one interesting non-implication missing is that a fortiori there consistently exist union ultrafilters that are not ordered union. However, the construction in [BH87] does not give any direct information on what it means to be an unordered union ultrafilter. In a manner of speaking it is a sledge hammer smashing orderedness so badly that is difficult to identify

Figure 1: Union ultrafilters



how orderedness actually fails. Because of this the main result is dedicated to understanding unordered union ultrafilters. In particular, our construction answers a question by Andreas Blass if there can be unordered union ultrafilters that map to Ramsey ultrafilters via max and min. The following result due to Andreas Blass will be needed later.

Theorem 2.4 (Homogeneity (Blass))
Let p_0, p_1 be non-isomorphic, selective ultrafilters and let $f \in 2^\omega$.

If $\mathfrak{P}(\omega)$ is partitioned into an analytic and coanalytic part, there are $X_i \in p_i$ $(i = 0, 1)$ such that the set of (ranges of) increasing sequences

$$(x_n)_{n\in\omega} \text{ with } x_n \in X_{f(n)}$$

*is homogeneous. We call such sequences f-*alternating.

Proof. This is [Bla88, Theorem 7] □

Regarding this theorem, the following folklore observation is very useful later; cf. [Bla87, Lemma 1.2].

Remark 2.5 Given any $f \in 2^\omega$ and non-isomorphic, selective ultrafilters p_0, p_1 and $A_i \in p_i$ $(i \in 2)$, there exists an f-alternating sequence in $A_0 \cup A_1$ such that its alternating parts are sets in p_0 and p_1 respectively.partition ω in intervals as follows:

Summary. We modify the argument from [Bla87, Lemma 1.2] for alternating sequences using a standard argument for not nearly coherent filters. ⊠

Proof. (1.) Pick $A_0 \in p_0$, $A_1 \in p_1$.

(2.) Let us say f *switches at* i if $f(i-1) = j$ and $f(i) = 1-j$ (for $j \in 2$).

(3.) It is easy to inductively partition ω into intervals I_n large enough so that both $|I_n \cap A_0|$ and $|I_n \cap A_1|$ are at least as large as the longest constant sequence in the range of f up to the $(2n)$th switch; in other words, we make the intervals long enough so that when we can build an f-alternating sequence with each alternating "block" contained in one of the I_n.

[[We will now thin out the ultrafilter sets so that they alternate (though not yet f-alternate) but with a twist: the thinned out sets will never meet the same interval I_n. After we accomplish this we can fill elements back in from the original A_i to get an f-alternating sequence. Since this only enlarges our sets, we stay in our filters.]]

(4.) Consider the interval collapsing map, mapping elements in I_n to n.

(5.) Since this collapsing map cannot map our non-isomorphic selectives to the same ultrafilter, we can find $B_0 \in p_0, B_1 \in p_1$ with $B_i \subseteq A_i$ ($i \in 2$) such that B_0, B_1 never meet the same interval I_n.

(6.) Next, consider the map on B_0 defined by taking $x \in B_0$ to the largest $y \in B_1$ with $y < x$; if this fails map x to 0.

(7.) Since p_0 is selective, this map becomes injective on a set $C_0 \in p_0$, $C_0 \subseteq B_0$. As a result, there must be at least one element from B_1 between every two elements in C_0.

(8.) The same argument for B_1 (comparing it to C_0) yields $C_1 \in p_1$, $C_1 \subseteq B_1$ such that there is at least one element from C_0 between every two elements in C_1.

(9.) This might, of course, have ruined said property of C_0, but we can safely fill in extra elements from B_1 to C_1 to reestablish it; we still call that possibly larger set C_1. In other words, C_0 and C_1 alternate.

(10.) Of course, this also means the intervals I_n that are met by C_0 and C_1 alternate.

(11.) Finally, by choice of the I_n, we can now form an f-alternating sequence that incluces $C_0 \cup C_1$.

   (a) Simply add elements from the original A_i to the C_i ($i \in 2$) in such a way that they still meet the same intervals but in a block large enough to become f-alternating.

   (b) Since $C_0 \in p_0$, $C_1 \in p_1$, this f-alternating sequence is as desired. □

3 Stability

Andreas Blass laid the foundation for all further research regarding union and hence strongly summable ultrafilters in [Bla87]. The final theorem from that paper gives a list of potent characterizations of the strongest notion, stable ordered union ultrafilters. (Un)fortunately, not all union ultrafilters are ordered. The first example was constructed in [BH87] and we will construct an example in the second part. However, all known constructions of union ultrafilter yield stable ones.

In this section we will discuss which of the characterizations for stable ordered union ultrafilters also hold for stable union ultrafilters. Because this requires a few definitions that are not relevant for the second half of this paper, we will proceed as follows. We will introduce the one notion that is also of interest for the second part and continue to prove the main result of this section. Following the proof we will discuss the other notions less formally since this does not require as much proof.

3.1 Stability and the Ramsey property for pairs

Definition 3.1 (Ramsey property for pairs) Consider $u \in \delta\mathbb{F}$.

Ramsey property for pairs

- We denote the ordered ordered pairs by $\mathbb{F}^2_<$, i.e.,

$$\mathbb{F}^2_< := \{(s,t) \in \mathbb{F}^2 \mid s < t\}.$$

  Often $(s < t)$ is a convenient notation for elements in $\mathbb{F}^2_<$.

- u has the *Ramsey property for pairs* if for any finite partition of $\mathbb{F}^2_<$ there exists $A \in u$ such that $A^2_<$ is homogeneous.

In [Bla87, Theorem 4.2] Andreas Blass showed that for ordered union ultrafilters the Ramsey property for pairs (and other properties we discuss later) is equivalent to stability. The following result shows that orderedness is not necessary for this equivalence.

However, it must be stressed that even though the formulation of the Ramsey property is the same, the result is quite different for the unordered case. For an ordered union ultrafilter we get homogeneity for all pairs from the generating sequence. In the unordered case, we do not get such a full property as we cannot check pairs of generators that mesh. We might try to blame this on our formulation of the Ramsey property. Why not ask for partitions of disjoint ordered pairs instead? Unfortunately, this is not possible as the partition of the disjoint pairs into ordered and unordered pairs yields a counterexample for all union ultrafilters, in fact, all idempotent ultrafilter in $\delta\mathbb{F}$. Every FU-set yields both ordered and unordered pairs no matter how nicely the generating sequence behaves.

Theorem 3.2
A union ultrafilters is stable if and only if it has the Ramsey property for pairs.

Summary. The argument (necessarily) follows the same strategy as the proof of [Bla87, Theorem 4.2]. The forward direction is similar to the proof of Ramsey's Theorem using a non-principal ultrafilter. To get a homogeneous set actually in the ultrafilter stability and a new kind of parity argument is applied.

The reverse conclusion is just as in the original proof by Andreas Blass. ⊠

Proof. (1.) The Ramsey property for pairs implies stability.

(a) Given any sequence $(FU(\mathbf{s}^\alpha))_{\alpha<\omega}$ in u consider the following set of ordered pairs

$$\{(v,w) \in \mathbb{F}^2_< \mid w \in \bigcap_{\alpha<\max(v)} FU(\mathbf{s}^\alpha)\}.$$

(b) Any $FU(\mathbf{t}) \in u$ will yield ordered pairs that are in the above set.

(i.) Pick any $v \in FU(\mathbf{t})$.

(ii.) Take $w > v$ from $FU(\mathbf{t}) \cap \bigcap_{\alpha<\max(v)} FU(\mathbf{s}^\alpha) \in u$.

(c) Therefore, by the Ramsey property for pairs, there must be a set $FU(\mathbf{s}) \in u$ such that all ordered pairs are included in the above set.

(d) Then $\mathbf{s} \sqsubseteq^* \mathbf{s}^\alpha$ for all $\alpha < \omega$.
  (i.) Given $\alpha < \omega$, pick s_i with $\max(s_i) > \alpha$.
  (ii.) Then all but finitely many s_j have $s_j > s_i$.
  (iii.) For such s_j of course $(s_i, s_j) \in FU(\mathbf{s})^2_<$, hence

$$s_j \in \bigcap_{\beta < \max(s_i)} FU(\mathbf{s}^\beta),$$

  (iv.) In particular $s_j \in FU(\mathbf{s}_\alpha)$ – as desired.

(2.) Stability implies the Ramsey property for pairs.
(a) Assume that $A_0 \dot\cup A_1 = \mathbb{F}^2_<$.
(b) Since u is an ultrafilter (in $\delta\mathbb{F}$)

Always pick one colour beyond x

$$(\forall x \in \mathbb{F})(\exists i)\{y \in \mathbb{F} \mid (x < y) \in A_i\} \in u.$$

(c) Since u is an ultrafilter it concentrates on one colour; without loss it is 0, i.e., there is $A \in u$ such that

A, C_x – almost always pick the same colour.

$$(\forall x \in A)\ C_x := \{y \in \mathbb{F} \mid (x < y) \in A_0\} \in u.$$

(d) Since u is union there are $FU(\mathbf{s}^\alpha) \in u$ (for $\alpha < \omega$) such that

$$FU(\mathbf{s}^\alpha) \subseteq \bigcap_{\max(x) \leq \alpha} C_x.$$

[[This small simplification ensures that for $x \in A$ we get $FU(\mathbf{s}^{\max(x)}) \subseteq C_x$ by choice of $\mathbf{s}^\alpha$.]]

(e) Since u is stable by assumption, there is $FU(\mathbf{s}) \in u$ such that

Stability – almost always pick from the same set

$$\mathbf{s} \sqsubseteq^* \mathbf{s}^\alpha \text{ for all } \alpha < \omega.$$

[[Next we introduce a function j that essentially just checks how many members of $\mathbf{s}$ are not included in $FU(\mathbf{s}^\alpha)$.]]

(f) Consider the following function

j – Counting where $\mathbf{s}$ fails

$$j : \omega \to \omega, \alpha \mapsto \max\{\max(s_i) \mid s_i \notin FU(\mathbf{s}^\alpha)\}.$$

Without loss, j is strictly increasing.

[[We can make j strictly increasing by replacing $FU(\mathbf{s}^\alpha)$ with $\bigcap_{\beta \leq \alpha} FU(\mathbf{s}^\beta)$ in the definition of j. Alternatively, intersecting $FU(\mathbf{s}^{\alpha+1})$ with $FU(\mathbf{s}^\alpha)$ when we defined them also guarantees $\mathbf{s}^{\alpha+1} \sqsubseteq \mathbf{s}^\alpha$ for all $\alpha < \omega$. In either case, the "losses" will at most increase with increasing α.]]

(g) Observe that for all $x \in FU(\mathbf{s})$

$$\min(x) > j(\alpha) \Rightarrow x \in FU(\mathbf{s}^\alpha).$$

  (i.) For s_i this follows by contraposition from the definition of j.
  (ii.) Therefore if $\min(x) > j(\alpha)$ this argument implies that all $s_i \subseteq x$ are in $FU(\mathbf{s}^\alpha)$.
  (iii.) In particular, so is their union, i.e., x.

[[After this observation the next goal is to construct $A' \in u$ for which $v < w$ in A' implies $\min(w) > j(\max(v))$. For then $w \in FU(\mathbf{s}^{\max(v)}) \subseteq C_v$. For this a new partition argument is needed.]]

Thinning out 1 – bounding j

(h) $\{x \in FU(\mathbf{s}) \mid j(\min(x)) < \max(x)\} \in u$.
 (i.) In any condensation of $\mathbf{s}$ there are x, x' and $x' \cup x$ such that $x < x'$ and $j(\min(x)) < \max(x')$.
 (ii.) But then calculate

$$j(\min(x \cup x')) = j(\min(x)) < \max(x') = \max(x \cup x').$$

 (iii.) Hence any set in u will intersect the above set; so it lies in u.

(i) In particular, there exists $FU(\mathbf{t}) \in u$ included in the above set.

Thinning out 2 – splitting points

(j) For $x \in FU(\mathbf{t})$ say that x *splits at* $n \in x$, whenever

$$x \cap (n+1), x \setminus (n+1) \in FU(\mathbf{t}) \text{ and}$$
$$(\exists t_k)\ x \cap (n+1) < t_k < x \setminus (n+1).$$

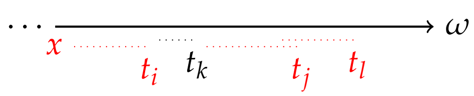


Figure 2: Splitting point – an example

Let $\pi(x)$ be the number of splitting points of x, i.e.,

$$\pi(x) := |\{n \in x \mid x \text{ splits at } n\}|$$

[[The splitting points tell how often x splits into two ordered parts (the one up to n and the one beyond n) – but more importantly with a gap in between.]]

(k) $\{x \in FU(\mathbf{t}) \mid \pi(x) = 1 \bmod 2\} \in u$.
 (i.) Any condensation of t will contain some $x < y < z$ and $x \cup z$.
 (ii.) In that case, the number of splitting points of $x \cup z$ is

$$\pi(x \cup z) = \pi(x) + \pi(z) + 1.$$

 (iii.) In particular, the number of splitting points for at least one of $x, z, x \cup z$ must be odd.

(l) In particular, there exists $FU(\mathbf{v}) \in u$ contained in the above set.

(m) For any $w_0 < w_1$ in $FU(\mathbf{v})$, there exists t_j with

$$w_0 < t_j < w_1$$

 (i.) Or else $\pi(w_0 \cup w_1) = \pi(w_0) + \pi(w_1)$ would be an even number of splitting points.

The conclusion

(n) $FU(\mathbf{v})$ is homogeneous, i.e., $FU(\mathbf{v})^2_< \subseteq A_0$.
 (i.) For this pick any $w, w' \in FU(\mathbf{v})$ with $w < w'$.

(ii.) By the last step, there exists some t_j with $w < t_j < w'$. Therefore

$$\min(w') > \max(t_j) > j(\min(t_j)) > j(\max(w))$$

where the third inequality holds because of step 2h, the last because j is strictly increasing.

(iii.) But as we noted just before step 2h, this implies $w' \in C_w$, i.e., $(w, w') \in A_0$ – as desired.

This concludes the proof. □

3.2 Stability and other partition properties

Let us now discuss the other properties from [Bla87, Theorem 4.2].

The Ramsey property for k-tuples It is straightforward to generalise the Ramsey property for pairs to k-tuples (with $k < \omega$) as follows. An ultrafilter u has the *Ramsey property for k-tuples* if for every partition of $\mathbb{F}^k_< := \{(s_0, \dots, s_{k-1}) \mid (\forall i < k-1) s_i < s_{i+1}\}$ we can find $A \in u$ such that $A^k_<$ is homogeneous. It is not difficult to show by an induction much like the induction used for Ramsey's Theorem for ω that the Ramsey property for pairs implies the Ramsey property for k-tuples for all $k < \omega$. An alternative argument follows from the property described in the next paragraph.

The infinitary Ramsey property In [Bla87, Theorem 4.2] Andreas Blass also discusses the infinitary analogue of the Ramsey properties. For this consider the set of ordered ω-sequences $\mathbb{F}^\omega_< := \{\mathbf{s} \in \mathbb{F}^\omega \mid (\forall i < \omega) s_i < s_{i+1}\}$. Then u has *the infinitary Ramsey property* if for every partition of $\mathbb{F}^\omega_<$ into an analytic and co-analytic part there exists $A \in u$ such that all ordered subsets of A are in the same part. It is not difficult to check that the proof in [Bla87] does not require the union ultrafilter to be ordered. It might be worthwhile to check that the strength of this infinitary partition property suffers even more than the finitary ones from dropping the ordered union requirement. For a stable ordered union ultrafilter not only do we get the infinitary Ramsey property, but the homogeneous set itself is generated by an ordered sequence, hence that ordered sequence is in that part of the partition. In the unordered case this statement simply does not make sense as the partition only covers ordered sequences.

Characterization via min The last two properties from [Bla87, Theorem 4.2] are formulated in terms of ultrapowers of ω. To keep our discussion short, we assume some basic knowledge of ultrapowers of ω; for a concise introduction cf. [Bla87, Section 1] (which is available at [Bla]). Given an ultrafilter V on a countable set I, we say that $f, g \in I^\omega$ are in the same *sky* if there exists $h, h' \in \omega^\omega$ such that on a set in V we have $g \leq h \circ f$ and

$f \le h' \circ g$. It is known that equivalently f, g are in the same sky if there exist finite-to-one h, h' such that $h \circ f = h' \circ g$. Skies are obviously order convex.

For lack of a better term, we say that u is *stable via* min if whenever $f \in \mathbb{F}^\omega, g \in \omega^\omega$ and $A \in u$ such that $f(s) < g \circ \min(s)$ for all $s \in A$, then there exists $h \in \omega^\omega$ such that $f(s) = h \circ \min(s)$ on some set in u; in other words, then the values of $f(s)$ only depend on $\min(s)$.[2]

In [Bla87, Theorem 4.2] it is essentially[3] shown that this notion is equivalent to stability for ordered union ultrafilters. For union ultrafilters we can show two things. On the one hand, the following observation shows that stability via min implies stability. On the other, the next section will include an example showing that (consistently) stability via min does not imply that a union ultrafilter is ordered.

Theorem 3.3
If a union ultrafilter is stable via min *then it has the Ramsey property for pairs. In particular, it is stable.*

Summary. We only sketch the argument since it is just a recombination of the argument for ordered union ultrafilters with a recent result by Andreas Blass. ⊠

Proof. (1.) Let u be a union ultrafilter that is stable via min.

(2.) The Ramsey property for pairs is easily seen to be equivalent to the statement that $\mathbb{F}^2_<$ and $(A \times A)_{A \in u}$ generate an ultrafilter on $\mathbb{F}^2$, namely the tensor product $u \otimes u$.

(3.) So take any ultrafilter V on $\mathbb{F}^2$ containing all these sets. We show that $V = u \otimes u$.

(4.) By a characterization of tensor products due to Puritz [Pur72], cf. also [Bla87, Section 1], it suffices to show that whenever $g_1, g_2 \in \omega^\omega$, then $g_1 \circ \pi_1$ lies in a lower sky than $g_2 \circ \pi_2$ (unless the latter is constant on a set in V).[4]

(5.) It suffices to compare $\max \circ \pi_1$ with $\min \circ \pi_2$.
- (a) $\pi_2(V) = u$ and by [BH87, Theorem 2] $\min(u)$ is a P-point, i.e., the sky of min contains exactly two skies, one of them the sky of constant functions.
- (b) Combining this with stability via min, we get that $\min \circ \pi_2$ is in the lowest non-standard sky for elements of the form $g_2 \circ \pi_2$.
- (c) Also, max is finite-to-one, so $\max \circ \pi_1$ is in the highest sky for elements of the form $g_1 \circ \pi_1$.

(6.) $\max \circ \pi_1$ is at most in the same sky as $\min \circ \pi_2$.
- (a) By assumption on V we have $\max \circ \pi_1 < \min \circ \pi_2$ on $\mathbb{F}^2_< \in V$.

(7.) $\max \circ \pi_1$ is not in the same sky as $\min \circ \pi_2$.

---

[2] In terms of the ultrapower this means that min generates an initial segment of the ultrapower.

[3] "Essentially" in the sense that the relevant part of [Bla87, Theorem 4.2] includes the condition that the image of the union ultrafilter under min is a P-point. This fact was established later for all union ultrafilters in [BH87, Theorem 2].

[4] Where π_i is the projection to the i-th coordinate ($i \in 2$).

(a) By [Bla09, Theorem 38], $\min(u), \max(u)$ are not near coherent filters, i.e., no two finite-to-one maps will map $\min(u)$ and $\max(u)$ to the same ultrafilter.
(b) But by [BH87, Theorem 2] $\min(u), \max(u)$ are both P-points.
(c) So no two maps will map $\min(u)$ and $\max(u)$ to the same non-principal ultrafilter.
(d) In other words, on any set in V, we have $h \circ \max \circ \pi_1 \neq h' \circ \min \circ \pi_2$ for any $h, h' \in \omega^\omega$ (unless both sides are constant).
(e) So $\max \circ \pi_1$ and $\min \circ \pi_2$ are not in the same sky.

(8.) Therefore $V = u \otimes u$, i.e., u is stable. □

The canonical partition property We say that that an ultrafilter $u \in \delta\mathbb{F}$ has *the canonical partition property* if for each $f : \mathbb{F} \to \omega$ there exists $A \in u$ such that $f \restriction A$ has one of the following properties:

- $f \restriction A$ is constant,
- $f \restriction A = g \circ \min \restriction A$ for some injective $g \in \omega^\omega$, in particular, the values only depend on $\min(s)$,
- $f \restriction A = g \circ (\min, \max) \restriction A$ for some injective $g : \omega^2 \to \omega$, in particular, the values of $f \restriction A$ depend only on $(\min(s), \max(s))$,
- $f \restriction A = g \circ \max \restriction A$ for some injective $g \in \omega^\omega$, in particular, the values of $f \restriction A$ depend only on $\max(s)$.
- $f \restriction A$ is injective.

Again, the proof of [Bla87, Theorem 4.2] not only shows that the canonical partition property implies stability via min, hence stability, for ordered union ultrafilters, but also for union ultrafilters in general. It remains open whether this property is equivalent to stability. Also, we do not know if it implies orderedness.

3.3 Stability and additive isomorphisms

We end this section with the following application of stability which will be useful later. The result is remininiscent of the role of P-points and Ramsey ultrafilters in the Rudin-Keisler order.

Lemma 3.4 (Stability and Additive Isomorphisms)
Every additively isomorphic image of a stable union ultrafilter is a stable union ultrafilter and every additively isomorphic image of a stable ordered union ultrafilter is a stable ordered union ultrafilter.

Summary. Stability is straightforward; for orderedness we use the Ramsey property of stable ordered union ultrafilters. ⊠

Proof. (1.) Let u' and u be additively isomorphic ultrafilters, i.e., there exist $FU(\mathbf{s}) \in u$, $FU(\mathbf{x}) \in u'$ such that

$$\pi : FU(\mathbf{s}) \to FU(\mathbf{x}), \prod_{i \in F} s_i \mapsto \prod_{i \in F} x_i$$

additionally has $\pi(u) = u'$.

(2.) If u is stable, so is u'.

(a) Given a sequence of condensations with FU-sets in u' we may assume without loss that all of them condense $\mathbf{x}$.

(b) But then the preimages under π form a sequence of condensations in u.

(c) Applying the stability of u yields a common almost condensation of the images.

(d) Then its image under π is exactly the desired common almost condensation in u'.

(3.) If u is stable ordered, so is u'.

(a) By step 2, we only need to show that u' is ordered. Pick any $A \in u'$; we may assume without loss $A \subseteq FU(\mathbf{x})$ and also that $\mathbf{s}$ (from the definition of π) is ordered.

(b) Consider

Working in u: a partition for orderedness.

$$X := \{(v, w) \in [\pi^{-1}[A]]^2_< \mid \max(\pi(v)) < \min(\pi(w))\}.$$

(c) By the Ramsey property from Theorem 3.2, there exists an ordered sequence t such that $FU(\mathbf{t}) \in u$ and $FU(\mathbf{t})$ is either included in or disjoint from X.

(d) But it cannot be disjoint from X.

(i.) Since π is injective, there exist $t_i < t_j$ such that $\pi(t_i) < \pi(t_j)$.

(ii.) In fact, given any t_i all but finitely many t_j have this property.

(iii.) Then $(t_i, t_j) \in X \cap FU(\mathbf{t})^2_<$.

(e) But this implies that $\pi[FU(\mathbf{t})] = FU(\pi[\mathbf{t}])$ is ordered – and of course in u' and refining A.

(f) In other words, u' is ordered union.

This concludes the proof. □

4 Unordered union ultrafilters

We now turn to our main result that selectivity of the image under min and max cannot indicate orderedness of a union ultrafilter. At first this is a negative result since the alternative would probably have involved a new partition theorem involving Ramsey ultrafilters. However, the construction of the counterexample offers an answer to the simple question: What does an unordered union ultrafilter look like? As mentioned earlier the construction in [BH87] does not really answer this question. Nevertheless, its proof represents a blueprint for constructions of (stable) union ultrafilters.

By definition, to be unordered means that there must be a "special" *FU*-set in the ultrafilter that will not be refined to an ordered *FU*-set in the ultrafilter.[5] In particular, the sequence generating the *FU*-set itself cannot be ordered. But if a sequence is not ordered, it is meshed in the sense that some of its members must mesh. Of course, such a sequence will be condensed again and again – and yet no ordered condensation can be allowed. So the question becomes: what might this meshing look like?

Let us do some handwaving arguments on some simple attempts that are doomed to fail. Since any union ultrafilter is in $\delta\mathbb{F}$ and our sequence is disjoint, there must be "arbitrarily late" meshing, i.e., if only finitely many elements of $\mathbf{s}$ mesh we have already lost. It is also easy to see that union ultrafilters concentrate on condensations that contain unions of many members of the sequence, e.g., because the sequence itself will not be in the union ultrafilter; therefore there cannot be a bound on the number of s_i which mesh. Finally, by parity arguments the meshing cannot be only of, e.g., the form $s_{2i} \sqcap s_{2i+1}$, since a union ultrafilter will concentrate on those with an even number of adjacent indices – so any union ultrafilter will condense such a sequence to an ordered sequence. Finally, the critical concern will have to be, whether there can be enough meshing while keeping the images under min and max Ramsey ultrafilters.

The main result of this section is as follows.

Theorem 4.1 (Stable Unordered Union Ultrafilters)
Assume CH. There exists a stable union ultrafilter u with $\min(u)$ *and* $\max(u)$ *selective, but there exists* $FU(\mathbf{s}) \in u$ *such that for every ordered sequence* $\mathbf{t}$

$$\mathbf{t} \sqsubseteq \mathbf{s} \Rightarrow FU(\mathbf{t}) \notin u.$$

In fact, any two non-isomorphic selective ultrafilters can be prescribed for min *and* max.

Note that the assumption of *CH* can be weakened to essentially iterated Cohen forcing; this will be discussed at the end of the section.

Fortunately, with the help of the lemma in the previous section this implies a stronger version guaranteeing rigidity under additive isomorphisms.

Corollary 4.2 (Unordered Union Ultrafilters)
Assume CH. There exists a stable union ultrafilter u with $\min(u)$ *and* $\max(u)$ *selective, but u is not additively isomorphic to an ordered union ultrafilter.*

In fact, any two non-isomorphic selective ultrafilters can be prescribed for min *and* max.

Proof. This follows from the above theorem and Lemma 3.4

[5] We will always have some ordered *FU*-sets in any (union) ultrafilter, e.g., $\mathbb{F} = FU((\{i\})_{i\in\omega})$.

4.1 The construction

Recall the goal: however the union ultrafilters u is constructed, it must include a set $FU(\mathbf{s})$, such that *any ordered condensation* $\mathbf{t} \sqsubseteq \mathbf{s}$ is excluded, i.e., $FU(\mathbf{t}) \notin u$.

The critical issue – a special set $FU(\mathbf{s})$

It is not difficult to put together a union ultrafilter with a base of unordered FU-sets. But this does not suffice, since there might be a different base of ordered FU-sets by accident.

To prevent this, no unordered condensation that is added in the inductive construction can accidentally be, at the same time, a condensation of some other, ordered condensation of the fixed $FU(\mathbf{s})$ (thus including that ordered condensation of $FU(\mathbf{s})$ in the ultrafilter u as well).

This means that every chosen sequence must eventually have a high degree of meshing not just in itself but due to the s_i that appear in its support. The following definition prepares for the right notion of meshing.

Definition 4.3 (The meshing graph) Given $\mathbf{s} = (s_i)_{i<N}$ (for some $N \leq \omega$) and some condensation $\mathbf{t} = (t_j)_{j<K}$ of $\mathbf{s}$ (for some $K \leq N$) we define the *meshing graph* G_t to be the graph on the vertices $\{t_j \mid j < K\}$ with edges

The meshing graph

$$E(G_t) = \{\{t_i, t_j\} \mid (\exists s_n \subseteq t_i, s_m \subseteq t_j)\ s_n \sqcap s_m\},$$

i.e., there is an edge whenever two t_j are meshed and this meshing is caused by two elements from $\mathbf{s}$.

This notion allows to discuss the degree of meshing in terms of the connectedness of the graph. On the one hand, it is an advantage to connect to graph theory and graph colourings. On the other hand, it is unclear how well connected the graph should be – and it is not trivial to get Ramsey-type theorems for graphs that allow a flexible degree of connectedness. Fortunately, it will be enough to work with complete graphs.

To begin the construction, a thoroughly meshed sequence is required. After all, in an inductive construction under CH, the critical FU-set must appear after countably many steps so it might as well appear right away. The general case without a preselected FU-set will be discussed later.

Remark 4.4 (Fix the meshed sequence s) From now on fix a sequence $\mathbf{s} = (s_i)_{i\in\omega}$ such that for any n there exist $i_0 < \ldots < i_n$ such that

The fixed sequence $\mathbf{s}$

$$G_{(s_{i_0},\ldots,s_{i_n})}$$

is a complete graph with $n+1$ vertices.[6]

This simply means that the sequence includes arbitrarily large segments that have the best meshing. As mentioned, for now it is enough to pick any such sequence (which is easy to construct inductively). It will be proved how to find such a sequence with respect to two prescribed selective ultrafilters at the end of the construction. It is useful to note that such a

[6]Here the meshing graph is computed with respect to $\mathbf{s}$ itself.

sequence might (and later will) be chosen "nearly ordered" in the sense that the increasingly large complete graphs appear in an ordered fashion.

The following definition tries to capture the right kind of meshing that is needed for condensations and more generally for sets that are suitable for the ultrafilter.

Definition 4.5 (s-meshed) A set $A \subseteq \mathbb{F}$ is called **s**-*meshed* if for any $n \in \omega$ there exist (disjoint) $\mathbf{t} = (t_i)_{i<n}$ such that

s-meshed

- $FU(\mathbf{t}) \subseteq (FU(\mathbf{s}) \cap A)$
- The meshing graph G_t is a complete graph.

We call such a finite sequence an *n-witness* of A.[7]

A set A is **s**-meshed if there are members of A that have a high degree of meshing and additionally the witnesses for the meshing are given by arbitrarily large, finite FU-sets where the members of the **s**-support mesh very much.

The following observation should support the claim that this is the right notion for this setting, i.e., such sets do not force us to add ordered condensations to an ultrafilter.

Proposition 4.6
*If A is **s**-meshed, then it is not included in $FU(\mathbf{t})$ for any ordered $\mathbf{t} \sqsubseteq \mathbf{s}$.*

Proof. (1.) To be an ordered condensation $\mathbf{t} \sqsubseteq \mathbf{s}$ means that G_t has no edges.

(2.) There are no disjoint elements in $FU(\mathbf{t})$ with a non-empty meshing graph.
   (a) Assume $v, w \in FU(\mathbf{t})$ have an edge, i.e., there exist $s_i \subseteq t_j \subseteq v, s_k \subseteq t_l \subseteq w$ with $s_i \sqcap s_j$.
   (b) Therefore $t_j \sqcap t_l$.
   (c) Since $\mathbf{t}$ is ordered, this implies $t_j = t_l$, i.e., $t_j \subseteq v \cap w \neq \emptyset$

(3.) Hence $FU(\mathbf{t})$ cannot include an **s**-meshed set. □

To be able to link the new notion with ultrafilters it needs to be partition regular. This requires the following classical result which is sometimes called "finite Hindman's Theorem" even though it historically preceded and motivated Hindman's Theorem.

Theorem 4.7 (Folkman-Rado-Sanders)
For any $n \in \omega$ there exists $h(n) \in \omega$ such that for any disjoint sequence $\mathbf{x} = (x_i)_{i<h(n)}$ in $\mathbb{F}$ the following holds:

Whenever $FU(\mathbf{x})$ is finitely partitioned, there exists a condensation of length n with a homogeneous FU-set.

Folkman-Rado-Sanders Theorem

[7] Note that an **s**-meshed set is compatible with $\delta\mathbb{F}$ since for any $v \in \mathbb{F}$ any disjoint sequence of length $\max(v) + 2$ must have an element in $\sigma(v)$.

Proof. The original discovery is attributed to Folkman and Sanders independently; it follows from Rado's Theorem and from the Graham-Rothschild Parameter Sets Theorem, [GR71, cf. Corollary 3]. For a proof from Hindman's Theorem by a compactness argument see [HS98, Theorem 5.15]; for a more recent overview on its combinatorial aspects cf. [PV90]. □

This allows for the proof of the first piece of the puzzle.

Lemma 4.8 (s-meshed partition regular)
The notion of being **s***-meshed is partition regular.*

In particular, any **s***-meshed set is included in an ultrafilter consisting only of sets that are* **s***-meshed.*

Summary. Given a finite partition of an **s**-meshed set, the Folkman-Rado-Sanders Theorem implies large homogeneous condensations. To get n-witnesses it turns out that a homogeneous condensation inherits a complete meshing graph. ⊠

Proof. (1.) Clearly, $\mathbb{F}$ itself is **s**-meshed since it contains $FU(\mathbf{s})$.

(2.) So fix an arbitrary **s**-meshed set A and any partition $A = A_0 \dot\cup A_1$.

(3.) Since A contains arbitrarily large witnesses, the Folkman-Rado-Sanders Theorem (plus the pigeon hole principle) implies that in either A_0 or A_1 there are arbitrarily large condensations of these witnesses.

(4.) Any condensation **v** of a witness **t** has a complete meshing graph.
  - (a) Given v_i, v_j in the condensation, there are $t_k \subseteq v_i, t_l \subseteq v_j$.
  - (b) Since **t** has a complete meshing graph, there exists $s_m \subseteq t_k \subseteq v_i$, $s_n \subseteq t_l \subseteq v_j$ with $s_m \sqcap s_n$.
  - (c) Therefore v_i, v_j are connected in the meshing graph $G_{\mathbf{v}}$.

(5.) Therefore, either A_0 or A_1 is **s**-meshed. □

The next step is to show that the ultrafilters containing **s**-meshed sets are algebraically rich.

Lemma 4.9 (The meshing semigroup)
The set

$$H := \{p \in FU^\infty(s) \cap \delta\mathbb{F} \mid (\forall A \in p)\ A \text{ is } \mathbf{s}\text{-meshed}\}$$

is a closed subsemigroup of $\delta\mathbb{F}$*.*

Proof. (1.) H is a closed subset of $\delta\mathbb{F}$ since it is defined by a constraint on all members of its elements.

(2.) Lemma 4.8 implies that it is not empty.

(3.) H is a subsemigroup.
  - (a) Pick arbitrary $p, q \in H$ and $V \in p$, $(W_v)_{v \in V}$ in q; in particular all these sets are **s**-meshed.
  - (b) Then $\bigcup_{v \in V}(v \cdot W_v)$ is **s**-meshed.
    - (i.) Pick any $n \in \omega$.
    - (ii.) By assumption on p there exists an n-witness $\mathbf{t} = (t_i)_{i<n}$ such that
$$FU(\mathbf{t}) \subseteq V.$$

(iii.) Similarly, by assumption on q, there exists an n-witness $\mathbf{t}' = (t'_i)_{i<n}$ such that

$$FU(\mathbf{t}') \subseteq \bigcap_{x \in FU(\mathbf{t}_0,\dots,t_n)} W_v \cap \sigma(\bigcup_{i \le n} t_i) \quad (\in q).$$

(iv.) But then for $\mathbf{v} = (v_i)_{i<n}$ with $v_i := t_i \cup t'_i$ in fact

$$FU(\mathbf{v}) \subseteq \bigcup_{v \in V} v \cdot W_v.$$

(v.) Additionally, G_v is a complete graph since G_t was (or since $G_{t'}$ was) – making the sets "fatter" only increases the chance of being meshed.

(vi.) In particular, the set is $\mathbf{s}$-meshed – as desired.

(c) Therefore, $p \cdot q \in H$.

This competes the proof. □

The next step is to show that the preimage filters under min and max are compatible with H, i.e., contain $\mathbf{s}$-meshed sets.

Lemma 4.10
If $A \cap \min[FU(\mathbf{s})]$, $B \cap \max[FU(\mathbf{s})]$ are both infinite, then

$$\min{}^{-1}[A] \cap \max{}^{-1}[B]$$

is $\mathbf{s}$-meshed.

Summary. Pick three sets of members of $\mathbf{s}$: one set to get the prescribed minimum, another set to get the meshing, and finally a set to get the prescribed maximum. ⊠

Proof. (1.) Given $n \in \mathbb{N}$ we pick three times n-many elements of the sequence $\mathbf{s} = (s_i)_{i \in \omega}$.

(2.) Since A is infinite, it is possible to pick $(s_{i_k})_{k<n}$ with $\min(s_{i_k}) \in A$.

(3.) By the meshing of $\mathbf{s}$, it is possible to pick $(s_{j_k})_{k<n}$ with a complete meshing graph but lying beyond everything chosen so far.[8]

(4.) Since B is infinite, it is possible to pick $(s_{l_k})_{k<n}$ with $\max(s_{l_k}) \in B$, again beyond everything chosen so far.

(5.) Then $(t_k)_{k<n}$ defined by $t_k := s_{i_k} \cup s_{j_k} \cup s_{l_k}$ is an n-witness for $\min^{-1}[A] \cap \max^{-1}[B]$. □

An easy corollary is the following.

Corollary 4.11
Let p_1 and p_2 be ultrafilters including $\min[FU(\mathbf{s})]$, $\max[FU(\mathbf{s})]$ *respectively. Then*

$$\overline{\min{}^{-1}(p_1)} \cap \overline{\max{}^{-1}(p_2)} \cap H \neq \emptyset.$$

[8] In other words, with minima greater than the greatest maximum so far.

Proof. By Lemma 4.10 all elements of the preimage filter are $\mathbf{s}$-meshed. A standard application of Zorn's Lemma (or equivalenty compactness) allows us to extend any such filter to an ultrafilter in H. □

Note also that $\overline{\min^{-1}(p)}$ is a right ideal, $\overline{\max^{-1}(p)}$ a left ideal in $\delta\mathbb{F}$ for any $p \in \beta\mathbb{N}$; in particular their intersection is a closed subsemigroup. This is easily checked, cf. also [Kra09, Section 2.3] and [HS98, Theorem 6.9].

4.2 Main lemma and theorem

After the preparations are complete it is possible to tackle the main lemma for the inductive construction. Let $\langle \min^{-1}(p_1) \cup \max^{-1}(p_2) \rangle$ denote the filter generated by the union of the coherent filters $\min^{-1}(p_1)$, $\max^{-1}(p_2)$.

Lemma 4.12 (Main Lemma)
Assume we are given non-isomorphic, selective ultrafilters p_1, p_2 with $\max[FU(\mathbf{s})] \in p_1$ and $\min[FU(\mathbf{s})] \in p_2$ as well as some $X \subseteq \mathbb{F}$.
For every $\alpha < \omega$ let $\mathbf{t}^\alpha = (t_i^\alpha)_{i\in\omega}$ be a sequence such that

$$\mathbf{t}^{\alpha+1} \sqsubseteq^* \mathbf{t}^\alpha$$
$$FU(\mathbf{t}^\alpha) \text{ is } \mathbf{s}\text{-meshed}$$
$$FU(\mathbf{t}^\alpha) \in \langle \min^{-1}(p_1) \cup \max^{-1}(p_2) \rangle.$$

Then there exists $\mathbf{z} = (z_i)_{i\in\omega}$ such that

$$\mathbf{z} \sqsubseteq^* \mathbf{t}^\alpha \text{ for every } \alpha < \omega,$$
$$FU(\mathbf{z}) \subseteq X \text{ or } FU(\mathbf{z}) \cap X = \emptyset,$$
$$FU(\mathbf{z}) \text{ is } \mathbf{s}\text{-meshed}$$
$$FU(\mathbf{z}) \in \langle \min^{-1}(p_1) \cup \max^{-1}(p_2) \rangle.$$

Summary. By a standard Galvin-Glazer argument there exists a common almost condensation of the given FU-sets and X (or its complement). Since all sets are $\mathbf{s}$-meshed, the condensation can be $\mathbf{s}$-meshed. The Homogeneity Theorem 2.4 ensures such a condensation can be found in $\min^{-1}(p_1) \cup \max^{-1}(p_2)$. ⊠

Proof. (1.) By the assumptions,

$$H \cap \overline{\min^{-1}(p_1)} \cap \overline{\max^{-1}(p_2)} \cap \bigcap_{\alpha<\omega} FU^\infty(\mathbf{t}^\alpha) \neq \emptyset.$$

(2.) As an intersection of closed semigroups it is a closed semigroup which therefore contains an idempotent $e \in \delta\mathbb{F}$. Without loss $X \in e$; in particular X is $\mathbf{s}$-meshed.

e – the helpful idempotent

[[The aim is to apply the Homogeneity Theorem 2.4.]]

(3.) Consider the following analytic set in $\mathfrak{P}(\omega)$. An analytic set

$$\begin{aligned}\{Y \subseteq \omega \mid (\exists \mathbf{z} = (z_i)_{i\in\omega})\ & Y = \min[FU(\mathbf{z})] \cup \max[FU(\mathbf{z})], \\ & (\forall \alpha < \omega)\ z \sqsubseteq^* \mathbf{t}^\alpha, \\ & FU(\mathbf{z}) \subseteq X, \\ & FU(\mathbf{z}) \text{ is } FU\text{-meshed}\}.\end{aligned}$$

This set encodes all the candidates for the claim.

(4.) Define $f \in 2^\omega$ inductively to have n-many 0's followed by n-many 1's for each n in increasing order of n's. 010011000111...

[[We will apply Theorem 2.4 to get an f-alternating sequence whose alternating blocks form sets in our selective ultrafilters. An f-alternating sequence alternates by picking n-many elements from both sets at the n-th step. It is useful to check that if we asked for alternating instead of f-alternating then we would always miss the analytic set, since any $\mathbf{z}$ with alternating minima and maxima must be ordered, so it isn't FU-meshed. On the other hand, a "nearly ordered" sequence such as $\mathbf{s}$ comes in ordered blocks of completely meshed finite sequences. The minima and maxima of such a sequence are precisely f-alternating which is why we choose f-alternating here.]]

(5.) By Theorem 2.4 there exist $Y_1 \in p_1$, $Y_2 \in p_2$ such that Using homogeneity

$$\{A \subseteq Y_1 \cup Y_2 \mid A\ f\text{-alternating}\}$$

is either contained in or disjoint from the analytic set.

[[If the above set is included in the analytic set then Remark 2.5 guarantees the existence of the set desired to complete the proof. Fortunately, given any Y_1, Y_2 a Galvin-Glazer argument shows that the set can never be disjoint.]]

(6.) But for every $Y_1 \in p_1$, $Y_2 \in p_2$ there exists $\mathbf{t} = (t_i)_{i\in\omega}$, a common almost condensation of $(\mathbf{t}^\alpha)_{\alpha<\omega}$ such that Applying Galvin-Glazer to find $\mathbf{t}$

$$FU(\mathbf{t}) \subseteq Z := X \cap \min{}^{-1}[Y_1] \cap \max{}^{-1}[Y_2] \cap FU(\mathbf{s});$$

additionally, $\mathbf{t}$ is $\mathbf{s}$-meshed and the minima and maxima are f-alternating.

(a) First note $Z \in e$; without loss $Z^\star = Z \in e$ (with respect to e). We construct the desired sequence by induction.

(b) At the inductive step n, having constructed $t_0, \ldots, t_k$ (where $k = \sum_{i=0}^{n-1} i$) we assume by induction hypothesis that the following intersection is in e

$$Z^\star \cap \bigcap_{x\in FU(\mathbf{t}_0,\ldots,t_k)} x^{-1}Z^\star \cap \sigma(\bigcup_{i<k} t_i) \cap \bigcap_{\alpha<n} FU(\mathbf{t}^\alpha).$$

(c) Pick an $\mathbf{s}$-meshing n-witness $t_k, \ldots, t_{n+k}$ from it.[9]

[9]For later reference, note the following. We have a lot of freedom at this point to impose other properties on these $n+1$-many elements of $\mathbf{t}$. In particular, we can first choose an ordered sequence of length $n+1$, then a sequence of meshing witnesses beyond those such that the union of the ith from the ordered part with the ith from the meshed part is just as good to continue our induction, i.e., all finite unions are still in Z. This kind of "late meshing" will be needed for an observation regarding stability via min at the end of this section.

(d) As usual in the Galvin-Glazer argument, the analogous intersection for $FU(t_0, \ldots .t_{k+n})$ is again in e.
(e) The resulting sequence is $\mathbf{s}$-meshed by construction.
(f) Note that for a sequence of length n with a complete meshing graph, all minima must come before all maxima. Since the witnesses are chosen in an ordered fashion, this implies that the entire sequence has f-alternating minima and maxima.

(7.) By Remark 2.5 there exists a sequence $\mathbf{z}$ for Y_1 and Y_2 themselves — and with all the desired properties to conclude the proof. □

Note that as promised, the constructed condensation is "nearly ordered". It is now easy to describe the CH-construction.

Theorem 4.13

Assume CH and let p_1, p_2 be non-isomorphic, selective ultrafilters containing $\min[FU(\mathbf{s})]$ and $\max[FU(\mathbf{s})]$ respectively. *Putting it all together*

Then there exists a stable union ultrafilter u with $FU(\mathbf{s}) \in u$, $\min(u) = p_1$, $\max(u) = p_2$ and such that every ordered $\mathbf{t} \sqsubseteq \mathbf{s}$ has $FU(\mathbf{t}) \notin u$.

Proof. (1.) Assuming CH, fix $(X_\alpha)_{\alpha<\omega_1}$, an enumeration of $\mathfrak{P}(\mathbb{F})$.
We argue by transfinite induction on $\beta < \omega_1$.

(2.) Assume for $\beta < \omega_1$ there are $(FU(\mathbf{t}^\alpha))_{\alpha<\beta}$ such that for all $\gamma < \alpha < \beta$

$$\begin{aligned}
&\mathbf{t}^\alpha \sqsubseteq \mathbf{s}\\
&\mathbf{t}^\alpha \sqsubseteq^* \mathbf{t}^\gamma\\
&FU(\mathbf{t}^\alpha) \subseteq X_\alpha \ \vee \ FU(\mathbf{t}^\alpha) \cap X_\alpha = \emptyset\\
&\min[FU(\mathbf{t}^\alpha)] \in p_1 \wedge \max[FU(\mathbf{t}^\alpha)] \in p_2\\
&FU(\mathbf{t}^\alpha)\ \mathbf{s}\text{-meshed}
\end{aligned}$$

(3.) Pick a cofinal sequence $(\alpha(n))_{n\in\omega}$ in β.

(4.) Applying Lemma 4.12 to $X := X_\beta$ and $(FU(\mathbf{t}^{\alpha(n)}))_{n\in\omega}$ there exists $\mathbf{t}^\beta$ sufficient to continue the induction.

(5.) It should not be difficult to check that the resulting sets will generate a union ultrafilter as desired. □

Finally is useful to realize that the choice of the sequence $\mathbf{s}$ is not all that special.

Corollary 4.14 (The main theorem)

Assume CH. For any two non-isomorphic, selective ultrafilters p_1, p_2 there exists a stable union ultrafilter u which is not ordered, such that $\min(u) = p_1$ and $\max(u) = p_2$. *The main theorem*

Summary. The preceding theorem can be applied after using Theorem 2.4 to make sure that there is an apropriate sequence. ⊠

Proof. (1.) To invoke the preceding theorem it is sufficient to generate a suitably meshed sequence $\mathbf{s}$ with $\min[FU(\mathbf{s})] \in p_1, \max[FU(\mathbf{s})] \in p_2$.

(2.) For this consider the analytic set

$$\{X \subseteq \omega \mid (\exists \mathbf{s})\ X = \max[FU(\mathbf{s})] \cup \min[FU(\mathbf{s})] \text{ and } FU(\mathbf{s}) \text{ is } \mathbf{s}\text{-meshed}\}.$$

(3.) By Theorem 2.4 there exist f-alternating $Y_i \in p_i$ $(i \in 2)$ such that set of f-alternating subsets of $Y_1 \cup Y_2$ is either included or disjoint from the analytic set.

(4.) There exist f-alternating $Y_i \in p_i$ $(i \in 2)$ such that $Y_0 \cup Y_1$ lies in the analytic set.

[[The argument is just as in the proof of the main lemma, i.e, it suffices to check that for any $Y_i \in p_i$ $(i \in 2)$ $\min^{-1}[Y_1] \cap \max^{-1}[Y_2]$ must include $FU(\mathbf{s})$ for some suitably meshed sequence.]]

(a) Let $Y_i \in p_i$ be as in the previous step.

(b) Recall that $\overline{\min^{-1}(p_1)} \cap \overline{\max^{-1}(p_2)}$ is a closed subsemigroup; so we can find an idempotent ultrafilter therein.

(c) Therefore there exists $FU(\mathbf{v}) \subseteq \min^{-1}[Y_1] \cap \max^{-1}[Y_2]$ by the Galvin-Glazer Theorem 1.8.

(d) Then there exists a condensation of v to an f-alternating, meshed sequence $\mathbf{s}$, i.e., with $FU(\mathbf{s})$ being $\mathbf{s}$-meshed (with respect to $\mathbf{x}$).

(i.) For the inductive step $n \in \omega$ assume that for $k = \sum_{i<n} i$ there are $(s_i)_{i<k}$ with increasingly meshed graphs of sizes 1 through $n-1$.

Pick $2n$-many elements from $FU(\mathbf{v})$ as follows:

(ii.) First pick $(v_{i_j})_{j<n}$ past everything so far and then pick $(v_{i_j})_{n-1<j<2n}$ past additionally the ones just chosen and define

$$s_{k+j} := v_{i_j} \cup v_{i_j+n}.$$

(iii.) Then $s_k, \ldots, s_{k+n}$ is an n-witness.

(iv.) By construction, $\min[\mathbf{s}] \cup \max[\mathbf{s}]$ is f-alternating.

(e) Therefore, the f-alternating subsets are never disjoint from the anaytic set.

(f) By Remark 2.5, we find Y_0, Y_1 as desired.

(5.) This completes the proof. □

Andreas Blass suggested an alternative proof for this last corollary sketched below.

Proof. (1.) Given selectives p_1, p_2 there exists a permutation of ω simultaneously mapping p_i to p_i' $(i \in 2)$ with $\min[FU(\mathbf{s})] \in p_1'$ and $\max[FU(\mathbf{s})] \in p_2'$; p_1', p_2' are again selective. (Here, $\mathbf{s}$ is the previously fixed sequence).

(2.) The main theorem now gives a suitable u' for p_1', p_2'.

(3.) But the natural extension of the permutation to $\mathbb{F}$ yields an additive isomorphism on $FU(\mathbf{s}) \in u'$ mapping u' to a union ultrafilter u with $\min(u) = p_1$ and $\max(u) = p_2$.

(4.) Since additive isomorphisms preserve all the desired properties, this completes the proof. □

We can modify our construction to yield the following.

Theorem 4.15
There exists an unordered union ultrafilter that is stable via min*. In particular, stability via* min *does not imply orderedness of a union ultrafilter.*

Proof. (1.) We can modify the proof of the main lemma in the spirit of the (first) proof of the last corollary; compare the footnote in the proof of the main lemma.

(2.) That is, in the inductive step of the Galvin-Glazer argument first choose an ordered sequence (of length $n+1$) followed by an $\mathbf{s}$-meshed witness (of length n) past this sequence. Finally add the elements from the ordered sequence to the $\mathbf{s}$-meshed witness just as in the proof of the corollary.

(3.) The ultrafilter u resulting from this modified construction is of course still stable; in particular, it has the Ramsey property for pairs.

(4.) To show that it is stable via min, let $f \in \mathbb{F}^\omega, g \in \omega^\omega$ with $f(s) < g \circ \min(s)$ on a set in u; for simplicity, we may assume that this set is all of $\mathbb{F}$.

(5.) Consider $\{(s < t) \mid f(s \cup t) = f(s)\}$

(6.) Then there exists $A \in u$ with $A^2_<$ included in this set.
   - (a) By the Ramsey property for pairs, we get a homogeneous set A.
   - (b) Fix $s \in A$
   - (c) All $t \in \sigma(s)$ have $f(s \cup t) < g \circ \min(s \cup t) = g \circ \min(s)$.
   - (d) So on some $B \in u$, $f(s \cup t)$ is constant.
   - (e) But now for any $t < t'$ in B, we get

$$f((s \cup t) \cup t') = f(s \cup (t \cup t')) = f(s \cup t).$$

   - (f) In other words, $(s \cup t, t')$ is in the above set.

(7.) So for ordered pairs, the value of f on A only depends on min.

(8.) By construction of u, we find $\alpha < \omega_1$ such that $FU(\mathbf{s}^\alpha) \subseteq A$.

(9.) Then $f(s)$ depends only on $\min(s)$ on $FU(\mathbf{s}^{\alpha+1})$.
   - (a) Check that due to the modified construction every element of $\mathbf{s}^{\alpha+1}$ is a union of elements in $\mathbf{s}^\alpha$ where the first part is ordered with respect to the other parts.
   - (b) Hence the value of f depends only on that first part, i.e., only on min. □

As promised the assumption of the continuum hypothesis can be weakened.

Remark 4.16 Dropping the prescribed selective ultrafilters in Lemma 4.12, the modified consequent can be derived using Cohen forcing in the form of finite condensations of $\mathbf{s}$; using Lemma 4.10 it is not difficult to do some additional bookkeeping to ensure that the min-image and the max-image of the constructed union ultrafilter will be selective.

Therefore, the above kinds of union ultrafilters already exist assuming $cov(\mathscr{M}) = \mathfrak{c}$ alone, in particular under weak versions of Martin's Axiom.

For a detailed argument very much like the sketch we just proposed see [BH87, Theorem 5].

To conclude this final section, we state some questions that remain open.

Question 4.17 (1.) It is known that min and max of a union ultrafilter are not-near coherent P-points, the max-image is rapid, cf. [Bla09, Theorem 38]. Given such P-points on ω, does there (say under CH) exist a union ultrafilter mapping to them via min and max?

(2.) More vaguely, do stronger assumptions hold for min and max?

(3.) Most importantly, do there exist union ultrafilters that are not stable?

(4.) Is stability via min equivalent to stability?

(5.) Does the canonical partition property imply orderedness?

The first and second question are obviously related. A partition theorem for P-points similar to Theorem 2.4 can be found in [Bla87] and strengthened, cf. [Kra09, Theorem 4.10]. This might be helpful in attacking the first question, especially if something can be improved regarding the second question. The third question seems to be an entirely different beast. It is much more difficult since the Galvin-Glazer Theorem so easily helps to construct almost condensation just as was done in the main result. It would seem to require a somewhat new proof of Hindman's Theorem to tackle stability.

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