BRAUER GROUP OF MODULI OF PRINCIPAL BUNDLES OVER A CURVE

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ABSTRACT. Let G be a semisimple linear algebraic group over the field $\mathbb{C}$, and let C be an irreducible smooth complex projective curve of genus at least three. We compute the Brauer group of the smooth locus of the moduli space of semistable principal G–bundles over C. We also compute the Brauer group of the moduli stack of principal G–bundles over C.

1. INTRODUCTION

The base field will always be $\mathbb{C}$. Let Y be a smooth quasiprojective variety or more generally a smooth algebraic stack. Using the isomorphism $\mathbb{C}^r \otimes \mathbb{C}^{r'} = \mathbb{C}^{rr'}$, we have a homomorphism $\mathrm{PGL}(r, \mathbb{C}) \times \mathrm{PGL}(r', \mathbb{C}) \longrightarrow \mathrm{PGL}(rr', \mathbb{C})$. So a principal $\mathrm{PGL}(r, \mathbb{C})$–bundle $\mathbb{P}$ and a principal $\mathrm{PGL}(r', \mathbb{C})$–bundle $\mathbb{P}'$ on Y together produce a principal $\mathrm{PGL}(rr', \mathbb{C})$–bundle on Y, which we will denote by $\mathbb{P} \otimes \mathbb{P}'$. The two principal bundles $\mathbb{P}$ and $\mathbb{P}'$ are called *equivalent* if there are vector bundles V and V' on Y such that the principal bundle $\mathbb{P} \otimes \mathbb{P}(V)$ is isomorphic to $\mathbb{P}' \otimes \mathbb{P}(V')$. The equivalence classes form a group which is called the *Brauer group* of Y. The addition operation is defined by the tensor product, and the inverse is defined to be the dual projective bundle. The Brauer group of Y will be denoted by $\mathrm{Br}(Y)$.

Let C be an irreducible smooth complex projective curve such that genus$(C) \geq 3$. Fix a line bundle L over C of degree d. Let $M(r, L)$ be the moduli space of stable vector bundles E over C of rank r with $\bigwedge^r E = L$. The Brauer group $\mathrm{Br}(M(r, L))$ is cyclic of order g.c.d.$(r, \mathrm{degree}(L))$ [BBGN]. A generator for $\mathrm{Br}(M(r, L))$ is obtained by restricting the universal projective bundle over $C \times M(r, L)$ by fixing a point of C. The Brauer group of the smooth locus of stable principal $\mathrm{PGL}(r, \mathbb{C})$–bundles over C was computed in [BHg].

Let G be a semisimple linear algebraic group. Let $\mathcal{M}_C(G)$ be the moduli stack of principal G–bundles on C. The coarse moduli space of semistable principal G–bundles on C will be denoted by $M_C^{\mathrm{ss}}(G)$. Let

$$M_C(G)^{\mathrm{rs}} \subset M_C^{\mathrm{ss}}(G)$$

be the smooth locus. It is known that $M_C(G)^{\mathrm{rs}}$ coincides with the locus of regularly stable principal G–bundles on C (a stable principal G–bundle E_G is called regularly stable if

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the natural homomorphism from the center of G to the group of automorphisms of E_G is surjective). Our aim here is to compute the Brauer groups of $M_C(G)^{\mathrm{rs}}$ and $\mathcal{M}_C(G)$.

We prove the following theorem (see Theorem 4.1 and Corollary 6.5):

Theorem 1.1. *Assume that G is semisimple and simply connected. Then the Brauer group of $\mathcal{M}_C(G)$ is trivial, and the Brauer group of $\mathrm{Br}(M_C(G)^{\mathrm{rs}})$ is the group of characters $Z_G^\vee$ of the center $Z_G \subset G$.*

If G is not simply connected, then the Brauer group of the moduli stack $\mathcal{M}_C(G)$ is computed in Theorem 7.4 (in conjunction with Proposition 7.3). The Brauer group of $M_C(G)^{\mathrm{rs}}$ is computed in Theorem 6.3.

We will describe a corollary of these computations.

Let $Z_{\widetilde{G}}$ be the center of the universal cover $\widetilde{G}$ of G. Define

$$\Psi(G) \ \subset \ \mathrm{Hom}(Z_{\widetilde{G}} \otimes_{\mathbb{Z}} Z_{\widetilde{G}}, \mathbb{Q}/\mathbb{Z})$$

as in (6.6). Given an element $\delta \in \pi_1(G) \subset Z_{\widetilde{G}}$, define

$$\mathrm{ev}_G^\delta \ : \ \Psi(G) \ \longrightarrow \ \mathrm{Hom}(Z_{\widetilde{G}}/\pi_1(G), \mathbb{Q}/\mathbb{Z})$$

as in (6.7). The connected components of $\mathcal{M}_C(G)$ parametrized by $\pi_1(G)$. For any $\delta \in \pi_1(G)$, let $\mathcal{M}_C(G)^\delta \subset \mathcal{M}_C(G)$ be the connected component corresponding to δ. Let $M_C(G)^{\delta,\mathrm{rs}}$ be the smooth locus of the connected component, corresponding to δ, of the moduli space $M_C^{\mathrm{ss}}(G)$.

We prove the following (see Corollary 7.6):

Theorem 1.2. *There is a short exact sequence*

$$0 \ \longrightarrow \ \mathrm{Coker}(\mathrm{ev}_G^\delta) \ \longrightarrow \ \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}}) \ \longrightarrow \ \mathrm{Br}(\mathcal{M}_C(G)^\delta) \ \longrightarrow \ 0 \, .$$

The Brauer groups for the moduli spaces of principal bundles with a classical group as the structure group are computed in Section 8.

We note that among the exceptional groups, G_2, F_4 and E_8 have the property that the center is trivial. Therefore, from Theorem 1.1 it follows that for these three groups

$$\mathrm{Br}(\mathcal{M}_C(G)) \ = \ 0 \ = \ \mathrm{Br}(M_C(G)^{\mathrm{rs}}) \, .$$

2. Preliminaries

As mentioned before, the base field is $\mathbb{C}$. For an algebraic stack Y, the torsion group $H^2_{\mathrm{\acute{e}t}}(Y, \mathbb{G}_m)_{\mathrm{torsion}}$ is called the *cohomological Brauer group* of Y. A theorem of Grothendieck says that if Y is a smooth variety, then $H^2_{\mathrm{\acute{e}t}}(Y, \mathbb{G}_m)$ is torsion [Mi]. If Y is a quasiprojective variety, then a theorem of Gabber says that $H^2_{\mathrm{\acute{e}t}}(Y, \mathbb{G}_m)_{\mathrm{torsion}}$ coincides with the Brauer group defined by the Morita equivalence classes of Azumaya algebras over Y.

As mentioned in the introduction, we are interested in computing the Brauer groups of the moduli stack of principal G–bundles and the smooth locus of the moduli space of

semistable principal G–bundles over a smooth projective curve. Since the smooth locus of the moduli space is a smooth quasiprojective variety, the above comments imply that all the three groups associated to it coincide. Similarly, since the moduli stack is smooth all the three groups associated to it also coincide.

Proposition 2.1. *Let Y be an algebraic stack satisfying the following two properties:*

- *any class in $H^2(Y, \mathbb{Z})$ is represented by a holomorphic line bundle on Y, and*
- *each holomorphic line bundle on Y admits an algebraic structure.*

Then there are isomorphisms

$$H^2_{\acute{e}t}(Y, \mathbb{G}_m)_{\text{torsion}} \cong H^2(Y, \mathcal{O}^*_{X,\text{an}})_{\text{torsion}} \cong H^3_B(Y, \mathbb{Z})_{\text{torsion}}.$$

Proof. For finite coefficients, we have a comparison isomorphism between the étale cohomology and the Betti cohomology. Let $\mu_n \subset \mathbb{C}^*$ be the group of n–th roots of 1. By the Kummer sequence, any element in $H^2_{\text{ét}}(Y, \mathbb{G}_m)_{\text{torsion}}$ (respectively, $H^2_{\text{an}}(Y, \mathcal{O}^*_{Y,\text{an}})_{\text{torsion}}$) is represented by a class in $H^2_{\text{ét}}(Y, \mu_n)$ (respectively, $H^2(Y, \mathbb{Z}/n\mathbb{Z})$) for some n. This implies the surjectivity of the homomorphism

$$H^2_{\text{ét}}(Y, \mathbb{G}_m)_{\text{torsion}} \longrightarrow H^2(Y, \mathcal{O}^*_{Y,\text{an}})_{\text{torsion}},$$

while its injectivity follows from the assumptions by a straight forward diagram chase.

The second isomorphism in the proposition is derived using the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{Y,\text{an}} \longrightarrow \mathcal{O}^*_{Y,\text{an}} \longrightarrow 0$$

associated to the homomorphism $f \longmapsto \exp(2\pi\sqrt{-1}f)$. $\square$

3. The ind–Grassmannian and $L_C(G)$

Let G be a connected semisimple linear algebraic group over $\mathbb{C}$. Let

$$\widetilde{G} \longrightarrow G$$

be the universal cover. The fundamental group of G will be denoted by $\pi_1(G)$; being abelian $\pi_1(G)$ is independent of the choice of base point in G. Let

$$L(G) := G((t)) \quad \text{and} \quad L^+(G) := G[[t]]$$

be the loop group and its naturally defined subgroup scheme respectively. Let

$$\mathcal{Q}_G := L(G)/L^+(G)$$

be the ind–Grassmannian; it is a direct limit of projective integral varieties.

Let C be an irreducible smooth complex projective curve with genus$(C) \geq 3$. Fix a base point $p_0 \in C$. Let

$$L_C(G) := G(\mathcal{O}(C - p_0)) \subset L(G)$$

be the sub–ind–group scheme.

Let $\mathcal{M}_C(G)$ be the moduli stack of principal G–bundles on C. The "Uniformization theorem" produces a canonical isomorphism

$$L_C(G)\backslash\mathcal{Q}_G \;\xrightarrow{\sim}\; \mathcal{M}_C(G)$$

[Fa], [Te], [BLS].

Proposition 3.1. *The cohomological Brauer group $H^2_{\acute{e}t}(\mathcal{Q}_G, \mathcal{O}^*_{\mathcal{Q}_G})_{\mathrm{torsion}}$ of $\mathcal{Q}_G$ is trivial.*

Proof. We begin by recalling a lemma from [BLS] (see [BLS, p. 185, Lemma 1.2]).

Lemma 3.2 ([BLS])**.** *The quotient map $L(G) \longrightarrow \mathcal{Q}_G$ induces a bijection*

$$\pi_1(G) \;=\; \pi_0(L(G)) \longrightarrow \pi_0(\mathcal{Q}_G)\,.$$

Each connected component of $\mathcal{Q}_G$ is isomorphic to $\mathcal{Q}_{\widetilde{G}}$, where $\widetilde{G}$ is the universal cover of G.

In view of Lemma 3.2, it is enough to prove the proposition for simply connected groups.

First assume that the group G is almost simple and simply connected.

Fix a Borel subgroup $B \subset G$ together with a maximal torus $T \subset B$ of G. The homomorphism of rings $\mathbb{C}[[t]] \longrightarrow \mathbb{C}$ defined by $t \longmapsto 0$ induces a homomorphism $L^+(G) \longrightarrow G$. Let

$$\mathcal{B} \subset L^+(G)$$

be the inverse image of B under this homomorphism. We recall that $\mathcal{B}$ is called the standard Borel subgroup of $L(G)$ associated to B. Let $N(T)$ be the normalizer of T in G. Let

$$\widetilde{W} \;:=\; \mathrm{Mor}(\mathbb{C}^*, N(T))/T$$

be the affine Weyl group containing the Weyl group $W = N(T)/T$ as a subgroup (the constant morphisms from $\mathbb{C}^*$ to $N(T)$ make $N(T)$ a subgroup of $\mathrm{Mor}(\mathbb{C}^*, N(T))$).

The ind–Grassmannian $\mathcal{Q}_G$ has the Bruhat decomposition

$$\mathcal{Q}_G \;=\; \bigcup_{w\in\widetilde{W}/W} \mathcal{B}w'L^+(G)/L^+(G)\,,$$

where $w' \in \widetilde{W}$ is any chosen representative of w. The quotient space $\widetilde{W}/W$ has a partial ordering inherited from the Bruhat partial ordering of the affine Weyl group $\widetilde{W}$. This makes $\mathcal{Q}_G$ a direct limit of projective varieties $\{Q_w\}_{w\in\widetilde{W}/W}$, where

$$(3.1) \qquad\qquad Q_w \;:=\; \bigcup_{v\leq w} \mathcal{B}vL^+(G)/L^+(G)\,.$$

For any $w \in \widetilde{W}/W$, consider the length of every element in the coset w. The smallest length among them will be denoted by $\ell(w)$.

The generalized Schubert variety Q_w in (3.1) has a Zariski open subset defined by $\mathcal{B}wL^+(G)/L^+(G)$ which is biregularly isomorphic with the affine space $\mathbb{A}^{\ell(w)}$, where $\ell(w)$ is defined above. As a result, Q_w equipped with the analytic topology has the structure

of a CW complex with only even dimensional cells. Hence $H^3(Q_w, \mathbb{Z}) = 0$. This implies that

$$H^3(\mathcal{Q}_G, \mathbb{Z}) = 0$$

because $\mathcal{Q}_G$ is a direct limit of these varieties Q_w. In particular,

$$(3.2) \qquad H^3(Q_w, \mathbb{Z})_{\text{torsion}} = 0 = H^3(\mathcal{Q}_G, \mathbb{Z})_{\text{torsion}} \,.$$

Similarly, we have

$$(3.3) \qquad H^1(\mathcal{Q}_G, \mathbb{Z}) = 0 \,.$$

Both Q_w and $\mathcal{Q}_G$ satisfy the two assumptions in Proposition 2.1 [KN, p. 157, Lemma 2.2]. Therefore, the proposition follows from Proposition 2.1 and (3.2) under the assumption that G is almost simple and simply connected.

In the general case where G is semisimple and simply connected, write G as a product

$$G = \prod_{k=1}^{s} G_i \,,$$

where each G_i is almost simple and simply connected. This enables us to write the ind–Grassmannian $\mathcal{Q}_G$ as a product

$$\mathcal{Q}_G = \prod_{k=1}^{s} \mathcal{Q}_{G_i} \,,$$

where $\mathcal{Q}_{G_i}$ is the ind–Grassmannian for G_i. In view of (3.2) and (3.3) for G_i, from the Künneth decomposition of $H^3(\prod_{k=1}^{s} \mathcal{Q}_{G_i}, \mathbb{Z})$ we conclude that $H^3(\mathcal{Q}_G, \mathbb{Z})_{\text{torsion}} = 0$. Now the proof of the proposition is completed using Proposition 2.1. $\qquad\square$

3.1. **Some properties of $L_C(G)$.** Let G be semisimple and simply connected. The topological properties of the ind–group scheme $L_C(G)$ that we need can be derived from the following theorem of Teleman ([Te, p. 8, Theorem 1]):

Theorem 3.3 ([Te]). *The natural map $L_C(G) \longrightarrow C^\infty(C \setminus \{p_0\}, G)$ defines a homotopy equivalence. Hence the homotopy type of $L_C(G)$ with the analytic topology is that of $G \times \Omega G^{2g}$.*

The connectedness and simply connectedness of $L_C(G)$ follow immediately from Theorem 3.3.

Proposition 3.4. *Let G be semisimple and simply connected. Let $BL_C(G)$ be the classifying space for $L_C(G)$. Then*

(1) $H^1(BL_C(G), \mathbb{Z}) = 0$,
(2) $H^2(BL_C(G), \mathbb{Z}/n\mathbb{Z}) = 0$, *and*
(3) $H^2(BL_C(G), \mathbb{C}^*) = 0$.

Proof. The group $H^1(BL_C(G), \mathbb{Z})$ parametrizes the space of all continuous homomorphisms from $L_C(G)$ to $\mathbb{Z}$. So $H^1(BL_C(G), \mathbb{Z})$ is trivial by the connectedness of $L_C(G)$.

The connectedness of $L_C(G)$ implies that $\pi_1(BL_C(G)) = \pi_0(L_C(G)) = 0$, and the simply connectedness of $L_C(G)$ implies that $\pi_2(BL_C(G)) = \pi_1(L_C(G)) = 0$. Hence (2) and (3) of the proposition follow from the Hurewicz's theorem.   □

4. BRAUER GROUP OF MODULI: G IS SIMPLY CONNECTED

Theorem 4.1. *Let G be simply connected and semisimple. Then* $\mathrm{Br}(\mathcal{M}_C(G)) = 0$.

Proof. The descent spectral sequence for the principal $L_C(G)$–bundle $\mathcal{Q}_G \longrightarrow \mathcal{M}_C(G)$ gives the following exact sequence in the analytic topology

$$(4.1) \qquad H^1(BL_C(G), \mathbb{C}^*) \hookrightarrow H^1(\mathcal{M}_C(G), \mathcal{O}^*) \xrightarrow{\theta} H^0(BL_C(G), H^1(\mathcal{Q}_G, \mathcal{O}^*))$$
$$\longrightarrow H^2(BL_C(G), \mathbb{C}^*) \longrightarrow \mathrm{kernel}(H^2(\mathcal{M}_C(G), \mathcal{O}^*) \longrightarrow H^0(BL_C(G), H^2(\mathcal{Q}_G, \mathcal{O}^*)))$$
$$\longrightarrow H^1(BL_C(G), H^1(\mathcal{Q}_G, \mathcal{O}^*)).$$

Let s be the number of almost simple factors in the product decomposition of G. We have $H^2_{\mathrm{\acute{e}t}}(\mathcal{Q}_G, \mathcal{O}^*_{\mathcal{Q}_G})_{\mathrm{torsion}} = 0$ by Proposition 3.1. Also, $H^1(\mathcal{Q}_G, \mathcal{O}^*) \cong \mathbb{Z}^s$ [BLS, p. 186, Lemma 1.4]. Therefore, from Proposition 3.4(1),

$$(4.2) \qquad H^0(BL_C(G), H^2(\mathcal{Q}_G, \mathcal{O}^*_{\mathcal{Q}_G})) = 0 = H^1(BL_C(G), H^1(\mathcal{Q}_G, \mathcal{O}^*_{\mathcal{Q}_G})).$$

The homomorphism of the Picard groups

$$(4.3) \qquad\qquad \theta : \mathrm{Pic}(\mathcal{M}_C(G)) \longrightarrow \mathrm{Pic}(\mathcal{Q}_G) \cong \mathbb{Z}^s$$

in (4.1) has a finite cokernel [BLS, p. 187, Proposition 1.5], in particular, this cokernel is a torsion group. Since $H^2(BL_C(G), \mathbb{C}^*) = 0$ (see Proposition 3.4(3)), from (4.1) it follows that θ in (4.3) is actually surjective. Now, using (4.2), from (4.1) we conclude that

$$H^2(\mathcal{M}_C(G), \mathcal{O}^*) = H^2(BL_C(G), \mathbb{C}^*) = 0.$$

Hence $\mathrm{Br}(\mathcal{M}_C(G)) = 0$.   □

Remark 4.2. It should be pointed out that the above arguments involving sheaves in analytic topology can be replaced by an argument which uses only constant sheaves in Euclidean topology. Using the exponential sequence, the Brauer group $\mathrm{Br}(X)$ for any X can be expressed as the quotient of $H^2(X, \mathbb{C}^*)$ by the image of $H^2(X, \mathbb{C})$. The isomorphism now

$$H^2(\mathcal{M}_C(G), \mathbb{C}) \cong H^2(\mathcal{Q}(G), \mathbb{C}),$$

simplifies the terms in the descent spectral sequence associated to the constant sheaf $\mathbb{C}^*$.

Let

$$Z_G \subset G$$

be the center. Let $M^{\mathrm{ss}}_C(G)$ be the coarse moduli space of semistable principal G–bundles over C. The smooth locus of $M^{\mathrm{ss}}_C(G)$ is the locus of regularly stable principal G-bundles over C [BHf2, Corollary 3.6]. We recall that a principal G–bundle E is called *regularly*

stable if E is stable, and the natural homomorphism $Z_G \longrightarrow \mathrm{Aut}(E)$ is surjective (the restriction to Z_G of the action of G on E produces this homomorphism). Let

$$\mathcal{M}_C(G)^{\mathrm{rs}} \subset \mathcal{M}_C(G)$$

be the open sub–stack defined by the regularly stable bundles [BHf1, Lemma 2.3]. Let

(4.4) $$p : \mathcal{M}_C(G)^{\mathrm{rs}} \longrightarrow M_C(G)^{\mathrm{rs}}$$

be the morphism to the coarse moduli space, so $M_C(G)^{\mathrm{rs}}$ is the moduli space of regularly stable principal G–bundles over C.

Proposition 4.3. *As before, G is semisimple.*

(1) *$\mathcal{M}_C(G)^{\mathrm{rs}}$ is an open sub–stack of the moduli stack $\mathcal{M}_C(G)$ such that the complement is of codimension at least two.*

(2) *The morphism p in (4.4) defines a gerbe over $M_C(G)^{\mathrm{rs}}$ banded by Z_G.*

Proof. See [Fa, Theorem II.6] or Theorem 2.4 of [BHf1] for a proof of the first part. The second part is proved in Section 6 of [BHf1]. □

Let

(4.5) $$\psi \in H^2(M_C(G)^{\mathrm{rs}}, Z_G)$$

be the cohomology class defined by the gerbe in (4.4).

Let Z be a finite abelian group, and let

(4.6) $$\alpha : \mathcal{M} \longrightarrow M$$

be a gerbe banded by Z, where M is irreducible. Take any line bundle L over $\mathcal{M}$. So L is given by a functor L_S from $\mathcal{M}(S)$ to the groupoid of line bundles on S for every $\mathbb{C}$–scheme S. In particular, L_S defines for every object $\mathcal{E}$ in $\mathcal{M}(S)$ a group homomorphism

$$L_{S,\mathcal{E}} : \mathrm{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \longrightarrow \mathrm{Aut}_{\mathcal{O}_S}(L_S(\mathcal{E})) = \Gamma(S, \mathcal{O}_S^*) \,.$$

The compatibility conditions ensure that the composition

$$Z(S) \xrightarrow{\iota_{\mathcal{E}}} \mathrm{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \xrightarrow{L_{S,\mathcal{E}}} \Gamma(S, \mathcal{O}_S^*)$$

defines a 1–morphism $Z \times \mathcal{M} \longrightarrow \mathbb{G}_m \times \mathcal{M}$ over $\mathcal{M}$. As M is connected and $\mathrm{Hom}(Z, \mathbb{G}_m)$ is discrete, this 1–morphism is the pullback of some character $\chi : Z \longrightarrow \mathbb{G}_m$. We call χ the *weight* of L. (See [BHf1, Section 6].)

Lemma 4.4. *Let $\beta \in H^2(M, Z)$ be the class of the gerbe in (4.6). Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(M) \xrightarrow{\alpha^*} \mathrm{Pic}(\mathcal{M}) \xrightarrow{\mathrm{wt}} \mathrm{Hom}(Z, \mathbb{G}_m) \xrightarrow{\beta_*} \mathrm{Br}(M) \xrightarrow{\alpha^*} \mathrm{Br}(\mathcal{M}) \,,$$

where β_* takes a homomorphism $\eta : Z \longrightarrow \mathbb{G}_m$ to the image of β under the homomorphism $H^2(M, Z) \longrightarrow H^2(M, \mathbb{G}_m)$ induced by η.

Proof. This is the Leray spectral sequence for $\mathcal{O}_{\mathcal{M}}^* \longrightarrow \mathcal{M} \xrightarrow{\alpha} M$. The only points to note are that

$$\alpha_* \mathcal{O}_{\mathcal{M}}^* = \mathcal{O}_M^*$$

and $R^1 \alpha_* \mathcal{O}_{\mathcal{M}}^*$ is the constant sheaf $\mathrm{Hom}(Z, \mathbb{G}_m)$.   □

Theorem 4.5. *There is an exact sequence*

$$0 \longrightarrow \mathbb{Z}^s \xrightarrow{p^*} \mathbb{Z}^s \xrightarrow{\mathrm{wt}} \mathrm{Hom}(Z_G, \mathbb{G}_m) \xrightarrow{\psi_*} \mathrm{Br}(M_C(G)^{\mathrm{rs}}) \longrightarrow 0 \,,$$

where s is the number of almost simple factors in G, p^* is the homomorphism of Picard groups induced by p in (4.4), wt is the weight map defined above, and ψ_* is constructed using the class ψ in (4.5).

Proof. From Theorem 4.1 and Proposition 4.3(1) it follows that $\mathrm{Br}(\mathcal{M}_C(G)^{\mathrm{rs}}) = 0$. Therefore, Lemma 4.4 applied to the gerbe in (4.4) produces the exact sequence.   □

5. THE TWISTED CASE

In this section the semisimple group G is assumed to be simply connected. But the moduli stack and moduli space will be twisted.

Fix a maximal torus T of G. As before, the center of G will be denoted by Z_G. Let σ be the rank of G, so σ is the dimension of T.

As before, for any m, let $\mu_m \subset \mathbb{C}^*$ be the group of all m–th roots of 1. We fix an isomorphism

$$(5.1) \qquad\qquad \rho : T \longrightarrow (\mathbb{G}_m)^\sigma \quad \text{with} \quad \rho(Z_G) = \prod_{i=1}^{\sigma} \mu_{r_i} \,.$$

Using this isomorphism, the homomorphism

$$(\mathbb{G}_m)^\sigma \longrightarrow (\mathbb{G}_m)^\sigma$$

defined by $\prod_{i=1}^{\sigma} z_i \longmapsto \prod_{i=1}^{\sigma}(z_i)^{r_i}$ produces an isomorphism

$$T/Z_G \xrightarrow{\sim} T \,.$$

Let $C(G)$ denote the quotient of $G \times T$ by Z_G for the diagonal action. The projections

$$q : C(G) \longrightarrow T/Z_G = T \quad \text{and} \quad p : C(G) \longrightarrow G/Z_G$$

induce morphisms of stacks

$$(5.2) \qquad\qquad \det : \mathcal{M}_C(C(G)) \longrightarrow \mathcal{M}_C(T)$$

and

$$(5.3) \qquad\qquad \mathcal{M}_C(C(G)) \longrightarrow \mathcal{M}_C(G/Z_G) \,.$$

For an element $\mathbf{d} = (d_1, \ldots, d_\sigma) \in \mathbb{Z}^\sigma$, let $\mathcal{O}_C(\mathbf{d}p)$ denote the rational point of $\mathcal{M}_C(T)$ defined by $(\mathcal{O}_C(d_1 p_0), \ldots, \mathcal{O}_C(d_\sigma p_0))$. Define

$$(5.4) \qquad\qquad \mathcal{M}_C^{\mathbf{d}}(G) := \det^{-1}(\mathcal{O}_C(\mathbf{d}p)) \subset \mathcal{M}_C(C(G)) \,,$$

where det is the morphism in (5.2). Let

(5.5) $$\Phi \, : \, \mathcal{M}_C^{\mathbf{d}}(G) \, \longrightarrow \, \mathcal{M}_C(G/Z_G)$$

be the restriction of the morphism in (5.3).

Take any $\delta \in Z_G$. Consider

$$\rho(\delta) \, = \, (a_1, \cdots, a_\sigma) \, \in \, (\mathbb{G}_m)^\sigma \, ,$$

where ρ is the homomorphism in (5.1). Take $\mathbf{d} \in \mathbb{Z}^\sigma$ such that there is an element

$$\mathbf{r} \, := \, (r_1, \ldots, r_\sigma) \, \in \, \mathbb{Z}^\sigma$$

satisfying the condition that $\exp(-2\pi\sqrt{-1}d_i/r_i) \, = \, a_i$ for every $i \in [1, \sigma]$.

The connected components of $\mathcal{M}_C(G/Z_G)$ are parametrized by $\pi_1(G/Z_G) \, = \, Z_G$ (see Lemma 3.2). Let

$$\mathcal{M}_C(G/Z_G)^\delta \, \subset \, \mathcal{M}_C(G/Z_G)$$

be the connected component corresponding to the element δ. Let

$$\mathcal{M}_C^\delta(G) \, := \, \Phi^{-1}(\mathcal{M}_C(G/Z_G)^\delta) \, \subset \, \mathcal{M}_C^{\mathbf{d}}(G)$$

be the open and closed sub–stack of $\mathcal{M}_C^{\mathbf{d}}(G)$, where Φ is the morphism in (5.5). This $\mathcal{M}_C^\delta(G)$ is called the *twisted* moduli stack (see [BLS, Section 2]).

The restriction

$$\Phi|_{\mathcal{M}_C^\delta(G)} \, : \, \mathcal{M}_C^\delta(G) \, \longrightarrow \, \mathcal{M}_C(G/Z_G)^\delta$$

is surjective.

The group G/Z_G will also be denoted by G_{ad}. The connected components of $L(G_{\mathrm{ad}})$ are parametrized by Z_G (see Lemma 3.2). Let

$$(LG_{\mathrm{ad}})^\delta \, \subset \, L(G_{\mathrm{ad}})$$

be the connected component corresponding to δ.

The following proposition is proved in [BLS] (see [BLS, p. 189, (2.4)]):

Proposition 5.1 ([BLS]). *For any $\delta \in Z_G$, and any $\zeta \in (LG_{\mathrm{ad}})^\delta(\mathbb{C})$, there is a natural isomorphism*

$$\mathcal{M}_C^\delta(G) \, \cong \, \zeta^{-1}L_C(G)\zeta \backslash \mathcal{Q}_{\widetilde{G}} \, .$$

Proposition 5.1 implies that the of proof of Theorem 4.1 goes through in this twisted case. So we have the following theorem:

Theorem 5.2. *For any $\delta \in Z_G$, the group $\mathrm{Br}(\mathcal{M}_C^\delta(G))$ is trivial.*

There is a coarse moduli space $M_C^\delta(G)^{\mathrm{ss}}$ for the open sub–stack of $\mathcal{M}_C^\delta(G)$ defined by the locus of semistable principal $C(G)$–bundles. There is also an open subscheme $M_C^\delta(G)^{\mathrm{rs}} \subset M_C^\delta(G)^{\mathrm{ss}}$ corresponding to the regularly stable principal $C(G)$–bundles.

Proposition 5.3. *The codimension of the complement of $\mathcal{M}_C^\delta(G)^{\mathrm{rs}}$ in $\mathcal{M}_C^\delta(G)$ is at least two.*

The smooth locus of $M_C^\delta(G)^{\mathrm{ss}}$ is $M_C^\delta(G)^{\mathrm{rs}}$.

Proof. Let $\mathcal{M}_C^\delta(G/Z_G)^{\mathrm{rs}}$ be the sub–stack of $\mathcal{M}_C^\delta(G/Z_G)$ corresponding to the regularly stable principal $C(G)$–bundles; that it is a sub–stack follows from [BHf1, Lemma 2.3]. By [Fa, Theorem II.6] or [BHf1, Theorem 2.4] it follows that the complement of $\mathcal{M}_C^\delta(G/Z_G)^{\mathrm{rs}}$ in $\mathcal{M}_C^\delta(G/Z_G)$ has codimension at least two. Taking inverse image for the morphism Φ, the first part of the proposition follows.

The proof that the smooth locus of $M_C^\delta(G)^{\mathrm{ss}}$ is $M_C^\delta(G)^{\mathrm{rs}}$ is identical to that of [BHf2, Corollary 3.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The morphism

$$(5.6) \qquad\qquad p \,:\, \mathcal{M}_C^\delta(G)^{\mathrm{rs}} \,\longrightarrow\, M_C^\delta(G)^{\mathrm{rs}}$$

defines a gerbe banded by Z_G. Let

$$(5.7) \qquad\qquad \psi \,\in\, H^2(M_C^\delta(G)^{\mathrm{rs}},\, Z_G)$$

be the class of this gerbe.

We now have the following moduli space version of Theorem 5.2.

Theorem 5.4. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(M_C^\delta(G)^{\mathrm{rs}}) \xrightarrow{\ p^*\ } \mathrm{Pic}(\mathcal{M}_C^\delta(G)^{\mathrm{rs}}) \xrightarrow{\ \mathrm{wt}\ } \mathrm{Hom}(Z_G, \mathbb{G}_m) \xrightarrow{\ \psi_*\ } \mathrm{Br}(M_C^\delta(G)^{\mathrm{rs}}) \longrightarrow 0\,,$$

where p is the morphism in (5.6), wt is the weight defined earlier, and ψ_* takes a homomorphism $\eta \,:\, Z_G \,\longrightarrow\, \mathbb{G}_m$ to the image of ψ under the homomorphism $H^2(M,\, Z) \longrightarrow H^2(M,\, \mathbb{G}_m)$ induced by η.

6. BRAUER GROUP OF MODULI: G IS NOT SIMPLY CONNECTED

In this section we compute the Brauer group of the moduli stack and the smooth locus of the moduli space of principal G-bundles when the group G is not necessarily simply connected.

Let $\widetilde{G}$ be the universal cover of the semisimple group G. The fundamental group $\pi_1(G)$ is a subgroup of the center

$$Z_{\widetilde{G}} \,\subset\, \widetilde{G}\,.$$

Fix a maximal torus $\widetilde{T} \subset \widetilde{G}$. We fix an isomorphism

$$\rho \,:\, \widetilde{T} \,\longrightarrow\, (\mathbb{G}_m)^\sigma \quad \text{with} \quad \rho(\pi_1(G)) \,=\, \prod_{i=1}^{\sigma} \mu_{r_i}\,.$$

There is a canonical isomorphism $\pi_0(\mathcal{M}_C(G)) \,=\, \pi_0(L(G)) \,\cong\, \pi_1(G)$. Take any $\delta \,\in\, \pi_1(G)$. Let

$$L(G)^\delta \,\subset\, L(G) \quad \text{and} \quad \mathcal{M}_C(G)^\delta \,\subset\, \mathcal{M}_C(G)$$

be the connected components corresponding to δ.

The above notation $\mathcal{M}_C(G)^\delta$ for a connected component should not be confused with the previous notation $\mathcal{M}_C^\delta(G)$ for a twisted moduli stack. It should also be clarified that

the twisted moduli stack (or the twisted moduli space) is defined only for simply connected groups.

The following proposition is proved in [BLS]; see [BLS, p. 186, Proposition 1.3].

Proposition 6.1. *Take any* $\delta \in \pi_1(G)$. *For any* $\zeta \in L(G)^{\delta}(\mathbb{C})$, *there is a natural isomorphism*

$$\mathcal{M}_C(G)^{\delta} \cong \zeta^{-1}L_C(G)\zeta\backslash\mathcal{Q}_{\widetilde{G}}\,.$$

Let

$$(6.1) \qquad\qquad q_G^{\delta} \,:\, \mathcal{Q}_{\widetilde{G}} \longrightarrow \mathcal{M}_C(G)^{\delta}$$

be the quotient morphism in Proposition 6.1.

Let

$$(6.2) \qquad\qquad \widehat{q}_{\widetilde{G}}^{\,\delta} \,:\, \mathcal{Q}_{\widetilde{G}} \longrightarrow \mathcal{M}_C^{\delta}(\widetilde{G})$$

be the quotient morphism in Proposition 5.1. There is a natural morphism of stacks

$$(6.3) \qquad\qquad \gamma \,:\, \mathcal{M}_C^{\delta}(\widetilde{G}) \longrightarrow \mathcal{M}_C(G)^{\delta}\,.$$

This morphism γ takes the semistable locus in $\mathcal{M}_C^{\delta}(\widetilde{G})$ to the semistable locus in $\mathcal{M}_C(G)^{\delta}$. Hence it induces a morphism

$$(6.4) \qquad\qquad \gamma_1 \,:\, M_C^{\delta}(\widetilde{G}) \longrightarrow M_C(G)^{\delta}$$

between the corresponding coarse moduli spaces of semistable bundles.

Let

$$M_C(G)^{\delta,\mathrm{rs}} \subset M_C(G)^{\delta}$$

be the smooth locus; we recall that $M_C(G)^{\delta,\mathrm{rs}}$ parametrizes the regularly stable bundles in $M_C(G)^{\delta}$ (see [BHf2, Corollary 3.6]). Define the finite group

$$\Gamma \,:=\, H^1(C, \pi_1(G))\,.$$

Lemma 6.2.   (1) *The morphism q_G^{δ} in (6.1) factors as follows:*

$$\mathcal{Q}_{\widetilde{G}} \xrightarrow{\widehat{q}_{\widetilde{G}}^{\,\delta}} \mathcal{M}_C^{\delta}(\widetilde{G}) \xrightarrow{\gamma} \mathcal{M}_C(G)^{\delta}\,,$$

where $\widehat{q}_{\widetilde{G}}^{\,\delta}$ and γ are defined in (6.2) and (6.3) respectively.
(2) *The restriction of the morphism γ_1 in (6.4) to the Zariski open subset*

$$\gamma_1^{-1}(M_C(G)^{\delta,\mathrm{rs}}) \subset M_C^{\delta}(\widetilde{G})^{\mathrm{rs}}$$

defines a principal Γ–bundle over $M_C(G)^{\delta,\mathrm{rs}}$.

Proof. See [BLS, p. 189, (2.4)]. $\square$

We can apply the Serre spectral sequence to the principal Γ–bundle in Lemma 6.2 and get the following:

Theorem 6.3. *The Brauer group computation for $M_C(G)^{\delta,\mathrm{rs}}$ and $M_C^\delta(\widetilde{G})^{\mathrm{rs}}$ are related by the following exact sequence:*

$$
\begin{aligned}
H^1(\Gamma, \mathbb{C}^*) &\hookrightarrow H^1(M_C(G)^{\delta,\mathrm{rs}}, \mathcal{O}^*) \longrightarrow H^1(M_C^\delta(\widetilde{G})^{\mathrm{rs}}, \mathcal{O}^*) \\
&\longrightarrow H^2(\Gamma, \mathbb{C}^*) \longrightarrow \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}}) \longrightarrow \mathrm{Br}(M_C^\delta(\widetilde{G})^{\mathrm{rs}}) \longrightarrow 0.
\end{aligned}
$$

Let $\mathcal{M}_C(G)^{\delta,\mathrm{rs}}$ be the open sub–stack of $\mathcal{M}_C(G)^\delta$ defined by the regularly stable bundles. The morphism to the coarse moduli space

$$(6.5) \qquad\qquad p : \mathcal{M}_C(G)^{\delta,\mathrm{rs}} \longrightarrow M_C(G)^{\delta,\mathrm{rs}}$$

is a gerbe banded by Z_G, and we have the induced homomorphism

$$p^* : \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}}) \longrightarrow \mathrm{Br}(\mathcal{M}_C(G)^\delta).$$

Now the exact sequence in Lemma 4.4 can be used in relating the Brauer group of the smooth locus of the moduli space to the Brauer group of the stack. The kernel of the above homomorphism p^* can be computed using a result of [BHf1] which will be recalled below.

Let $\Psi \subset \mathrm{Hom}(Z_{\widetilde{G}} \otimes_{\mathbb{Z}} Z_{\widetilde{G}}, \mathbb{Q}/\mathbb{Z})$ be the abelian group of all symmetric bilinear maps

$$b : Z_{\widetilde{G}} \times Z_{\widetilde{G}} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

that come from an even W–invariant symmetric bilinear form

$$\Lambda_{\mathrm{coroots}} \times \Lambda_{\mathrm{coroots}} \longrightarrow \mathbb{Z}.$$

Let

$$(6.6) \qquad\qquad \Psi(G) \subset \Psi$$

be the subgroup of elements b such that $b(\pi_1(G) \times \pi_1(G)) = 0$.

Given an element $\delta \in \pi_1(G)$, let

$$(6.7) \qquad\qquad \mathrm{ev}_G^\delta : \Psi(G) \longrightarrow \mathrm{Hom}(Z_{\widetilde{G}}/\pi_1(G), \mathbb{Q}/\mathbb{Z})$$

be the evaluation map that sends any b to $b(\delta, -) : Z_G \longrightarrow \mathbb{Q}/\mathbb{Z}$.

The following proposition (Proposition 7.4 in [BHf1]) computes the weight map.

Proposition 6.4. *The kernel of the homomorphism*

$$p^* : \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}}) \longrightarrow \mathrm{Br}(\mathcal{M}_C(G)^\delta)$$

is given by the $\mathrm{Coker}(\mathrm{ev}_G^\delta)$.

This and Theorem 4.5 together imply the following:

Corollary 6.5. *If G is almost simple and simply connected, then $\mathrm{Br}(M_C(G)^{\mathrm{rs}}) = Z_G^\vee$.*

7. A direct approach

In this section we pursue the earlier spectral sequence argument.

Let $\widetilde{G} \longrightarrow G$ be the universal cover with G semisimple. The kernel of this homomorphism is identified with $\pi_1(G)$. Let

$$\Gamma := H^1(C, \pi_1(G))$$

be the finite group.

Lemma 7.1. *There is a short exact sequence*

$$0 \longrightarrow L_C(\widetilde{G})/\pi_1(G) \longrightarrow L_C(G) \longrightarrow \Gamma \longrightarrow 0.$$

The quotient space $L_C(\widetilde{G})/\pi_1(G)$ has the homotopy type of $G \times \Omega(\widetilde{G})^{2g-1}$

Proof. The short exact sequence is established in [BLS, p. 185, (1.2b)]. The statement on homotopy type is a consequence of the fact that $L_C(\widetilde{G})$ itself has the homotopy type of $\widetilde{G} \times \Omega(\widetilde{G})^{2g-1}$ (see Theorem 3.3). $\square$

Lemma 7.2. *We have*

(1) $H^1(B(L_C(\widetilde{G})/\pi_1(G)), \mathbb{C}^*) = 0$*, and*
(2) $H^2(B(L_C(\widetilde{G})/\pi_1(G)), \mathbb{C}^*) = \pi_1(G)^\vee$.

Proof. Since $L_C(\widetilde{G})$ is connected (see Lemma 3.2),

$$\pi_1(B(L_C(\widetilde{G})/\pi_1(G)) = \pi_0(L_C(\widetilde{G})/\pi_1(G)) = 0.$$

Hence the first statement follows.

To prove the second statement, consider the product decomposition

$$B(L_C(\widetilde{G})/\pi_1(G)) = BG \times B(\Omega\widetilde{G})^{2g-1},$$

and apply the Künneth decomposition to it. The individual cohomology computations are done as follows. We have $H^1(BG, \mathbb{C}^*) = 0$ and $H^1(B(\Omega\widetilde{G}), \mathbb{C}^*) = 0$ because $\pi_1(BG) = \pi_0(G) = 0$; also,

$$\pi_1(B(\Omega\widetilde{G})) = \pi_0(\Omega\widetilde{G}) = \pi_1(\widetilde{G}) = 0.$$

These and Hurewicz isomorphism imply that

$$H^2(B(\Omega\widetilde{G}), \mathbb{C}^*) \cong \mathrm{Hom}(\pi_2(B(\Omega\widetilde{G})), \mathbb{C}^*) \quad \text{and} \quad H^2(BG, \mathbb{C}^*) \cong \mathrm{Hom}(\pi_2(BG), \mathbb{C}^*).$$

Since $\pi_2(B(\Omega\widetilde{G})) = \pi_1(\Omega\widetilde{G}) = \pi_2(\widetilde{G}) = 0$, and $\pi_2(BG) = \pi_1(G)$, the second statement in the proposition follows. $\square$

As before, $\Gamma := H^1(C, \pi_1(G))$. The following is a generalization of Proposition 3.4.

Proposition 7.3. *With the above notation,*

(1) $H^1(BL_C(G), \mathbb{Z}) = 0$*, and*
(2) *there is a short exact sequence*

(7.1) $$0 \longrightarrow H^2(\Gamma, \mathbb{C}^*) \longrightarrow H^2(BL_C(G), \mathbb{C}^*)) \longrightarrow \pi_1(G)^\vee \longrightarrow 0.$$

Proof. The first part follows from the fact that

$$\pi_1(BL_C(G)) = \pi_0(L_C(G)) = \Gamma$$

is a finite group.

For the second part we use Lemma 7.1 to realize the space $B(L_C(\widetilde{G})/\pi_1(G))$ as a principal Γ–bundle over $BL_C(G)$. More precisely,

$$B(L_C(\widetilde{G})/\pi_1(G)) = (EL_C(G))/(L_C(\widetilde{G})/\pi_1(G)) \longrightarrow EL_C(G)/L_C(G) = BL_C(G).$$

The Serre spectral sequence gives the following exact sequence

$$H^0(\Gamma, H^1(B(L_C(\widetilde{G}/\pi_1(G))), \mathbb{C}^*)) \longrightarrow H^2(\Gamma, \mathbb{C}^*) \longrightarrow \mathrm{Ker}(H^2(BL_C(G), \mathbb{C}^*) \to$$

$$H^0(\Gamma, H^2(B(L_C(\widetilde{G}/\pi_1(G))), \mathbb{C}^*))) \longrightarrow H^1(\Gamma, H^1(B(L_C(\widetilde{G}/\pi_1(G))), \mathbb{C}^*)).$$

By Lemma 7.2, this reduces to the one in the lemma.   □

The analogue of Theorem 4.1 for a general semisimple group is the following.

Theorem 7.4. *For any $\delta \in \pi_1(G)$, the following exact sequence computes the Brauer group of the moduli stack of principal G-bundles over C:*

$$\mathrm{Pic}(\mathcal{M}_C(G)^\delta) \longrightarrow \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}}) \longrightarrow H^2(BL_C(G), \mathbb{C}^*) \longrightarrow \mathrm{Br}(\mathcal{M}_C(G)^\delta) \longrightarrow 0\,;$$

the above group $H^2(BL_C(G), \mathbb{C}^*)$ is computed by the exact sequence in (7.1).

Proof. The proof of the theorem follows the same steps as in the proof of Theorem 4.1, namely the descent spectral sequence and the cohomology computations — which are now provided by Lemma 7.3.   □

As the last step in the computation we prove the following.

Proposition 7.5. *Consider the morphism p in (6.5). The induced homomorphism*

$$p^* : \mathrm{Br}(M_C^\delta(G)^{\mathrm{rs}}) \longrightarrow \mathrm{Br}(\mathcal{M}_C^\delta(G)^{\mathrm{rs}}) = \mathrm{Br}(\mathcal{M}_C^\delta(G))$$

is surjective.

Proof. The above equality $\mathrm{Br}(\mathcal{M}_C^\delta(G)^{\mathrm{rs}}) = \mathrm{Br}(\mathcal{M}_C^\delta(G))$ follows from Proposition 4.3.

Let $\widetilde{G} \longrightarrow G$ be its universal covering homomorphism with the kernel $\pi_1(G)$. We get an inclusion $\pi_1(G) \subset Z_{\widetilde{G}}$, and the quotient is identified with Z_G. We have a canonical inclusion of

$$Z_G \hookrightarrow L_C(\widetilde{G})/\pi_1(G) \hookrightarrow L_C(G)$$

(see Lemma 7.1).

Let

$$\mathcal{Q}_1 \subset \mathcal{Q}_{\widetilde{G}}$$

be the open sub–stack defined by the inverse image of $\mathcal{M}_C(G)^{\delta,\mathrm{rs}}$ under the morphism q_G^δ in (6.1). Choosing an element $\zeta \in (LG)^\delta(\mathbb{C})$ as in Proposition 6.1 we get an action of the group $L_C(G)/Z_G$ on $\mathcal{Q}_1$ whose quotient is $M_C(G)^{\delta,\mathrm{rs}}$.

Since the pullback of $\mathcal{Q}_1 \longrightarrow M_C(G)^{\delta,\mathrm{rs}}$ by p in (6.5) is identified with $\mathcal{Q}_1 \times Z(G)$, we conclude, by taking cohomologies, that the diagram

$$
\begin{array}{ccc}
H^2(BL_C(G),\, \mathbb{C}^*) & \longrightarrow & \mathrm{Br}(\mathcal{M}_C(G)^{\delta,\mathrm{rs}}) \\
\uparrow & & \uparrow \\
H^2(B(L_C(G)/Z(G)),\, \mathbb{C}^*) & \longrightarrow & \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}})
\end{array}
$$

commutes. The upper horizontal arrow is surjective by Theorem 7.4. Hence to complete the proof of the proposition it suffices to show that the homomorphism

$$
H^2(BL_C(G),\, \mathbb{C}^*) \longrightarrow H^2(B(L_C(G)/Z(G)),\, \mathbb{C}^*)
$$

is surjective. But this surjectivity follows by applying Lemma 7.3 to $BL_C(G)$ and $B(L_C(G)/Z(G))$ and observing that the terms in the exact sequences match. This completes the proof of the proposition. $\qquad\square$

Combining the above proposition with Proposition 6.4 and Lemma 4.4 we get the following.

Corollary 7.6. *For any semisimple G, there is a short exact sequence*

$$
0 \longrightarrow \mathrm{Coker}(\mathrm{ev}_G^\delta) \longrightarrow \mathrm{Br}(M_C(G)^{\delta,\mathrm{rs}}) \longrightarrow \mathrm{Br}(\mathcal{M}_C(G)^\delta) \longrightarrow 0\,.
$$

8. Computations for classical groups

8.1. **The cases of SL_n and PGL_n.** Take a positive integer n, and take any $d \in [0,\, n-1]$. Fix a line bundle L_d on C of degree d. The twisted moduli stack $\mathcal{M}_C^d(\mathrm{SL}_n)$ is the moduli stack of vector bundles on C of rank n and determinant isomorphic to L_d. Theorem 4.1 and Theorem 5.2 imply that

$$
\mathrm{Br}(\mathcal{M}_C^d(\mathrm{SL}_n)) = 0
$$

for any d.

By Proposition 6.4, the Brauer group of $M_C^d(\mathrm{SL}_n)^{\mathrm{rs}}$ coincides with the cokernel of the homomorphism ev_G^δ, and this can be computed to be $\mathbb{Z}/\mathrm{g.c.d.}(n,d)\mathbb{Z}$ by Table 1 of [BHf1]. This recovers the results of [BBGN].

For the case of PGL_n, the exact sequence given in Theorem 6.3 or Theorem 7.4 reproduces Theorem 1 of [BHg].

The above approach also works for quotients of SL_n by finite subgroups of Z_{SL_n}.

8.2. **The cases Sp_{2n} and PSp_{2n}.** We have $\mathrm{Br}(\mathcal{M}_C^d(\mathrm{Sp}_{2n})) = 0$ for $d = 0,\, 1$ by Theorem 4.1 and Theorem 5.2.

From Corollary 6.5 we have $\mathrm{Br}(M_C^0(\mathrm{Sp}_{2n})^{\mathrm{rs}}) = \mathbb{Z}/2\mathbb{Z}$.

Proposition 8.1. *Let $d = 1$.*

(1) *Assume that $n \geq 3$ is odd. Then*

$$
\mathrm{Br}(M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}}) = 0\,.
$$

The square of the generating line bundle on $\mathcal{M}_C^1(\mathrm{Sp}_{2n})$ descends to $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ and generates $\mathrm{Pic}(M_C^1(\mathrm{Sp}_{2n}))^{\mathrm{rs}}$.

(2) *Assume that* $n \geq 3$ *is even. Then* $\mathrm{Br}(M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}}) = \mathbb{Z}/2\mathbb{Z}$. *The generating line bundle on* $\mathcal{M}_C^1(\mathrm{Sp}_{2n})$ *descends to* $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ *and generates* $\mathrm{Pic}(M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}})$.

Proof. This follows by the combination of exact sequences in Theorem 5.4 and Proposition 6.4. The cokernel of the homomorphism ev_G^δ is 0 when n is odd, and it is $\mathbb{Z}/2\mathbb{Z}$ when n is even by Table 1 of [BHf1]. $\qquad\square$

Remark 8.2. Let $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{ss}}$ be the twisted moduli space of semistable bundles. Proposition 8.1 has the interesting consequence that $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{ss}}$ is locally factorial for odd n, and it is not locally factorial for even n. This is because the Picard group of $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{ss}}$ is always generated by the descent of the square of the generating line bundle on $\mathcal{M}_C^1(\mathrm{Sp}_{2n})$ (see [BLS, p. 209, Proposition 11.2(a)]).

Note that $\pi_1(\mathrm{PSp}_{2n}) = Z_{Sp_{2n}}$. For $d \in \pi_1(\mathrm{PSp}_{2n})$, let $\mathcal{M}_C(\mathrm{PSp}_{2n})^d \subset \mathcal{M}_C(\mathrm{PSp}_{2n})$ be the connected component corresponding to d. Similarly,

$$M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}} \subset M_C(\mathrm{PSp}_{2n})^{\mathrm{rs}}$$

is the connected component corresponding to d. The locus $\mathcal{M}_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$ in $\mathcal{M}_C(\mathrm{PSp}_{2n})^d$ of regularly stable bundles is an open sub–stack such that the complement is of codimension at least two. Hence the Brauer groups of $\mathcal{M}_C(\mathrm{PSp}_{2n})^d$ and $\mathcal{M}_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$ coincide.

Define

$$\Gamma := H^2(C, \mu_2)$$

(the 2–torsion points of the Jacobian of C).

Proposition 8.3. *Let* $d = 0$ *or* 1. *If* $n \geq 3$ *is odd, then* $\mathrm{Br}(\mathcal{M}_C(\mathrm{PSp}_{2n})^d) \cong H^2(\Gamma, \mathbb{C}^*)$. *If* $n \geq 3$ *is even, then* $\mathrm{Br}(\mathcal{M}_C(\mathrm{PSp}_{2n})^d) \cong H^2(\Gamma, \mathbb{C}^*) \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. Let $U^{d,\mathrm{rs}} \subset M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ be the Zariski open subset defined by the inverse image of $M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$. The codimension of the complement of $U^{d,\mathrm{rs}}$ is at least two. To prove the proposition one can use both the approaches described used here. For example, Theorem 6.3 gives the following exact sequence

$$(8.1) \qquad 0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow H^2(\Gamma, \mathbb{C}^*) \longrightarrow \mathrm{Br}(M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}) \longrightarrow \mathrm{Br}(U^{d,\mathrm{rs}}) \longrightarrow 0,$$

where m is the smallest power of the generating line bundle on $U^{d,\mathrm{rs}}$ which descends to the moduli space $M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$.

Since $M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ is smooth, and the codimension of the complement of $U^{d,\mathrm{rs}}$ is at least two, the homomorphisms

$$\mathrm{Pic}(M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}) \longrightarrow \mathrm{Pic}(U^{d,\mathrm{rs}}) \quad \text{and} \quad \mathrm{Br}(M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}) \longrightarrow \mathrm{Br}(U^{d,\mathrm{rs}})$$

induced by the inclusion $U^{d,\mathrm{rs}} \hookrightarrow M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ are isomorphisms.

When n is even, by [BLS, p. 191, Proposition 4.2], the generating line bundle on the affine Grassmannian $\mathcal{Q}_{\mathrm{Sp}_{2n}}$ descends all the way to $M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$. Hence in this case we have $m = 1$ in (8.1).

When $d = 0$ with n odd, the generating line bundle on the affine Grassmannian descends to $M_C^0(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ (see Proposition 8.1), and not to $M_C(\mathrm{PSp}_{2n})^{0,\mathrm{rs}}$ by [BLS, Proposition 4.2]. Hence in this case we have $m = 2$ in (8.1).

If $d = 1$ with n odd, the smallest power of the generating line bundle on the affine Grassmannian which descends to $M_C^1(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ and $M_C(\mathrm{PSp}_{2n})^{0,\mathrm{rs}}$ is 2 (again by Proposition 8.1 and [BLS, Proposition 4.2]). Hence $m = 1$ in this case.

If $d = 1$ with n is odd, then the above observation and Proposition 8.1 together imply the proposition.

In the remaining cases we get a short exact sequence of the form

$$(8.2) \qquad 0 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^b \longrightarrow \mathrm{Br}(M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \,.$$

To complete the proof of the proposition it suffices to show that the sequence in (8.2) splits. To show that the sequence splits, it is enough to prove that the generating Brauer class of $M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ (when it is non–trivial) is the pull back of a two–torsion class in $\mathrm{Br}(M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}})$.

Let $\mathcal{M}_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}} \subset \mathcal{M}_C^d(\mathrm{Sp}_{2n})$ be the sub–stack defined by the regularly stable bundles. The Brauer group of $M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ (when it is non–trivial) is generated by the class of the generic gerbe

$$(8.3) \qquad \mathcal{M}_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}} \longrightarrow M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$$

which is banded by μ_2. The action of $\Gamma = H^2(C, \mu_2)$ on the moduli space $\mathcal{M}_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ naturally lifts to define an action of Γ on the stack $\mathcal{M}_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$. The corresponding quotient is a gerbe over $\mathcal{M}_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}} = M_C(\mathrm{PSp}_{2n})^{d,\mathrm{rs}}$ banded by μ_2 whose pull back to $M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ is the gerbe in (8.3). Consequently, the generating Brauer class of $M_C^d(\mathrm{Sp}_{2n})^{\mathrm{rs}}$ (when it is non–trivial) is the pull back of a two–torsion class in $\mathrm{Br}(M_C^d(\mathrm{PSp}_{2n})^{\mathrm{rs}})$. Hence the exact sequence in (8.2) splits. $\qquad \square$

8.3. **The cases of** Spin_n, SO_n **and** PSO_n. The center Z_{Spin_n} of Spin_n is $\mathbb{Z}/2\mathbb{Z}$ when n odd, and it is $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ depending on whether n is of the form $4l + 2$ or $4l$. Take any $\delta \in Z_{\mathrm{Spin}_n}$. Since the Dynkin index of the standard representation of Spin_n in SL_n is 2, the Picard group of the twisted moduli stack $\mathcal{M}_C^\delta(\mathrm{Spin}_n)$ is generated by the Pffafian line bundle P whose square is the determinant bundle of cohomology.

By Theorem 4.1 and Theorem 5.2,

$$\mathrm{Br}(\mathcal{M}_C^\delta(\mathrm{Spin}_n)) = 0 \,.$$

Proposition 8.4. (1) *If $\delta \in Z_{\mathrm{Spin_n}}$ is zero, then $\mathrm{Br}(M_C^0(\mathrm{Spin}_n)^{\mathrm{rs}}) = Z_{\mathrm{Spin}_n}^\vee$.*
(2) *Assume that $n \geq 4$ is odd. Then $\mathrm{Br}(M_C^1(\mathrm{Spin}_n)^{\mathrm{rs}}) = \mathbb{Z}/2\mathbb{Z}$. The Pffafian bundle descends to $M_C^1(\mathrm{Spin}_n)^{\mathrm{rs}}$ and generates the Picard group $\mathrm{Pic}(M_C^1(\mathrm{Spin}_n)^{\mathrm{rs}})$.*
(3) *Assume that $n = 4l + 2 \geq 8$ and $\delta = 1$ or $3 \mod (4)$. Then $\mathrm{Br}(M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}) = 0$. The fourth power of the Pffafian descends to $M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$ and it generates the Picard group $\mathrm{Pic}(M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}})$.*

(4) *Assume that* $n = 4l+2 \geq 8$ *and* $\delta = 2 \bmod (4)$. *Then* $\mathrm{Br}(M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}) = \mathbb{Z}/2\mathbb{Z}$.
   The square of the Pffafian descends to $M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$ *generating* $\mathrm{Pic}(M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}})$.
(5) *Assume that* $n = 4l \geq 8$ *and* $\delta \neq 0$. *Then* $\mathrm{Br}(M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}) = \mathbb{Z}/2\mathbb{Z}$. *The square*
   of the Pffafian descends to $M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$ *generating the Picard group.*

Proof. In view of Proposition 6.4, this is a straightforward calculation using Table 1 in
[BHf1]. $\qquad\square$

For the case of SO_n, note that $\pi_1(\mathrm{SO}_n) \subset Z_{\mathrm{Spin}_n}$. For any $\delta \in \pi_1(\mathrm{SO}_n)$, let

$$\mathcal{M}_C(\mathrm{SO}_n)^\delta \subset \mathcal{M}_C(\mathrm{SO}_n)$$

be the connected component corresponding to δ. Similarly, let

$$M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}} \subset M_C(\mathrm{SO}_n)^{\mathrm{rs}}$$

be the connected component corresponding to δ.

Define

$$\Gamma' := H^1(C, \pi_1(\mathrm{SO}_n)^\vee).$$

Proposition 8.5. *Let* $n \geq 8$. *Take any* $\delta \in \pi_1(\mathrm{SO}_n)$.

(1) *If* $\delta = 0$, *then*

$$\mathrm{Br}(M_C(\mathrm{SO}_n)^{0,\mathrm{rs}}) = H^2(\Gamma', \mathbb{C}^*) \oplus Z_{\mathrm{Spin}_n}^\vee.$$

(2) *If* $\delta \neq 0$, *then* $\mathrm{Br}(M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}}) \cong H^2(\Gamma', \mathbb{C}^*) \oplus \mathbb{Z}/2\mathbb{Z}$.
(3) *We have* $\mathrm{Br}(\mathcal{M}_C(\mathrm{SO}_n)^\delta) \cong H^2(\Gamma', \mathbb{C}^*) \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof is very similar to that for Sp_n as done in Proposition 8.1.

For any $\delta \in \pi_1(\mathrm{SO}_n)$, let

$$U^{\delta,\mathrm{rs}} \subset M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$$

be the Zariski open subset given by the inverse image of $M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}}$. The codimension
of the complement of $U^{\delta,\mathrm{rs}}$ in $M_C^\delta(\mathrm{Spin}_n)$ is at least two. Now Theorem 6.3 gives the exact
sequence

(8.4) $\qquad 0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow H^2(\Gamma', \mathbb{C}^*) \longrightarrow \mathrm{Br}(M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}}) \longrightarrow \mathrm{Br}(U^{\delta,\mathrm{rs}}) \longrightarrow 0,$

where m is the smallest power of the generating line bundle on $U^{\delta,\mathrm{rs}}$ which descends to
$M_C(\mathrm{SO}_2)^{\delta,\mathrm{rs}}$.

We will show that the integer m defined in (8.4) is 1. This amounts to showing that the
generating line bundle on $M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$ descend to the quotient $M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}} = U^{\delta,\mathrm{rs}}/\Gamma'$.

If $\delta = 0$, this follows from the existence of the Pffafian bundle.

Assume that $\delta \neq 0$. We note that Proposition 8.4 gives the smallest power of the
Pffafian which descends to $M_C^\delta(\mathrm{Spin}_n)^{\mathrm{rs}}$ from the moduli stack $\mathcal{M}_C^\delta(\mathrm{Spin}_n)$. Hence it
is enough to compute the power of the Pffafian on the moduli stack $\mathcal{M}_C^\delta(\mathrm{Spin}_n)$ which
descends to $M_C(\mathrm{SO}_n)^{\delta,\mathrm{rs}}$. To calculate this, we again use Proposition 6.4 and the Table 1 of

[BHf1], and explicitly check that the power of the Pffafian which descends to $M_C(SO_n)^{\delta,rs}$ coincide with the power that descends to $M_C^{\delta}(Spin_n)^{rs}$.

Let

$$(8.5) \qquad 0 \longrightarrow H^2(\Gamma', \mathbb{C}^*) \longrightarrow Br(M_C(SO_n)^{\delta,rs}) \longrightarrow Z_{Spin_n} \longrightarrow 0 \,,$$

be the exact sequence obtained from (8.4).

To conclude parts (1) and (2) of the proposition, we still need to show that the exact sequence in (8.5) splits. The proof of this splitting is identical to the case for Sp_n; we omit the details.

For the last part of the proposition, we use the existence of Pffafian to conclude that the homomorphism

$$H^2(BL_C(SO_n), \mathbb{C}^*) \longrightarrow Br(\mathcal{M}_C(SO_n)^{\delta})$$

in Theorem 7.4 is an isomorphism. Now Proposition 7.3 gives an exact sequence

$$0 \longrightarrow H^2(\Gamma', \mathbb{C}^*) \longrightarrow Br(\mathcal{M}_C(SO_n)^{\delta}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \,.$$

This sequence splits by using the argument for splitting of the (8.5) and Proposition 7.5. $\qquad \square$

For the case of PSO_{2n}, The components of the moduli spaces are parametrized by the center $Z_{Spin_{2n}}$. Since PSO_{2n} is of adjoint type, the Brauer groups of $M_C(PSO_{2n})^{\delta,rs}$ and $\mathcal{M}_C(PSO_{2n})^{\delta}$ can be identified as before. Define

$$\Gamma_1 := H^1(C, Z_{Spin_{2n}}^{\vee}) \quad \text{and} \quad B = H^2(\Gamma_1, \mathbb{C}^*) \,.$$

Using the above methods and description of the Picard and the Brauer groups as in Proposition 8.4 and [BLS, Proposition 5.5] we get the following proposition.

Proposition 8.6. *Let $n \geq 4$. We have $Br(M_C(PSO_{2n})^{\delta,rs}) = B/A_{2n,\delta} \oplus Z^{\vee}$, where the groups $A_{2n,\delta}$ is computed as follows: $A_{4n,\delta} = \mathbb{Z}/2\mathbb{Z}$ (respectively, 0) if $\delta = 0$ (respectively, $\neq 0$), and $A_{4n+2,\delta} = \mathbb{Z}/4\mathbb{Z}$ (respectively, $\mathbb{Z}/2\mathbb{Z}$) if δ is 0 mod (4) (respectively, 2 mod (4)), and $A_{4n+2,\delta} = 0$ otherwise.*

There is a remaining classical group Ω_{4n} for $n \geq 3$ defined by taking quotient of $Spin_{4n}$ by a central subgroup of order 2 which is not the one which defines SO_{4n}. There are two choices for such a sub-group and they define isomorphic groups.

We can use the methods described above to determine the Picard and the Brauer groups of the moduli spaces of Ω_{4n} bundles and we get the following.

Proposition 8.7.   (1) *We have $Br(M_C(\Omega_{4n})^{\delta,rs}) \cong H^2(\Gamma, \mathbb{C}^*) \oplus \mathbb{Z}/2\mathbb{Z}$. The square of the Pffafian descends to $M_C(\Omega_{4n})^{\delta,rs}$ generating the Picard group.*
   (2) *We have $Br(\mathcal{M}_C(\Omega_{4n})^{\delta}) \cong H^2(\Gamma, \mathbb{C}^*) \oplus A_{4n,d}$, where $A_{4n,d} = 0$ if $d = 0$ or n is odd, and $A_{4n,d} = \mathbb{Z}/2\mathbb{Z}$ otherwise. The Picard group of $\mathcal{M}_C(\Omega_{4n}^{\delta})$ is generated by the descent of the Pffafian if n is even with $d \neq 0$, it is generated by the descent of the square of the Pffafian in the other cases.*

The two exceptional groups E_6 and E_7 have nontrivial center. Using Theorem 7.4 and Corollary 7.6 the Brauer group of the moduli for corresponding adjoint type groups can be computed.

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