Information Geometric Modeling of Scattering Induced Quantum Entanglement

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We present an information geometric analysis of quantum entanglement generated by an s-wave scattering event between two minimum uncertainty Gaussian wave packets. We conjecture that the pre and post-collisional quantum dynamical scenarios related to an elastic head-on collision are macroscopic manifestations emerging from specific underlying microscopic statistical structures. Then we describe them by uncorrelated and correlated Gaussian statistical models, respectively. This allows us to express the entanglement strength, quantified by purity, in terms of scattering potential and incident particle energies. Furthermore, we show how the entanglement duration can be related to the scattering potential and incident particle energies. Finally, we uncover a quantitative relation between entanglement and information geometric complexity of motion.

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One of the most important features of composite quantum mechanical systems is their ability to become entangled [1, 2]. In general, quantum entanglement is described by quantum correlations among the distinct subsystems of the entire composite quantum system. For such correlated quantum systems, it is not possible to specify the quantum state of any subsystem independently of the remaining subsystems [1]. Apart from these remarks, the fundamental meaning of quantum entanglement is still a widely debated issue [3].

From a conceptual point of view, the simplest and most realistic mechanism of generating entanglement between two particles is via scattering processes [4, 5]. The two particles can become entangled as they approach each other as a consequence of mutual interactions. For instance, for interaction potentials with a strong repulsive core, quantum interference between incident and reflected waves can generate transient entanglement. After the collision, the two particles may still be entangled and share forms of quantum information in the final scattered state. Quantum entanglement can also be generated during inelastic collisions between the dissipative walls of a container and the quantum system confined within it [6]. Entanglement may also be induced in multi-atom systems confined in a harmonic trap interacting via a delta interaction potential [7].

In order to obtain a clear and detailed understanding of entanglement, it is first necessary to quantify it. It is known that for maximally entangled states it is not possible to specify the quantum state of any subsystem, while for separable states it is. Thus, one is led to consider the von Neumann entropy of the reduced state, measuring its degree of mixedness, as an entanglement measure. This turns out to be correct for pure bipartite states [8] (the case we are considering in this Letter), while for more general states other entanglement measures should be invoked [9, 10].

Apart from the above presented remarks, a great deal remains unclear about the physical interpretation of entanglement measures [11] and much remains unsatisfactory about our understanding of scattering-induced quantum entanglement, especially with regard to how interaction potentials and particle energies control the entanglement [4]. Finally, our knowledge of the connections between entanglement and complexity of motion remains far from complete [12].

In this Letter we investigate the potential utility of the *Information Geometric Approach to Chaos* (IGAC) [13, 14] in characterizing the quantum entanglement produced by a head-on elastic collision between two Gaussian wave packets interacting via a scattering process [15].

IGAC is a theoretical framework developed to study the complexity of informational geodesic flows on curved statistical manifolds underlying the probabilistic description of physical, biological or chemical systems. IGAC is the information geometric analogue of conventional geometrodynamical approaches [16, 17] where the classical configuration space is replaced by a statistical manifold with the additional possibility of considering complex dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat). For recent applications of the IGAC to quantum physics we refer to [18, 19]. In [19], for instance, we proposed a novel information-geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field and conjectured that our findings might find some potential physical applications in quantum energy level statistics. Here we conjecture that the scattering induced quantum entanglement is a macroscopic manifestation emerging from specific statistical microstructures. Specifically, using information geometric techniques [20] and inductive inference methods [21, 22], we propose that the pre and post-collisional scenarios are modelled by an uncorrelated [23] and correlated Gaussian statistical model [24], respectively. We present an analytical connection between the entanglement strength - quantified in terms of purity - to the scattering potential and incident particle energies. Furthermore, we relate the entanglement duration to the scattering potential and incident particle energies. Finally, we uncover a quantitative relation between quantum entanglement and the information geometric complexity of motion [25].

Before describing the physical system being studied, we recall that spatially localized Gaussian wave packets are especially useful to describe naturally occurring quantum states and they are easy to handle since many important properties of these states can often be obtained in an analytic fashion [26]. Furthermore, the Wigner distribution of Gaussian wave packets is positive-definite and therefore Gaussian states could be tagged as essentially classical [27].

The physical system being considered consists of two interacting spin-0 particles of equal mass m. For such a system, a complete set of commuting observables is furnished by the momentum operators of each particle [28]. In terms of the center of mass and relative coordinates, the Hamiltonian $\mathcal{H}$ of the system becomes

$$\mathcal{H} = \mathcal{H}_{\text{cm}} + \mathcal{H}_{\text{rel}}, \tag{1}$$

with

$$\mathcal{H}_{\text{cm}} = \frac{P^2}{2M} \text{ and } \mathcal{H}_{\text{rel}} = \frac{p^2}{2\mu} + V\left(r\right), \tag{2}$$

where $M \equiv 2m$ is the total mass and $\mu \equiv \frac{m}{2}$ is the reduced mass. The interaction potential $V\left(r\right)$ is isotropic and has a short range d such that $V\left(r\right) \approx 0$ for $r > d$. Before colliding, the two particles are in the form of disentangled Gaussian wave packets, each characterized by a width σ_0 in momentum space. The initial distance between the two particles is R_0 and their average initial momenta - setting the Planck constant $\hbar$ equal to one - are $\mp k_0$, respectively. From [15], after some straightforward algebra it follows that the initial (pre-collisional) two-particle square wave amplitude in momentum space is given by

$$P_{\text{pre}}^{(\text{QM})}\left(k, k_0, \sigma_0\right) = \frac{1}{\sqrt{\pi}\sigma_0} \exp\left[-\frac{\left(k-k_0\right)^2}{\sigma_0^2}\right], \tag{3}$$

where we have made use of the center of mass and relative coordinates $K \equiv k_1 + k_2$ and $k \equiv \frac{1}{2}\left(k_1 - k_2\right)$, respectively. The choice $K = 0$ is a natural one representing the two-particle system experiencing an exact head-on collision, with each particle's momentum having equal magnitude but opposite sign. As a side remark we point out that recent research suggests that quantum entanglement may be an observer-dependent concept in non-inertial frames [29]. In this context, we observe that the frame in which $K = 0$ is inertial and non-accelerating. For this reason, the possible observer-dependence of entanglement is not an issue in the present work. Similarly, following [15] and after some tedious algebra it turns out that the final (long time limit, post-collisional) two-particle square wave amplitude in momentum space is given by

$$P_{\text{post}}^{(\text{QM})}\left(k, k_0, \sigma_0; r_{\text{QM}}\right) = \frac{1}{\sqrt{\pi}\sigma_0\left[1 + \frac{2\sqrt{2}}{\sqrt{3}} r_{\text{QM}} + \frac{1}{\sqrt{2}} r_{\text{QM}}^2\right]} \exp\left(-\frac{\left(k-k_0\right)^2}{\sigma_0^2}\right)\left[1 + r_{\text{QM}} \exp\left(-\frac{\left(k-k_0\right)^2}{2\sigma_0^2}\right)\right]^2, \tag{4}$$

with

$$r_{\text{QM}} = r_{\text{QM}}\left(k_0, R_0, \theta\left(k_0\right)\right) \approx -\frac{\theta\left(k_0\right)}{k_0 R_0} \approx -\frac{\left|f_0\left(k_0\right)\right|}{R_0}, \tag{5}$$

where $\theta\left(k\right)$ and $f_0(k) \equiv \frac{\exp[2i\theta(k)]-1}{2ik}$ are the s-wave scattering phase shift and scattering amplitude, respectively, and we are considering them around $k = k_0$ and in the limit of low-energy scattering, i.e. $\theta\left(k\right) \ll 1$.

As pointed out earlier, in order to properly analyze entanglement, the entanglement entropy obtained from the long time limit post-collisional wave function is required. In most cases however, this must be performed numerically. Thus, to approach the problem analytically and simultaneously gain insights into the problem, it is convenient to make use of the linearized version of the entropy of the system, i.e. of the purity of the system [15]. The purity function is defined as

$$\mathcal{P} \overset{\text{def}}{=} \text{Tr}\left(\rho_A^2\right), \tag{6}$$

where $\rho_A \equiv \text{Tr}_B\left(\rho_{AB}\right)$ is the reduced density matrix of particle A and ρ_{AB} is the two-particle density matrix associated with the post-collisional two-particle wave function. For pure two-particle states, the smaller the value of $\mathcal{P}$ the higher the entanglement. That is, the loss of purity provides an indicator of the degree of entanglement. Hence, a disentangled product state corresponds to $\mathcal{P} = 1$. We emphasize that the purity has been used as a measure of the degree of entanglement in various physical situations [30], especially in atomic physics in order to characterize the two-body correlations in dynamical atomic processes [31, 32]. Under the assumption that the two particles are well separated both initially (before collision) and finally (after collision) [33] and assuming that the colliding Gaussian wave packets are very narrow in the momentum space ($\sigma_0 \ll 1$ such that the phase shift can be treated as a constant $\theta\left(k_0\right)$), it follows that the purity of the post-collisional two-particle wave function is approximately given by [15]

$$\mathcal{P} \approx 1 - \frac{S_0\left(k_0\right)}{\pi}\sigma_{\text{collision}}^2, \tag{7}$$

where $S_0(k_0) \equiv 4\pi |f_0(k_0)|^2$ is the scattering cross section and $\sigma_{\text{collision}}^{-1}$ is the spatial width of the wave packets at the collision time $t_{\text{collision}} = \frac{R_0}{k_0}$ with

$$\sigma_{\text{collision}} \equiv \frac{\sigma_0}{\sqrt{1+\left(\frac{\sigma_0^2 R_0}{2k_0}\right)^2}}. \tag{8}$$

Equation (7) implies that the loss of purity is solely controlled by the ratio of the scattering cross section $S_0(k_0)$ to the transverse area $\sigma_{\text{collision}}^{-2}$of the wave packets. Although very interesting, Wang's analysis [15] does not address the problem of how the interaction potentials and particle energies control the scattering-induced entanglement and it does not discuss any possible connection between the entanglement generated in the scattering process and the complexity of the motion related to the pre and post-collisional quantum dynamical scenarios.

In this Letter, we attempt to provide satisfying answers to such unsolved relevant issues. We conjecture that the pre and post-collisional quantum dynamical scenarios characterized by (3) and (4), respectively, and describing the quantum entanglement (quantified in terms of the purity $\mathcal{P}$ in (7)) produced by a head-on collision between two Gaussian wave packets are macroscopic manifestations emerging from specific underlying microscopic statistical structures. Specifically, we propose that $P_{\text{pre}}^{(\text{QM})}(k, k_0, \sigma_0)$ can be interpreted as a limiting case (initial time limit) arising from a Gaussian probability distribution $P_{\text{pre}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma)$,

$$P_{\text{pre}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma) \overset{\text{def}}{=} \frac{\exp\left\{-\frac{1}{2\sigma^2}\left[(k_1-\mu_{k_1})^2+(k_2-\mu_{k_2})^2\right]\right\}}{2\pi\sigma^2}. \tag{9}$$

As a matter of fact, using the center of mass and relative coordinates K and k and choosing $K=0$, we obtain that $k_1 = k$, $k_2 = -k$. Finally, setting $\mu_{k_1} = k_0$, $\mu_{k_2} = -k_0$ and $\sigma = \sigma_0$, we obtain

$$P_{\text{pre}}^{(\text{QM})}(k, k_0, \sigma_0) = P_{\text{pre}}^{(\text{IG})}(k, k_0, \sigma_0). \tag{10}$$

Similarly, we propose that $P_{\text{post}}^{(\text{QM})}(k, k_0, \sigma_0; r_{\text{QM}})$ can be viewed as a limiting case (final or long time limit) arising from a Gaussian probability distribution $P_{\text{post}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma; r_{\text{IG}})$,

$$P_{\text{post}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma; r_{\text{IG}}) = \frac{\exp\left\{-\frac{1}{2\sigma^2\left(1-r_{\text{IG}}^2\right)}\left[(k_1-\mu_{k_1})^2 - 2r_{\text{IG}}(k_1-\mu_{k_1})(k_2-\mu_{k_2}) + (k_2-\mu_{k_2})^2\right]\right\}}{2\pi\sigma^2\sqrt{1-r_{\text{IG}}^2}}, \tag{11}$$

where $r_{\text{IG}} \overset{\text{def}}{=} \frac{\langle k_1 k_2\rangle - \langle k_1\rangle\langle k_2\rangle}{\sigma^2}$ is the correlation coefficient. Using the center of mass and relative coordinates K and k and choosing $K=0$, we obtain that $k_1 = k$, $k_2 = -k$. Finally, setting $\mu_{k_1} = k_0$, $\mu_{k_2} = -k_0$ and $\sigma = \sigma_0$, we obtain

$$P_{\text{post}}^{(\text{IG})}(k, k_0, \sigma_0; r_{\text{IG}}) = \frac{1}{\sqrt{\pi}\sigma_0\sqrt{1-r_{\text{IG}}}} \exp\left[-\frac{(k-k_0)^2}{\sigma_0^2(1-r_{\text{IG}})}\right]. \tag{12}$$

In this case it turns out that when both the weak correlation ($r_{\text{IG}} \ll 1$) and the weak scattering conditions ($r_{\text{QM}} \ll 1$) are satisfied, we obtain an excellent overlapping between (4) and (12),

$$P_{\text{post}}^{(\text{QM})}(k, k_0, \sigma_0; r_{\text{QM}}) \approx P_{\text{post}}^{(\text{IG})}(k, k_0, \sigma_0; r_{\text{IG}}), \text{ for } r_{\text{IG}} \ll 1 \text{ and } r_{\text{QM}} \ll 1, \tag{13}$$

assuming that k_0, σ_0, r_{QM} and r_{IG} are fixed numerical constants and letting k assume values in the neighborhood of k_0. At this stage our conjecture is only mathematically sustained by the formal identities (10) and (13). To render our conjecture also physically motivated, recall that s-wave scattering can also be described in terms of a scattering potential $V(r)$ and the scattering phase shift $\theta(k)$. For the problem under consideration, $V(r)$ equals the constant value V for $0 \leq r \leq d$ while it is vanishing for $r > d$. The quantities V and d denote the height (for $V > 0$; repulsive potential) or depth (for $V < 0$; attractive potential) and range of the potential, respectively. Integrating the radial part of Schrdinger equation with this potential for the scattered wave and imposing the matching condition at $r = d$ for its solution and its first derivative leads to [34, 35]

$$k_{\text{in}} \cot(k_{\text{in}} d) = k_{\text{out}} \cot(k_{\text{out}} d + \theta), \tag{14}$$

with $k_{\text{in}} = \sqrt{2\mu(T-V)}$ for $0 \leq r \leq d$ and $k_{\text{out}} = \sqrt{2\mu T}$ for $r > d$. The quantities μ and T are the reduced mass and kinetic energy of the two-particle system in the relative coordinates, respectively; k_{in} and k_{out} represent the conjugate-coordinate wave vectors inside and outside the potential region, respectively. Equation (14) indicates that the scattering potential $V(r)$ shifts the phase of the scattered wave at points beyond the scattering region. We will show that equation (14) allows to exploit our information geometric modeling and relate the s-wave scattering phase shift θ to the correlation coefficient r_{IG}.

Our information geometric modeling may be briefly described in the following way. The pre-collisional scenario is characterized by the information geometric dynamics on the curved statistical manifold $\mathcal{M}_s^{(\text{uncorr.})}$ of uncorrelated Gaussian probability distributions $P_{\text{pre}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma)$ given in (9). Furthermore, the post-collisional scenario is characterized by the information geometric dynamics on the curved statistical manifold $\mathcal{M}_s^{(\text{corr.})}$ of correlated Gaussian probability distributions $P_{\text{post}}^{(\text{IG})}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma; r_{\text{IG}})$ given in (11). Omitting technical details that will appear elsewhere [36], it turns out that in the limit of low energy s-wave scattering ($k_0 d \ll 1$) and low correlations ($r_{\text{IG}} \ll 1$), the matching condition (14) in the information geometric framework leads to

$$\theta(k_0) \approx -\frac{1}{3} r_{\text{IG}} d^3 k_0^3, \tag{15}$$

where

$$r_{\text{IG}} = \frac{V}{T} = \frac{2\mu V}{k_0^2}. \tag{16}$$

Combining (15) and (16), we obtain

$$\theta(k_0) \approx -\frac{2}{3} \mu V d^3 k_0. \tag{17}$$

Equation (17), obtained via information geometric dynamical methods, is in perfect agreement with the result presented in [37] where standard Schrodinger's quantum dynamics was employed. This is the first significant finding of this Letter and allows to state that our conjecture is also physically motivated.

As a consequence of (7) and (15), we find that when both low energy and weak correlation regimes occur, the purity $\mathcal{P}$ of the system becomes

$$\mathcal{P} \approx 1 - \frac{4}{9} d^6 k_0^4 \sigma_{\text{collision}}^2 r_{\text{IG}}^2. \tag{18}$$

Equation (18) implies that the purity $\mathcal{P}$ can be expressed in terms of physical quantities such as the scattering potential $V(r)$ and the initial quantities k_0, σ_0 and R_0 via (8) and (16). This is the second significant finding obtained within our hybrid approach (quantum dynamical results combined with information geometric modeling techniques) that allows to explain how the interaction potential $V(r)$ and the incident particle energies T control the strength of the entanglement. The role played by r_{IG} in the quantities $\mathcal{P}$ and V suggests that the physical information about quantum scattering and therefore about quantum entanglement is encoded in the statistical correlation coefficient, specifically in the covariance term $\text{Cov}(k_1, k_2) \overset{\text{def}}{=} \langle k_1 k_2 \rangle - \langle k_1 \rangle \langle k_2 \rangle$ appearing in the definition of r_{IG}.

An additional interesting finding uncovered by our approach is the entanglement duration Δ,

$$\Delta(k_0, \sigma_0, r_{\text{IG}}) \equiv \tau_{\text{corr.}} - \tau_{\text{uncorr.}} \propto \left| \ln \left\{ 1 - \left[(1 - r_{\text{IG}})^{-1/2} - 1 \right] \cdot \eta_\Delta \right\} \right|, \tag{19}$$

where $\tau_{\text{corr.}}$ and $\tau_{\text{uncorr.}}$ are the temporal intervals required for a particle to reach the same value of momentum k_0 from 0 in the post-collisional scenario, in presence and in the absence of correlations r_{IG}, respectively, and η_Δ is given by

$$\eta_\Delta = \left(\frac{k_0}{\sigma_0}\right)^2 \exp\left[\left(\frac{\sigma_0}{k_0}\right)^2 - \frac{3}{4}\left(\frac{\sigma_0}{k_0}\right)^4 + \mathcal{O}\left[\left(\frac{\sigma_0}{k_0}\right)^6\right]\right] \text{ for } \frac{\sigma_0}{k_0} \ll 1. \tag{20}$$

Here, we can find the upper bound value of r_{IG} by means of (19) and (20),

$$r_{\text{IG}} < \frac{2}{\eta_\Delta}. \tag{21}$$

For example, with $\sigma_0/k_0 \sim 10^{-3}$ we have $r_{\text{IG}} < 2 \times 10^{-6}$. We observe that the entanglement duration can be controlled via the initial parameters k_0, σ_0 and the correlations r_{IG} (therefore via the incident particle energies and the scattering potential due to (16)). Also, we notice that in the absence of correlations, i.e. $r_{\text{IG}} \to 0$, $\Delta \to 0$. It is anticipated that the maximum duration would be obtained when r_{IG} is the greatest and the ratio σ_0/k_0 is the smallest.

Our final finding uncovers an interesting quantitative connection between quantum entanglement quantified by the purity $\mathcal{P}$ in (18) and the information geometric complexity of motion on the uncorrelated and correlated curved statistical manifolds

$\mathcal{M}_s^{(\text{uncorr.})}$ and $\mathcal{M}_s^{(\text{corr.})}$, respectively. The information geometric complexity as defined in [25] represents the volume of the effective parametric space explored by the system in its evolution between the chosen initial and final macrostates. The volume itself is in general given in terms of a multidimensional fold-integral over the geodesic paths connecting the initial and final macrostates. A geodesic on a curved statistical manifold represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates. For additional details, we refer to one of our latest presentations appearing in [25]. Here, omitting technical details and following the works presented in [23, 24, 36], one finds that

$$\mathcal{C}_{\text{IG}}^{(\text{corr.})} = \sqrt{\frac{1-r_{\text{IG}}}{1+r_{\text{IG}}}}\mathcal{C}_{\text{IG}}^{(\text{uncorr.})}, \tag{22}$$

where $\mathcal{C}_{\text{IG}}^{(\text{corr.})}$ and $\mathcal{C}_{\text{IG}}^{(\text{uncorr.})}$ denotes the information geometric complexities of motion on the chosen statistical manifolds. As a side remark, we point out that (22) confirms that an increase in the correlational structure of the dynamical equations for the statistical variables labelling a macrostate of a system implies a reduction in the complexity of the geodesic paths on the underlying curved statistical manifolds [38, 39]. In other words, making macroscopic predictions in the presence of correlations is easier than in their absence. Combining (18) and (22) it follows that

$$\mathcal{P} \approx 1 - \frac{4}{9} d^6 k_0^4 \sigma_{\text{collision}}^2 \left\{ \frac{\left[\mathcal{C}_{\text{IG}}^{(\text{uncorr.})}\right]^2 - \left[\mathcal{C}_{\text{IG}}^{(\text{corr.})}\right]^2}{\left[\mathcal{C}_{\text{IG}}^{(\text{uncorr.})}\right]^2 + \left[\mathcal{C}_{\text{IG}}^{(\text{corr.})}\right]^2} \right\}^2. \tag{23}$$

From (23) it is evident that the scattering-induced quantum entanglement and the information geometric complexity of motion are connected. In particular, when purity goes to unity (entanglement-free scenario), the difference between the correlated and uncorrelated information geometric complexities approaches zero.

In conclusion, using information geometric techniques and inductive inference methods, we have proposed that the pre and post-collisional scenarios describing the quantum entanglement produced by a head-on elastic collision between two Gaussian wave packets are modelled by an uncorrelated and correlated Gaussian statistical model, respectively. We showed that our conjecture is physically motivated by Equation (15). It allowed us to connect the entanglement strength quantified in terms of purity to the scattering potential and incident particle energies (Eqs. (16) and (18)). We were also capable of relating the entanglement duration Δ to the scattering potential $V(r)$ and incident particle energies T (Eqs. (16) and (19)). Finally, we uncovered a quantitative relation between quantum entanglement measured by the purity and the information geometric complexity of motion (Eq. (23)). As a final remark, we point out that our analysis allows us to interpret quantum entanglement as a perturbation of statistical space geometry in analogy to the interpretation of gravitation as perturbation of flat spacetime. The nature of the perturbation of statistical geometry is global. This, together with the time-independence of the geometry, leads to the notion of non-locality. The perturbation of statistical geometry is associated with the scattering phase shift in the momentum space.

We are confident that the present work represents significant progress toward the goal of understanding the relationship between statistical microcorrelations and quantum entanglement on the one hand and the effect of microcorrelations on the dynamical complexity of informational geodesic flows on the other. It is our hope to build upon the techniques employed in this work to ultimately establish a sound information-geometric interpretation of quantum entanglement.

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