VECTOR GROUPOIDS

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ABSTRACT. The main purpose of this paper is to study the vector groupoids. This is an algebraic structure which combines the concepts of Brandt groupoid and vector space such that these are compatible. The new concept of vector groupoid has applications in geometry and other areas. [1]

1 INTRODUCTION

A groupoid, also known as a *virtual group* [16], is an algebraic structure introduced by H. Brandt [1]. A groupoid (in the sense of Brandt) can be thought as a set with a partially defined multiplication, for which the usual properties of a group hold whenever they make sense.

A generalization of Brandt groupoid has appeared in [9]. C. Ehresmann added further structures (topological and differentiable as well as algebraic) to groupoids.

Groupoids and its generalizations (topological groupoids, Lie groupoids, measure groupoids, sympectic groupoids etc.) are mathematical structures that have proved to be useful in many areas of science [algebraic topology ([3], [8]), harmonic analysis and operators algebras ([8], [18], [22]), differential geometry and its applications ([4], [6], [14], [17], [21]), noncommutative geometry ([5]), algebraic and geometric combinatorics ([13], [20]), dynamics of networks ([7], [11], [19] and more].

It is remarkable to note that according to A. Connes [5], Heisenberg was discovered quantum mechanics by considering the groupoid of quantum transitions rather than the group of symmetry.

The paper is organized as follows. In Section 2 we define groupoids and useful properties of them are presented. In Section 3 we introduce the concept of vector groupoid and its properties are established. In Section 3 we give some algebraic constructions of vector groupoids.

2 BRANDT GROUPOIDS

We recall the minimal necessary backgrounds on groupoids for our developments (for further details see e.g. [2], [10], [12], [15] and references therein for more details).

Definition 2.1. *([6]) A **groupoid** G **over** G_0 (**in the sense of Brandt**) is a pair (G, G_0) of nonempty sets such that $G_0 \subseteq G$ endowed with two*

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surjective maps $\alpha, \beta : G \to G_0$ *(called* **source**, *respectively* **target**, *a partially binary operation (called* **multiplication***)* $m : G_{(2)} \to G,\ (x,y) \longmapsto m(x,y) := x{\cdot}y$, *where* $G_{(2)} := G \times_{(\beta,\alpha)} G = \{(x,y) \in G \times G \mid \beta(x) = \alpha(y)\}$ *is the* **set of composable pairs** *and a map* $\iota : G \to G,\ x \longmapsto \iota(x) := x^{-1}$ *(called* ***inversion****), which verify the following conditions:*

*(G) (***associativity***):* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *in the sense that if one of two products* $(x \cdot y) \cdot z$ *and* $x \cdot (y \cdot z)$ *is defined, then the other product is also defined and they are equals;*

*(G2) (***units***): for each* $x \in G \ \Rightarrow\ (\alpha(x), x),\ (x, \beta(x)) \in G_{(2)}$ *and we have* $\alpha(x) \cdot x = x \cdot \beta(x) = x$;

*(G3) (***inverses***): for each* $x \in G \ \Rightarrow\ (x, x^{-1}),\ (x^{-1}, x) \in G_{(2)}$ *and we have* $x^{-1} \cdot x = \beta(x),\ \ x \cdot x^{-1} = \alpha(x)$.

A groupoid G over G_0 with the *structure functions* α, β, m, ι is denoted by $(G, \alpha, \beta, m, \iota, G_0)$ or (G, α, β, G_0) or (G, G_0). The element $\alpha(x)$ respectively $\beta(x)$ is called the *left unit* respectively *right unit* of x; G_0 is called the *unit set* of G. The map (α, β) defined by:

$$(\alpha, \beta) : G \to G_0 \times G_0, \quad (\alpha, \beta)(x) := (\alpha(x), \beta(x)),\ x \in G,$$

is called the *anchor map* of G. For each $u \in G_0$, the set $G_u := \alpha^{-1}(u)$ (resp. $G_u := \beta^{-1}(u)$) is called $\alpha-$ *fibre* (resp. $\beta-$ *fibre*) of G at $u \in G_0$. If $u, v \in G_0$ we will write $G_v^u = \alpha^{-1}(u) \cap \beta^{-1}(v)$.

A groupoid (G, G_0) is said to be *transitive*, if its anchor map is surjective.

Convention. (1) We write sometimes xy for $m(x,y)$, if $(x,y) \in G_{(2)}$.

(2) Whenever we write a product in a given groupoid, we are assuming that it is defined. $\square$

In the following proposition we summarize some basic rules of algebraic calculation in a Brandt groupoid obtained directly from definitions.

Proposition 2.1. *([12]) In a groupoid* $(G, \alpha, \beta, m, \iota, G_0)$ *the following assertions hold :*

(i) $\alpha(u) = \beta(u) = u, \quad u \cdot u = u \quad$ *and* $\quad \iota(u) = u,\ \forall u \in G_0$;

(ii) $\alpha(x \cdot y) = \alpha(x) \quad$ *and* $\quad \beta(x \cdot y) = \beta(y),\ \forall (x,y) \in G_{(2)}$;

(iii) $\alpha\left(x^{-1}\right) = \beta(x) \quad$ *and* $\quad \beta\left(x^{-1}\right) = \alpha(x),\ \forall x \in G$;

(iv) **(cancellation law)** *If for* $x, y_1, y_2, z \in G$ *we have* $(x, y_1), (x, y_2), (y_1, z), (y_2, z) \in G_{(2)}$*, then:*

$$(a) \quad x \cdot y_1 = x \cdot y_2 \ \Rightarrow\ y_1 = y_2; \qquad (b) \quad y_1 \cdot z = y_2 \cdot z \ \Rightarrow\ y_1 = y_2.$$

(v) For each $x \in G$ *we have* $(x^{-1})^{-1} = x$.

(vi) If $(x,y) \in G_{(2)}$*, then* $(y^{-1}, x^{-1}) \in G_{(2)}$ *and the equality holds:*

$$(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}.$$

(vii) For all $(x,y) \in G_{(2)}$*, the following equalities hold:*

$$x^{-1} \cdot (x \cdot y) = y \quad \text{and} \quad (x \cdot y) \cdot y^{-1} = x.$$

In a groupoid (G, G_0) for any $u \in G_0$, the set $G(u) := \alpha^{-1}(u) \cap \beta^{-1}(u) = \{x \in G \mid \alpha(x) = \beta(x) = u \}$ is a group under the restriction of the partial multiplication m to $G(u)$, called the *isotropy group at* u of G.

Proposition 2.2. *([12]) Let* $(G, \alpha, \beta, m, \iota, G_0)$ *be a groupoid. Then:*

(i) $\alpha \circ \iota = \beta$, $\beta \circ \iota = \alpha$ *and* $\iota \circ \iota = Id_G$.

(ii) $\varphi : G(\alpha(x)) \to G(\beta(x))$, $\varphi(z) := x^{-1}zx$ *is an isomorphism of groups.*

(iii) If (G, G_0) *is transitive, then all isotropy groups are isomorphes.*

A *group bundle* is a groupoid (G, G_0) with the property that $\alpha(x) = \beta(x)$ for all $x \in G$. Moreover,a group bundle is the union of its isotropy groups $G(u) = \alpha^{-1}(u), u \in G_0$ (here, two elements may be composed iff they lie in the same fiber $\alpha^{-1}(u)$).

If (G, α, β, G_0) is a groupoid then $Is(G) := \{x \in G \mid \alpha(x) = \beta(x)\}$ is a group bundle, called the *isotropy group bundle* of G.

Example 2.1. *(i)* Any group G having e as unity, is a groupoid over $G_0 = \{e\}$ with the structure functions α, β, m, ι given by:
$\alpha(x) = \beta(x) = e$, $\iota(x) = x^{-1}$ for all $x \in G$ and $m(x, y) = xy$ for all $x, y \in G$.

(ii) Any set X can be endowed with a *nul groupoid* structure over itself. For this we take: $\alpha = \beta = \iota = Id_X$; $x, y \in X$ are composable iff $x = y$ and we define $x \cdot x = x$.

(iii) The Cartesian product $G := X \times X$ has a structure of groupoid over $\Delta_X = \{(x, x) \in X \times X \mid x \in X\}$ by taking the structure functions as follows: $\widetilde{\alpha}(x, y) := (x, x)$, $\widetilde{\beta}(x, y) := (y, y)$; the elements (x, y) and (y', z) are composable in $G := X \times X$ iff $y' = y$ and we define $(x, y) \cdot (y, z) = (x, z)$ and the inverse of (x, y) is defined by $(x, y)^{-1} := (y, x)$. This is usually called the *pair* or *coarse groupoid*. Its unit set is $G_0 := \Delta_X$. The isotropy group $G(u)$ at $u = (x, x)$ is the nul group $\{(u, u)\}$.

Example 2.2. *(i) The symmetry groupoid* $\mathcal{SG}(X)$. Let X be a nonempty set and consider

$G := \mathcal{SG}(A, X) = \{f : A \to A \mid \emptyset \neq A \subseteq X,\ f \text{ is bijective } \}$ and

$G_0 := \{Id_A \mid \emptyset \neq A \subseteq X\}$, where Id_A is the identity map on A.

Let $G_{(2)} := \{(f, g) \in G \times G | D(f) = D(g)\}$, where $D(f)$ denotes the domain of f. The structure functions $\alpha, \beta : G \to G_0$, $\iota : G \to G$ and the multiplication $m : G_{(2)} \to G$ are given by:

$\alpha(f) := Id_{D(f)}$, $\beta(f) := Id_{D(f)}$, $\iota(f) := f^{-1}$ and $m(f, g) := f \circ g$.

Then (G, G_0) is a groupoid, called the *groupoid of bijective functions from the subsets* A *of* X *onto* A or the *symmetry groupoid of the set* X.

The isotropy group at $u = Id_A$ is the symmetry group of the set A, i.e. $G(u) = \{f : A \to A \mid f \text{ is bijective } \}$.

In particular, the symmetry groupoid of a finite set $X = \{x_1, x_2, \ldots, x_n\}$, is called the *symmetry groupoid of degree* n and is denoted by $\mathcal{SG}_n$. Its unit set is $\mathcal{SG}_{n,0} = \{Id_A \mid \emptyset \neq A \subseteq \{x_1, x_2, \ldots, x_n\}\}$. The cardinals of these finite sets are given by:

$$\mid \mathcal{SG}_n \mid \; = \; \sum_{k=1}^{n} k! \binom{n}{k}, \qquad \mid \mathcal{SG}_{n,0} \mid \; = \; 2^n - 1.$$

(ii) The Galois groupoid $\mathcal{G}al(\mathcal{E}/K)$. Let F/K be an extension field of a field K, i.e. K is a subfield of F. We consider an indexed family $\mathcal{E} := (E_i)_{i\in I}$ of intermediate fields E_i, that is $K \subseteq E_i \subseteq F$ for each $i \in I$. Let

$\Gamma := \mathcal{G}al(\mathcal{E}/K) = \{\varphi : E_i \to E_i \mid \varphi \text{ is a K -automorphism } \}$ and

$\Gamma_0 := \mathcal{G}al(\mathcal{E}/K)_0 = \{Id_{E_I} \mid i \in I\}$.

Let $\Gamma_{(2)} := \{(\varphi, \psi) \in \Gamma \times \Gamma | D(\varphi) = D(\psi)\}$. The structure functions $\overline{\alpha}, \overline{\beta} : \Gamma \to \Gamma_0$, $\overline{\iota} : \Gamma \to \Gamma$ and $\overline{m} : \Gamma_{(2)} \to \Gamma$ are given by:

$$\overline{\alpha}(\varphi) := Id_{D(\varphi)}, \quad \overline{\beta}(\varphi) := Id_{D(\varphi)}, \quad \overline{\iota}(\varphi) := \varphi^{-1} \quad \text{and} \quad \overline{m}(\varphi, \psi) := \varphi \circ \psi.$$

Then $\mathcal{G}al(\mathcal{E}/K)$ is a groupoid over $\mathcal{G}al(\mathcal{E}/K)_0$, called the *Galois groupoid associated to* $\mathcal{E}$. The isotropy group at $u = Id_{E_i}$ is the Galois group $Gal(E_i/K)$.

Definition 2.2. *([6]) By* ***morpfism of groupoids*** *or* **groupoid morphism** *between the groupoids* $(G, \alpha, \beta, m, \iota, G_0)$ *and* $(G', \alpha', \beta', m', \iota', G'_0)$, *we mean a map* $f : G \to G'$ *which verifies the following conditions:*

(i) $\forall\ (x, y) \in G_{(2)} \implies (f(x), f(y)) \in G'_{(2)}$;

(ii) $f(m(x, y)) = m'(f(x), f(y)),\ \forall\ (x, y) \in G_{(2)}$.

Proposition 2.3. *If* $f : G \longrightarrow G'$ *is a morpfism of groupoids, then:*

$$(a)\ \ f(u) \in G'_0, \quad \forall\ u \in G_0; \qquad (b)\ \ f\left(x^{-1}\right) = (f(x))^{-1},\ \forall\ x \in G.$$

From Proposition 2.3(a) follows that a groupoid morphism $f : G \to G'$ induces a map $f_0 : G_0 \to G'_0$ taking $f_0(u) := f(u)$, $(\forall)u \in G_0$, i.e. the map f_0 is the restriction of f to G_0. We say that $(f, f_0) : (G, G_0) \to (G', G'_0)$ is a morphism of groupoids.

If $G_0 = G'_0$ and $f_0 = Id_{G_0}$, we say that $f : G \to G'$ is a G_0 - *morphism* of groupoids over G_0.

A groupoid morphism (f, f_0) is said to be *isomorphism of groupoids* or *groupoid isomorphism*, if f and f_0 are bijective maps.

Proposition 2.4. *([12]) Let* $(G, \alpha, \beta, m, \iota, G_0)$ *and* $(G', \alpha', \beta', m', \iota', G'_0)$ *be two groupoids. The pair* $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ *where* $f : G \longrightarrow G'$ *and* $f_0 : G_0 \longrightarrow G'_0$, *is a groupoid morphism if and only if the following conditions are verified:*

(i) $\alpha' \circ f = f_0 \circ \alpha$ *and* $\beta' \circ f = f_0 \circ \beta$;

(ii) $f(m(x, y)) = m'(f(x), f(y)), \quad \forall\ (x, y) \in G_{(2)}$.

Remark 2.1. Applying Propositions 2.3 and 3.4 we can conclude that a groupoid morphism $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ is linked with the structure functions by the relations :

$$\alpha' \circ f = f_0 \circ \alpha, \quad \beta' \circ f = f_0 \circ \beta, \quad m' \circ (f \times f) = f \circ m, \quad \iota' \circ f = f \circ \iota \quad (2.1)$$

where $(f \times f)(x, y) := (f(x), f(y)), \ \forall\, x, y \in G \times G$.

Definition 2.3. *([8]) A groupoid morphism* $(f, f_0) : (G, G_0) \longrightarrow (G', G'_0)$ *satisfying the following condition:*

$$\forall\, x, y \in G \ \text{ such that } \ (f(x), f(y)) \in G'_{(2)} \quad \Rightarrow \quad (x, y) \in G_{(2)} \qquad (2.2)$$

will be called **strong morphism** *or* **homomorphism of groupoids**.

Example 2.3. Let the symmetry groupoid $\mathcal{SG}_n$ of the finite set $X = \{x_1, x_2, \ldots, x_n\}$ and the multiplicative group $\{+1, -1\}$ (regarded as groupoid over $\{+1\}$). We define the map

$$sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\}, \ f \in \mathcal{SG}_n \longmapsto sgn^\sharp(f) := sgn(f),$$

where $sgn(f)$ is the signature of the permutation f of degree $k = |D(f)|$.

We have that $sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\}$ *is a groupoid morphism.*

Indeed, let $f, g \in G_{(2)}$, where $G = \mathcal{SG}(A, X)$ such that $D(f) = D(g) := A_k := \{x_{j_1}, \ldots, x_{j_k}\} \subseteq X, \ 1 \leq k \leq n$. Then f and g are permutations of A_k and $f \circ g$ is also a permutation of A_k. It is clearly that the condition (i) from Definition 2.2 is verified. Also, it is well known that $sgn(f \circ g) = sgn(f) \cdot sgn(g)$. Hence $sgn^\sharp(m(f, g)) = sgn^\sharp(f) \cdot sgn^\sharp(g)$. Therefore the condition (ii) from Definition 2.2 holds.

The map $sgn^\sharp : \mathcal{SG}_n \to \{+1, -1\}$ *is not a groupoid homomorphism.*

Indeed, for $X = \{x_1, x_2, x_3, x_4\}$ we consider the permutations $f, g \in \mathcal{SG}_4$, where $f = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix}$ and $g = \begin{pmatrix} x_1 & x_3 & x_4 \\ x_4 & x_3 & x_1 \end{pmatrix}$. Then $sgn^\sharp(f) = +1, \ sgn^\sharp(g) = -1$ and $(sgn^\sharp(f), sgn^\sharp(g)) \in \{+1, -1\} \times \{+1, -1\}$. But f and g are not composable in $\mathcal{SG}_4$, since $D(f) \neq D(g)$.

3 VECTOR GROUPOIDS

Definition 3.1. *A* **vector groupoid over a field** K*, is a groupoid* $(V, \alpha, \beta, \odot, \iota, V_0)$ *such that:*

(3.1.1) V *is a vector space over* K*, and the units set* V_0 *is a subspace of* V.
(3.1.2) The source and the target maps α *and* β *are linear maps.*
(3.1.3) The inversion $\iota : V \longrightarrow V, \ x \longmapsto \iota(x) := x^{-1}$ *is a linear map and the following condition is verified:*

(1) $\qquad x + x^{-1} = \alpha(x) + \beta(x),$ *for all* $x \in V$.

(3.1.4) The map $m : V_{(2)} := \{(x, y) \in V \times V \mid \alpha(y) = \beta(x)\} \to V$, $(x, y) \longmapsto m(x, y) := x \odot y$, *satisfy the following conditions :*

1. $x \odot (y + z - \beta(x)) = x \odot y + x \odot z - x$, *for all* $x, y, z \in V$, *such that* $\alpha(y) = \beta(x) = \alpha(z)$.

2. $x \odot (ky + (1 - k)\beta(x)) = k(x \odot y) + (1 - k)x$, *for all* $x, y \in V$, *such that* $\alpha(y) = \beta(x)$.

3. $(y + z - \alpha(x)) \odot x = y \odot x + z \odot x - x$, *for all* $x, y, z \in V$, *such that* $\alpha(x) = \beta(y) = \beta(z)$.

4. $(ky + (1 - k)\alpha(x)) \odot x = k(y \odot x) + (1 - k)x$ *for all* $x, y \in V$, *such that* $\alpha(x) = \beta(y)$.

When there can be no confusion we put xy or $x \cdot y$ instead of $x \odot y$.
From Definition 3.1 follows the following corollary.

Corollary 3.1. *Let* $(V, \alpha, \beta, \odot, \iota, V_0)$ *be a vector groupoid. Then:*
(i) *The source and target* $\alpha, \beta : V \to V_0$ *are linear epimorphisms.*
(ii) *The inversion* $\iota : V \to V$ *is a linear automorphism.*
(iii) *The fibres* $\alpha^{-1}(0)$ *and* $\beta^{-1}(0)$ *and the isotropy group* $V(0) := \alpha^{-1}(0) \cap \beta^{-1}(0)$ *are vector subspaces of the vector space* V.

Example 3.1. Let V be a vector space over a field K. If we define the maps $\alpha_0, \beta_0, \iota_0 : V \longrightarrow V$, $\alpha_0(x) = \beta_0(x) = 0$, $\iota_0(x) = -x$, and the multiplication law $m_0(x, y) = x + y$, then $(V, \alpha_0, \beta_0, m_0, \iota_0, V_0 = \{0\})$ is a vector groupoid called *vector groupoid with a single unit*. We will denote this vector groupoid by $(V, +)$. Therefore, each vector space V over K can be regarded as vector groupoid over $V_0 = \{0\}$. □

Example 3.2. Let V be a vector space over a field K. Then V has a structure of null groupoid over V (see Example 2.1(ii)). In this case the structure functions are $\alpha = \beta = \iota = Id_V$ and $x \odot x = x$ for all $x \in V$. We have that $V_0 = V$ and the maps α, β, ι are linear. Since $x + \iota(x) = x + x$ and $\alpha(x) + \beta(x) = x + x$ imply that the condition 3.1.3(1) holds. It is easy to verify the conditions 3.1.4(1)- 3.1.4(4) from Definition 3.1. Then V is a vector groupoid, called the *null vector groupoid* associated to V. □

Example 3.3. Let V be a vector space over a field K. We consider the pair groupoid $(V \times V, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{\iota}, \Delta_V)$ associated to V (see Example 2.1(iii)). We have that $V \times V$ is a vector space over K and the source $\widetilde{\alpha}$ and target $\widetilde{\beta}$ are linear maps. Also, the inversion $\widetilde{\iota} : V \times V \to V \times V$ is a linear isomorphism. Therefore it follows that the conditions (3.1.1) $-$ (3.1.3) are satisfied. By a direct computation we verify that the relations 3.1.4(1) - 3.1.4(4) from Definition 3.1 hold. Hence $V \times V$ is a vector groupoid called the *coarse vector groupoid* or *pair vector groupoid* associated to V. □

Example 3.4. The vector groupoid $V^2(p, q)$. Let V be a vector space over a field K and let $p, q \in K$ such that $pq = 1$. The maps $\alpha, \beta, \iota : V^2 \longrightarrow V^2$, $\alpha(x, y) := (x, px)$, $\beta(x, y) := (qy, y)$, $\iota(x, y) := (qy, px)$ together with the multiplication law given on $V^2_{(2)} := \{((x,y),(qy,z)) \mid x,y,z \in V\} \subset V^2 \times V^2$, by $(x,y)\cdot(qy,z) := (x,z)$ determine on V^2 a structure of vector groupoid. This is called the *pair* or the *coarse vector groupoid of type* (p,q) and it is denoted by $V^2(p,q)$.

If $p = q = 1$, then the vector groupoid $V^2(1,1)$ coincide with the pair vector groupoid associated to V(see Example 3.3).

If n is a prime number and $p,q \in \mathbb{Z}_n$, such that $pq = 1$, then $\mathbb{Z}^2_n(p,q)$ is called the *modular* or *cryptographic vector groupoid.* □

Example 3.5. Let V be vector space over a field K. One consider the maps $\alpha,\beta,\iota : V^3 \longrightarrow V^3$, $\alpha(x_1,x_2,x_3) := (x_1,x_1,0)$, $\beta(x_1,x_2,x_3) := (x_2,x_2,0)$, $\iota(x_1,x_2,x_3) := (x_2,x_1,-x_3)$ together with the multiplication law given on $V^3_{(2)} = \{((x_1,x_2,x_3),(x_2,y_2,y_3)) \mid x_1,x_2,x_3,y_2,y_3 \in V\} \subset V^3 \times V^3$ by $(x_1,x_2,x_3) \odot (x_2,y_2,y_3) := (x_1,y_2,x_3+y_3)$.

Then $(V^3,\alpha,\beta,\iota,\odot,V^3_0)$, where $V^3_0 = \{(x,x,0) \mid x \in V\}$, is a vector groupoid. □

In the following proposition, we give, in addition to those in Proposition 2.1, other rules of algebraic calculation in a vector groupoid.

Proposition 3.1. *In a vector groupoid* $(V,\alpha,\beta,\odot,\iota,V_0)$ *the following assertions hold :*

(i) $0\cdot x = x,\ \forall\ x \in \alpha^{-1}(0)$;

(ii) $x\cdot 0 = x,\ \forall\ x \in \beta^{-1}(0)$;

(iii) For all $x,\ y \in \beta^{-1}(0)$, *we have* $x - \alpha(x) = y - \alpha(y) \Longrightarrow x = y$;

(iv) for all $x,y \in \alpha^{-1}(0)$, *we have* $x - \beta(x) = y - \beta(y) \Longrightarrow x = y$.

Proof. (i) If $x \in \alpha^{-1}(0)$, then $\alpha(x) = 0 = \beta(0)$. So $(0,x) \in V_{(2)}$ and, using the condition *(G2)* from Definition 2.1, one obtains that $0\cdot x = \alpha(x)\cdot x = x$.
(iv) Let $x,y \in \alpha^{-1}(0)$ such that $x-\beta(x) = y-\beta(y)$. Then $\alpha(x) = \alpha(y) = 0$ and $x - y = \beta(x) - \beta(y)$. Since α is linear map and applying Proposition 2.1 (i), one obtains that $0 = \alpha(x)-\alpha(y) = \alpha(x-y) = \alpha(\beta(x)-\beta(y)) = \beta(x)-\beta(y) = x-y$, and so $x = y$.

Similarly, we prove that the assertions (ii) and (iii) hold. □

Proposition 3.2. *Let* $(V,\alpha,\beta,\odot,\iota,V_0)$ *be a vector groupoid. Then:*

(i) $t_\beta : \alpha^{-1}(0) \longrightarrow \beta^{-1}(0)$, $t_\beta(x) := \beta(x) - x$ *is a linear isomorphism.*

(ii) $t_\alpha : \beta^{-1}(0) \longrightarrow \alpha^{-1}(0)$, $t_\alpha(x) := \alpha(x) - x$ *is a linear isomorphism.*

Proof. *(i)* Let be $x_1,x_2 \in V$ and $k_1,k_2 \in K$. Then $t_\beta(k_1x_1+k_2x_2) =$
$= \beta(k_1x_1+k_2x_2) - (k_1x_1+k_2x_2) = k_1(\beta(x_1)-x_1) + k_2(\beta(x_2)-x_2) =$
$= k_1t_\beta(x_1) + k_2t_\beta(x_2)$. Hence t_β is a linear map.

Let now $x,y \in \alpha^{-1}(0)$ such that $t_\beta(x) = t_\beta(y)$. Applying Proposition 3.1(iv), one obtains $x = y$, and so the map t_β is injective.

For any $y \in \beta^{-1}(0)$ we take $x = \alpha(y) - y$. Clearly $x \in \alpha^{-1}(0)$. We have $t_\beta(x) = \beta(\alpha(y)-y)-(\alpha(y)-y) = \alpha(y)-\beta(y)-\alpha(y)+y = y$, since $\beta(y) = 0$. Hence the map t_β is surjective. Therefore t_β is a linear isomorphism.

(ii) Similarly we prove that t_α is a linear isomorphism. $\square$

Proposition 3.3. *Let* $(V, +, \cdot, \alpha, \beta, \odot, \iota, V_0)$ *be a vector groupoid over* K *and* $u \in V_0$ *any unit of* V*. The following assertions hold.*

(i) *The isotropy group* $V(u) := \{x \in V \mid \alpha(x) = \beta(x) = u\}$ *endowed with the laws* $\boxplus : V \times V \to V$ *and* $\boxtimes : K \times V \to V$ *given by:*

$$x \boxplus y = x + y - u, \quad \forall\ x, y \in V(u) \tag{3.1}$$

$$k \boxtimes x = kx + (1-k)u, \quad \forall\ k \in K,\ x \in V(u), \tag{3.2}$$

has a structure of vector space over K*.*

(ii) *The vector space* $(V(u), \boxplus, \boxtimes)$ *together with the restrictions of structure functions* α, β, ι *to* $V(u)$ *and the multiplication* $\boxdot : V(u)_{(2)} = V(u) \times V(u) \to V(u)$ *given by:*

$$x \boxdot y = (x-u) \odot (y-u) + u, \quad \forall\ x, y \in V(u) \tag{3.3}$$

has a structure of vector groupoid with a single unit over K*.*

Proof. (i) Using the linearity of the functions α and β we verify that the laws $\boxplus$ and $\boxtimes$ given by (3.1) and (3.2) are well-defined. For instance, for $x, y \in V(u)$ we have $\alpha(x \boxplus y) = \alpha(x+y-u) = \alpha(x)+\alpha(y)-\alpha(u) = u$, since $\alpha(x) = \alpha(y) = \alpha(u) = u$. Similarly, $\beta(x \boxplus y) = u$. Hence $x \boxplus y \in V(u)$. It is easy to verify that $(V(u), \boxplus)$ is a commutative group. Its null vector is the element $u \in V(u)$. The opposite $\boxminus x$ of $x \in V(u)$ is $\boxminus x = 2u - x$.

For any $x, y \in V(u)$ and $k, k_1, k_2 \in K$, the law $\boxtimes$ verify the following relations:

(a) $k \boxtimes (x \boxplus y) = (k \boxtimes x) \boxplus (k \boxplus y)$,

(b) $(k_1 + k_2) \boxtimes x = (k_1 \boxtimes x) \boxplus (k_2 \boxtimes x)$,

(c) $k_1 \boxtimes (k_2 \boxtimes x) = (k_1 k_2) \boxtimes x$,

(d) $1 \boxtimes x = x$ (here 1 is the unit of the field K).

Indeed, we have $k \boxtimes (x \boxplus y) = k(x \boxplus y) + (1-k)u = k(x+y) + (1-2k)u$ and $(k \boxtimes x) \boxplus (k \boxplus y) = (k \boxtimes x) + (k \boxplus y) - u = k(x+y) + (1-2k)u$. Hence the equality (a) holds.

In the same manner we prove that the equalities (b) - (d) hold. Therefore $(V, \boxplus, \boxtimes)$ is a vector space.

(ii) From the above assertion follows that the condition (3.1.1) from Definition 3.1 is satisfied.

The restrictions of the linear maps α and β to $V(u)$ are linear maps, and so the condition (3.1.2) from Definition 3.1 holds.

Also, the restriction of the linear maps ι to $V(u)$ is linear map. Applying the equality 3.1.3(1) from Definition 3.1, for any $x \in V(u)$ we have $x \boxplus \iota(x) = x + \iota(x) - u = \alpha(x) + \beta(x) - u = \alpha(x) \boxplus \beta(x)$. Therefore the condition (3.1.3) from Definition 3.1 holds.

Let $x, y \in V(u)$. Applying the properties of maps α and β we have $\alpha(x \boxdot y) = \alpha((x-u) \odot (y-u) + u) = \alpha((x-u) \odot (y-u)) + \alpha(u) =$
$= \alpha(x-u) + \alpha(u) = \alpha(x) = u$ and $\beta(x \boxdot y) = u$ and so $x \boxdot y \in V(u)$. Hence the law $\boxdot$ given by the relation (3.3) is well-defined.

If $x, y, z \in V(u)$ then the following equality holds:

$$(e) \quad x \boxdot (y \boxplus z \boxplus (\boxminus\beta(x))) = (x \boxdot y) \boxplus (x \boxdot z) \boxplus (\boxminus x)).$$

Indeed, we have

$(e.1)$ $x \boxdot (y \boxplus z \boxplus (\boxminus\beta(x))) = x \boxdot (y \boxplus z \boxplus (\boxminus u)) = x \boxdot (y \boxplus z \boxplus u) =$
$= x \boxdot (y \boxplus z) = (x-u) \odot (y \boxplus z - u) + u = (x-u) \odot ((y-u) + (z-u)) + u.$

Replacing in the equality 3.4.1(1) the elements $x, y, z \in V(u)$ respectively with $x-u, y-u, z-u \in V(u)$, we obtain the following equality

$$(f) \quad (x-u) \odot ((y-u)+(z-u)) = (x-u) \odot (y-u) + (x-u) \odot (z-u) - (x-u),$$

since $\beta(x-u) = 0$.

Using the relation (f), the equality $(e.1)$ becomes

$$(e.2) \quad x \boxdot (y \boxplus z \boxplus (\boxminus\beta(x))) = (x-u) \odot (y-u) + (x-u) \odot (z-u) + 2u - x.$$

On the other hand we have

$(e.3)$ $(x \boxdot y) \boxplus (x \boxdot z) \boxplus (\boxminus x)) = ((x \boxdot y) \boxplus (x \boxdot z)) \boxplus (2u - x) =$
$= (x \boxdot y + x \boxdot z - u) \boxplus (2u - x) = x \boxdot y + x \boxdot z - x =$
$= (x-u) \odot (y-u) + (x-u) \odot (z-u) + 2u - x.$

Using (e.2) and (e.3) we obtain the equality (e). Hence, the relation 3.4.1(1) from Definition 3.1 holds.

In the same manner we can prove that the relations 3.1.4(2) - 3.1.4(4) from Definition 3.1 are verified. $\square$

We call $(V(u), \boxplus, \boxtimes, \alpha, \beta, \boxdot, \iota, V_0(u) = \{u\})$ the *isotropy vector groupoid* at $u \in V_0$ of V, when one refers to the above structure given on it.

Definition 3.2. *Let $(V_1, \alpha_1, \beta_1, V_{1,0})$ and $(V_2, \alpha_2, \beta_2, V_{2,0})$ be two vector groupoids.*

A groupoid morphism (resp. groupoid homomorphism) $f : V_1 \longrightarrow V_2$ with property that f is a linear map, is called **vector groupoid morphism** *(resp.* **vector groupoid homomorphism** *).*

Example 3.6. Let $(V, \alpha, \beta, \odot, \iota, V_0)$ be a vector groupoid. We consider the pair vector groupoid $(V_0 \times V_0, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{m}, \widetilde{\iota}, \Delta_{V_0})$. Then *the anchor map* $(\alpha, \beta) : V \to V_0 \times V_0$ *is a homomorphism of vector groupoids between the vector groupoids* V *and* $V_0 \times V_0$.

Indeed, if we denote $(\alpha, \beta) := f$ and consider the elements $x, y \in G$ such that $(f(x), f(y)) \in (V_0 \times V_0)_{(2)}$, then $\widetilde{\beta}(f(x)) = \widetilde{\alpha}(f(y))$ and we have $\widetilde{\beta}(\alpha(x), \beta(x)) = \widetilde{\alpha}(\alpha(y), \beta(y)) \Rightarrow (\beta(x), \beta(x)) = (\alpha(y), \alpha(y)) \Rightarrow \beta(x)) = \alpha(y)$, i.e. $(x, y) \in V_{(2)}$. Therefore the condition (i) from Definition 2.2 holds.

For $(x, y) \in V_{(2)}$ we have

$f(m(x, y)) = f(xy) = (\alpha(xy), \beta(xy)) = (\alpha(x), \beta(y))$ and

$\widetilde{m}(f(x), f(y)) = \widetilde{m}((\alpha(x), \beta(x)), (\alpha(y), \beta(y))) = (\alpha(x), \beta(y))$.

Hence the equality (ii) from Definition 2.2 is verified.

Let now two elements $x, y \in V$ such that $(f(x), f(y)) \in (V_0 \times V_0)_{(2)}$. Then $\widetilde{\beta}(f(x)) = \widetilde{\alpha}(f(y))$. Since $f(x) = (\alpha(x), \beta(x))$ and $f(y) = (\alpha(y), \beta(y))$ we deduce that $(\beta(x), \beta(x)) = (\alpha(y), \alpha(y))$. Therefore $\beta(x) = \alpha(y)$ and $(x, y) \in G_{(V)}$. Therefore the condition (2.2) from Definition 2.3 is satisfied.

Hence $f : V \to V_0 \times V_0$ is a groupoid homomorphism.

Let $x, y \in V$ and $a, b \in K$. Since α, β are linear maps, we have

$f(ax + by) = (\alpha(ax + by), \beta(ax + by)) = (a\alpha(x) + b\alpha(y), a\beta(x) + b\beta(y)) =$
$= a(\alpha(x), \beta(x)) + b(\alpha(y), \beta(y)) = af(x) + bf(y)$, i.e. f is a linear map.

Therefore, the conditions from Definition 3.2 are verified. Hence f is a vector groupoid homomorphism. $\square$

4 ALGEBRAIC CONSTRUCTIONS OF VECTOR GROUPOIDS

In this section we shall give some important ways of building up new vector groupoids.

1. Direct product of two vector groupoids. Let given the vector groupoids $(V, \alpha_V, \beta_V, \odot_V, \iota_V, V_0)$ and $(W, \alpha_W, \beta_W, \odot_W, \iota_W, W_0)$. We have that $V_0 \times W_0$ is a vector subspace of the direct product $V \times W$ of vector spaces V and W.

We can easy prove that $V \times W$ endowed with the structure functions $\alpha_{V\times W}, \beta_{V\times W}, \odot_{V\times W}$ and $\iota_{V\times W}$ given by

$\alpha_{V\times W}(v, w) := (\alpha_V(v), \alpha_W(w))$, $\beta_{V\times W}(v, w) := (\beta_V(v), \beta_W(w))$,
$(v_1, w_1)\odot_{V\times W}(v_2, w_2) := (v_1\odot_V v_2, w_1\odot_W w_2)$, $\iota_{V\times W}(v, w) := (\iota_V(v), \iota_W(w))$
for all $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$, is a vector groupoid over $V_0 \times W_0$.

This vector groupoid is called the *direct product of vector groupoids* (V, V_0) and (W, W_0).

By a direct computation we can verify that the projections
$pr_V : V \times W \to V$ and $pr_W : V \times W \to W$ are morphisms of vector groupoids, called the *canonical projections* of the vector groupoid $V \times W$ onto vector groupoid V and W, respectively. The following assertion holds

The direct product of two transitive vector groupoids is also a transitive vector groupoid.

2. Trivial vector groupoid $\mathcal{TVG}(V,W)$. Let W be a vector subspace of a vector space V over K. The set $\mathcal{V} := W \times V \times W$ has a natural structure of vector space. The set $\mathcal{V}_0 := \{(w,0,w) \in \mathcal{V} \mid w \in W\}$ is a vector subspace of $\mathcal{V}$ (here 0 is the null vector of V). We introduce on $\mathcal{V} := W \times V \times W$ the structure functions $\alpha_{\mathcal{V}}, \beta_{\mathcal{V}}, \odot_{\mathcal{V}}$ and $\iota_{\mathcal{V}}$ as follows.

For all $(w_1, v, w_2) \in \mathcal{V}$, the source and target $\alpha_{\mathcal{V}}, \beta_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}_0$ are defined by

$\alpha_{\mathcal{V}}(w_1, v, w_2) := (w_1, 0, w_1); \quad \beta_{\mathcal{V}}(w_1, v, w_2) := (w_2, 0, w_2).$

The partially multiplication $\odot_{\mathcal{V}} : \mathcal{V}_{(2)} \to \mathcal{V}$, where

$\mathcal{V}_{(2)} = \{((w_1, v_1, w_2), (w'_2, v_2, w_3)) \in \mathcal{V} \times \mathcal{V} \mid w_2 = w'_2\}$ and the inversion map $\iota_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ are given by

$(w_1, v_1, w_2) \odot_{\mathcal{V}} (w_2, v_2, w_3) := (w_1, v_1 + v_2, w_3); \; \iota_{\mathcal{V}}(w_1, v, w_2) := (w_2, -v, w_1).$

It is easy to verify that the conditions of Definition 2.1 are satisfied. Then $(\mathcal{V}, \alpha_{\mathcal{V}}, \beta_{\mathcal{V}}, \odot_{\mathcal{V}}, \iota_{\mathcal{V}}, \mathcal{V}_0)$ is a groupoid. Also, the condition (3.1.1) from Definition 3.1 is verified.

Let now two elements $x, y \in \mathcal{V}$ and $a, b \in K$ where $x = (w_1, v_1, w_2)$ and $y = (w_3, v_2, w_4)$. We have

$\alpha_{\mathcal{V}}(ax + by) = \alpha_{\mathcal{V}}(aw_1 + bw_3, av_1 + bv_2, aw_2 + bw_4) =$
$= (aw_1 + bw_3, 0, aw_1 + bw_3) = a(w_1, 0, w_1) + b(w_3, 0, w_3) = a\alpha_{\mathcal{V}}(w_1, v_1, w_2) +$
$+ b\alpha_{\mathcal{V}}(w_3, v_2, w_4) = a\alpha_{\mathcal{V}}(x) + b\alpha_{\mathcal{V}}(y).$

It follows that $\alpha_{\mathcal{V}}$ is a linear map. Similarly we prove that $\beta_{\mathcal{V}}$ is a linear map. Therefore the conditions (3.1.2) from Definition 3.1 hold.

For $x = (w_1, v_1, w_2) \in \mathcal{V}$ and $y = (w_3, v_2, w_4) \in \mathcal{V}$ and $a, b \in K$, we have

$\iota_{\mathcal{V}}(ax + by) = \iota_{\mathcal{V}}(aw_1 + bw_3, av_1 + bv_2, aw_2 + bw_4) =$
$= (aw_2 + bw_4, -av_1 - bv_2, aw_1 + bw_3) = a(w_2, -v_1, w_1) + b(w_4, -v_2, w_3) =$
$= a\iota_{\mathcal{V}}(w_1, v_1, w_2) + b\iota_{\mathcal{V}}(w_3, v_2, w_4) = a\iota_{\mathcal{V}}(x) + b\iota_{\mathcal{V}}(y).$

It follows that $\iota_{\mathcal{V}}$ is a linear map. Also

$x + \iota_V(x) = (w_1, v_1, w_2) + (w_2, -v_1, w_1) = (w_1 + w_2, 0, w_1 + w_2) =$
$= (w_1, 0, w_1) + (w_2, 0, w_2) = \alpha_V(x) + \beta_V(x).$

Hence the condition (3.1.3) from Definition 3.1 holds.

For to verify the relation 3.1.4(1) from Definition 3.1 we consider the arbitrary elements $x, y, z \in \mathcal{V}$ where $x = (w_1, v_1, w_2), y = (w_3, v_2, w_4)$ and $z = (w_5, v_3, w_6)$ such that $\alpha_{\mathcal{V}}(y) = \beta_{\mathcal{V}}(x) = \alpha_{\mathcal{V}}(z)$. Then $w_2 = w_3 = w_5$ and follows $x = (w_1, v_1, w_2), y = (w_2, v_2, w_4)$ and $z = (w_2, v_3, w_6)$.

For all $k \in K$ we have

$(i) \quad x \odot_{\mathcal{V}} (y + z - \beta_{\mathcal{V}}(x)) = (w_1, v_1, w_2) \odot_{\mathcal{V}} ((w_2, v_2, w_4) +$
$+ (w_2, v_3, w_6) - (w_2, 0, w_2)) = (w_1, v_1, w_2) \odot_{\mathcal{V}} (w_2, v_2 + v_3, w_4 + w_6 - w_2) =$
$= (w_1, v_1 + v_2 + v_3, w_4 + w_6 - w_2)$ and

$(ii) \quad x \odot_{\mathcal{V}} y + x \odot_{\mathcal{V}} z - x = (w_1, v_1, w_2) \odot_{\mathcal{V}} (w_2, v_2, w_4) +$
$+ (w_1, v_1, w_2) \odot_{\mathcal{V}} (w_2, v_3, w_6) - (w_1, v_1, w_2) = (w_1, v_1 + v_2, w_4) +$

$+(w_1, v_1+v_3, w_6) - (w_1, v_1, w_2) = (w_1, v_1+v_2+v_3, w_4+w_6-w_2)$.

Using (i) and (ii) we obtain $x \odot_{\mathcal{V}} (y+z-\beta_{\mathcal{V}}(x)) = x \odot_{\mathcal{V}} y + x \odot_{\mathcal{V}} z - x$. Hence the condition 3.1.4 (1) from Definition 3.1 holds.

Let now $x=(w_1,v_1,w_2), y=(w_2,v_2,w_4)$ and $k \in K$. We have

(iii) $x \odot_{\mathcal{V}} (ky+(1-k)\beta_{\mathcal{V}}(x)) = (w_1,v_1,w_2) \odot_{\mathcal{V}} (k(w_2,v_2,w_4)+$
$+(1-k)(w_2,0,w_2)) = (w_1,v_1,w_2) \odot_{\mathcal{V}} (w_2, kv_2, kw_4+(1-k)w_2) =$
$= (w_1, v_1+kv_2, kw_4+(1-k)w_2)$ and

(iv) $k(x \odot_{\mathcal{V}} y) + (1-k)x = k((w_1,v_1,w_2) \odot_{\mathcal{V}} ((w_2,v_2,w_4))+$
$+(1-k)(w_1,v_1,w_2) = k(w_1,v_1+v_2,w_4)+(1-k)(w_1,v_1,w_2) =$
$= (w_1, v_1+kv_2, kw_4+(1-k)w_2)$

Using the equalities (iii) and (iv) we obtain that the condition 3.1.4 (2) from Definition 3.1 holds.

In the same manner we prove that the conditions 3.1.4 (3) and 3.1.4 (4) hold. Hence $\mathcal{V} := W \times V \times W$ is a vector groupoid over $\mathcal{V}_0$. Its set of units can be identified with the vector subspace W of V.

The vector groupoid $(\mathcal{V} := W \times V \times W, \alpha_{\mathcal{V}}, \beta_{\mathcal{V}}, \odot_{\mathcal{V}}, \iota_{\mathcal{V}}, \mathcal{V}_0)$ is called the *trivial vector groupoid* associated to pair of vector spaces (V, W) with $W \subseteq V$. This vector groupoid is denoted by $\mathcal{TVG}(V,W)$. The isotropy group at $u=(w,0,w) \in \mathcal{V}_0$ is $V(u) = \{(w,v,w) \mid v \in V\}$ which identify with the group $(V,+)$.

3. Whitney sum of two vector groupoids over the same base. Let $(V, \alpha_V, \beta_V, \odot_V, \iota_V, V_0)$ and $(V', \alpha_{V'}, \beta_{V'}, \odot_{V'}, \iota_{V'}, V_0)$ be two vector groupoids over the same base (i.e. V and V' have the same units). The set $V \oplus V' := \{\ (v,v') \in V \times V' \mid \alpha_V(v) = \alpha_{V'}(v'), \beta_V(v) = \beta_{V'}(v')\ \}$ has a natural structure of vector space. It is clearly that $\Delta_{V_0} = \{(u,u) \in V_0 \times V_0 \mid u \in V_0\} \subseteq V \oplus V'$ is a vector subspace.

We introduce on $\mathcal{W} := V \oplus V'$ the structure functions $\alpha_{\mathcal{W}}, \beta_{\mathcal{W}}, \odot_{\mathcal{W}}$ and $\iota_{\mathcal{W}}$ as follows.

The source and target $\alpha_{\mathcal{W}}, \beta_{\mathcal{W}} : \mathcal{W} \to \Delta_{V_0}$ are defined by

$$\alpha_{\mathcal{W}}(v,v') := (\alpha_V(v), \alpha_V(v)); \quad \beta_{\mathcal{W}}(v,v') := (\beta_V(v), \beta_V(v)), \quad (v,v') \in \mathcal{W}.$$

The partially multiplication $\odot_{\mathcal{W}} : \mathcal{W}_{(2)} \to \mathcal{W}$, where $\mathcal{W}_{(2)} = \{((v_1,v_1'),((v_2,v_2')) \in \mathcal{W} \times \mathcal{W} \mid \beta_V(v_2) = \alpha_V(v_1)\}$ and the inversion map $\iota_{\mathcal{W}} : \mathcal{V} \to \mathcal{W}$ are given by

$$(v_1,v_1') \odot_{\mathcal{W}} (v_2,v_2') := (v_1 \odot_V v_2, v_1' \odot_{V'} v_2'); \quad \iota_{\mathcal{W}}(v,v') := (\iota_V(v), \iota_{V'}(v')).$$

By a direct computation we prove that the conditions of Definition 2.1 are satisfied. Then $(\mathcal{W} := V \oplus V', \alpha_{\mathcal{W}}, \beta_{\mathcal{W}}, \odot_{\mathcal{W}}, \iota_{\mathcal{W}}, \Delta_{V_0})$ is a groupoid. Also, the condition (3.1.1) from Definition 3.1 is verified.

Let now two elements $x, y \in \mathcal{W}$ and $a, b \in K$ where $x = (v_1, v_1')$ and $y = (v_2, v_2')$. We have

$\alpha_{\mathcal{W}}(ax+by) = \alpha_{\mathcal{W}}(av_1+bv_2, av'_1+bv'_2) = (\alpha_V(av_1+bv_2), \alpha_V(av_1+bv_2)) = (a\alpha_V(v_1) + b\alpha_V(v_2), a\alpha_V(v_1) + b\alpha_V(v_2))$ and

$a\alpha_{\mathcal{W}}(x) + b\alpha_{\mathcal{W}}(y) = a\alpha_{\mathcal{W}}(v_1, v'_1) + b\alpha_{\mathcal{W}}(v_2, v'_2) = a(\alpha_V(v_1), \alpha_V(v_1)) + b(\alpha_V(v_2), \alpha_V(v_2)) = (a\alpha_V(v_1) + b\alpha_V(v_2), a\alpha_V(v_1) + b\alpha_V(v_2))$ since α_V is a linear map. It follows that $\alpha_{\mathcal{W}}$ is a linear map.

Similarly we obtain that $\beta_{\mathcal{W}}$ is a linear map. Therefore the conditions (3.1.2) from Definition 3.1 hold.

For $x = (v_1, v'_1) \in \mathcal{W}$ and $y = (v_2, v'_2) \in \mathcal{W}$ and $a, b \in K$, we have successively

$\iota_{\mathcal{W}}(ax+by) = \iota_{\mathcal{W}}(av_1+bv_2, av'_1+bv'_2) = (\iota_V(av_1+bv_2), \iota_{V'}(av'_1+bv'_2)) = (a\iota_V(v_1)+b\iota_V(v_2), a\iota_{V'}(v'_1)+b\iota_{V'}(v'_2)) = a(\iota_V(v_1), \iota_{V'}(v'_1))+b(\iota_V(v_2), \iota_{V'}(v'_2)) = a\iota_W(v_1, v'_1)+b\iota_W(v_2, v'_2) = a\iota_W(x)+b\iota_W(y)$, since ι_V and $\iota_{V'}$ are linear map.

Using the equalities 3.1.3(1) for the inversion maps ι_V and $\iota_{V'}$ we have
$x + \iota_W(x) = (v, v') + (\iota_V(v), \iota_{V'}(v')) = (v + \iota_V(v), v' + \iota_{V'}(v')) =$
$= (\alpha_V(v) + \beta_V(v), \alpha_{V'}(v') + \beta_{V'}(v')) = (\alpha_V(v) + \beta_V(v), \alpha_V(v) + \beta_V(v)) =$
$= \alpha_W(v, v') + \beta_W(v, v') = \alpha_W(x) + \beta_W(x)$ for any $x = (v, v') \in W$.

Hence the conditions (3.1.3) from Definition 3.1 hold.

For to verify the relation 3.1.4(1) from Definition 3.1 we consider the arbitrary elements $x, y, z \in \mathcal{W}$ where $x = (v_1, v'_1), y = (v_2, v'_2)$ and $z = (v_3, v'_3)$. We assume that $\alpha_{\mathcal{W}}(y) = \beta_{\mathcal{W}}(x) = \alpha_{\mathcal{W}}(z)$.

Applying the properties of the structure functions of the vector groupoids V and V', we have

$y + z - \beta_{\mathcal{W}}(x) = (v_2, v'_2) + (v_3, v'_3) - \beta_{\mathcal{W}}(v_1, v'_1) =$
$= (v_2+v_3, v'_2+v'_3)-(\beta_V(v_1), \beta_V(v_1)) = (v_2+v_3-\beta_V(v_1), v'_2+v'_3-\beta_V(v_1)) =$
$= (v_2 + v_3 - \beta_V(v_1), v'_2 + v'_3 - \beta_{V'}(v'_1))$ and

(a) $x\odot_{\mathcal{W}}(y+z-\beta_{\mathcal{W}}(x)) = (v_1, v'_1)\odot_{\mathcal{W}}(v_2+v_3-\beta_V(v_1), v'_2+v'_3-\beta_{V'}(v'_1)) =$
$= (v_1 \odot_V (v_2 + v_3 - \beta_V(v_1), v'_1 \odot_{V'} (v'_2 + v'_3 - \beta_{V'}(v'_1))$.

On the other hand we have

(b) $x \odot_{\mathcal{W}} y + x \odot_{\mathcal{W}} z - x = (v_1, v'_1) \odot_{\mathcal{W}} (v_2, v'_2) + (v_1, v'_1) \odot_{\mathcal{W}} (v_3, v'_3)-$
$-(v_1, v'_1) = (v_1 \odot_V v_2, v'_1 \odot_{V'} v'_2) + (v_1 \odot_V v_3, v'_1 \odot_{V'} v'_3) - (v_1, v'_1) =$
$= (v_1 \odot_V v_2 + v_1 \odot_V v_3 - v_1, v'_1 \odot_{V'} v'_2 + v'_1 \odot_{V'} v'_3 - v'_1)$.

Using now the relations (a), (b) and the relations 3.1.4(1) for V and V', we obtain the equality $x \odot_{\mathcal{W}} (y + z - \beta_{\mathcal{W}}(x)) = x \odot_{\mathcal{W}} y + x \odot_{\mathcal{W}} z - x$. Hence the condition 3.1.4 (1) holds.

We verify now the relation 3.1.4(4). For this, let $x = (v_1, v'_1) \in \mathcal{W}$, $y = (v_2, v'_2) \in \mathcal{W}$ such that $\alpha_{\mathcal{W}}(y) = \beta_{\mathcal{W}}(x)$ and $k \in K$. We have

(c) $(ky + (1-k)\alpha_{\mathcal{W}}(x)) \odot_{\mathcal{W}} x =$
$= (kv_2 + (1-k)\alpha_V(v_1), kv'_2 + (1-k)\alpha_{V'}(v'_1)) \odot_{\mathcal{W}} (v_1, v'_1) =$
$= ((kv_2 + (1-k)\alpha_V(v_1)) \odot_V v_1, (kv'_2 + (1-k)\alpha_{V'}(v'_1)) \odot_{V'} v'_1)$ and

(d) $k(y \odot_{\mathcal{W}} x) + (1-k)x = k((v_2, v'_2) \odot_{\mathcal{W}} (v_1, v'_1)) + (1-k)(v_1, v'_1) =$
$= (k(v_2 \odot_V v_1) + (1-k)v_1, k(v'_2 \odot_{V'} v'_1) + (1-k)v'_1)$.

Using the equalities (c) and (d) and the relations 3.1.4(4) for V and V', we obtain that the condition 3.1.4 (4) holds.

In the same manner we prove that the conditions 3.1.4 (2) and 3.1.4 (3) hold. Hence $V \oplus V'$ is a vector groupoid.

The vector groupoid $(\mathcal{W} := V \oplus V', \alpha_{\mathcal{W}}, \beta_{\mathcal{W}}, \odot_{\mathcal{W}}, \iota_{\mathcal{W}}, \Delta_{V_0})$ is called the *Whitney sum* of the vector groupoids (V, V_0) and (V', V_0). The base of this vector groupoid can be identified with V_0.

Proposition 4.1. *If (V, V_0) and (V', V_0) are transitive vector groupoids, then the Whitney sum $(V \oplus V', \Delta_{V_0})$ is a transitive vector groupoid.*

Proof. It must prove that the anchor $(\alpha_{\mathcal{W}}, \beta_{\mathcal{W}}) : \mathcal{W} \to \Delta_{V_0} \times \Delta_{V_0}$ is surjective. □

If $(V \oplus V', \Delta_{V_0})$ is the Whitney sum of vector groupoids (V, V_0) and (V', V_0), then the projections maps $p : V \oplus V' \to V$ and $p' : V \oplus V' \to V'$ defined by $p(v, v') = v$ and $p'(v, v') = v'$ are morphisms of vector groupoids.

Theorem 4.1. *Let (V, V_0) and (V', V_0) be two vector groupoids. The triple $(V \oplus V', p, p')$ verifies the* **universal property of the Whitney sum***:*

for all triple (U, q, q') composed by vector groupoid $(U, \alpha_U, \beta_U, \odot_U, \iota_U, V_0)$ and two morphisms of vector groupoids $V' \xleftarrow{q'} U \xrightarrow{q} V$, there exists a unique morphism of vector groupoids $\varphi : U \to V \oplus V'$ such that the following diagram:

$$\begin{array}{ccccc} V' & \xleftarrow{p'} & V \oplus V' & \xrightarrow{p} & V \\ & {}^{q'}\nwarrow & \uparrow\varphi & \nearrow q & \\ & & U & & \end{array}$$

is commutative.

Proof. We consider the map $\varphi : U \to V \oplus V'$ by taking $\varphi(u) := (q(u), q'(u))$ for all $u \in U$. By hypothesis the maps $q : U \to V$ and $q' : U \to V'$ are vector groupoid morphisms. Then $(\alpha_V \circ q)(u) = \alpha_U(u)$ and $(\alpha_{V'} \circ q')(u) = \alpha_U(u)$, for all $u \in U$. It follows that $\alpha_V(q(u)) = \alpha_{V'}(q'(u))$. Similarly $\beta_V(q(u)) = \beta_{V'}(q'(u))$. Therefore $\varphi(u) \in W := V \oplus V'$. Hence φ is well-defined.

Let now $x, y \in U$ such that $(x, y) \in U_{(2)}$, i.e. $\beta_U(y) = \alpha_U(x)$. Also we have $(q(x), q(y)) \in V_{(2)}$, i.e. $\beta_V(q(y)) = \alpha_V(q(x))$, since q is a groupoid morphism. Then $(\varphi(x), \varphi(y)) \in W_{(2)}$. Indeed, $\beta_W(\varphi(y)) = \beta_W(q(y), q'(y)) = (\beta_V(q(y)), \beta_V(q(y))) = (\alpha_V(q(x)), \alpha_V(q(x)) = \alpha_W(q(x), q'(x)) = \alpha_W(\varphi(x))$.

For $x, y \in U$ such that $(x, y) \in U_{(2)}$ we have $\varphi(x \odot_U y) =$
$= (q(x \odot_U y), q'(x \odot_U y)) = (q(x) \odot_V q(y), q'(x) \odot_{V'} q'(y)) = \varphi(x) \odot_W \varphi(y)$.

Using the linearity of q and q' it is easy to verify that φ is a linear map. Therefore, φ is a vector groupoid morphism. We have $p \circ \varphi = q$ and $p' \circ \varphi = q'$.

In a standard manner we prove that φ is a unique morphism of vector groupoids such that the above diagram is commutative. □

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