SUBRESULTANTS IN MULTIPLE ROOTS

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Abstract. We extend our previous work on Poisson-like formulas for subresultants in roots to the case of polynomials with multiple roots in both the univariate and multivariate case, and also explore some closed formulas in roots for univariate polynomials in this multiple roots setting.

1. Introduction

In [DKS2006] we presented Poisson-like formulas for multivariate subresultants in terms of the roots of the system given by all but one of the input polynomials, provided that all the roots were *simple*, i.e. that the ideal generated by these polynomials is zero-dimensional and radical. Multivariate resultants were mainly introduced by Macaulay in [Mac1902], after earlier work by Euler, Sylvester and Cayley, while multivariate subresultants were first defined by Gonzalez-Vega in [GLV1990, GLV1991], generalizing Habicht's method [Hab1948]. The notion of subresultants that we use in this text was introduced by Chardin in [Cha1995].

Later on, in [DHKS2007, DHKS2009], we focused on the classical univariate case and reworked the relation between subresultants and double Sylvester sums, always in the simple roots case (where double sums are actually well-defined). As one of the referees of the MEGA'2007 conference pointed out, working out these results for the case of polynomials with multiple roots would also be interesting.

This paper is a first attempt in that direction. We succeed in describing Poisson like formulas for univariate and multivariate subresultants in the presence of multiple roots, as well as to obtain formulas in roots in the univariate setting for some extremal cases. However, it is still not clear which is the correct way of generalizing Sylvester double sums in the multiple roots case.

The paper is organized as follows: In Section 2 we recall the definitions of the classical univariate subresultants and Sylvester double sums. We then introduce the generalized Wronskian and Vandermonde matrices, and show how the Poisson formulas obtained in [Hon1999] for the case of simple roots extend to the multiple roots setting by means of these generalized matrices.

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We also obtain formulas in roots for subresultants in two extremal non-trivial cases. In Section 3 we present Poisson-like formulas for multivariate subresultants in the case of multiple roots, generalizing our previous results described in [DKS2006].

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2. Univariate Case: Subresultants in multiple roots

2.1. **Notation.** We first establish a notation that will make the presentation of the problem and the state of the art simpler.

Set $d, e \in \mathbb{N}$ and let

$$A := \big(\alpha_1, \dots, \alpha_d\big) \quad \text{and} \quad B := \big(\beta_1, \dots, \beta_e\big)$$

be two (ordered) sets of d and e different indeterminates respectively.

For $m, n \in \mathbb{N}$, set $(d_1, \dots, d_m) \in \mathbb{N}^m$ and $(e_1, \dots, e_n) \in \mathbb{N}^n$ such that

$$d_1 + \cdots + d_m = d \quad \text{and} \quad e_1 + \cdots + e_n = e$$

and let

$$\overline{A} := \big((\alpha_1, d_1); \dots ; (\alpha_m, d_m)\big) \quad \text{and} \quad \overline{B} := \big((\beta_1, e_1); \dots ; (\beta_n, e_n)\big)$$

(these will be regarded as "limit sets" of A and B when roots are packed following the corresponding multiplicity patterns).

We associate to A and B the monic polynomials of degrees d and e respectively

$$f(x) := \prod_{i=1}^{d} (x - \alpha_i) \quad \text{and} \quad g(x) := \prod_{j=1}^{e} (x - \beta_j)$$

and the set

$$R(A, B) = \prod_{1 \le i \le d, 1 \le j \le e} (\alpha_i - \beta_j) = \prod_{1 \le i \le d} g(\alpha_i)$$

with natural limits when the roots are packed

$$\overline{f}(x) := \prod_{i=1}^{m} (x - \alpha_i)^{d_i} \quad \text{and} \quad \overline{g}(x) := \prod_{j=1}^{n} (x - \beta_j)^{e_j},$$

$$R(\overline{A}, \overline{B}) = \prod_{1 \le i \le m, 1 \le j \le n} (\alpha_i - \beta_j)^{d_i e_j} = \prod_{1 \le i \le m} \overline{g}(\alpha_i)^{d_i}.$$

2.2. **Subresultants and Sylvester double sums.** We recall that for $0 \le t \le d < e$ or $0 \le t < d = e$, the t-th subresultant of the polynomials $f = a_d x^d + \cdots + a_0$ and $g = b_e x^e + \cdots + b_0$, introduced by J.J. Sylvester in [Sylv1853], is defined as

$$\operatorname{Sres}_t(f,g) := \det \overbrace{\left[\begin{array}{ccccc} a_d & \cdots & \cdots & a_{t+1-(e-t-1)} & x^{e-t-1}f(x) \\ & \ddots & & \vdots & \vdots \\ & & a_d \cdots & a_{t+1} & x^0 f(x) \\ \hline b_e & \cdots & \cdots & b_{t+1-(d-t-1)} & x^{d-t-1}g(x) \\ & \ddots & & \vdots & \vdots \\ & & b_e \cdots & b_{t+1} & x^0 g(x) \end{array}\right]}^{d+e-2t} \begin{array}{l} \Big\} e-t \\ \\ \Big\} d-t \end{array}$$

with $a_\ell = b_\ell = 0$ for $\ell < 0$. When $t = 0$ we have $\operatorname{Sres}_t(f,g) = \operatorname{Res}(f,g)$.

In the same article [Sylv1853] Sylvester also introduced for $0 \le p \le d, 0 \le q \le e$ the following *double-sum* expression in A and B,

$$\operatorname{Sylv}^{p,q}(A,B;x) := \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'|=p,\, |B'|=q}} R(x,A')\, R(x,B')\, \frac{R(A',B')\, R(A\backslash A', B\backslash B')}{R(A',A\backslash A')\, R(B',B\backslash B')},$$

where by convention $R(A',B') = 1$ if $A' = \emptyset$ or $B' = \emptyset$. For instance

$$\operatorname{Sylv}^{0,0}(A,B;x) = R(A,B) = \prod_{1\le i\le d, 1\le j\le e} (\alpha_i - \beta_j) = \operatorname{Res}(f,g). \tag{1}$$

We note that $\operatorname{Sylv}^{p,q}(A,B;x)$ only makes sense when $\alpha_i \ne \alpha_j$ and $\beta_i \ne \beta_j$ for $i \ne j$, since otherwise some denominators in $\operatorname{Sylv}^{p,q}(A,B;x)$ would vanish.

The following relation between these double sums and the subresultants (for monic polynomials with simple roots f and g) was described by Sylvester: for any choice of $0 \le p \le d$ and $0 \le q \le e$ such that $t := p + q$ satisfies $t < d \le e$ or $t = d < e$, one has

$$\operatorname{Sres}_t(f,g) = (-1)^{p(d-t)} \binom{t}{p}^{-1} \operatorname{Sylv}^{p,q}(A,B;x). \tag{2}$$

This gives an expression for the subresultant in terms of the differences of the roots —generalizing the well-known formula (1)— *in case f and g have only simple roots*. However, when the roots are packed, i.e. when we deal with $\overline{A}$ and $\overline{B}$, the expression for the resultant is stable, i.e.

$$\operatorname{Res}(\overline{f},\overline{g}) = \prod_{1\le i\le m,\, 1\le j\le n} (\alpha_i - \beta_j)^{d_i e_j},$$

while not only there is no simple expression of what $\operatorname{Sres}_t(\overline{f},\overline{g})$ is in terms of differences of roots but moreover there is no simple definition of what $\operatorname{Sylv}^{p,q}(\overline{A},\overline{B};x)$ should be in order to preserve Identity (2). Of course, since

$\mathrm{Sres}_t(\overline{f},\overline{g})$ is defined anyway, $\mathrm{Sylv}^{p,q}(\overline{A},\overline{B};x)$ could be defined as the result

$$\mathrm{Sylv}^{p,q}(\overline{A},\overline{B};x) := (-1)^{p(d-t)}\binom{t}{p}\mathrm{Sres}_t(\overline{f},\overline{g})$$

but this is not quite satisfactory because on one hand this does not clarify how Sres_t behaves in terms of the roots when these are packed, and on the other hand, $\mathrm{Sylv}^{p,q}(\overline{A},\overline{B};x)$ is defined for every $0 \le p \le d$ and $0 \le q \le e$ while Sres_t is only defined for $t := p+q \le \min\{d,e\}$.

The goal of this section it to describe in terms of roots some particular cases of the subresultant of two univariate polynomials, when these polynomials have multiple roots. These are very partial answers to the questions raised above, since we were not able to give a right expression for what the Sylvester double sums should be, even in the particular cases we could consider. Nevertheless the results we obtained give a hint of how complex it can be to give complete general answers, at least in terms of double or multiple sums, see Theorem 2.8 below.

2.3. **Generalized Vandermonde and Wronskian matrices.** We need to recall some facts on generalized Vandermonde and Wronskian matrices.

Notation 2.1. *Set $u \in \mathbb{N}$. The* generalized Vandermonde *or* confluent *(non-necessarily square) $u \times d$ matrix $V_u(\overline{A})$ associated to $\overline{A} = \big((\alpha_1,d_1);\dots;(\alpha_m,d_m)\big)$,* [Kal1984]*, is*

$$V_u(\overline{A}) = V_u\big((\alpha_1,d_1);\dots;(\alpha_m,d_m)\big) := \underbrace{\begin{array}{|c|c|c|}\hline V_u(\alpha_1,d_1) & \dots & V_u(\alpha_m,d_m) \\ \hline\end{array}}_{d}\; u\ ,$$

where

$$V_u(\alpha_i,d_i) := \overbrace{\left|\begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \\ \alpha_i & 1 & 0 & \dots & 0 \\ \alpha_i^2 & 2\alpha_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_i^{u-1} & (u-1)\alpha_i^{u-2} & \binom{u-1}{2}\alpha_i^{u-3} & \dots & \binom{u-1}{d_i-1}\alpha_i^{u-d_i} \end{array}\right|}^{d_i}\; u$$

with the convention that when $k<j$, $\binom{j}{k}\alpha_i^{k-j} = 0$.
When $d_i = 1$ for all i, this gives the usual Vandermonde matrix $V_u(A)$. When $u = d$, we omit the sub-index u and write $V(\overline{A})$ and $V(A)$.

For example

$$V\big((\alpha,3);(\beta,2)\big) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ \alpha & 1 & 0 & \beta & 1 \\ \alpha^2 & 2\alpha & 1 & \beta^2 & 2\beta \\ \alpha^3 & 3\alpha^2 & 3\alpha & \beta^3 & 3\beta^2 \\ \alpha^4 & 4\alpha^3 & 6\alpha^2 & \beta^4 & 4\beta^3 \end{array}\right]$$

and

$$V_3((\alpha,3);(\beta,2)) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ \alpha & 1 & 0 & \beta & 1 \\ \alpha^2 & 2\alpha & 1 & \beta^2 & 2\beta \end{bmatrix}.$$

The determinant of a square confluent matrix is non-zero, and satisfies, [Ait1939],

$$\det\big(V(\overline{A})\big) = \prod_{1\le i<j\le m} (\alpha_j - \alpha_i)^{d_i d_j}.$$

In the same way that the usual Vandermonde matrix $V(A)$ is related to the Lagrange Interpolation Problem on A, the generalized Vandermonde matrix $V(\overline{A})$ is associated with the Hermite Interpolation Problem on $\overline{A}$ (see for instance [KTO1997]):

Given $\{y_{i,j_i}, 1 \le i \le m, 0 \le j_i < d_i\}$, there exists a unique polynomial p of degree $\deg(p) < d$ which satisfies the following conditions:

$$\begin{cases} p(\alpha_1) = 0!\, y_{1,0}, & p'(\alpha_1) = 1!\, y_{1,1}, \quad \dots, \quad p^{(d_1-1)}(\alpha_1) = (d_1-1)!\, y_{1,d_1-1}, \\ \quad\vdots & \quad\vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ p(\alpha_m) = 0!\, y_{m,0}, & p'(\alpha_m) = 1!\, y_{m,1}, \quad \dots, \quad p^{(d_m-1)}(\alpha_m) = (d_m-1)!\, y_{m,d_m-1}. \end{cases}$$

This polynomial $p = a_0 + a_1x + \cdots + a_{d-1}x^{d-1}$ is given by the only solution of

$$(a_0\ a_1\ \dots\ a_{d-1}) \cdot V(\overline{A}) = (y_{1,0}\ y_{1,1}\ \cdots\ y_{m,d_m-1})$$

(here the right vector is indexed by the pairs (i, j_i) for $1 \le i \le m, 0 \le j_i < d_i$) and satisfies

$$\det\big(V(\overline{A})\big)\, p(x) = -\det \left[\begin{array}{c|c} V(\overline{A}) & \begin{matrix} 1 \\ x \\ \vdots \\ x^{d-1} \end{matrix} \\ \hline y_{1,0}\ \ y_{1,1}\ \ \cdots\ \ y_{m,d_m-1} & 0 \end{array}\right]. \tag{3}$$

We can also view p in a more suitable basis, where the corresponding "Vandermonde" matrix has more structure. We introduce the d polynomials in this basis.

Notation 2.2. *For $1 \le i \le m$ we set*

$$\overline{f}_i := \prod_{j\ne i} (x-\alpha_j)^{d_j}$$

and, for $0 \le k_i < d_i$,

$$\overline{f}_{i,k_i} := \frac{\overline{f}}{(x-\alpha_i)^{d_i-k_i}} \;=\; (x-\alpha_i)^{k_i}\overline{f}_i.$$

It it useful to compute the derivatives of f_{i,k_i} specialized into α_ℓ.

Lemma 2.3. *Fix* $1 \le i \le m$ *and* $0 \le k < d_i$. *Then*

$$\overline{f}_{i,k}^{(q)}(\alpha_j) = \begin{cases} 0 & \text{for } \ j \neq i \ \text{ and } \ 0 \le q < d_j, \\ 0 & \text{for } \ j = i \ \text{ and } \ 0 \le q < k, \\ \frac{q!}{(q-k)!}\,\overline{f}_i^{(q-k)}(\alpha_i) & \text{for } \ j = i \ \text{ and } \ k \le q < d_i. \end{cases}$$

Proof. It is obvious for $j \neq i$ and $0 \le q < d_j$ since α_j is a root of multiplicity d_j of $\overline{f}_{i,k}$. Also for $j = i$ and $0 \le q < k$ since α_i is a root of multiplicity k of $\overline{f}_{i,k}$. Finally, for $k \le q < d_i$, by Leibniz rule for the derivative of the product,

$$\overline{f}_{i,k}^{(q)} = \left((x-\alpha_i)^k\,\overline{f}_i\right)^{(q)} = \sum_{j=0}^{q} \binom{q}{j} \left((x-\alpha_i)^k\right)^{(j)} \overline{f}_i^{(q-j)},$$

and therefore

$$\overline{f}_{i,k}^{(q)}(\alpha_i) = \frac{q!}{(q-k)!}\,\overline{f}_i^{(q-k)}(\alpha_i),$$

since $\left((x-\alpha_i)^k\right)^{(j)}(\alpha_i) = 0$ if $j \neq k$ and $\left((x-\alpha_i)^k\right)^{(k)}(\alpha_i) = k!$. □

This implies that $p = \sum_{i,k_i} a_{i,k_i} \overline{f}_{i,k_i}$ is given by the only solution of

$$(a_{1,0}\ a_{1,1}\ \dots\ a_{m,d_m-1}) \cdot V'(\overline{A}) = (y_{1,0}\ y_{1,1}\ \dots\ y_{m,d_m-1})$$

where

$$V'(\overline{A}) := \begin{array}{c|c|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{d_1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{d_m} & \\ \cline{2-4} & V'(\alpha_1,d_1) & \mathbf{0} & \mathbf{0} & d_1 \\ \cline{2-4} & \mathbf{0} & \ddots & \mathbf{0} & \\ \cline{2-4} & \mathbf{0} & \mathbf{0} & V'(\alpha_m,d_m) & d_m \\ \cline{2-4} \end{array},$$

with

$$V'(\alpha_i,d_i) := \overbrace{\left|\begin{array}{cccc} \overline{f}_i(\alpha_i) & \overline{f}_i'(\alpha_i) & \dots & \frac{\overline{f}_i^{(d_i-1)}(\alpha_i)}{(d_i-1)!} \\ 0 & \overline{f}_i(\alpha_i) & \dots & \frac{\overline{f}_i^{(d_i-2)}(\alpha_i)}{(d_i-2)!} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \overline{f}_i(\alpha_i) \end{array}\right|}^{d_i}\ d_i$$

and satisfies

$$p(x) = -\det\left(V'(\overline{A})\right)^{-1} \det \begin{array}{|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{1} & \\ \hline & \overline{f}_{1,0} & \\ & \overline{f}_{1,1} & \\ V'(\overline{A}) & \vdots & d \\ & \overline{f}_{m,d_m-1} & \\ \hline y_{1,0}\quad y_{1,1}\quad \dots \quad y_{m,d_m-1} & 0 & 1 \\ \hline \end{array}.$$

We note that

$$\begin{aligned}\det(V'(\overline{A})) &= \prod_{1\le i\le m} \overline{f}_i(\alpha_i)^{d_i}\\ &= (-1)^{\frac{m(m-1)}{2}}\Big(\prod_{1\le i<j\le m}(\alpha_j-\alpha_i)^{d_id_j}\Big)^2 = (-1)^{\frac{m(m-1)}{2}}\det(V(\overline{A}))^2.\end{aligned}$$

In particular $p=\sum_{i,j_i} y_{i,j_i}p_{i,j_i}$ where for $1\le i\le m$, $0\le j_i<d_i$, the *basic Hermite polynomials* p_{i,j_i} is the unique polynomial of degree $\deg(p_{i,j_i})<d$ determined by the conditions for $1\le \ell\le m$, $0\le q_\ell<d_\ell$,

$$\begin{cases} p_{i,j_i}^{(q_\ell)}(\alpha_\ell)=j_i! \ \text{ for } \ \ell=i \text{ and } q_\ell=j_i,\\ p_{i,j_i}^{(q_\ell)}(\alpha_\ell)=0 \ \text{ otherwise.}\end{cases} \tag{4}$$

When $\overline{A}=A=(\alpha_1,\dots,\alpha_d)$, then, denoting $f_i:=\overline{f}_i$, we have

$$p_{i,0}=\prod_{\ell\ne i}\frac{x-\alpha_\ell}{\alpha_i-\alpha_\ell}=\frac{1}{f_i(\alpha_i)}\,f_i \quad\text{for}\quad 1\le i\le d,$$

while for $\overline{A}=(\alpha,d)$,

$$p_{1,j}=(x-\alpha)^j=\overline{f}_{1,j}\quad\text{for}\quad 0\le j<d.$$

The following proposition generalizes these two extremal formulas.

Proposition 2.4. *Fix $1\le i\le m$ and $0\le j<d_i$. Then*

$$p_{i,j}=\frac{1}{\overline{f}_i(\alpha_i)}\sum_{k=0}^{d_i-1-j}(-1)^k\left(\sum_{k_1+\ldots+\widehat{k_i}+\ldots+k_m=k}\ \prod_{\ell\ne i}\frac{\binom{d_\ell-1+k_\ell}{k_\ell}}{(\alpha_i-\alpha_\ell)^{k_\ell}}\right)\overline{f}_{i,j+k}$$

where $k_1+\cdots+\widehat{k_i}+\cdots+k_m$ denotes the sum without k_i. (When $m=1$, the right expression under brackets is understood to equal 1 for $k=0$ and 0 otherwise.)

Proof. We first prove that

$$p_{i,j}=\sum_{k=0}^{d_i-1-j}\frac{1}{k!}\left(\frac{1}{\overline{f}_i}\right)^{(k)}(\alpha_i)\,\overline{f}_{i,j+k} \tag{5}$$

verifying that the right-hand side expression satisfies Conditions (4). By Lemma 2.3, for $\ell\ne i$ and $0\le q<d_\ell$ or $\ell=i$ and $0\le q<j$, $p_{i,j}^{(q)}(\alpha_\ell)=0$. It remains to show that $p_{i,j}^{(j)}(\alpha_i)=j!$ and $p_{i,j}^{(q)}(\alpha_i)=0$ for $j<q<d_i$.
But

$$\overline{f}_{i,j+k}^{(q)}(\alpha_i)=\binom{q}{j+k}(j+k)!\,\overline{f}_i^{(q-j-k)}(\alpha_i)\ \text{ when } j+k\le q, \text{ i.e. when } \ k\le q-j,$$

while $\overline{f}^{(q)}_{i,j+k}(\alpha_i) = 0$ when $k > q - j$. This implies

$$\begin{aligned}
p^{(q)}_{i,j}(\alpha_i) &= \sum_{k=0}^{d_i-1-j} \frac{1}{k!}\left(\frac{1}{\overline{f}_i}\right)^{(k)}(\alpha_i)\,\overline{f}^{(q)}_{i,j+k}(\alpha_i) \\
&= \sum_{k=0}^{q-j} \frac{q!}{k!(q-j-k)!}\left(\frac{1}{\overline{f}_i}\right)^{(k)}(\alpha_i)\,\overline{f}_i^{(q-j-k)}(\alpha_i) \\
&= \frac{q!}{(q-j)!}\sum_{k=0}^{q-j}\binom{q-j}{k}\left(\frac{1}{\overline{f}_i}\right)^{(k)}(\alpha_i)\,\overline{f}_i^{(q-j-k)}(\alpha_i) \\
&= \frac{q!}{(q-j)!}\left(\frac{\overline{f}_i}{\overline{f}_i}\right)^{(q-j)}(\alpha_i) \quad = \quad \begin{cases} j! & \text{if} \quad q = j \\ 0 & \text{if} \quad q > j \end{cases}.
\end{aligned}$$

To conclude the proof, we apply again Leibnitz rule for the derivative of a product:

$$\left(\frac{1}{\overline{f}_i}\right)^{(k)} = (-1)^k\,k! \sum_{k_1+\ldots+\widehat{k_i}+\ldots+k_r=k}\ \prod_{\ell\neq i}\frac{\binom{d_\ell-1+k_\ell}{k_\ell}}{(x-\alpha_\ell)^{d_\ell+k_\ell}}.$$

The statement follows from plugging this expression, specialized into α_i, in Identity (5). □

The basic Hermite polynomials enable us to compute the inverse of the confluent matrix $V(\overline{A})$: for $1 \le i \le r$ define the following $(d_i \times d)$-matrix V_i^* as

$$V_i^* := \overbrace{\boxed{\begin{matrix}\text{coefficients of } p_{i,1} \\ \vdots \\ \text{coefficients of } p_{i,d_i}\end{matrix}}}^{d}\ d_i\ ,$$

where the coefficients of $p_{i,j_i}(x)$ are written in the monomial basis $1, x, \ldots x^{d-1}$. Then a simple verification shows that

$$V(\overline{A})^{-1} = \overbrace{\begin{array}{|c|}\hline V_1^* \\ \hline \vdots \\ \hline V_m^* \\ \hline\end{array}}^{d}\begin{matrix} d_1 \\ \\ d_m\end{matrix}\ .$$

Now we set the notation for a slight modification of a case of generalized Wronskian matrices.

Notation 2.5. *Set $u \in \mathbb{N}$. Given a polynomial $h(z)$, we define the* generalized Wronskian *(non-necessarily square) $u \times d$ matrix $W_{h,u}(\overline{A})$ associated to $\overline{A} = \big((\alpha_1, d_1); \ldots; (\alpha_m, d_m)\big)$ as*

$$W_{h,u}(\overline{A}) = W_{h,u}\big((\alpha_1, d_1); \dots; (\alpha_m, d_m)\big) := \overset{d}{\boxed{\begin{array}{c|c|c} W_{h,u}(\alpha_1, d_1) & \dots & W_{h,u}(\alpha_m, d_m) \end{array}}}\ {\scriptstyle u}\ ,$$

where

$$W_{h,t}(\alpha_i, d_i) := \overset{d_i}{\boxed{\begin{array}{cccc} h(\alpha_i) & f'(\alpha_i) & \dots & \frac{h^{(d_i-1)}(\alpha_i)}{(d_i-1)!} \\ (zh)(\alpha_i) & (zh)'(\alpha_i) & \dots & \frac{(zh)^{(d_i-1)}(\alpha_i)}{(d_i-1)!} \\ \vdots & \vdots & & \vdots \\ (z^{u-1}h)(\alpha_i) & (z^{u-1}h)'(\alpha_i) & \dots & \frac{(z^{u-1}h)^{(d_i-1)}(\alpha_i)}{(d_i-1)!} \end{array}}}\ {\scriptstyle u}\ .$$

When $u = d$, we omit the sub-index u and write $W_h(\overline{A})$.

For example for $h(z) = x - z$ and $\overline{A} = (\alpha, 3)$,

$$W_{x-z}(\alpha, 3) = W_{x-z,3}(\alpha, 3) = \overset{3}{\boxed{\begin{array}{ccc} x - \alpha & -1 & 0 \\ \alpha x - \alpha^2 & x - 2\alpha & -1 \\ \alpha^2 x - \alpha^3 & 2\alpha x - 3\alpha^2 & x - 3\alpha \end{array}}}\ {\scriptstyle 3}\ .$$

The determinant of a square Wronskian matrix is easily obtainable performing row operations in the case of one block, and by induction in the size of the matrix in general:

$$\det\big(W_h(\overline{A})\big) = \Big(\prod_{1\le i<j\le m} (\alpha_j - \alpha_i)^{d_i d_j}\Big) h(\alpha_1)^{d_1} \cdots h(\alpha_m)^{d_m}.$$

2.4. **Subresultants in multiple roots.** In this section, we describe some explicit formulas we can get for subresultants in terms of both sets of roots of $\overline{f} = (x - \alpha_1)^{d_1} \cdots (x - \alpha_m)^{d_m}$ and $\overline{g} = (x - \beta_1)^{e_1} \cdots (x - \beta_n)^{e_n}$ with $d = \sum_{i=1}^m d_i$ and $e = \sum_{j=1}^n e_j$. The main drawback of following the approach used in [DHKS2007, DHKS2009] is the fact that submatrices of generalized Vandermonde matrices are not always generalized Vandermonde matrices, so in general their determinants cannot be expressed as products of differences. This is why the search for nice formulas in double sums in the case of multiple roots is more challenging. The following claim is a generalization of [Hon1999, Th. 3.1] and [DHKS2007, Lem. 2] which includes the multiple roots case. It is also strongly related to a multiple roots case version of [DHKS2009, Th. 1].

Theorem 2.6. *Set $0 \le t \le d < e$ or $0 \le t < d = e$. Then*

$$\operatorname{Sres}_t(\overline{f},\overline{g}) = (-1)^{d-t}\det\big(V(\overline{A})\big)^{-1}\det \begin{array}{|c|c|l} \multicolumn{1}{c}{t} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \begin{matrix}1\\ \vdots \\ x^t\end{matrix} & t+1 \\ \hline W_{\overline{g},d-t}(\overline{A}) & \mathbf{0} & d-t \\ \hline \end{array}$$

$$= (-1)^{c}\det\big(V(\overline{A})\big)^{-1}\det\big(V(\overline{B})\big)^{-1}\det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \mathbf{0} & \begin{matrix}1\\ \vdots \\ x^t\end{matrix} & t+1 \\ \hline V_{d+e-t}(\overline{A}) & V_{d+e-t}(\overline{B}) & \mathbf{0} & d+e-t \\ \hline \end{array},$$

where $c := \max\{e \pmod 2, d-t \pmod 2\}$.

Proof. We use repeatedly that when computing determinants, exchanging two consecutive blocks of d and e rows (or columns) in a matrix changes the sign by $(-1)^{de}$.

Let $\overline{f} = \sum_{i=0}^{d} a_i x^i$, where $a_d = 1$, and $\overline{g} = \sum_{j=0}^{e} b_j x_i$, where $b_e = 1$. As in [DHKS2007], we define the following matrices

$$M_{\overline{f}} := \begin{array}{|ccccc|l} \multicolumn{5}{c}{d+e-t} & \\ \hline a_0 & \dots & a_d & & & \\ & \ddots & & \ddots & & e-t \\ & & a_0 & \dots & a_d & \\ \hline \end{array}, \qquad M_{\overline{g}} := \begin{array}{|ccccc|l} \multicolumn{5}{c}{d+e-t} & \\ \hline b_0 & \dots & b_e & & & \\ & \ddots & & \ddots & & d-t \\ & & b_0 & \dots & b_e & \\ \hline \end{array}.$$

and

$$S_t := \begin{array}{|c|l} \multicolumn{1}{c}{d+e-t} & \\ \hline M_{x-z} & t \\ \hline M_{\overline{f}} & e-t \\ \hline M_{\overline{g}} & d-t \\ \hline \end{array} \quad \text{where} \quad M_{x-z} := \begin{array}{|cccccc|l} \multicolumn{6}{c}{d+e-t} & \\ \hline x & -1 & 0 & \dots & \dots & 0 & \\ \ddots & \ddots & \ddots & & & \vdots & t \\ & x & -1 & 0 & \dots & 0 & \\ \hline \end{array}.$$

We have ([DHKS2007, Lem. 1]):

$$\operatorname{Sres}_t(\overline{f},\overline{g}) = (-1)^{(e-t)(d-t)}\det(S_t).$$

Also, as in the proof of [DHKS2007, Lem. 2],

$$\begin{array}{r|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{d+e-t} \\ \cline{2-2} t & M_{x-z} \\ \cline{2-2} e-t & M_{\overline{f}} \\ \cline{2-2} d-t & M_{\overline{g}} \\ \cline{2-2} \end{array} \begin{array}{|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e-t} & \\ \hline V_{d+e-t}(\overline{A}) & \mathbf{0} & d \\ \cline{2-2} & \mathrm{Id}_{e-t} & e-t \\ \hline \end{array} = \begin{array}{|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e-t} & \\ \hline W_{x-z,t}(\overline{A}) & * & t \\ \hline \mathbf{0} & M'_{\overline{f}} & e-t \\ \hline W_{\overline{g},d-t}(\overline{A}) & * & d-t \\ \hline \end{array},$$

because $M_{x-z} \cdot V_{d+e-t} = W_{x-z,t}$, $M_{\overline{f}} \cdot V_{d+e-t} = W_{\overline{f},e-t} = \mathbf{0}$ since $\overline{f}(\alpha_i) = \dots = \overline{f}^{(d_i-1)}(\alpha_i) = 0$, and $M_{\overline{g}} \cdot V_{d+e-t} = W_{\overline{g},d-t}$. Finally $M'_{\overline{f}}$ is a lower triangular matrix with diagonal entries $a_d = 1$.

This implies first the generalization of [DHKS2007, Lem. 2] to the multiple roots case:

$$\det\big(V(\overline{A})\big)\,\mathrm{Sres}_t(\overline{f},\overline{g}) = \det \begin{array}{|c|l} \multicolumn{1}{c}{d} & \\ \hline W_{x-z,t}(\overline{A}) & t \\ \hline W_{\overline{g},d-t}(\overline{A}) & d-t \\ \hline \end{array} = (-1)^{d-t}\det \begin{array}{|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \begin{matrix}1\\ \vdots\\ x^t\end{matrix} & t+1 \\ \hline W_{\overline{g},d-t}(\overline{A}) & \mathbf{0} & d-t \\ \hline \end{array} ,$$

where the second equality is a consequence of obvious row and column operations coming from the row and column equivalences:

$$\det \begin{array}{|c|l} \multicolumn{1}{c}{d} & \\ \hline W_{x-z,t}(\overline{A}) & t \\ \hline W_{\overline{g},d-t}(\overline{A}) & d-t \\ \hline \end{array} = (-1)^{d}\det \begin{array}{|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{1} & \\ \hline \mathbf{0} & 1 & 1 \\ \hline W_{z-x,t}(\overline{A}) & \mathbf{0} & t \\ \hline W_{\overline{g},d-t}(\overline{A}) & \mathbf{0} & d-t \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|l} \multicolumn{1}{c}{d_i} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\alpha_i,d_i) & \begin{matrix}1\\ \vdots\\ x^t\end{matrix} & t+1 \\ \hline \end{array} \equiv_r \begin{array}{|c|c|l} \multicolumn{1}{c}{d_i} & \multicolumn{1}{c}{1} & \\ \hline 1\ \ 0\ \cdots\ 0 & 1 & 1 \\ \hline W_{z-x,t}(\alpha_i,d_i) & \mathbf{0} & t \\ \hline \end{array} \equiv_c \begin{array}{|c|c|l} \multicolumn{1}{c}{d_i} & \multicolumn{1}{c}{1} & \\ \hline \mathbf{0} & 1 & 1 \\ \hline W_{z-x,t}(\alpha_i,d_i) & \mathbf{0} & t \\ \hline \end{array} .$$

Next, we get as in the proof of [DHKS2007, Lem. 3],

$$\det\big(V(\overline{A})\det\big(V(\overline{B})\big)\,\mathrm{Sres}_t(\overline{f},\overline{g}) = (-1)^{d-t+e+(d-t)e}\det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \mathbf{0} & \begin{matrix}1\\ \vdots\\ x^t\end{matrix} & t+1 \\ \hline V_e(\overline{A}) & V_e(\overline{B}) & \mathbf{0} & e \\ \hline W_{\overline{g},d-t}(\overline{A}) & \mathbf{0} & \mathbf{0} & d-e \\ \hline \end{array}$$

$$= (-1)^c \det\left(\begin{array}{r|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{t+1} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{d-t} \\ \cline{2-4} t+1 & \mathrm{Id}_{t+1} & \mathbf{0} & \mathbf{0} \\ \cline{2-4} e & \mathbf{0} & \mathrm{Id}_e & \mathbf{0} \\ \cline{2-4} d-t & \mathbf{0} & \multicolumn{2}{c|}{M_{\overline{g}}} \\ \cline{2-4} \end{array} \quad \begin{array}{|c|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \mathbf{0} & \begin{matrix}1\\ \vdots\\ x^t\end{matrix} & t+1 \\ \hline V_{d+e-t}(\overline{A}) & V_{d+e-t}(\overline{B}) & \mathbf{0} & d+e-t \\ \hline \end{array} \right)$$

$$= (-1)^c \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{1} & \\ \hline V_{t+1}(\overline{A}) & \mathbf{0} & \begin{matrix}1\\ \vdots\\ x^t\end{matrix} & t+1 \\ \hline V_{d+e-t}(\overline{A}) & V_{d+e-t}(\overline{B}) & \mathbf{0} & d+e-t \\ \hline \end{array} ,$$

since the first matrix of the second row is lower triangular with 1 in the diagonal.
We note that starting from

$$\det\big(V(\overline{A})\big)\,\mathrm{Sres}_t(\overline{f},\overline{g}) = \det \begin{array}{|c|l} \multicolumn{1}{c}{d} & \\ \hline W_{x-z,t}(\overline{A}) & t \\ \hline W_{\overline{g},d-t}(\overline{A}) & d-t \\ \hline \end{array}$$

and using the same arguments as above, we also get very simply
(6)

$$\det\big(V(\overline{A})\det\big(V(\overline{B})\big)\operatorname{Sres}_t(\overline{f},\overline{g}) = (-1)^{(d-t)e}\det \begin{array}{cc|c|c|c} & & d & e & \\ \cline{3-4} & & W_{x-z}(\overline{A}) & \mathbf{0} & t \\ \cline{3-4} & & V_{d+e-t}(\overline{A}) & V_{d+e-t}(\overline{B}) & d+e-t \\ \cline{3-4} \end{array}$$

□

As mentioned above, when $t=0$ the formula in roots for $\operatorname{Sres}_0(f,g)$ specializes well when considering $\operatorname{Sres}_0(\overline{f},\overline{g})$. When $t=d<e$, the formula $\operatorname{Sres}_d(f,g)=\prod_{1\le i\le d}(x-\alpha_i)$ also specializes well as $\operatorname{Sres}_d(\overline{f},\overline{g})=\prod_{1\le i\le m}(x-\alpha_i)^{d_i}$. Our purpose now is to understand formulas in roots for the following extremal subresultants, i.e for Sres_1 and $\operatorname{Sres}_{d-1}$, in case of multiple roots.

• The case $t=d-1<e$: When f has simple roots, it is known (or can easily be derived for instance from Sylvester's Identity (2) for $p=d-1$ and $q=0$) that

$$\operatorname{Sres}_{d-1}(f,\overline{g}) = \sum_{i=1}^{d}\overline{g}(\alpha_i)\Big(\prod_{j\neq i}\frac{x-\alpha_j}{\alpha_i-\alpha_j}\Big) = \sum_{i=1}^{d}\overline{g}(\alpha_i)\,p_i,$$

where p_i is the basic Lagrange interpolation polynomial of degree strictly bounded by d such that $p_i(\alpha_i)=1$ and $p_i(\alpha_j)=0$ for $j\neq i$. In other words, $\operatorname{Sres}_{d-1}(f,\overline{g})$ is the Lagrange interpolation polynomial of degree strictly bounded by d which coincides with $\overline{g}$ in the d values $\alpha_1,\dots,\alpha_d$. This formula does not apply when f has multiple roots, but we can show that we get the natural generalization of this fact, that is, that $\operatorname{Sres}_{d-1}(\overline{f},\overline{g})$ is the Hermite interpolation polynomial of degree strictly bounded by d which coincides with $\overline{g}$ and its derivatives up to the corresponding orders in the m values $\alpha_1,\dots,\alpha_m$:

Proposition 2.7.

$$\operatorname{Sres}_{d-1}(\overline{f},\overline{g}) = \sum_{i=1}^{m}\sum_{j_i=0}^{d_i-1}\frac{\overline{g}^{(j_i)}(\alpha_i)}{j_i!}\,p_{i,j_i},$$

where p_{i,j_i} is the basic Hermite interpolation polynomial defined by Condition (4) or Proposition 2.4 for $\overline{A}$.

Proof. In this case, applying the first statement of Theorem 2.6 we get

$$\operatorname{Sres}_{d-1}(\overline{f},\overline{g}) = -\det\big(V(\overline{A})\big)^{-1}\det \begin{array}{|c|c|c} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{1} & \\ \hline V_d(\overline{A}) & \begin{array}{c}1\\ \vdots\\ x^{d-1}\end{array} & d \\ \hline W_{\overline{g},1}(\overline{A}) & \mathbf{0} & 1 \\ \hline \end{array}$$

where when following the subindex notation of Formula (3), we note that

$$\big(W_{\overline{g},1}(\overline{A})\big)_{i,j_i} = \frac{\overline{g}^{(j_i)}(\alpha_i)}{j_i!}.$$

Therefore by Formula (3), $\mathrm{Sres}_{d-1}(\overline{f},\overline{g})$ is the stated Hermite interpolation polynomial. □

For example, when $\overline{A}=(\alpha,d)$, we get

$$\mathrm{Sres}_{d-1}((x-\alpha)^d,\overline{g})=\sum_{j=0}^{d-1}\frac{\overline{g}^{(j)}(\alpha)}{j!}\,(x-\alpha)^j,$$

the Taylor expansion of $\overline{g}$ up to order $d-1$.

• The case $t=1<d$: We keep Notation 2.2. When f has simple roots, it is known (or can easily be derived for instance from Sylvester's Identity (2) for $p=1$ and $q=0$) that

$$\begin{aligned}\mathrm{Sres}_1(f,\overline{g}) &= (-1)^{d-1}\sum_{i=1}^{d}\Big(\prod_{j\neq i}\frac{\overline{g}(\alpha_j)}{\alpha_i-\alpha_j}\Big)(x-\alpha_i)\\ &= (-1)^{d-1}\sum_{i=1}^{d}\frac{\prod_{j\neq i}\overline{g}(\alpha_j)}{f_i(\alpha_i)}\,(x-\alpha_i).\end{aligned}\tag{7}$$

The general situation is a bit less obvious, but in any case we can get an expression of $\mathrm{Sres}_1(\overline{f},\overline{g})$ by using the coefficients of the Hermite interpolation polynomial, in this case of the whole data

$$\overline{A}\cup\overline{B}:=\big((\alpha_1,d_1);\ldots;(\alpha_m,d_m);(\beta_1,e_1);\ldots;(\beta_n,e_n)\big).$$

We note that

$$\det\big(V(\overline{A}\cup\overline{B})\big)=\det\big(V(\overline{A})\big)\det\big(V(\overline{B})\big)R(\overline{B},\overline{A})$$

which holds even when $\alpha_i=\beta_j$ for some i,j.

Theorem 2.8.

$$\begin{aligned}\mathrm{Sres}_1(\overline{f},\overline{g}) = \sum_{i=1}^{m}(-1)^{d-d_i}\Big(\frac{\prod_{j\neq i}\overline{g}(\alpha_j)^{d_j}}{\overline{f}_i(\alpha_i)}\Big)\overline{g}(\alpha_i)^{d_i-1}\Big((x-\alpha_i)\cdot \\ \sum_{\substack{k_1+\cdots+\widehat{k}_i+\cdots\\ \cdots+k_{m+n}=d_i-1}}\ \prod_{j\neq i}\frac{\binom{d_j-1+k_j}{k_j}}{(\alpha_i-\alpha_j)^{k_j}}\prod_{\ell=1}^{n}\frac{\binom{e_\ell-1+k_{m+\ell}}{k_{m+\ell}}}{(\alpha_i-\beta_\ell)^{k_{m+\ell}}}\\ +\min\{1,d_i-1\}\sum_{\substack{k_1+\cdots+\widehat{k}_i+\cdots\\ \cdots+k_{m+n}=d_i-2}}\ \prod_{j\neq i}\frac{\binom{d_j-1+k_j}{k_j}}{(\alpha_i-\alpha_j)^{k_j}}\prod_{\ell=1}^{n}\frac{\binom{e_\ell-1+k_{m+\ell}}{k_{m+\ell}}}{(\alpha_i-\beta_\ell)^{k_{m+\ell}}}\Big).\end{aligned}$$

Proof. Setting $t=1$ in Expression (6) we get

$$\det\big(V(\overline{A})\big)\det\big(V(\overline{B})\big)\,\mathrm{Sres}_1(\overline{f},\overline{g}) = (-1)^{(d-1)e}\det\begin{array}{|c|c|}\multicolumn{1}{c}{d} & \multicolumn{1}{c}{e}\\ \hline W_{x-z,1}(\overline{A}) & \mathbf{0}\\ \hline V_{d+e-1}(\overline{A}) & V_{d+e-1}(\overline{B})\\ \hline\end{array}\begin{array}{c}\\ 1\\ d+e-1\end{array}$$

where

$$W_{x-z,1}=(\underbrace{x-\alpha_1,-1,0,\ldots,0}_{d_1},\ldots,\underbrace{x-\alpha_m,-1,0\ldots,0}_{d_m}).$$

We expand the determinant w.r.t. the first row, and observe that when we delete the first row and column j, the matrix that survives coincides with $V(\overline{A}\cup\overline{B})_{(d+e,j)}$, the submatrix of $V(\overline{A}\cup\overline{B})$ obtained by deleting the last row and column j. Therefore,

$$\det\begin{array}{|c|c|}\hline W_{x-z,1}(\overline{A}) & \mathbf{0}\\ \hline V_{d+e-1}(\overline{A}) & V_{d+e-1}(\overline{B})\\ \hline\end{array}$$
$$=\sum_{j=1}^{m}(-1)^{\phi(j)-1}\Big(\det\big(V(\overline{A}\cup\overline{B})|_{(d+e,\phi(j))}\big)(x-\alpha_j)+\det\big(V(\overline{A}\cup\overline{B})|_{(d+e,\phi'(j))}\big)\Big),$$

where $\phi(i)$ equals the number of the column corresponding to $(1,\alpha_i,\ldots,\alpha_i^{d+e-1})$ in $V(\overline{A}\cup\overline{B})$, and $\phi'(i)=\phi(i)+1$ if $d_i>1$ and 0 otherwise.
Now, from

$$\det\big(V(\overline{A}\cup\overline{B})|_{(d+e,\phi(j))}\big)=(-1)^{d+e-\phi(j)}\det\big(V(\overline{A}\cup\overline{B})\big)\,V(\overline{A}\cup\overline{B})^{-1}_{\phi(j),d+e},$$
$$\det\big(V(\overline{A}\cup\overline{B})|_{(d+e,\phi'(j))}\big)=(-1)^{d+e-\phi'(j)}\det\big(V(\overline{A}\cup\overline{B})\big)\,V(\overline{A}\cup\overline{B})^{-1}_{\phi'(j),d+e}$$

(by the cofactor expression for the inverse) and

$$\det\big(V(\overline{A}\cup\overline{B})\big)=(-1)^{de}\det\big(V(\overline{A})\big)\det\big(V(\overline{B})\big)R(\overline{A},\overline{B})$$

we first get, since $R(\overline{A},\overline{B})=\prod_{1\le i\le m}\overline{g}(\alpha_i)^{d_i}$, that

$$\mathrm{Sres}_1(\overline{f},\overline{g})=(-1)^{d-1}\Big(\prod_{1\le i\le m}\overline{g}(\alpha_i)^{d_i}\Big)\left(\sum_{i=1}^{m}V(\overline{A},\overline{B})^{-1}_{\phi(i),d+e}(x-\alpha_i)-\sum_{i=1}^{m}V(\overline{A},\overline{B})^{-1}_{\phi'(i),d+e}\right).$$

We set $\overline{h}:=\overline{f}\,\overline{g}$, and for $i=1,\ldots,m$, $\overline{h}_i:=\overline{h}/(x-\alpha_i)^{d_i}$. In [Cs1975, Id. 9], it is shown that

$$V(\overline{A},\overline{B})^{-1}_{\phi(i),d+e}=\frac{1}{(d_i-1)!}\left(\frac{1}{\overline{h}_i}\right)^{(d_i-1)}(\alpha_i),$$

and when $d_i>1$,

$$V(\overline{A},\overline{B})^{-1}_{\phi'(i),d+e}=\frac{1}{(d_i-2)!}\left(\frac{1}{\overline{h}_i}\right)^{(d_i-2)}(\alpha_i).$$

Therefore, we obtain the statement by applying Leibniz rule

$$\left(\frac{1}{\overline{h}_i}\right)^{(k)}=(-1)^k\,k!\sum_{k_1+\cdots+\widehat{k_i}+\cdots+k_{m+n}=k}\ \prod_{\substack{1\le j\le m\\ j\ne i}}\frac{\binom{d_j-1+k_j}{k_j}}{(x-\alpha_j)^{d_j+k_j}}\prod_{1\le\ell\le n}\frac{\binom{e_\ell-1+k_{m+\ell}}{k_{m+\ell}}}{(x-\beta_\ell)^{e_\ell+k_{m+\ell}}}.$$

□

Note that in the case that f has simple roots we immediately recover Identity (7) while when $\overline{f}=(x-\alpha)^d$ for $d\ge2$, we recover Proposition 3.2 of [DKS2009]:

$$\begin{aligned}\mathrm{Sres}_1((x-\alpha)^d,\overline{g}) &= \overline{g}(\alpha)^{d-1}\sum_{k_1+\cdots+k_n=d-1}\Big(\prod_{\ell=1}^{n}\frac{\binom{e_\ell-1+k_\ell}{k_\ell}}{(\alpha-\beta_\ell)^{k_\ell}}\Big)(x-\alpha)+\\ &\qquad\sum_{k_1+\cdots+k_s=d-2}\ \prod_{\ell=1}^{n}\frac{\binom{e_\ell-1+k_\ell}{k_\ell}}{(\alpha-\beta_\ell)^{k_\ell}}.\end{aligned}$$

3. Multivariate Case: Poisson-like formulas for Subresultants

We turn to the multivariate case, considering the definition of subresultants introduced in [Cha1995]. Our goal is to generalize Theorem 3.2 in [DKS2006] to the non-generic case when the polynomials have multiple roots. We briefly recall here this statement, and refer the reader to [DKS2006] for more background on the topic.

3.1. **Notation.** Fix $n \in \mathbb{N}$ and set $D_i \in \mathbb{N}$ for $1 \le i \le n+1$. Let

$$f_i := \sum_{|\boldsymbol{\alpha}| \le D_i} a_{i,\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \ \in \ K[x_1, \ldots, x_n],$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\ge 0})^n$, $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, and K is a field of characteristic zero, that we can assume without loss of generality to be algebraically closed.
Fix $t \in \mathbb{N}$. Let $k := \mathcal{H}_{D_1 \ldots D_{n+1}}(t)$ be the Hilbert function at t of a regular sequence of $n+1$ homogeneous polynomials in $n+1$ variables of degrees $D_1, \ldots, D_{n+1}$, i.e.

$$k = \#\{\boldsymbol{x}^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \le t, \, \alpha_i < D_i, \, 1 \le i \le n, \text{ and } t - |\boldsymbol{\alpha}| < D_{n+1}\}.$$

We set

$$\mathcal{S} := \{\boldsymbol{x}^{\boldsymbol{\gamma}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{\gamma}_k}\} \subset K[\boldsymbol{x}]_t$$

a set of k monomials of degree bounded by t, and

$$\Delta_{\mathcal{S}}(f_1, \ldots, f_{n+1}) := \Delta^{(t)}_{\mathcal{S}^h}(f_1^h, \ldots, f_{n+1}^h),$$

the *order t subresultant of $f_1^h, \ldots, f_{n+1}^h$ with respect to*

$$\mathcal{S}^h := \{\boldsymbol{x}^{\boldsymbol{\gamma}_1} x_{n+1}^{t-|\boldsymbol{\gamma}_1|}, \ldots, \boldsymbol{x}^{\boldsymbol{\gamma}_k} x_{n+1}^{t-|\boldsymbol{\gamma}_k|}\}.$$

defined in [Cha1995], see also [DKS2006]. Here, f_i^h denotes the homogenization of f_i by the variable x_{n+1}.
We recall that the subresultant $\Delta_{\mathcal{S}}$ is a polynomial in the coefficients of the f_i^h of degree $\mathcal{H}_{D_1 \ldots D_{i-1} D_{i+1} \ldots D_{n+1}}(t - D_i)$ for $1 \le i \le n+1$, having the following universal property: $\Delta_{\mathcal{S}}$ vanishes at a particular coefficient specialization of $f_1^h, ..., f_{n+1}^h \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ if and only if $I_t \cup \mathcal{S}^h$ does not generate the space of all forms of degree t. Here, I_t is the degree t part of the ideal generated by the specialized f_i^h's.
We set $\rho := (D_1 - 1) + \cdots + (D_n - 1)$ and for $j \ge 0$, $\tau_j := \mathcal{H}_{D_1 \ldots D_n}(j)$, the Hilbert function at j of a regular sequence of n homogeneous polynomials in n variables of degrees $D_1, \ldots, D_n$. Also

$$(8) \quad \mathcal{T}_j := \begin{cases} \textit{any} \text{ set of } \tau_j \text{ monomials of degree } j & \text{if } j \ge \max\{0, t - D_{n+1} + 1\}, \\ \{\boldsymbol{x}^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| = j, \alpha_i < D_i \text{ for } 1 \le i \le n\} & \text{if } 0 \le j < t - D_{n+1} + 1. \end{cases}$$

Set $D := D_1 \cdots D_n$ for the *Bézout number*, the number of common solutions of $f_1, \ldots, f_n$ in $\overline{K}^n$.
We denote $\mathcal{T} := \cup_{j \ge 0} \mathcal{T}_j$ and $\mathcal{T}^* := \cup_{j=t+1}^{\rho} \mathcal{T}_j$, and we note that $|\mathcal{T}| = D$. Let $\mathcal{T} := \{\boldsymbol{x}^{\boldsymbol{\alpha}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{\alpha}_D}\}$ and assume that for $s := |\mathcal{T}^*|$ we have $\mathcal{T}^* = \{\boldsymbol{x}^{\boldsymbol{\alpha}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{\alpha}_s}\}$. Also set
(9)

$$\mathcal{R} := \{\boldsymbol{x}^{\boldsymbol{\beta}_1}, \ldots, \boldsymbol{x}^{\boldsymbol{\beta}_r}\} = \{\boldsymbol{x}^{\boldsymbol{\alpha}} : |\boldsymbol{\alpha}| \le t, \alpha_i < D_i, 1 \le i \le n, \ t - |\boldsymbol{\alpha}| \ge D_{n+1}\}.$$

Finally, for $1 \le i \le n$, let $\widetilde{f}_i$ be the homogeneous component of degree D_i of f_i, and $\widetilde{\Delta}_{\mathcal{T}_j} := \Delta^{(j)}_{\mathcal{T}_j}(\widetilde{f}_1, \dots, \widetilde{f}_n)$ be the order j subresultant of $\widetilde{f}_1, \dots, \widetilde{f}_n$ with respect to $\mathcal{T}_j$.

3.2. **Poisson-like formula for subresultants: the generic case.**
Let $Z := \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_D\}$ be the set of all common roots of $f_1, \dots, f_n$ in K^n *which we assume for now are all simple.* We denote by

$$V_{\mathcal{T}}(Z) := \begin{array}{|ccc|} \boldsymbol{\xi}_1^{\boldsymbol{\alpha}_1} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\alpha}_1} \\ \vdots & & \vdots \\ \boldsymbol{\xi}_1^{\boldsymbol{\alpha}_D} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\alpha}_D} \end{array} \in K^{D\times D} \tag{10}$$

the Vandermonde matrix associated to $\mathcal{T}$. In [DKS2006] we defined

$$\mathcal{O}_{\mathcal{S}}(Z) := \overbrace{\begin{array}{|ccc|} \boldsymbol{\xi}_1^{\boldsymbol{\gamma}_1} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\gamma}_1} \\ \vdots & & \vdots \\ \boldsymbol{\xi}_1^{\boldsymbol{\gamma}_k} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\gamma}_k} \\ \hline \boldsymbol{\xi}_1^{\boldsymbol{\alpha}_1} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\alpha}_1} \\ \vdots & & \vdots \\ \boldsymbol{\xi}_1^{\boldsymbol{\alpha}_s} & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\alpha}_s} \\ \hline \boldsymbol{\xi}_1^{\boldsymbol{\beta}_1} f_{n+1}(\boldsymbol{\xi}_1) & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\beta}_1} f_{n+1}(\boldsymbol{\xi}_D) \\ \vdots & & \vdots \\ \boldsymbol{\xi}_1^{\boldsymbol{\beta}_r} f_{n+1}(\boldsymbol{\xi}_1) & \cdots & \boldsymbol{\xi}_D^{\boldsymbol{\beta}_r} f_{n+1}(\boldsymbol{\xi}_D) \end{array}}^{D} \begin{array}{l} \\ k \\ \\ \\ s \\ \\ \\ r \\ \\ \end{array} \in K^{D\times D}, \tag{11}$$

and we proved the following result:

Theorem 3.1. [DKS2006, Th. 3.2] *For any $t \in \mathbb{Z}_{\ge 0}$ and for any $\mathcal{S} = \{\boldsymbol{x}^{\boldsymbol{\gamma}_1}, \dots, \boldsymbol{x}^{\boldsymbol{\gamma}_k}\} \subset K[\boldsymbol{x}]_t$ of cardinality $k = \mathcal{H}_{D_1 \dots D_{n+1}}(t)$, the order t subresultant $\Delta_{\mathcal{S}}(f_1, \dots, f_{n+1})$ satisfies:*

$$\Delta_{\mathcal{S}}(f_1, \dots, f_{n+1}) = \pm \left(\prod_{j=t-D_{n+1}+1}^{t} \widetilde{\Delta}_{\mathcal{T}_j} \right) \frac{\det \mathcal{O}_{\mathcal{S}}(Z)}{\det V_{\mathcal{T}}(Z)}. \tag{12}$$

3.3. **Poisson-like formula for subresultants: the non-generic case.** For multivariate systems with multiple roots the coordinates of the roots and their multiplicities do not uniquely determine the defining ideal. Thus, in order to obtain an expression for the subresultant in terms of the roots of the first n polynomials $f_1, \dots, f_n$ as in Theorem 3.1, we need to introduce notions of the multiplicity structure of the roots that are sufficient to define $(f_1, \dots, f_n)$.

To be more precise, let $I = (f_1, \dots, f_n)$ be the zero-dimensional ideal generated by $f_1, \dots, f_n$ in $K[\boldsymbol{x}]$, and denote $A := K[\boldsymbol{x}]/I$. If all the roots $Z = \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_D\}$ of I are simple, then the entries of the matrix $V_{\mathcal{T}}(Z)$ defined in (10) are the evaluation maps $\mathrm{ev}_{\boldsymbol{\xi}_i} : A \to K$ applied to the monomials in $\mathcal{T}$ for $1 \le i \le D$. It is a well-known fact that the set $\{\mathrm{ev}_{\boldsymbol{\xi}_i}\}_{1 \le i \le D}$ is a basis of A^*, the dual of the quotient ring A as a K-vector space, and $\mathcal{T}$ is a basis of A if and only if $\det(V_{\mathcal{T}}(Z)) \neq 0$. In the case of multiple roots, the family of evaluation maps is still linearly independent but does not generate the whole dual space, and hence other forms must be considered in order to describe A^* and to get a non-singular matrix generalizing $V_{\mathcal{T}}(Z)$.

All along this section we will use the language of dual algebras to generalize Theorem 3.1 for the multiple roots case (see for instance in [KK1987, BCRS1996] and the references therein). In Theorem 3.4 below we show that any basis of the dual A^* gives rise to generalizations of Theorem 3.1, as long as we assume that $\mathcal{T}$ is a basis of A. This is the most general setting where a generalization of Theorem 3.1 will hold; however, this version of the Theorem, using general elements of the dual, does not give a formula for the subresultant in terms of the roots.

In order to obtain these expressions, we need to consider a specific basis of A^* which contains the evaluation maps described above. It turns out that one can define a basis for A^* in terms of linear combinations of higher order derivative operators evaluated at roots of I. This is the content of the so called theory of "inverse systems" introduced by Macaulay in [Mac1916], and developed in a context closer to our situation under the name of "Gröbner duality " in [Gr1970, MMM1995, EM2007] among others.

First, we give a generalization of Theorem 3.1 to the multiple roots setting, or of Theorem 2.6 to the multivariate case, by replacing the set $\{\mathrm{ev}_{\boldsymbol{\xi}_i}\}_{1\le i\le D}$ by an arbitrary basis of A^*.

We assume that A has dimension D over K, i.e. there are no roots at infinity and that the monomials in $\mathcal{T}$ are a basis of A.

The following is a multivariate analogue of Definition 2.5:

Definition 3.2. *Let $\boldsymbol{\Lambda} := \{\Lambda_1, \dots, \Lambda_D\}$ be a basis of A^* as a K-vector space. Given any set $E = \{\boldsymbol{x}^{\boldsymbol{\alpha}_1}, \dots, \boldsymbol{x}^{\boldsymbol{\alpha}_u}\}$ of u monomials and given any polynomial $h(\boldsymbol{x})$ the* generalized Vandermonde matrix $V_E(\boldsymbol{\Lambda})$ *and the* generalized Wronskian matrix $W_{h,E}(\boldsymbol{\Lambda})$ *corresponding to E, $\boldsymbol{\Lambda}$ and h are the following $u \times D$ matrices:*

$$V_E(\boldsymbol{\Lambda}) = \overbrace{\begin{pmatrix} \Lambda_1(\boldsymbol{x}^{\boldsymbol{\alpha}_1}) & \cdots & \Lambda_D(\boldsymbol{x}^{\boldsymbol{\alpha}_1}) \\ \vdots & & \vdots \\ \Lambda_1(\boldsymbol{x}^{\boldsymbol{\alpha}_u}) & \cdots & \Lambda_D(\boldsymbol{x}^{\boldsymbol{\alpha}_u}) \end{pmatrix}}^{D} u, \quad W_{h,E}(\boldsymbol{\Lambda}) = \overbrace{\begin{pmatrix} \Lambda_1(\boldsymbol{x}^{\boldsymbol{\alpha}_1}h) & \cdots & \Lambda_D(\boldsymbol{x}^{\boldsymbol{\alpha}_1}h) \\ \vdots & & \vdots \\ \Lambda_1(\boldsymbol{x}^{\boldsymbol{\alpha}_u}h) & \cdots & \Lambda_D(\boldsymbol{x}^{\boldsymbol{\alpha}_u}h) \end{pmatrix}}^{D} u.$$

We modify the definition of the matrix $\mathcal{O}_{\mathcal{S}}(Z)$ in (11) as follows:

Definition 3.3. *Let $\mathcal{S} = \{\boldsymbol{x}^{\gamma_1}, \dots, \boldsymbol{x}^{\gamma_k}\} \subset K[\boldsymbol{x}]_t$ be of cardinality $k = \mathcal{H}_{D_1\dots D_{n+1}}(t)$, $\mathcal{T}^* := \cup_{j=t+1}^{\rho}\mathcal{T}_j$ as in (8) and $\mathcal{R}$ as in (9). Then*

$$\mathcal{O}_{\mathcal{S}}(\boldsymbol{\Lambda}) := \overbrace{\begin{pmatrix} V_{\mathcal{S}}(\boldsymbol{\Lambda}) \\ \hline V_{\mathcal{T}^*}(\boldsymbol{\Lambda}) \\ \hline W_{f_{n+1},\mathcal{R}}(\boldsymbol{\Lambda}) \end{pmatrix}}^{D} \begin{matrix} k \\ s \\ r \end{matrix} \quad \in \; K^{D\times D}.$$

Note that by our assumption on $\mathcal{T}$ being a basis of A and $\boldsymbol{\Lambda}$ being a basis of A^*, we have $\det(V_{\mathcal{T}}(\boldsymbol{\Lambda})) \neq 0$. The following is the extension of Theorem 3.1 to the multiple roots case.

Theorem 3.4. *Let $(f_1, \dots, f_n) \subset K[\boldsymbol{x}]$ be a zero-dimensional ideal of dimension D, $\mathcal{T}$ be a basis of $K[\boldsymbol{x}]/(f_1, \dots, f_n)$ and $\boldsymbol{\Lambda}$ be a basis of its dual as* *a K-vector space. For any $t \in \mathbb{Z}_{\geq 0}$ and for any $\mathcal{S} = \{\boldsymbol{x}^{\gamma_1}, \ldots, \boldsymbol{x}^{\gamma_k}\} \subset K[\boldsymbol{x}]_t$ of cardinality $k = \mathcal{H}_{D_1 \ldots D_{n+1}}(t)$, the order t subresultant $\Delta_{\mathcal{S}}(f_1, \ldots, f_{n+1})$ satisfies*

$$\Delta_{\mathcal{S}}(f_1, \ldots, f_{n+1}) = \pm \left(\prod_{j=t-D_{n+1}+1}^{t} \widetilde{\Delta}_{\mathcal{T}_j} \right) \frac{\det \mathcal{O}_{\mathcal{S}}(\boldsymbol{\Lambda})}{\det V_{\mathcal{T}}(\boldsymbol{\Lambda})}. \tag{13}$$

Proof of Theorem 3.4. The proof is similar to the proof of Theorem 3.2 in [DKS2006], to which we refer for notations and details. Extra care must be taken however, as we are not considering the f_i's to be generic anymore in this case.

Let $N^* := \dim(K[\boldsymbol{x}]_t) + \dim(\mathcal{T}^*)$ and $\mathcal{B}$ be a monomial basis for the vector space generated by $K[\boldsymbol{x}]_t + \mathcal{T}^*$. Assume that the columns of the square matrix

$$M_{\mathcal{S}} := \begin{array}{|c|} \hline I_{\mathcal{S} \cup \mathcal{T}^*} \\ \hline M_{f_1} \\ \vdots \\ M_{f_n} \\ \hline M_{f_{n+1}} \\ \hline \end{array} \in K^{N^* \times N^*}$$

defined in [DKS2006] are written in the basis $\mathcal{B}$. Here $I_{\mathcal{S} \cup \mathcal{T}^*}$ is the transpose of the matrix of the immersion of the vector space generated by $\mathcal{S} \cup \mathcal{T}^*$ into the vector space generated by $K[\boldsymbol{x}]_t + \mathcal{T}^*$ in the basis $\mathcal{B}$. We saw in [DKS2006] that

$$\det(M_{\mathcal{S}}) = \pm\, \mathcal{E}(t)\, \Delta_{\mathcal{S}}(f_1, \ldots, f_{n+1}),$$

where $\mathcal{E}(t)$ denotes the extraneous factor (see [DKS2006] for its definition). We define the square matrix

$$V_{N^*} := \overset{\qquad D \qquad\quad N^*-D}{\begin{array}{|c|c|} \hline V_{\mathcal{B}}(\boldsymbol{\Lambda}) & I_{\mathcal{B} \setminus \mathcal{T}} \\ \hline \end{array}}\ {\scriptstyle N^*} \quad \in \ K^{N^* \times N^*}$$

where $I_{\mathcal{B} \setminus \mathcal{T}}$ is the matrix of the immersion of the vector space generated by $\mathcal{B} \setminus \mathcal{T}$ into the one generated by $\mathcal{B}$, and thus $\det(V_{N^*}) = \pm \det(V_{\mathcal{T}}(\boldsymbol{\Lambda}))$. Also, for the matrix product $M_{\mathcal{S}}\, V_{N^*}$ we get

$$\begin{array}{|c|} \hline I_{\mathcal{S} \cup \mathcal{T}^*} \\ \hline M_{f_1} \\ \vdots \\ M_{f_n} \\ \hline M_{f_{n+1}} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline V_{\mathcal{B}}(\boldsymbol{\Lambda}) & I_{\mathcal{B} \setminus \mathcal{T}} \\ \hline \end{array} = \begin{array}{|c|c|} \hline V_{\mathcal{S}}(\boldsymbol{\Lambda}) & * \\ \hline V_{\mathcal{T}^*}(\boldsymbol{\Lambda}) & * \\ \hline & M'_{f_1} \\ \mathbf{0} & \vdots \\ & M'_{f_n} \\ \hline W_{f_{n+1}, \mathcal{R}}(\boldsymbol{\Lambda}) & * \\ \hline \end{array}. \tag{14}$$

The block of zeroes comes from applying the operators Λ_i to elements of the ideal I (monomial multiples of the f_i's, $1 \leq i \leq n$). Setting

$$M' := \boxed{\begin{matrix} M'_{f_1} \\ \vdots \\ M'_{f_n} \end{matrix}}, \tag{15}$$

we conclude that

$$\pm\,\mathcal{E}(t)\,\Delta_{\mathcal{S}}(f_1,\ldots,f_{n+1})\,\det(V_{\mathcal{T}}(\mathbf{\Lambda})) = \det(M_S\,V_{N^*}) = \pm\det(M')\det(\mathcal{O}_{\mathcal{S}}(\mathbf{\Lambda})).$$

In [DKS2006] we showed that

$$\det(M') = \pm\mathcal{E}(t)\left(\prod_{j=t-D_{n+1}+1}^{t} \widetilde{\Delta}_{\mathcal{T}_j}\right),$$

so if $\mathcal{E}(t) \neq 0$, the claim is proved. If the latter does not happen, we consider a perturbation "à la Canny" as in [Can1992], i.e. we replace f_i by $f_{i,\lambda} := f_i + \lambda\, x_i^{D_i}$ for $1 \leq i \leq n$. Here, λ is a new parameter, and we regard our initial system as having coefficients in $K(\lambda)$. It is easy to see that the dimension of the quotient ring $K(\lambda)[x_1,\ldots,x_n]/(f_{1,\lambda},\ldots,f_{n,\lambda})$ is also equal to D, and hence we can regard the family $\{f_{i,\lambda}\}_{1\leq i\leq n}$ as a flat deformation of the input sequence $f_1,\ldots,f_n$.
It can also be shown (see [Can1992]) that $\mathcal{E}_\lambda(t)$, the extraneous factor in Macaulay's formulation applied to the polynomials $f_{i,\lambda}$, $i = 1,\ldots,n$, does not vanish. Indeed, if E_t is the matrix whose determinant gives $\mathcal{E}(t)$ with rows and columns ordered properly, it is easy to see that the perturbed matrix is equal to $E_t + \lambda\,I$, I being the identity matrix. So its determinant is not zero in $K(\lambda)$.
Hence, we are actually in the non-singular setting and (13) reads in the perturbed situation as follows:

$$\Delta_{\mathcal{S}}(f_{1,\lambda},\ldots,f_{n,\lambda},f_{n+1}) = \pm\left(\prod_{j=t-D_{n+1}+1}^{t} \widetilde{\Delta}_{\mathcal{T}_j,\lambda}\right)\frac{\det\mathcal{O}_{\mathcal{S}}(\mathbf{\Lambda}_\lambda)}{\det V_{\mathcal{T}}(\mathbf{\Lambda}_\lambda)}$$

for *any* basis $\mathbf{\Lambda}_\lambda$ of the dual of $A_\lambda := K(\lambda)[x_1,\ldots,x_n]/(f_{1,\lambda},\ldots,f_{n,\lambda})$ as a $K(\lambda)$-vector space. Here, $\widetilde{\Delta}_{\mathcal{T}_j,\lambda} = \Delta_{\mathcal{T}_j}(\widetilde{f}_{1,\lambda},\ldots,\widetilde{f}_{n,\lambda})$. As subresultants are stable by evaluating $\lambda \to 0$, the claim will be proven if we can show that there exists a basis $\mathbf{\Lambda}_\lambda$ of A_λ^* which becomes $\mathbf{\Lambda}$ when setting $\lambda \to 0$. This can be done as follows: recall that $\mathcal{T} = \{\boldsymbol{x}^{\boldsymbol{\alpha}_1},\ldots,\boldsymbol{x}^{\boldsymbol{\alpha}_D}\}$ is a monomial basis of A. By flatness, we have that $\mathcal{T}$ is also a monomial basis of A_λ. Then there exist bases $\{\boldsymbol{y}_{\boldsymbol{\alpha}_1},\ldots,\boldsymbol{y}_{\boldsymbol{\alpha}_D}\}$ and $\{\boldsymbol{y}_{\boldsymbol{\alpha}_1,\lambda},\ldots,\boldsymbol{y}_{\boldsymbol{\alpha}_D,\lambda}\}$ for A^* and A_λ^*, respectively, such that for $1 \leq i,j \leq D$,

$$\boldsymbol{y}_{\boldsymbol{\alpha}_i,\lambda}(\boldsymbol{x}^{\boldsymbol{\alpha}_j}) = \begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \lim_{\lambda\to 0}\boldsymbol{y}_{\boldsymbol{\alpha}_i,\lambda} = \boldsymbol{y}_{\boldsymbol{\alpha}_i}.$$

For each $\Lambda_i \in \mathbf{\Lambda}$ we write $\Lambda_i = \sum_{j=1}^{D} c_{ij}\boldsymbol{y}_{\boldsymbol{\alpha}_j}$, where $c_{ij} \in K$, and set $\Lambda_{\lambda,i} := \sum_{j=1}^{D} c_{ij}\boldsymbol{y}_{\boldsymbol{\alpha}_j,\lambda}$, $1 \leq i \leq D$. As the matrix $(c_{ij})_{1\leq i,j\leq D}$ is invertible, we then have that $\mathbf{\Lambda}_\lambda := \{\Lambda_{1,\lambda}, \dots, \Lambda_{D,\lambda}\}$ is a basis of A^*_λ and the fact that $\mathbf{\Lambda}_\lambda \to \mathbf{\Lambda}$ when $\lambda \to 0$ holds straightforwardly.

□

As we mentioned before, for an arbitrary basis $\mathbf{\Lambda}$ of A^* the expression in Theorem 3.4 may not provide a formula in terms of the roots of $f_1, \dots, f_n$. In order to obtain one, we recall here the notion of Gröbner duality from [MMM1995].

For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ define the differential operator

$$\partial_{\boldsymbol{\alpha}} := \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

and consider the ring $K[[\partial]] := \{\sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} a_{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} : a_{\boldsymbol{\alpha}} \in K\}$.

For $1 \le i \le n$ define the K-linear map

$$\sigma_i : K[[\partial]] \to K[[\partial]];\ \sigma_i(\partial_{\boldsymbol{\alpha}}) = \begin{cases} \partial_{(\alpha_1,\dots,\alpha_i-1,\dots,\alpha_n)} & if\ \alpha_i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ define $\sigma_{\boldsymbol{\beta}} = \sigma_1^{\beta_1} \circ \cdots \circ \sigma_n^{\beta_n}$.

A K-vector space $V \subset K[[\partial]]$ is *closed* if $\dim_K(V)$ is finite and for all $\boldsymbol{\beta} \in \mathbb{N}^n$ and $\mathbf{D} \in V$ we have $\sigma_{\boldsymbol{\beta}}(\mathbf{D}) \in V$. Note that $K[[\partial]]$ and its closed subspaces have a natural $K[\boldsymbol{x}]$-module structure given by $\boldsymbol{x}^{\boldsymbol{\beta}}\mathbf{D}(f) := \mathbf{D}(\boldsymbol{x}^{\boldsymbol{\beta}} f) = \sigma_{\boldsymbol{\beta}}(\mathbf{D})(f)$.

Let $\boldsymbol{\xi} \in K^n$. For a closed subspace $V \subset K[[\partial]]$ define

$$\nabla_{\boldsymbol{\xi}}(V) := \{f \in K[\boldsymbol{x}] \ :\ \mathbf{D}(f)(\boldsymbol{\xi}) = 0,\ \ \forall\, \mathbf{D} \in V\} \ \subset\ K[\boldsymbol{x}].$$

Let $\mathbf{m}_{\boldsymbol{\xi}} \subset K[\boldsymbol{x}]$ be the maximal ideal defining $\boldsymbol{\xi}$. For an ideal $J \subset \mathbf{m}_\xi$ define

$$\Delta_{\boldsymbol{\xi}}(J) := \{\mathbf{D} \in K[[\partial]] \ :\ \mathbf{D}(f)(\boldsymbol{\xi}) = 0,\ \ \forall\, f \in J\} \ \subset\ K[[\partial]].$$

Then the following theorem gives the so called Gröbner duality:

Theorem 3.5 ([Gr1970, MMM1995]). *Fix $\boldsymbol{\xi} \in K^n$. The correspondences between closed subspaces $V \subset K[[\partial]]$ and $\mathbf{m}_\xi$-primary ideals Q, $V \mapsto \nabla_{\boldsymbol{\xi}}(V)$ and $Q \mapsto \Delta_{\boldsymbol{\xi}}(Q)$ are 1-1 and satisfy $V = \Delta_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{\xi}}(V))$ and $Q = (\nabla_{\boldsymbol{\xi}}(\Delta_{\boldsymbol{\xi}}(Q))$. Moreover,*

$$\dim_K(\Delta_{\boldsymbol{\xi}}(Q)) = \mathrm{mult}(Q) \quad and \quad \mathrm{mult}(\nabla_{\boldsymbol{\xi}}(V)) = \dim_K(V).$$

We set $Z := \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_D\}$ for the set of all common roots of $f_1, \dots, f_n$ in K^n and $\mathbf{m}_{\boldsymbol{\xi}_i} \subset K[\boldsymbol{x}]$ for the maximal ideal corresponding to $\boldsymbol{\xi}_i$ for $1 \le i \le m$.

Example 3.6. ([EM2007, Exemple 7.37])

Let $f_1 = 2x_1x_2^2 + 5x_1^4$, $f_2 = 2x_1^2x_2 + 5x_2^4 \ \in \mathbb{C}[x, y]$.

Then $Z = \{\mathbf{0}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4, \boldsymbol{\xi}_5\}$ where $\mathbf{0} = (0, 0)$ has multiplicity eleven and $\boldsymbol{\xi}_i = (\frac{-2}{5\xi^{2i}}, \frac{-2}{5\xi^{3i}})$ where ξ is a primitive 5-th root of unity, are all simple, $1 \le i \le 5$.

Denote by $Q_{\mathbf{0}}$, $Q_{\boldsymbol{\xi}_i}$, $1 \le i \le 5$, the primary ideals corresponding to the roots, then $\Delta_{\boldsymbol{\xi}_i}(Q_{\boldsymbol{\xi}_i}) = \langle 1 \rangle$ for $1 \le i \le 5$ and if $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the canonical basis of $\mathbb{Z}^2$,

$$\begin{aligned}\Delta_{\mathbf{0}}(Q_{\mathbf{0}}) =& \langle 1, \partial_{\boldsymbol{e}_1}, \partial_{\boldsymbol{e}_2}, \partial_{2\boldsymbol{e}_1}, \partial_{\boldsymbol{e}_1+\boldsymbol{e}_2}, \partial_{2\boldsymbol{e}_2}, \partial_{3\boldsymbol{e}_1}, \partial_{3\boldsymbol{e}_2}, \\ & (4\partial_{4\boldsymbol{e}_1} - 5\partial_{\boldsymbol{e}_1+2\boldsymbol{e}_2}), (4\partial_{4\boldsymbol{e}_2} - 5\partial_{2\boldsymbol{e}_1+\boldsymbol{e}_2}), (3\partial_{2\boldsymbol{e}_1+3\boldsymbol{e}_2} - \partial_{5\boldsymbol{e}_1} - \partial_{5\boldsymbol{e}_2})\rangle.\end{aligned}$$

Using Gröbner duality, we are now able to give an expression for the subresultant in terms of the roots of $f_1, \dots, f_n$. For $\mathbf{D} \in K[[\partial]]$and $\boldsymbol{\xi} \in K^n$, we denote by $\mathbf{D}|_{\boldsymbol{\xi}}$ the element of A^* defined as $\mathbf{D}|_{\boldsymbol{\xi}}(f) = \mathbf{D}(f)(\boldsymbol{\xi})$. In particular, under this notation, $1|_{\boldsymbol{\xi}} = \mathrm{ev}_{\boldsymbol{\xi}}$.

Corollary 3.7. *Using our previous assumptions, let $I = (f_1, \dots, f_n)$ and*

$$I = Q_1 \cap \cdots \cap Q_m$$

be the primary decomposition of I, where Q_i is a $\mathbf{m}_{\boldsymbol{\xi}_i}$-primary ideal with $d_i := \mathrm{mult}(Q_i)$. For $1 \leq i \leq m$ let $V_i := \Delta_{\boldsymbol{\xi}_i}(Q_i) \subset K[[\partial]]$ be the corresponding closed subspace, and fix a basis $\{\mathbf{D}_{i,1}, \dots, \mathbf{D}_{i,d_i}\}$ for V_i such that $\mathbf{D}_{i,1} = 1$. Then

$$\boldsymbol{\Lambda} := \{\mathbf{D}_{1,1}|_{\boldsymbol{\xi}_1}, \dots, \mathbf{D}_{1,d_1}|_{\boldsymbol{\xi}_1}, \dots, \mathbf{D}_{m,1}|_{\boldsymbol{\xi}_m}, \dots, \mathbf{D}_{m,d_m}|_{\boldsymbol{\xi}_m}\}$$

is a basis of A^* over K.

Note that the above choice for the dual basis $\boldsymbol{\Lambda}$ contains the evaluation maps for the roots of I, and using this $\boldsymbol{\Lambda}$ in Theorem 3.4 gives an expression for the subresultant in terms of the roots of I.

Example 3.8. This is a very simple example containing an expression for a subresultant in terms of the roots.
Let $f_1 := x_1x_2,\ f_2 := x_1^2 + (x_2 - 1)^2 - 1,\ f_3 := c_0 + c_1x_1 + c_2x_2$, with $c_0, c_1, c_2 \in \mathbb{C}$. Then $Z = \{(0,0), (0,2)\}$ where $(0,0)$ has multiplicity 3 and $(0,2)$ is simple. By computing explicitly, we check that $\mathcal{T} := \{1, x_1, x_2, x_2^2\}$ is a basis of $A = \mathbb{C}[x_1, x_2]/(f_1, f_2)$ and that

$$\boldsymbol{\Lambda} := \left\{1|_{(0,0)}, \partial_{\boldsymbol{e}_1}|_{(0,0)}, (\partial_{\boldsymbol{e}_2} + 2\partial_{2\boldsymbol{e}_1})|_{(0,0)}, 1|_{(0,2)}\right\}$$

is a basis of A^*. We will use these bases to express the degree $t = \rho = 2$ subresultant $\Delta_{x_2^2}(f_1, f_2, f_3)$ with $\mathcal{S} = \{x_1^2\}$ in terms of the roots of f_1, f_2. First, $\Delta_{x_1^2}(f_1, f_2, f_3)$ is equal to the following 6×6 determinant (since here the extraneous factor is 1):

$$\Delta_{x_1^2}(f_1, f_2, f_3) = \det M_{\mathcal{S}} = \det \begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c_0 & c_1 & c_2 & 0 & 0 & 0 \\ 0 & c_0 & 0 & c_1 & c_2 & 0 \\ 0 & 0 & c_0 & 0 & c_1 & c_2 \end{vmatrix} = c_0^3 + 2c_0^2c_2.$$

On the other hand, Theorem 3.4 gives the following expression:

$$\left(\prod_{j=2}^{2} \widetilde{\Delta}_{\mathcal{T}_j}\right) \frac{\det \mathcal{O}_{\mathcal{S}}(\boldsymbol{\Lambda})}{\det V_{\mathcal{T}}(\boldsymbol{\Lambda})} = \frac{\det \begin{vmatrix} 0 & 0 & 2 & 0 \\ c_0 & c_1 & c_2 & c_0 + 2c_2 \\ 0 & c_0 & 2c_1 & 0 \\ 0 & 0 & c_0 & 2c_0 + 4c_2 \end{vmatrix}}{\det \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{vmatrix}} = \frac{4(c_0^3 + 2c_0^2c_2)}{4},$$

using that $\mathcal{T}_2 = \{x_2^2\}$ is the degree 2 part of $\mathcal{T}$, and

$$\widetilde{\Delta}_{\mathcal{T}_2}(\widetilde{f}_1, \widetilde{f}_2) = \det \begin{array}{|ccc|} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} = 1.$$

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