OPTIMAL SETS FOR A CLASS OF MINIMIZATION PROBLEMS WITH CONVEX CONSTRAINTS

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ABSTRACT. We look for the minimizers of the functional $J_\lambda(\Omega) = \lambda|\Omega| - P(\Omega)$ among planar convex domains constrained to lie into a given ring. We prove that, according to the values of the parameter λ, the solutions are either a disc or a polygon. In this last case, we describe completely the polygonal solutions by reducing the problem to a finite dimensional optimization problem. We recover classical inequalities for convex sets involving area, perimeter and inradius or circumradius and find a new one.

1. INTRODUCTION

Shape optimization problems for geometric functionals as the volume and the perimeter have always aroused a large interest; the most famous examples are inequalities of the isoperimetric type. In particular in the classical isoperimetric inequality one looks for a set minimizing the perimeter among all the sets of fixed area or, equivalently, for a set maximizing the area among all the sets of fixed perimeter. On the other hand one can consider reverse isoperimetric type inequalities. Of course, this makes sense only working with supplementary constraints like convexity or involving inradius and/or circumradius in order to avoid degenerate solutions. Namely one can maximize the perimeter among convex sets with fixed volume contained in some given ball or, analogously, minimize the volume among sets of fixed perimeter which contain a given ball. The analysis of such classical problems naturally leads to the study of critical points of functionals of the type

$$J_\lambda(\Omega) = \lambda|\Omega| - P(\Omega), \tag{1.1}$$

where $|\cdot|$ is the area, $P(\cdot)$ is the perimeter and λ stands for some Lagrange multiplier.

Another motivation is to get geometric inequalities for convex sets like in [5] or [8] (see [10] for a good overview of such inequalities). In particular in [5] J. Favard investigated some functionals of the area and the perimeter which are homogeneous in P and $|\cdot|^{1/2}$; in particular he studied the maximum for the functional $P(\Omega)/\sqrt{|\Omega|}$ among convex sets contained in an annular ring and he proved that the optimal set is a polygon which is inscribed in the exterior ball and all of its sides, except at most one, are tangent to the interior disk. The same functional had been investigated by K. Ball in [1] where he presents a reverse isoperimetric inequality in the N-dimensional case substituting the constraints on the inradius and circumradius by considering classes of affine equivalent convex bodies, rather than individual bodies. In particular he proved that for any convex set $K \subseteq \mathbb{R}^N$ there exists an affine image $F(K)$ for which

$$\frac{P(F(K))}{|F(K)|^{\frac{N-1}{N}}},$$

is no larger than the corresponding expression for a regular N-dimensional tetrahedron.

In this paper we choose to consider the following minimization problem for every value of the parameter $\lambda \geqslant 0$:

$$\min_{\Omega \in \mathscr{C}_{a,b}} \lambda|\Omega| - P(\Omega), \tag{1.2}$$

where:

$$\mathscr{C}_{a,b} = \{K \subseteq \mathbb{R}^2 \ \ K \text{ convex}, D_a \subseteq K \subseteq D_b\};$$

(here and later D_r is the ball of radius r with center at the origin). Notice that the class $\mathscr{C}_{a,b}$ is compact with respect to the Hausdorff distance, moreover the functional $\lambda|\Omega| - P(\Omega)$ is bounded

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from below by $\lambda|D_a|-P(D_b)$, and continuous thanks to the convexity constraint (see e.g. [6]); hence the minimum in (1.2) is in fact achieved for every value of $\lambda \geqslant 0$. For a more general existence result for minimum problems in the class of convex sets, we refer to [3].

In the paper we present a description of optimal sets to Problem (1.2); more precisely we prove the following result.

Theorem 1.1. *For every $\lambda \geqslant 0$ there exists an optimal set Ω_λ which solves Problem (1.2). In particular*
- *if $0 \leqslant \lambda \leqslant \frac{1}{2b}$ then $\Omega_\lambda = D_b$;*
- *if $\frac{1}{2b} < \lambda < \frac{2}{a}$ then Ω_λ is a polygon;*
- *if $\lambda > \frac{2}{a}$ then $\Omega_\lambda = D_a$.*

The proof of this result can be found in Corollary 2.2 for the case $\frac{1}{2b} < \lambda < \frac{2}{a}$, and in Theorem 2.13 for $\lambda \leqslant \frac{1}{2b}$ or $\lambda \geqslant \frac{2}{a}$. The case of $\lambda = 2/a$ is discussed in details in Remark 2.10. A further description of the optimal polygon(s) is presented in Section 3. Notice that, obviously, the functional is invariant under rotations, thus there is no uniqueness of solution. Nevertheless we will see that, except for a finite number of values for λ, the solution is unique up to rotation.

In order to prove that solutions to Problem (1.2) are either polygons or the given balls D_a or D_b, the idea is to analyse optimality conditions for (1.2) either from a geometric or from an analytic point of view. In particular the notion of *support function* of the set K will be useful: $h = h_K$ is the function $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$h_K(u) \quad = \quad \sup_{x\in K} < x; u > \qquad \text{for every } u \in \mathbb{R}^2.$$

We consider the functional J_λ defined in (2.3), on the class of convex subsets of $\mathbb{R}^2$; hence Problem (1.2) can be rewritten as

$$\min_{\Omega\in\mathscr{C}_{a,b}} J_\lambda(\Omega).$$

Moreover, the functional J_λ can be rewritten in terms of its support function as follows:

$$J_\lambda(\Omega) = \frac{\lambda}{2}\int_0^{2\pi} (h^2 - h'^2)\, d\theta - \int_0^{2\pi} h\, d\theta.$$

Recalling that the convexity of a set K can be expressed in terms of its support function as $h_K''+h_K \geqslant 0$, the class $\mathscr{C}_{a,b}$ is reduced to

$$\mathscr{C}_{a,b} = \{K \subseteq \mathbb{R}^2 \ : \ a \leqslant h_K \leqslant b,\ h_K'' + h_K \geqslant 0 \text{ for every } \theta \in [0, 2\pi]\}.$$

A fundamental preliminary result is expressed in theorem below, which is due to J. Lamboley and A. Novruzi (see [9, Theorem 2.1]). They considered generic functionals of the form

$$\int_0^{2\pi} G(\theta, u(\theta), u'(\theta))d\theta,$$

where u stands either for the support function or the gauge function of a planar convex domain, and they proved that, under a concavity property of $G(\theta, u, p)$ solutions to the associated minimum problem are (locally) polygons. Applying their result to the formulation of J_λ in terms of support function, we get the following.

Theorem 1.2 ([9]). *For every $\lambda \geqslant 0$, if Ω_λ is a solution to (1.2) then Ω_λ is locally a polygon in the interior of the annulus $D_b \setminus D_a$.*

Moreover, using [9, Theorem 2.2], it is possible to get a range of values of λ for which solutions are polygons. However, the application of their result yields a range of value $\frac{1}{b} \leqslant \lambda \leqslant \frac{1}{a}$ while we are able to get the same result for $\frac{1}{2b} < \lambda < \frac{2}{a}$. The reason is the following: we actually consider more general perturbations of a convex set that they did. Namely in the proof of Theorem 2.1 we consider perturbations of a generic set Ω of the form Ω^η, expressed by the support functions as

$$h_{\Omega^\eta}(\theta) = h_\Omega(\theta) + w(\theta, \eta),$$

with

$$w(\theta, \eta) = \Big(h_{T_\eta}(\theta) - h_\Omega(\theta)\Big)\chi_{(0,\eta)}(\theta) \quad \text{or} \quad w(\theta, \eta) = \Big(h_{S_\eta}(\theta) - h_\Omega(\theta)\Big)\chi_{(0,2\eta)}(\theta),$$

where T_η is the triangle of vertices $(0,0), (b,0), (b\cos\eta, b\sin\eta)$ and S_η is the quadrilateral of vertices $(0,0), (a,0), (a, a\tan\eta), (a\cos 2\eta, a\sin 2\eta)$ (see Figure 2 for details). These kind of perturbations are not of the simple type $h_{\Omega^\eta}(\theta) = h_\Omega(\theta) + t\eta(\theta)$ considered in [9].

In Section 3 a detailed characterization of optimal polygons is presented. In particular it is shown that optimal polygons are either inscribed in the exterior ball D_b or circumscribed to the interior ball D_a. This is proved via refinements of a natural geometric argument of "anti-symmetrization". It is in fact evident that an optimal polygon Ω cannot contain two consecutive *free* sides, that is two consecutive sides which are neither a chord of D_b nor tangent to D_a. Otherwise the perturbation in Figure 1 would be possible, in contradiction with the optimality of the set Ω. More precisely, assume there exist two free sides $\overline{AB}, \overline{BC}$; we consider the set Ω_t obtained as a perturbation of the set Ω by moving the vertex B in the direction $\nu = \overrightarrow{AC}$ for a time $t \in \mathbb{R}$ (notice that all the other vertices are fixed). This is a so called *parallel chord movement*, as Ω_t is obtained from Ω by moving

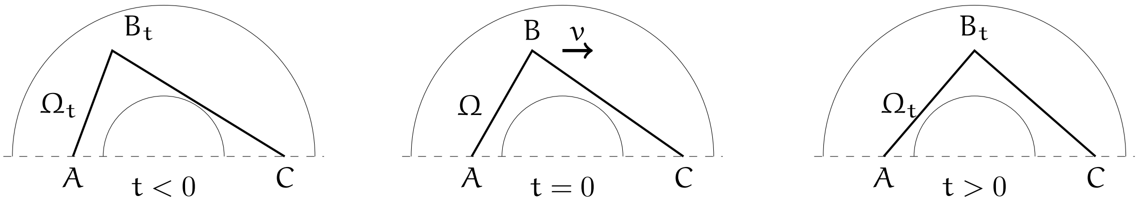


FIGURE 1. A parallel chord movement: optimal sets cannot have "free" sides.

its lines (only those contained into the half plane determined by the line AC and the point B), along the direction ν. For small times the set Ω_t is still a convex set and in particular it still belongs to the class $\mathscr{C}_{a,b}$. Moreover it is clear that $|\Omega_t| = |\Omega|$ for every $t \in \mathbb{R}$ and that there exists $\bar{t}$ such that $P(\Omega_{\bar{t}}) > P(\Omega)$; hence Ω cannot be optimal.

2. MAIN RESULTS

2.1. **First characterizations.**

Theorem 2.1. *Let Ω_λ be a minimizer of (1.2), then for $1/2b < \lambda < 2/a$, $\partial\Omega_\lambda$ does not contain neither arcs of D_a nor arcs of D_b.*

Corollary 2.2. *For every $1/2b < \lambda < 2/a$ minimizers to (1.2) are polygons.*

Proof. By Theorem 1.2 for every value of $\lambda \geqslant 0$ a minimizer can be composed only by segments and arcs of D_a and D_b. We will prove in Corollary 2.12 that the number of segments is necessarily finite. Thus using Theorem 2.1 the thesis follows. □

Proof of Theorem 2.1. We split the proof into two steps.

Step 1: if $\lambda > 1/2b$ then $\partial\Omega_\lambda$ does not contain arcs of ∂D_b.

Let $\Omega \in \mathscr{C}_{a,b}$ and assume that it contains an arc of ∂D_b on its boundary, that is there exists a subinterval of $[0, 2\pi)$ (which for simplicity is assumed to be $(0,\gamma)$ for some $\gamma > 0$), such that

$$\{\theta \in [0, 2\pi) \ : \ h_\Omega(e^{i\theta}) = b\} \supseteq (0, \gamma).$$

Let $\eta \in (0, \gamma/2)$ be such that $\cos\eta \geqslant a/b$ and consider Ω^η obtained from Ω by cutting a part of the arc by a chord of central angle η (see Figure 2 (a)). Notice that, as we choose $\cos\eta \geqslant a/b$, the new set Ω^η still belongs to the class $\mathscr{C}_{a,b}$.

We want to show that $J_\lambda(\Omega) > J_\lambda(\Omega^\eta)$; we get

$$J_\lambda(\Omega) - J_\lambda(\Omega^\eta) = \frac{b}{2}(\eta - \sin\eta\cos\eta)\left(\lambda 2b - 4\frac{\eta - \sin\eta}{\eta - \sin\eta\cos\eta}\right), \tag{2.1}$$

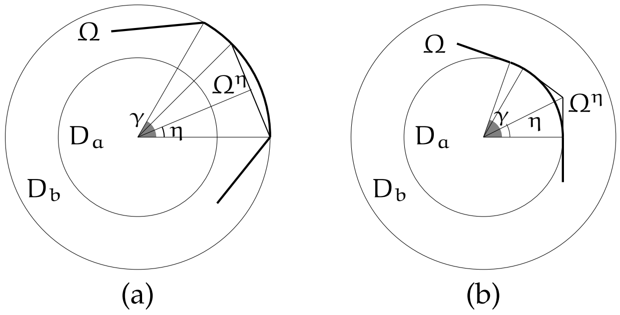


FIGURE 2. The constructions in Step 1 and Step 2 respectively.

for every $\eta \in (0, \gamma/2)$ sufficiently small. As $\lim_{\eta\to 0} 4\frac{\eta-\sin\eta}{\eta-\sin\eta\cos\eta} = 1$ and $\lambda > 1/2b$, for η sufficiently small we get

$$\lambda 2b - 4\frac{\eta-\sin\eta}{\eta-\sin\eta\cos\eta} > 0,$$

which gives the desired result.

Step 2: if $\lambda < 2/a$ then $\partial\Omega_\lambda$ does not contain arcs of ∂D_a.

Consider $\Omega \in \mathscr{C}_{a,b}$ and assume that $\partial\Omega$ contains an arc of ∂D_a, that is there exists an subinterval of $[0, 2\pi)$ (which for simplicity is assumed to be $(0, \gamma)$ for some $\gamma > 0$), such that

$$\{\theta \in [0, 2\pi) \;:\; h_\Omega(e^{i\theta}) = a\} \supseteq (0, \gamma).$$

Let $\eta \in (0, \gamma/2)$ be such that $\cos\eta \geqslant a/b$ and consider Ω^η obtained from Ω by cutting a part of the arc of D_a of width equals to 2η by two tangent lines to D_a, as shown in Figure 2 (b). Notice that, choosing $\eta > 0$ such that $\cos\eta \geqslant a/b$, the set Ω^η still belongs to the class $\mathscr{C}_{a,b}$. Moreover, comparing $J_\lambda(\Omega^\eta)$ and $J_\lambda(\Omega)$ we obtain

$$J_\lambda(\Omega) - J_\lambda(\Omega^\eta) = -a^2(\tan\eta - \eta)(\lambda - \frac{2}{a}),$$

which is positive as $\lambda < 2/a$ and hence $\partial\Omega_\lambda$ cannot contain arcs of D_a for every $\lambda < 2/a$. □

2.2. **Reduction to an optimization problem of finite dimension.** We define three classes of segments which will be useful in what follows. In particular it will turn out that the sides of an optimal polygon necessarily belong to these classes; as already noticed, in fact, free sides are not allowed for an optimal polygon. We here prove that in fact they are necessarily either chord of D_b or tangent side to D_a.

A similar representation for convex sets in terms of their central angles has been used also for other type of functionals in [4].

Definition 2.3. *The class $\mathscr{L}^a$ represents the class of tangent sides to D_a which are not chords of D_b. In particular if P_iP_j and P_jP_k are segments tangent to D_a, with $P_i, P_k \in \partial D_a$, the segments P_iP_j and P_jP_k are identified in the class $\mathscr{L}^a$ as the same element (and hence they are counted only once).*

The class $\mathscr{L}_b$ represents the class of segments which are chords of D_b not tangent to D_a. In particular the elements of $\mathscr{L}_b$ are half chords and each couple of half chords is in fact identified in the same element of $\mathscr{L}_b$. Hence for each chord P_iP_j of D_b if Q_i is its medium point, the segments P_iQ_i and Q_iP_j are identified in class $\mathscr{L}_b$.

The class $\mathscr{L}_b^a$ represents the class of segments which are at the same time tangent to D_a and chords of D_b. In particular a segment P_iP_j belongs to $\mathscr{L}_b^a$ if $P_i \in \partial D_b$ and $P_j \in \partial D_a$. Again we will count these segments in couples (it will be clear later that in fact the number of these segments is always even).

In an analogous way we define the corresponding classes of central angles.

Definition 2.4. *The class $\mathscr{A}^a$ is the class of angles which determine a segment in $\mathscr{L}^a$.*

The class $\mathscr{A}_b$ is the class of angles which determine a segment in $\mathscr{L}_b$.

The class $\mathscr{A}_b^a$ is the class of angles which determine a segment in $\mathscr{L}_b^a$.

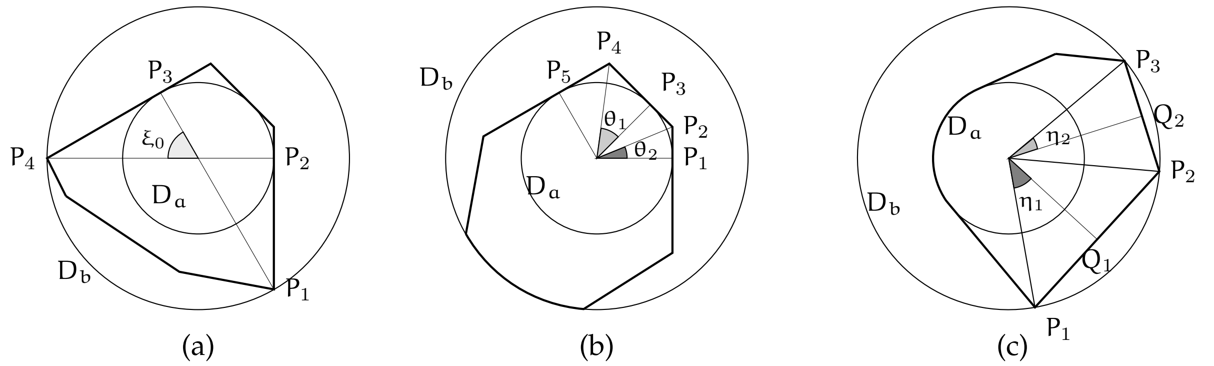


FIGURE 3. The classes of segments $\mathscr{L}_b^a$, $\mathscr{L}^a$, $\mathscr{L}_b$ and the corresponding classes of angles $\mathscr{A}_b^a$, $\mathscr{A}^a$, $\mathscr{A}_b$.

Remark 2.5. *Figure 3, (a), represents elements ξ_0 in $\mathscr{A}_b^a$ and the corresponding segments $P_1P_2 \equiv P_3P_4$ in $\mathscr{L}_b^a$; in particular each couple of segments and angles are identified, so that in the example it holds $|\mathscr{A}_b^a| = |\mathscr{L}_b^a| = 1$.*

Figure 3, (b), represents elements θ_i in the class $\mathscr{A}^a$ and the corresponding segments $P_1P_2 \equiv P_2P_3, P_3P_4 \equiv P_4P_5$ in the class $\mathscr{L}^a$; in the example it holds $|\mathscr{A}^a| = |\mathscr{L}^a| = 2$.

Figure 3, (c), represents elements η_j in the class $\mathscr{A}_b$ and the corresponding segments P_kQ_k in the class $\mathscr{L}_b$; as each couple of segments P_iQ_i, Q_iP_{i+1} is identified, in the example it holds $|\mathscr{A}_b| = |\mathscr{L}_b| = 2$.

Notice that all the segments in the class $\mathscr{L}_b^a$ have the same length equal to $\sqrt{b^2 - a^2}$ and analogously each angle $\xi_0 \in \mathscr{A}_b^a$ has the same value:

$$\sin \xi_0 = \frac{\sqrt{b^2 - a^2}}{b}, \qquad \cos \xi_0 = \frac{a}{b}. \tag{2.2}$$

Moreover for every $L_i \in \mathscr{L}^a$ there exists $\theta_i \in \mathscr{A}^a$ such that $L_i = a \tan \theta_i$ with $\theta_i < \xi_0$, while for $L_j \in \mathscr{L}_b$ there exists $\eta_j \in \mathscr{A}_b$ such that $L_j = b \sin \eta_j$ and $\eta_j < \xi_0$.

By construction it always holds

$$0 < \theta_i, \eta_j < \xi_0 < \frac{\pi}{2},$$

moreover by convexity $\sum_{x \in \mathscr{A}^a \cup \mathscr{A}_b \cup \mathscr{A}_b^a} x \leqslant \pi$ and $\sum_{l \in \mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b} l \leqslant P(\Omega)/2$. More precisely for an optimal polygon Ω, equality holds in the previous expressions, as shown in the following crucial theorem.

Theorem 2.6. *Let Ω_λ be a solution to (1.2) then its boundary can be decomposed into unions of arches of ∂D_a and ∂D_b and segments L_i belonging to $\mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$.*

Thanks to this result an optimal polygon Ω can be characterized by its classes of segments $\mathscr{L}_b^a, \mathscr{L}^a, \mathscr{L}_b$ or, analogously, by its classes of central angles $\mathscr{A}_b^a, \mathscr{A}^a, \mathscr{A}_b$. In particular by construction it turns out that if $\partial\Omega$ is composed only by arcs of D_a and D_b and segments in the classes $\mathscr{L}_b^a, \mathscr{L}^a, \mathscr{L}_b$, then the number of segments which have one vertex on ∂D_b and the other one on ∂D_a (that is the segments which identify the class $\mathscr{L}_b^a$), is even and hence we are allowed to identify segments of the type $\sqrt{b^2 - a^2}$ in couple.

Definition 2.7. *We define the class $\mathscr{K}_{a,b}$ as the class of sets Ω such that $D_a \subseteq \Omega \subseteq D_b$ and $\partial\Omega = \cup_{i \in I} L_i$, with $L_i \in \mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$.*

Hence, for every $\Omega \in \mathscr{K}_{a,b}$, the functional $J_\lambda(\Omega)$ can be expressed as:

$$\begin{aligned} J_\lambda(\Omega) \;=\; & \lambda\Big(\sum_{\xi_0 \in \mathscr{A}_b^a} a^2 \tan \xi_0 + a^2 \sum_{\theta_i \in \mathscr{A}^a} \tan \theta_i + b^2 \sum_{\eta_j \in \mathscr{A}_b} \sin \eta_j \cos \eta_j \Big) \\ & -2\Big(\sum_{\xi_0 \in \mathscr{A}_b^a} a \tan \xi_0 + a \sum_{\theta_i \in \mathscr{A}^a} \tan \theta_i + b \sum_{\eta_j \in \mathscr{A}_b} \sin \eta_j \Big). \end{aligned} \tag{2.3}$$

Notice that $\mathscr{K}_{a,b} \subseteq \mathscr{C}_{a,b}$, that is each Ω in the class $\mathscr{K}_{a,b}$ is a convex polygon. Hence by Corollary 2.2 and Theorem 2.6 it follows

$$\min_{\Omega\in\mathscr{C}_{a,b}} J_\lambda(\Omega) = \min_{\Omega\in\mathscr{K}_{a,b}} J_\lambda(\Omega),$$

for every $1/2b < \lambda < 2/a$. In particular for such values of λ the minimum problem can be expressed as:

$$\min_{\Omega\in\mathscr{K}_{a,b}} J_\lambda(\Omega) =$$

$$\min\left\{ J_\lambda(\Omega) \mid \Omega \in \mathscr{K}_{a,b};\ \sum_{\xi_0\in\mathscr{A}_b^a} \xi_0 + \sum_{\theta_i\in\mathscr{A}^a} \theta_i + \sum_{\eta_j\in\mathscr{A}_b} \eta_j = \pi;\quad 0<\theta_i,\eta_j<\xi_0 \right\}. \tag{2.4}$$

Notice that the classes $\mathscr{A}_b^a, \mathscr{A}^a, \mathscr{A}_b$ do not identify a unique shape of polygon, as shown in Figure 4. However the value of J_λ only depends on the values of the angles and their belonging

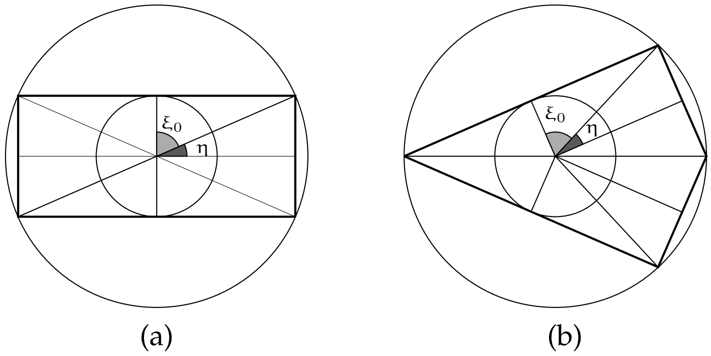


FIGURE 4. Two different polygons corresponding to the same classes of central angles. For them the value of the functional J_λ is the same

to a certain class; indeed these possible different polygons are equivalent for the minimization problem. Hence in what follows we will refer to a certain polygon Ω regarding only its classes of central angles (or equivalently its classes of segments).

Proof of Theorem 2.6. Thanks to Theorem 1.2 it is enough to prove that each segment of $\partial\Omega_\lambda$ belongs to $\mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$. Assume there exists a side PQ which is neither tangent to D_a nor a chord of D_b with $Q \in \text{int}D_b \setminus \overline{D_a}$. We define the point $H \in \partial\Omega_\lambda$ such that $HQ \in \partial\Omega_\lambda$ and $OH \perp HQ$, as shown in Figure 5. Let η be the angle determined by the normal lines to HQ and QP, respectively.

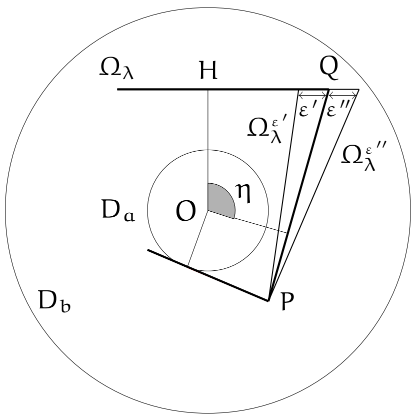


FIGURE 5. Segments of optimal polygons necessarily belong to $\mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$.

We consider $\Omega_\lambda^\varepsilon$ a perturbation of Ω_λ obtained slightly moving the vertex Q in a position Q^ε, which belongs to the same line HQ and which is at distance ε from Q (see Figure 5).

In the case of a perturbation with positive ε, we have

$$J_\lambda(\Omega_\lambda^\varepsilon) - J_\lambda(\Omega_\lambda) = \varepsilon \sin\eta \left(\frac{\lambda}{2}\,\overline{QP} - \frac{1-\cos\eta}{\sin\eta} + \frac{o(\varepsilon)}{\varepsilon}\right),$$

which implies, by the optimality of Ω_λ,

$$\frac{\lambda}{2} \geqslant \frac{\tan\eta/2}{\overline{QP}}. \tag{2.5}$$

In an analogous way, for $\varepsilon < 0$ we get

$$J_\lambda(\Omega_\lambda^\varepsilon) - J_\lambda(\Omega_\lambda) = -\varepsilon \sin\eta \left(\frac{\lambda}{2}\,\overline{QP} - \frac{1-\cos\eta}{\sin\eta} + \frac{o(\varepsilon)}{\varepsilon}\right),$$

which entails

$$\frac{\lambda}{2} \leqslant \frac{\tan\eta/2}{\overline{QP}},$$

and hence, by condition (2.5) we get, as a necessary condition for the optimality of Ω_λ,

$$\lambda = 2\,\frac{\tan\eta/2}{\overline{QP}}.$$

Let us now show that, even in this case, such a set Ω_λ cannot be a minimizer.

Fix $\bar\lambda = 2\ \tan\frac{\eta}{2} /\ \overline{QP}$. We consider the same perturbation as before, for $\varepsilon > 0$ and again we assume ε small enough in such a way that $\Omega_{\bar\lambda}^\varepsilon$ still belongs to $\mathscr{C}_{a,b}$. We compute $J_{\bar\lambda}(\Omega_{\bar\lambda}^\varepsilon) - J_{\bar\lambda}(\Omega_{\bar\lambda})$ in order to show that $J_{\bar\lambda}(\Omega_{\bar\lambda}^\varepsilon) < J_{\bar\lambda}(\Omega_{\bar\lambda})$, and hence that $\Omega_{\bar\lambda}$ cannot be a minimizer.

$$\begin{aligned} J_{\bar\lambda}(\Omega_{\bar\lambda}^\varepsilon) - J_{\bar\lambda}(\Omega_{\bar\lambda}) &= \sin\eta\,\frac{1-\cos\eta}{\sin\eta\,\overline{QP}}\,\overline{QQ^\varepsilon}\,\overline{QP} - \overline{QQ^\varepsilon} - \overline{Q^\varepsilon P} + \overline{QP} \\ &= \overline{QP} - \cos\eta\overline{QQ^\varepsilon} - \sqrt{\overline{QP}^2 + \overline{QQ^\varepsilon}^2 - 2\cos\eta\,\overline{QP}\,\overline{QQ^\varepsilon}}, \end{aligned} \tag{2.6}$$

notice that the quantity (2.6) is always negative for every positive ε as, if $\overline{QP} - \cos\eta\,\overline{QQ^\varepsilon}$ is non negative, it holds

$$\begin{aligned} \overline{QP} - \cos\eta\,\overline{QQ^\varepsilon} &= \sqrt{\overline{QP}^2 - 2\cos\eta\,\overline{QP}\,\overline{QQ^\varepsilon} + \cos^2\eta\,\overline{QQ^\varepsilon}} \\ &< \sqrt{\overline{QP}^2 + \overline{QQ^\varepsilon}^2 - 2\cos\eta\,\overline{QP}\,\overline{QQ^\varepsilon}}. \end{aligned}$$

□

Remark 2.8. *Notice that, as highlighted in the introduction about the proof of Theorem 2.1, the perturbations considered in the above proof are not of the linear form*

$$h_{\Omega_t}(\theta) = h_\Omega(\theta) + t\nu(\theta).$$

This allows us to get more information about the optimal domains.

As already noticed, the class $\mathscr{A}_b^a$ is composed by copies of the same angle ξ_0 which depends only on the data a, b: $\cos\xi_0 = a/b$. Hence $\mathscr{A}_b^a$ has at most π/ξ_0 elements which in particular implies that it is finite. Regarding $\mathscr{A}^a$ and $\mathscr{A}_b$ the following theorem holds which implies in particular that $\mathscr{A}^a$ and $\mathscr{A}_b$ are also finite sets (see Corollary 2.12).

Theorem 2.9. *Let Ω_λ be an optimal set belonging to the class $\mathscr{K}_{a,b}$ then*

1. *for $\lambda \neq 2/a$ there exists $\theta \in (0, \frac{\pi}{2})$ such that if $\mathscr{A}^a$ is not empty, then $\mathscr{A}^a = \{\theta\}$;*
2. *there exist $x, y \in (0, \frac{\pi}{2})$ such that if $\mathscr{A}_b$ is not empty, then either it is a singleton or $\mathscr{A}_b = \{x, ..., x\}$ or $\mathscr{A}_b = \{x, ..., x, y\}$ with $x > y$ and $\cos x + \cos y = 1/b\lambda$.*

Remark 2.10. *In the case $\lambda = 2/a$ the boundary of an optimal set Ω_λ only contains arcs of D_a or segments tangent to D_a as it follows by Theorem 1.2, Step 1 in Theorem 2.1, Theorem 2.6 and Lemma 2.15. Hence, for $\lambda = 2/a$, either $\Omega_\lambda = D_a$ or Ω_λ is a circumscribed figure to D_a which possibly has both tangent segments and arcs. Indeed for a polygons Ω circumscribed to D_a we have $|\Omega| = P(\Omega)a/2$ hence $J_{\frac{2}{a}}(\Omega) = 0$; more generally the same arrives if Ω is circumscribed to D_a and it contains arcs of D_a. Hence either $\mathscr{A}^a$ is empty or $\mathscr{A}^a = \{\theta_1, ..., \theta_m\}$ for some m such that $\sum_{i=1}^m \theta_i \leqslant \pi$ and $\cos\theta_i > a/b$.*

Proof of Theorem 2.9. We analyze first and second order optimality conditions for Problem (2.4). By the formulation (2.3) the functional J_λ can in fact be considered as a function of the angles $\xi_0 \in \mathscr{A}_b^a, \theta_i \in \mathscr{A}^a, \eta_j \in \mathscr{A}_b$. As their sum is finite and each θ_i, η_j is positive, the sets $\mathscr{A}^a$ and $\mathscr{A}_b$ have at most countably many elements, while $\mathscr{A}_b^a$ is finite.

Consider Ω_λ and assume $\mathscr{A}_b^a = \{\xi_0, ..., \xi_0\}$ with $|\mathscr{A}_b^a| = p$, $\mathscr{A}^a = \{\theta_1, ..., \theta_i, ...\}$ with $|\mathscr{A}^a| = q_a$, $\mathscr{A}_b = \{\eta_1, ..., \eta_j, ...\}$ with $|\mathscr{A}_b| = q_b$; let $N = p + q_a + q_b$, possibly infinity. Let us indicate by $X \in \mathbb{R}^N$ the sequence of angles

$$X = (\xi_0, ..., \xi_0, \theta_1, ..., \theta_i, ..., \eta_1, ..., \eta_j, ...) = (x_k)_{k=1,...,N},$$

and let $\bar{X}$ be the vector corresponding to the optimal set Ω_λ. With abuse of notation we write $J_\lambda(X)$ meaning $J_\lambda(\Omega)$, where Ω is the set corresponding to X. As $\Omega \in \mathscr{K}_{a,b}$, $J_\lambda(\Omega)$ can be expressed in the form (2.3), under the constraints in (2.4), namely

$$g_k(X) = x_k - \xi_0 < 0 \qquad \text{and} \qquad h(X) = \sum_{i=1}^N x_i - \pi = 0.$$

By the first order optimality conditions there exist Lagrange multipliers $\mu_0 \in \mathbb{R}$, $\mu_k \in \mathbb{R}^+$ for $k = 1, ..., p$ such that

$$\begin{cases} DJ_\lambda(\bar{X}) = \mu_0 Dh(\bar{X}) + \sum_{k=1}^p \mu_k Dg_k(\bar{X}), \\ \sum_{k=1}^p \mu_k g_k(\bar{X}) = 0; \end{cases} \tag{2.7}$$

this is equivalent to

$$\begin{cases} b^2(\lambda - \frac{2}{a}) = \mu_0 + \mu_k & \text{for } k = 1, ..., p \\ a^2(\lambda - \frac{2}{a})\frac{1}{\cos^2\theta_i} = \mu_0 & \text{for every } \theta_i \in \mathscr{A}^a \\ \lambda b^2 \cos 2\eta_j - 2b\cos\eta_j = \mu_0 & \text{for every } \eta_j \in \mathscr{A}_b. \end{cases} \tag{2.8}$$

From the second condition in (2.8) it easily follows $\theta_i = \theta_j$, $i, j = 1, ..., q_a$, and hence if $\mathscr{A}^a$ is not empty then it contains only copies of the same angle θ and hence $\mathscr{A}^a$ is finite.

Let us consider the third condition in (2.8); for $\eta_i, \eta_j \in \mathscr{A}_b$ it holds

$$\lambda\, b(\cos\eta_i - \cos\eta_j)(\cos\eta_i + \cos\eta_j) = \cos\eta_i - \cos\eta_j,$$

which implies either $\eta_i = \eta_j$ or $\eta_i \neq \eta_j$ with

$$\cos\eta_i + \cos\eta_j = \frac{1}{b\lambda}. \tag{2.9}$$

Hence $\mathscr{A}_b$ contains at most two different angles; let us call them x, y and assume $x > y$. This implies that also $\mathscr{A}_b$ is a finite set.

By the second order optimality conditions we have that for every $d \in \mathbb{R}^N$ which belongs to the critical cone associated to $\bar{X}$, that is such that d verifies

$$\begin{cases} \langle DJ_\lambda(\bar{X}); d\rangle & \leqslant 0, \\ \langle Dg_k(\bar{X}); d\rangle & \leqslant 0, \qquad \text{for } k = 1, ..., p \\ \langle Dh(\bar{X}); d\rangle & = 0, \end{cases} \tag{2.10}$$

it holds

$$\langle D^2 J_\lambda(\bar{X})d, d\rangle \geqslant 0, \tag{2.11}$$

where $D^2 J_\lambda$ is the diagonal matrix

$$[D^2 J_\lambda(X)]_{ii} = \begin{cases} 2\frac{b^2}{a^2}\sqrt{b^2-a^2}\,(a\lambda-2) & \text{if } i=1,...,p \\ 2a(a\lambda-2)\,\frac{\sin\theta}{\cos^3\theta} & \text{if } i=p+1,...,p+q_a \\ 2b(-b\lambda\sin 2\eta_j+\sin\eta_j) & \text{if } i=N-q_b+1,...,N. \end{cases} \tag{2.12}$$

Assume $q_a = |\mathscr{A}^a| \geqslant 2$ and let d be a vector in the critical cone with $d_i = 0$ if $i = 1,...,p, N - q_b + 1,...,N$ (that is d has non null components only corresponding to the elements of the class $\mathscr{A}^a$). Hence

$$\langle D^2 J_\lambda(\bar{X})d; d\rangle = 2a^2\left(\lambda - \frac{2}{a}\right)\frac{\sin\theta}{\cos^3\theta}\sum_{i=p+1}^{p+q_a} d_i^2,$$

which is negative and hence contradicts (2.11). This proves **1**.

Assume there exist $\eta_j = \eta_k = z \in \mathscr{A}_b$ and consider $d \in \mathbb{R}^N$ such that $d_j = -d_k$ and $d_i = 0$ for $i \neq j,k$. Hence d belongs to the critical cone (2.10) and hence (2.11) holds, that is

$$2bd_j^2\sin z(1-2b\lambda\cos z) \geqslant 0,$$

which entails $\cos z \leqslant 1/2b\lambda$. Analogously, assume $\eta_j = y, \eta_k = x$ with $\cos x + \cos y = 1/b\lambda$ by (2.9); consider the same d as before. Condition (2.11) gives

$$2b\, d_k^2(\sin y - \sin x)(1-2b\lambda\cos y) \geqslant 0,$$

which implies that if $x > y$ then $\cos y \geqslant 1/2b\lambda$ (and hence by (2.9) $\cos x \leqslant 1/2b\lambda$).

Assume $\mathscr{A}_b$ contains the set $\{x,y,y\}$ with $x > y$; then it holds $\cos y = 1/2b\lambda$ which implies $x = y$ by (2.9). Hence the thesis holds true. □

Remark 2.11. *The Hessian matrix* D^2J_λ *is the diagonal matrix given in (2.12). Since the critical (tangent) cone is here an hyperplane, three situations can occur:*

- *all the eigenvalues of* D^2J_λ *are non negative and the second order optimality condition is automatically fulfilled;*
- *there exist* at least *two negative eigenvalues and the quadratic form cannot be non negative on a hyperplane, thus the second order optimality condition is not satisfied;*
- *there exists one and only one negative eigenvalue. In this case, as explained in* [7, Corollary 4.6], *the quadratic form with eigenvalues* $\lambda_1 < 0 < \lambda_2 \leqslant \lambda_3 \leqslant \dots \lambda_N$ *will be non negative on the hyperplane* $H = (x_1, x_2, \dots x_N)^\perp$ *if and only if*

$$\sum_{i=1}^{N} \frac{x_i^2}{\lambda_i} \leqslant 0. \tag{2.13}$$

In our situation, to each angle $\theta \in \mathscr{A}^a$ *or* $y \in \mathscr{A}_b$ *corresponds a negative eigenvalue of* D^2J_λ. *This is the reason why we cannot have more than one of such angles. Moreover, as soon as one of these angles* $\theta \in \mathscr{A}^a$ *or* $y \in \mathscr{A}_b$ *exists, the inequality (2.13) gives an information which will be useful in the sequel, see Section 3.*

As pointed out in **2.** *of Theorem 2.9 if there exist two different angles* $x > y$ *then* $\cos x \leqslant 1/2\lambda b$. *More precisely this holds true also if the class* $\mathscr{A}_b$ *is composed only by copies of a same angle* x. *Indeed if* $\mathscr{A}_b \supseteq \{x,x\}$, *the eigenvalue of* $D^2J_\lambda(\Omega_\lambda)$ *associated to* x *has to be non negative, that is* $2b\sin x(1-2b\lambda\cos x) \geqslant 0$, *which gives* $\cos x \leqslant 1/2\lambda b$.

As already noticed in the proof of Theorem 2.9, the following holds.

Corollary 2.12. *Let* $\Omega_\lambda \in \mathscr{K}_{a,b}$ *be an optimal set such that* $\partial\Omega = \cup_{i\in I} L_i$, *with* $L_i \in \mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$. *Then* I *is a finite set of indices and hence for* $1/2b < \lambda < 2/a$ *the set* Ω_λ *is a polygon.*

This implies that for $1/2b < \lambda < 2/a$ the minimum Problem (1.2) can be explicitly rewritten as a function of the central angles of the polygon. In particular if Ω is an N-gone, we define X its vector of central angles such that $X = (\xi_0,...,\xi_0,\theta,x,...,x,y)$ that is x_i corresponds to the elements of the classes $\mathscr{A}_b^a, \mathscr{A}^a, \mathscr{A}_b$ for $i = 1,...,p$, $i = N - q_b + 1,...,N$, respectively; where $|\mathscr{A}_b^a| = p$, $|\mathscr{A}^a| = 1$, $|\mathscr{A}_b| = q_b$ with $p + 1 + q_b = N$. Recalling (2.4) we have

$$\min_{\Omega\in\mathscr{C}_{a,b}} J_\lambda(\Omega) = \min_{X\in\mathscr{A}} J_\lambda(X),$$

where

$$\mathscr{A} = \{X \in \mathbb{R}^N \quad \text{such that} \quad \sum_{i=1}^{N} x_i = \pi, \quad x_i < \xi_0 \text{ for } i = p+1, ..., N\},$$

and

$$J_\lambda(X) \;=\; \lambda\Big(\sum_{i=1}^{p+1} a^2 \tan x_i + b^2 \sum_{j=N-q_b}^{N} \sin x_j \cos x_j\Big) - 2\Big(\sum_{x_i=1}^{p+1} a \tan \xi_0 + b \sum_{x_j=N-q_b}^{N} \sin x_j\Big).$$

2.3. **Optimal shape for extremal values of** λ**.** We here analyse the case of extremal values of λ. In the limit cases $\lambda = 0$ or $\lambda = +\infty$ the solution to (1.2) is evident to be the exterior ball D_b and the interior one D_a, respectively. It is in fact the same also for values of λ near to these limit cases.

Theorem 2.13. *Let* Ω_λ *be a minimizer to (1.2);*

1. *if* $\lambda \leqslant 1/2b$ *then* Ω_λ *is unique and* $\Omega_\lambda = D_b$*;*
2. *if* $\lambda > 2/a$ *then* Ω_λ *is unique and* $\Omega_\lambda = D_a$*.*

In order to prove this result some preliminary steps are needed. They are collected in the following lemmas.

Lemma 2.14. *For every* $\lambda \leqslant 2/(a+b)$*,* Ω_λ *does not contain tangent sides to* D_a *which are not chord of* D_b*.*

Lemma 2.15. *For every* $\lambda \geqslant 1/a$*,* Ω_λ *does not contain chords of* D_b *which are not tangent to* D_a*.*

Proof of Theorem 2.13. This proof is in fact analogous and at the same time opposite to the proof of Theorem 2.1. Indeed we here consider the same constructions as before, to prove the exact complement: for $\lambda \leqslant 1/2b$ and $\lambda > 2/a$, the set Ω_λ does not contains segments.

Proof of part **1**.
As $\lambda < 2/(a+b)$, by Theorem 2.1, Lemma 2.14 and Theorem 2.6 we have that if $\partial\Omega$ contains a segment, then it is a chord of D_b. Let AB be one of these chords, $A = be^{i\theta_A}$, $B = be^{i\theta_B}$. We define Ω^η starting from Ω and substituting the chord AB with the corresponding arc on D_b; with $\eta = (\theta_A - \theta_B)/2$. We compare $J_\lambda(\Omega)$ with $J_\lambda(\Omega^\eta)$ getting

$$J_\lambda(\Omega^\eta) - J_\lambda(\Omega) = \frac{b}{2}(\eta - \sin\eta\cos\eta)\left(\lambda 2b - 4\frac{\eta - \sin\eta}{\eta - \sin\eta\cos\eta}\right),$$

which is negative as $\lambda 2b \leqslant 1$ and

$$\frac{\eta - \sin\eta}{\eta - \sin\eta\cos\eta} > \frac{1}{4}, \qquad \text{for every } \eta > 0.$$

Hence $\partial\Omega_\lambda$ does not contain chords of D_b, which implies $\Omega_\lambda = D_b$ since by step 2 in Theorem 2.1, $\partial\Omega_\lambda$ does not contain neither arcs of D_a.

Proof of part **2**.
As $\lambda > 1/a$, by Lemma 2.15 and Theorem 2.6 we have that Ω_λ can be composed only by arcs of ∂D_a and tangent segments to D_a; let AB, BC, with $A, C \in \partial D_a$, $A = ae^{i\theta_A}$, $C = ae^{i\theta_C}$, be some of them. Let η be such that $\tan\eta = \overline{AB}/a = \overline{BC}/a$ and let us consider the set Ω^η obtained from Ω substituting the segments $\overline{AB}, \overline{BC}$ by the corresponding arc of D_a, AC. Computing $J_\lambda(\Omega)$, and $J_\lambda(\Omega^\eta)$ we get

$$J_\lambda(\Omega) - J_\lambda(\Omega^\eta) = a^2(\tan\eta - \eta)(\lambda - \frac{2}{a}),$$

which is positive and hence Ω_λ cannot contain tangent segments to D_a. This entails that $\Omega_\lambda = D_a$. □

We now give the proof of Lemmas 2.14 and 2.15. Notice that we here use non-local perturbations of Ω.

Proof of Lemma 2.14. Let Ω be a set in the class $\mathscr{C}_{a,b}$ with $x \in \mathscr{A}^a$; let PQ, QR be the corresponding tangent sides to D_a. Notice that we can assume $R, P \in \partial D_b$ as by Theorem 2.9 there exists at most one angle in the class $\mathscr{A}^a$.

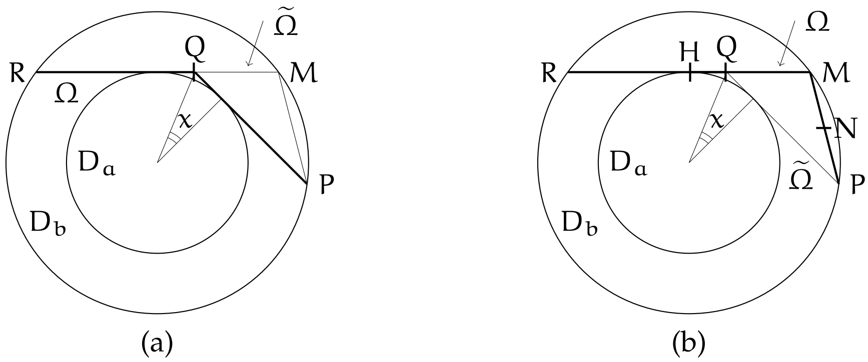


FIGURE 6. For $\lambda \leqslant 2/(a+b)$, $\mathscr{L}^a = \emptyset$; for $\lambda \geqslant 1/a$, $\mathscr{L}_b = \emptyset$

Consider a set $\widetilde{\Omega}$ as in Figure 6 (a), obtained from Ω by moving the point Q along the line RQ, up to the point M on the boundary of D_b. Hence,

$$J_\lambda(\widetilde{\Omega}) - J_\lambda(\Omega) = \frac{\lambda}{2}\overline{QM}\,\overline{QP}\sin 2x - (\overline{QM} + \overline{PM} - \overline{QP}).$$

As $\overline{QM} = \sqrt{b^2 - a^2} - a\tan x$, and $\overline{QP} = \sqrt{b^2 - a^2} + a\tan x$, we obtain $\overline{PM} = 2b\sin x$ and hence

$$J_\lambda(\widetilde{\Omega}) - J_\lambda(\Omega) = \frac{\sin x}{\cos x}(b^2\cos x - a^2)\left(\lambda - \frac{2}{b\cos x + a}\right),$$

which is negative since $\lambda \leqslant 2/(a+b) < 2/(b\cos x + a)$. This shows that if $\lambda \leqslant 2/(a+b)$ then the class $\mathscr{L}^a$ has to be empty. □

Proof of Lemma 2.15. Assume that $\partial\Omega$ contains a chord MP, with $M, P \in \partial D_b$, with MP not tangent to D_a. By Lemma 3.3 we can assume $\mathscr{L}_b = \{MN\}$, where N is the middle point of MP; then there exists a side MR which touches D_a at a point $H \in \partial\Omega \cap \partial D_a$.

Consider the set $\widetilde{\Omega}$ obtained from Ω by moving the point M along the line HM up to the point Q such that QP is tangent to D_a (see Figure 6 (b)). As in the proof of Lemma 2.14 we get

$$J_\lambda(\Omega) - J_\lambda(\widetilde{\Omega}) = \frac{\sin x}{\cos x}(b^2\cos x - a^2)\left(\lambda - \frac{2}{b\cos x + a}\right),$$

which is positive for $\lambda \geqslant 1/a$ as $b\cos x + a > 2a$. This shows that if $\lambda \geqslant 1/a$ then $\mathscr{L}_b$ has to be empty. □

3. FURTHER CHARACTERIZATIONS

This section is devoted to a more precise analysis of optimal sets, in particular regarding the total number of sides, and further properties of the classes $\mathscr{L}_b^a, \mathscr{L}^a, \mathscr{L}_b$. These results are useful if one wants to describe the optimal sets for a given value of a, b as shown in Section 4.

3.1. **Analysis of large values of λ.**

In this section we give a complete characterization of optimal sets for sufficiently large values of λ. In particular in Theorem 3.1 we give the exact number of sides of an optimal polygon together with a description of its classes of central angles for $1/b \leqslant \lambda < 2/a$.

Theorem 3.1. *For $1/b \leqslant \lambda < 2/a$ optimal sets Ω_λ are polygons in the class $\mathscr{K}_{a,b}$ with a minimum number of sides. In particular let p_0 be the largest integer such that $p_0\xi_0 \leqslant \pi$, where ξ_0 is defined as in (2.2); then $|\mathscr{A}_b^a| = p_0$ and either $\mathscr{A}_b$ is empty or so is $\mathscr{A}^a$.*

More precisely let $x = \pi - p_0\xi_0$; if $x \neq 0$ then Ω_λ is inscribed into D_b for $1/b \leqslant \lambda \leqslant 2/(b\cos x + a)$ while it is circumscribed to D_a for $2/(b\cos x + a) \leqslant \lambda < 2/a$ and Ω_λ has either p_0 or $p_0 + 1$ sides.

Before giving the proof we present some preliminary results, namely Lemma 3.2 and Lemma 3.3.

Lemma 3.2. *Let Ω_λ be an optimal set with $\lambda \geqslant 1/b$. Then its class $\mathscr{A}_b^a$ is not empty.*

Proof. Let $p = |\mathscr{A}_b^a|$, $q = |\mathscr{A}_b|$. Assume $q \geqslant 2$ hence we have: either $\mathscr{A}_b = \{x, y\}$ or $\mathscr{A}_b$ contains at least two copies of the same angle x.

As pointed out in Remark 2.11, in both cases optimality conditions (2.8), (2.10), (2.11) imply

$$\cos x \leqslant \frac{1}{2\lambda b}. \tag{3.1}$$

that is $x \geqslant x_0$, where $\cos x_0 = 1/2\lambda b$. Hence $p\xi_0 + (q-1)x \geqslant p\xi_0 + (q-1)x_0$ which entails

$$p\xi_0 + (q-1)x_0 < \pi.$$

If $p = 0$ then $q \geqslant 3$ as $\mathscr{A}_b^a$ empty implies $\mathscr{A}^a$ is empty as well. In particular if $q \geqslant 4$ then previous argument implies $x_0 < \pi/3$ which contradicts the fact that $\cos x_0 = 1/2\lambda b \leqslant 1/2$. If $q = 3$ then Ω_λ is an isosceles triangle identified by its central angles as $\{x, x, \pi - 2x\}$ with $x \in [\frac{\pi}{3}, \xi_0]$. By direct computations it turns out that the functional J_λ is monotone decreasing as function of x, for every $\lambda \geqslant \frac{1}{b}$ and hence the optimal isosceles triangle is determined for $x = \xi_0$, which contradicts the fact that $\mathscr{A}_b^a$ is empty. □

Lemma 3.3. *For $\lambda \geqslant \min\{1/2a, 1/b\}$ any optimal set Ω_λ has $|\mathscr{L}_b| \leqslant 1$.*

Proof. Assume $|\mathscr{L}_b| = |\mathscr{A}_b| \geqslant 2$, then as shown in the proof of Lemma 3.2 the optimality conditions (2.10), (2.11) implies (3.1), which gives $\lambda < 1/2a$ as by construction $\cos x > a/b$. Hence if $\lambda \geqslant 1/2a$ it holds $|\mathscr{A}_b| \leqslant 1$.

Assume now $\lambda \geqslant 1/b$; as we have already proved the thesis for $\lambda \geqslant 1/2a$, it is sufficient to consider the case $b > 2a$. Let $p = |\mathscr{A}_b^a|$, $q = |\mathscr{A}_b|$ and N be the total number of sides of Ω_λ. as $b > 2a$, it holds $\xi_0 > \pi/3$. Assume $\mathscr{A}_b$ contains two different angles x, y with $x > y$, hence by Theorem 2.9 $\mathscr{A}_b$ contains $(q-1)$ copies of x and one copy of y with $q \geqslant 2$ and it holds $p\xi_0 + (q-1)x + y \leqslant \pi$ (where equality holds if there does not exist an angle $\theta \in \mathscr{A}^a$). Notice that as $\cos x \leqslant 1/(2\lambda\, b)$ and $\lambda > 1/b$, it holds $x > \pi/3$. Moreover, by construction, $\xi_0 > x$, which gives $\pi > (p+q-1)\pi/3$ which implies $p+q < 4$ that is $p+q \leqslant 3$ and hence the only possibility is $p+q = 3$ either with $\mathscr{A}^a$ empty, or with $\mathscr{A}^a = \{\theta\}$ for some θ.

The case $\mathscr{A}^a$ not empty cannot be optimal as it implies $q = 2, p = 1$ and by translation we can easily obtain a new domain $\widetilde{\Omega}$ whose sides do not belong to $\mathscr{L}_b^a \cup \mathscr{L}^a \cup \mathscr{L}_b$ such that the value of J_λ is unchanged. As $\widetilde{\Omega}$ cannot be optimal, so is not Ω. In the case $\mathscr{A}^a = \emptyset$ we only have two candidates: the triangles T' and T'' determined by their sets of angles as $\{\xi_0, x, y\}$ and $\{z, z, u\}$, respectively. By a direct computation we obtain that neither T' nor T'' are optimal; in fact J_λ can be seen as a function of x, z, respectively, which decreases for $x, z \in (0, \xi_0)$ for $\lambda \geqslant b/(b^2 + ab - 2a^2)$, which is the case for $\lambda \geqslant 1/b$ (or $\lambda \geqslant 1/2a$). Hence,

$$J_\lambda(T'), J_\lambda(T'') > J_\lambda(T),$$

where T is the triangle determined by the angles $\{\xi_0, \xi_0, \pi - 2\xi_0\}$.

Assume now that the class $\mathscr{A}_b$ only contains copies of a same angle x, so that $p\xi_0 + qx \leqslant \pi$. We want to prove that $q \leqslant 1$. Indeed if $q \geqslant 2$ then the optimality conditions (2.10), (2.11) implies (3.1) (see Remark 2.11). In particular for $\lambda \leqslant 1/b$ we obtain $\cos x \leqslant 1/2$ that is $x \geqslant \pi/3$ and this gives $q \leqslant 2$ and then $q = 2$. We then have

$$\pi \geqslant p\xi_0 + 2x > (p+2)\frac{\pi}{3},$$

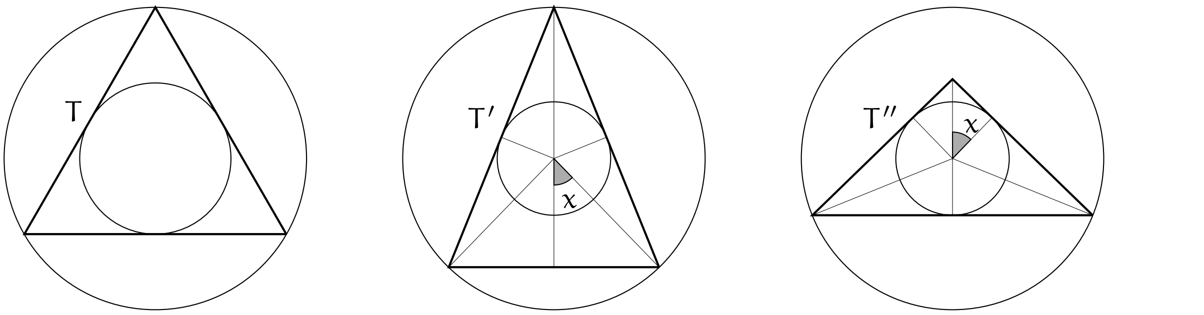
which entails $p < 1$ that is $p = 0$ and hence $N = q = 2$, which is absurd.

Hence Ω_λ is a triangle with the max number of sides which are at the same time tangent to D_a and chord of D_b and hence $|\mathscr{L}_b| \leqslant 1$. □

We finally present the proof of Theorem 3.1.

Proof of Theorem 3.1. We are going to prove that $|\mathscr{A}^a| \cdot |\mathscr{A}_b| = 0$; we split the proof in the cases $b \lessgtr 2a$.

Case $b \geqslant 2a$. Assume both $\mathscr{A}^a$ and $\mathscr{A}_b$ not empty. By the proof of Lemma 3.3 we have that Ω_λ is necessarily a triangle hence we have $\mathscr{A}_b^a = \{\xi_0\}$, $\mathscr{A}^a = \{\theta\}$, $\mathscr{A}_b = \{x\}$, which is not optimal as already noticed in the above proof since it can be translated. Hence either $\mathscr{A}^a$ is empty and we get the triangles T' with $\mathscr{A}_b = \{x\}$, or so is $\mathscr{A}_b$ and we obtain T'' with $\mathscr{A}^a = \{\theta\}$, respectively with $x = \theta = \pi - 2\xi_0$.



(a) $b = 2a$; (b) $b > 2a$, $x \in \mathscr{A}_b$; (c) $b > 2a$, $x \in \mathscr{A}^a$.

FIGURE 7. The triangles T, T', T'', respectively.

Otherwise both $\mathscr{A}^a$ and $\mathscr{A}_b$ are empty, hence T is the regular triangle of side $\sqrt{b^2 - a^2}$. By explicit computation we obtain that $\Omega_\lambda = T$ if $b = 2a$ (notice that in this case T is the only triangle which belongs to the class $\mathscr{K}_{a,b}$), while for $b > 2a$ we have

$$\Omega_\lambda = \begin{cases} T' & \text{for } \frac{1}{b} < \lambda \leqslant \frac{2b}{(b-a)(b+2a)}, \\ T'' & \text{for } \frac{2b}{(b-a)(b+2a)} \leqslant \lambda < \frac{2}{a}. \end{cases}$$

Case $b < 2a$. If both $\mathscr{A}^a$ and $\mathscr{A}_b$ are not empty, by Lemma 3.3 and Theorem 2.9 it holds $\mathscr{A}^a = \{\theta\}$, $\mathscr{A}_b = \{x\}$. Let $p = |\mathscr{A}_b^a|$. By construction $p + 2 \geqslant 4$ hence

$$\pi = p\xi_0 + \theta + x > (p+1)x \geqslant 3x,$$

which gives $x < \pi/3$, and hence $\cos x > 1/2$. Consider the second order optimality conditions (2.10), (2.11) and let $d = (0, ..., 0, -k, k) \in \mathbb{R}^N$ be a vector in the critical cone, where the last two components correspond to the element of $\mathscr{A}^a$ and $\mathscr{A}_b$ respectively. Computing the second derivatives of J_λ we get

$$\langle D^2 J_\lambda(\Omega_\lambda)d; d\rangle = k^2\Big(2a(a\lambda - 2)\frac{\sin\theta}{\cos^3\theta} + 2b\sin x(1 - 2b\lambda\cos x)\Big),$$

which is negative as we showed that $\cos x > 1/2$. This is a contradiction.

Hence Ω_λ is either inscribed into D_b or circumscribed to D_a with at most one side which does not belong to $\mathscr{L}_b^a$. This means that Ω_λ is a polygon composed by the maximum number of segment in $\mathscr{L}_b^a$ which are completed by a last segment determined by a central angle which belongs either to $\mathscr{A}^a$ or to $\mathscr{A}_b$. More precisely, Ω_λ can be represented by its central angles as the set of p copies of $\xi_0 \in \mathscr{A}_b^a$ with a last angle $x = \pi - p\xi_0$ such that $x < \xi_0$. Denote by ${\Omega_\lambda}^a$ the set corresponding to $x \in \mathscr{A}^a$ and ${\Omega_\lambda}^b$ that corresponding to $b \in \mathscr{A}_b$.

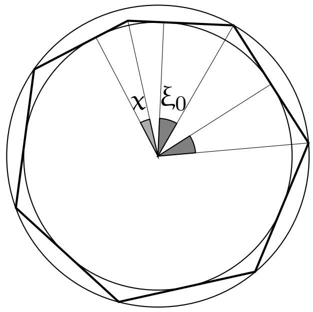
By a direct computation we get

$$J_\lambda({\Omega_\lambda}^a) - J_\lambda({\Omega_\lambda}^b) = \frac{\sin x}{\cos x}(a - b\cos x)\big(\lambda(a + b\cos x) - 2\big),$$

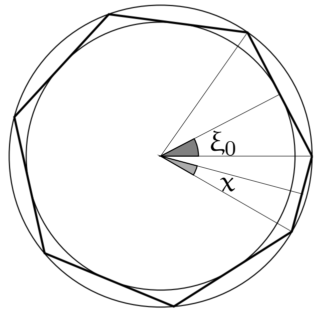
and hence

$$\Omega_\lambda = \begin{cases} {\Omega_\lambda}^b & \text{for } \frac{1}{b} < \lambda \leqslant \frac{2}{b\cos x + a}, \\ {\Omega_\lambda}^a & \text{for } \frac{2}{b\cos x + a} \leqslant \lambda < \frac{2}{a}. \end{cases}$$

□



(a) The angle x belongs to $\mathscr{A}^a$



(b) The angle x belongs to $\mathscr{A}_b$

FIGURE 8. The sets ${\Omega_\lambda}^a$ and ${\Omega_\lambda}^b$ in the proof of Theorem 3.1.

3.2. **Analysis of small values of λ.** By Lemma 2.14 and Corollary 2.2 for $1/2b < \lambda < 2/(a+b)$ an optimal set is a polygon inscribed into D_b with possible tangent sides to D_a. In particular by Lemma 3.3 there exists at least one chord which is tangent to the interior ball, for $\lambda \geqslant \min\{1/2a, 1/b\}$

The following proposition expresses the fact that if λ is sufficiently small (but sufficiently large to have a polygonal solution), then optimal sets are strictly inscribed into D_b.

Proposition 3.4. *Let Ω_λ be an optimal set, with $1/2b < \lambda < 1/(a+b)$. Then the classes $\mathscr{L}_b^a$ and $\mathscr{L}^a$ are empty.*

Proof. Let Ω be a polygon inscribed into D_b; assume that there exists a chord PQ of D_b which is tangent to D_a, that is $PM \in \mathscr{L}_b^a$ where M is the middle point of PQ. As shown in Figure 9 we consider the set Ω^ε obtained as a perturbation of Ω constructing two new points $P_\varepsilon, Q_\varepsilon$ on ∂D_b, such that $\overline{PP_\varepsilon} = \overline{QQ_\varepsilon} = \varepsilon$ (and hence $Q_\varepsilon P_\varepsilon$ is parallel to PQ) with $P_\varepsilon Q_\varepsilon \cap \Omega = \emptyset$. Let us denote by $\eta = \eta(\varepsilon)$ the angle between PQ and $P_\varepsilon P$.

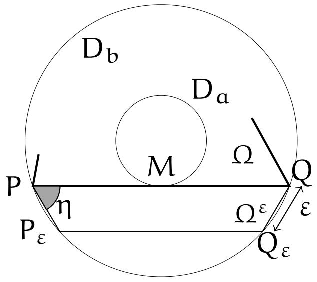


FIGURE 9. The construction of Ω^ε: for $\frac{1}{2b} < \lambda < \frac{1}{a+b}$, $\mathscr{A}_b^a = \emptyset$.

Again we want to show that in fact $J_\lambda(\Omega^\varepsilon) < J_\lambda(\Omega)$. Consider

$$J_\lambda(\Omega^\varepsilon) - J_\lambda(\Omega) = \varepsilon \sin\eta (2\sqrt{b^2 - a^2} - \varepsilon\cos\eta)\Big(\lambda - \frac{2\tan(\eta/2)}{2\sqrt{b^2-a^2} - \varepsilon\cos\eta}\Big);$$

and notice that

$$\lim_{\varepsilon\to 0} \frac{2\tan(\eta/2)}{2\sqrt{b^2-a^2} - \varepsilon\cos\eta} = \frac{1}{a+b},$$

since $\lim_{\varepsilon\to 0}\eta(\varepsilon) = \xi_0$. Hence, as $\lambda < 1/(a+b)$, there exists $\varepsilon > 0$ (and hence $\eta > 0$) such that $J_\lambda(\Omega^\varepsilon) - J_\lambda(\Omega) < 0$. ☐

As already noticed for small values of λ optimal polygons are inscribed into D_b. In particular for $1/2b < \lambda < 1/b$ either Ω_λ contains tangent sides to D_a, or it is either regular or "quasi-regular", where quasi-regular means that it has all the sides of equal length, except one. It would be interesting to investigate when each of the cases arrives.

Now let us consider the case of quasi-regular polygons. Notice that not for every values of λ, a, b, N an optimal quasi-regular N-gone can be constructed in the class $\mathcal{K}_{a,b}$. In particular some estimates of the possible number of sides of an optimal polygon holds.

Proposition 3.5. *Let Ω_λ be an optimal polygon, with $1/2b < \lambda \leqslant 1/b$ and let $p = |\mathscr{A}_b^a|$, $q = |\mathscr{A}_b|$. It holds*

$$p_0 + 1 - p \leqslant q \leqslant \frac{\pi}{x_0} - p\frac{\xi_0}{x_0} + 1, \tag{3.2}$$

where ξ_0 is defined in (2.2), $\cos x_0 = \frac{1}{2\lambda b}$ and $p_0 = [\frac{\pi}{\xi_0}]$.

In particular if $\mathscr{A}_b \supseteq \{x, y\}$ it also holds

$$\frac{\pi - p\xi_0 + \sqrt{(\pi - p\xi_0)^2 - \frac{9}{2}x_1}}{2x_1} + 1 \leqslant q \leqslant \frac{\pi}{x_1} - p\frac{\xi_0}{x_1} + 1.$$

Proof. Notice that p_0 represents the maximum number of copies of the angle ξ_0 that a polygon in the class $\mathcal{K}_{a,b}$ can have as central angle. That is $p_0\xi_0 \leqslant \pi < (p_0+1)\xi_0$. Hence the minimum number of sides is always at least p_0, and equality holds only in the case $p_0 = \pi/\xi_0$. In the general case $\pi/\xi_0 \notin \mathbb{N}$, it holds in fact $N \geqslant p_0 + 1$, that is

$$q \geqslant p_0 + 1 - p.$$

In what follows we assume $p_0 < \pi/\xi_0$, in order to treat a more general situation.

Notice that, by optimality conditions, if $\mathscr{A}_b$ contains a couple of equal angles $\{x, x\}$ or a couple of angles $\{x, y\}$, it holds $\cos x \leqslant 1/2\lambda b$ (see Theorem 2.9 for the case $\{x, y\} \subseteq \mathscr{A}_b$ and Remark 2.11 for the case $\{x, x\} \subseteq \mathscr{A}_b$). Hence if $q \geqslant 2$, and $\mathscr{A}_b$ has at least $(q-1)$ copies of an angle x, it holds $x_0 \leqslant x \leqslant \xi_0$ with $\cos x_0 = 1/2\lambda b$.

Assume that $\mathscr{A}_b$ only contains q copies of the same angle x, such that $p\xi_0 + qx = \pi$; we have

$$q \leqslant \frac{\pi}{x_0} - p\frac{\xi_0}{x_0}.$$

If $\mathscr{A}_b$ contains a couple of angles $\{x, y\}$, that is $\mathscr{A}_b = \{x, ..., x, y\}$, we have $x > y$ with $p\xi_0 + (q-1)x + y = \pi$, which gives

$$q \leqslant \frac{\pi}{x_0} - p\frac{\xi_0}{x_0} + 1,$$

and hence (3.2) is proved.

Moreover in this case the set Ω_λ can be optimal only if it satisfies the optimality conditions which appears in Theorem 2.9. More precisely by Corollary 4.6 in [7] (see (2.13)) it must hold

$$q - 1 \leqslant -\frac{\mu_x}{\mu_y},$$

where μ_x and μ_y are the eigenvalues of $D^2 J_\lambda(\Omega)$ corresponding to the angles x and y respectively: $\mu_x = 2b\sin x(1 - 2\lambda b\cos x)$, $\mu_y = 2b\sin y(1 - 2\lambda b\cos y)$. Indeed Ω_λ can be seen as an optimal polygon for the minimization problem in the class of $(p+q)$-gones with p fixed central angles equal to ξ_0, and hence Corollary 4.6 in [7] applies to the q eigenvalues $\mu_x, ..., \mu_x, \mu_y$.

We get the following necessary conditions for the optimality of the quasi-regular N-gone:

$$\begin{cases} (q-1)x + y = \pi - p\xi_0, \\ \cos x + \cos y = \dfrac{1}{\lambda b}, \\ \sin x - (q-1)\sin y \geqslant 0, \\ x - y > 0. \end{cases} \tag{3.3}$$

Notice that this corresponds to find the intersections between the graph of the function

$$\phi_\lambda(x) = \arccos\Big(\frac{1}{\lambda b} - \cos x\Big), \tag{3.4}$$

and the straight line $y = \pi - (q-1)x - p\xi_0$, which belong to a certain subset of the first octant, as shown in Figure 10.

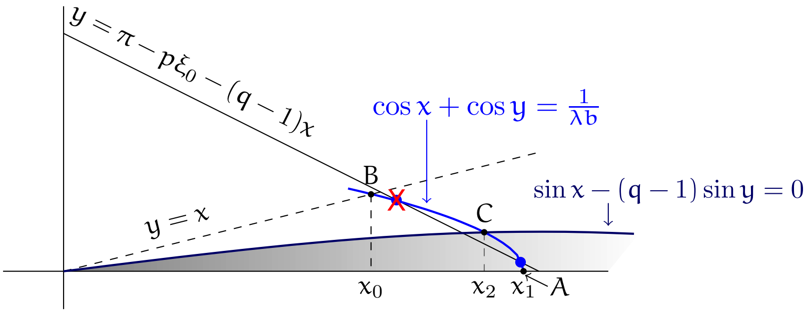


FIGURE 10. Conditions for the existence of a quasi-regular optimal polygon with $|\mathscr{A}_b^a| = p, |\mathscr{A}_b| = q$.

We denote by A, B, C the points indicated in the figure: $A \equiv (x_1, 0)$, $B \equiv (x_0, x_0)$, $C \equiv (x_2, y_2)$ such that

$$\cos x_0 = \frac{1}{2\lambda b}, \qquad \cos x_1 = \frac{1}{\lambda b} - 1, \tag{3.5}$$

$$\begin{cases} \cos x_2 + \cos y_2 = \dfrac{1}{\lambda b}, & \sin x_2 = (q-1)\sin y_2. \end{cases} \tag{3.6}$$

Hence we are interested in finding the zeros of the function

$$\psi_\lambda(x) = \phi_\lambda(x) - \pi + p\xi_0 + (q-1)x,$$

which belong to the interval $[x_2, x_1)$. Notice that the curve $\cos x + \cos y = 1/\lambda b$ being concave (for $x > y > 0$), so is the function $\psi_\lambda(x)$. In particular $\psi'_\lambda(x)$ has a unique zero at the point x_2, since

$$\phi'_\lambda(x_2) = -\frac{\sin x_2}{\left(1 - \left(\frac{1}{\lambda b} - \cos x_2\right)^2\right)^{\frac{1}{2}}} = -\frac{\sin x_2}{\left(1 - \cos^2 y_2\right)^{\frac{1}{2}}} = -\frac{\sin x_2}{\sin y_2} = -(q-1),$$

and the function ψ_λ is increasing for every $x \in (x_0, x_2)$ while it decreases for $x \in (x_2, x_1)$. Hence there exists a zero for ψ_λ in $[x_2, x_1)$ if and only if $\psi_\lambda(x_2) \geqslant 0$ and $\psi_\lambda(x_1) \leqslant 0$, that is

$$\phi_\lambda(x_2) - \pi + p\xi_0 + (q-1)x_2 \geqslant 0 \qquad \text{and} \tag{3.7}$$

$$\phi_\lambda(x_1) - \pi + p\xi_0 + (q-1)x_1 \leqslant 0. \tag{3.8}$$

As $\phi_\lambda(x_1) = 0$, condition (3.8) yields

$$q \leqslant \frac{\pi}{x_1} - p\frac{\xi_0}{x_1} + 1,$$

which gives an upper bound to the number of possible chords (non-tangent to D_a) of an optimal polygon.

In order to find a lower bound for q using (3.7), we need to estimate the value of $y_2 = \phi_\lambda(x_2)$, which can explicitly be found solving the system (3.6):

$$y_2 = \arccos\left(\sqrt{1 + \frac{1}{b^2\lambda^2}\frac{(q-1)^2}{q^2(q-2)^2}} - \frac{1}{b\lambda}\frac{1}{q(q-2)}\right).$$

By algebraic computations one can prove that

$$y_2 \leqslant \arccos\left(1 - \frac{9}{16(q-1)^2}\right) \leqslant \frac{3}{2}\sqrt{\frac{9}{16(q-1)^2}} = \frac{9}{8(q-1)},$$

and hence by (3.7) and the fact that $x_2 < x_1$, we have

$$\frac{9}{8(q-1)} - \pi + p\xi_0 + (q-1)x_1 \geqslant y_2 - \pi + p\xi_0 + (q-1)x_2 \geqslant 0,$$

which implies

$$q \geqslant \frac{\pi - p\xi_0 + \sqrt{(\pi - p\xi_0)^2 - \frac{9}{2}x_1}}{2x_1} + 1.$$

□

Corollary 3.6. *For* $1/2b < \lambda \leqslant 1/(a+b)$ *there exists at most one* $N \in \mathbb{N}$ *such that an optimal polygon is a quasi-regular* N*-gone.*

Proof. By Proposition 3.4 and Proposition 3.5 we have

$$\frac{\pi + \sqrt{\pi^2 - \frac{9}{2}x_1}}{2x_1} + 1 \leqslant q \leqslant \frac{\pi}{x_1} + 1.$$

Consider the difference between the upper and lower bounds:

$$\begin{aligned} \frac{\pi}{x_1} + 1 - \Big(\frac{\pi + \sqrt{\pi^2 - \frac{9}{2}x_1}}{2x_1} + 1\Big) &= \frac{\pi}{2x_1} - \frac{\pi}{2x_1}\sqrt{1 - \frac{9}{2\pi^2}x_1} \\ &\leqslant \frac{\pi}{2x_1} - \frac{\pi}{2x_1}\Big(1 - \frac{27}{8\pi^2}x_1\Big) = \frac{27}{16\,\pi} < 1, \end{aligned}$$

where the first inequality follows from the fact that $\sqrt{1-u} \geqslant 1 - \frac{3}{4}u$ for $u \leqslant \frac{8}{9}$. Hence there exists at most one value of N such that a quasi-regular optimal N-gone exists. □

As shown in the above proposition, quasi-regular optimal N-gones exist only for at most a specific value of N. Hence in most cases the solution will be a regular polygon. In the following proposition we analyze more in details this situation. Notice that by Corollary 3.6 and Proposition 3.7 below we can characterize the number of sides of an optimal polygon, for λ close to $1/2b$. In particular the number of sides tends to infinity as λ tends to $1/2b$. This shows that we have some kind of continuity of the solutions of Problem (1.2) when $\lambda \to 1/2b$ and this is in contrast with the situation for $\lambda \to 2/a$. Indeed, as explained in Theorem 3.1, for $\lambda > 2/a$ the optimal solution Ω_λ has the minimum number of sides while for $\lambda > 2/a$ it is the ball D_a.

Proposition 3.7. *Let* $1/2b < \lambda < 1/b$ *and consider the minimum Problem (1.2) in the class*

$$\mathscr{K}_{a,b} \cap \{\Omega \text{ regular polygon }\}.$$

There exists a decreasing sequence $\{\hat\beta_n\}$ *which tends to* $1/4$*, such that for* $\lambda b/2 \in [\hat\beta_N, \hat\beta_{N-1}]$ *the optimum is either the polygon* P_N *(if* $P_N \in \mathscr{K}_{a,b}$*) or the polygon* P_m *with* m *the minimum such that* $P_m \in \mathscr{K}_{a,b}$ *(if* $P_N \notin \mathscr{K}_{a,b}$*).*

Proof. Let $1/2b < \lambda < 1/b$ and let P_N be a regular N-gone inscribed into D_b, we want to analyse the minimum of $J_\lambda(P_N)$ with respect to N and the value of λ, where

$$J_\lambda(P_N) = \pi b\Big(\lambda\,\frac{b}{2}\frac{\sin(2\pi/N)}{2\pi/N} - \frac{\sin(\pi/N)}{\pi/N}\Big).$$

Let us denote $x = \pi/N$, and let $\beta = \lambda b/2$; with abuse of notation we will write $J_\lambda(x)$ meaning $J_\lambda(P_N)$. Computing the derivatives of $J_\lambda(x)$, we define $h(\beta, x) = x J_\lambda'(x)$:

$$h(\beta, x) = -\beta(\sin x \cos x - x\cos 2x) + \sin x - x\cos x.$$

In order to study the minima of $J_\lambda(x)$, we are interested in the zeros of h for $1/4 < \beta < 1/2$, and $0 < x \leqslant \pi/3$. We define the sequence β_n such that $h(\beta_n, \pi/n) = 0$, that is

$$\beta_n = \frac{\sin(\pi/n) - \pi/n\,\cos(\pi/n)}{\sin(\pi/n)\cos(\pi/n) - \pi/n\,\cos(2\pi/n)}. \tag{3.9}$$

Notice that $\{\beta_n\}_{n\in\mathbb{N}}$ is a decreasing sequence which tends to $1/4$ as n tends to infinity.

Consider $\beta_{n+1} < \beta < \beta_n$, then $h(\beta, \pi/n)$ is positive while $h(\beta, \pi/(n+1))$ is negative hence J_λ has a minimum for $x \in [\pi/(n+1), \pi/n]$, which means that either the optimal number of sides is n or it is $(n+1)$. In particular there exists $\hat\beta_n \in [\beta_{n+1}, \beta_n)$ such that $J_{2\beta/b}(\pi/n)$ is minimum for $\beta \in [\hat\beta_n, \hat\beta_{n-1}]$ where

$$\hat\beta_n = \Big(\frac{\sin(\pi/n)}{\pi/n} - \frac{\sin(\pi/(n+1))}{\pi/(n+1)}\Big) \Big/ \Big(\frac{\sin(2\pi/n)}{2\pi/n} - \frac{\sin(2\pi/(n+1))}{2\pi/(n+1)}\Big),$$

and $J_{\hat\lambda_n}(P_n) = J_{\hat\lambda_{n+1}}(P_{n+1})$ for $\hat\lambda_n = 2\hat\beta_n/b$.

Hence let $\bar n$ be the minimum number of sides such that $P_{\bar n}$ belongs to the class $\mathscr{K}_{a,b}$ and consider $1/2b < \lambda < 1/b$. Let $n \in \mathbb{N}$ be such that $2\lambda/b \in [\hat\beta_n, \hat\beta_{n-1}]$. If $n \geqslant \bar n$, then P_n minimizes J_λ among all regular polygons, if $n < \bar n$ then the optimal is $P_{\bar n}$. □

Notice that this result implies that in the case $b \geqslant 2a$, and $\hat\beta_3 \leqslant 2b\lambda \leqslant 1/2$ the optimal regular polygon is the equilateral triangle.

More generally in the case $b \geqslant 2a$ and $1/(a+b) \leqslant \lambda \leqslant 1/b$, we are going to show that only triangle can be optimal sets.

Proposition 3.8. *Let* $b > 2a$ *and* $\frac{1}{a+b} \leqslant \lambda \leqslant \frac{1}{b}$*; then* Ω_λ *is a triangle.*

Proof. As $\lambda \leqslant 1/b \leqslant 2/(a+b)$, the class $\mathscr{A}^a$ is empty by Lemma 2.14. We split the proof in two parts, considering the two cases $\mathscr{A}^a_b = \emptyset$ and $\mathscr{A}^a_b \neq \emptyset$.

Assume Ω_λ have no tangent sides to D_a (that is $\mathscr{A}^a_b = \emptyset$) and that Ω_λ is a quasi-regular polygon; hence condition (3.3) must hold true. Consider the curve $\cos x + \cos y = 1/b\lambda$; as $1/(a+b) \leqslant \lambda \leqslant 1/b$ and $b > 2a$, it holds

$$1 \leqslant \frac{1}{\lambda b} \leqslant \frac{3}{2}.$$

We compare the graphs of the functions $y = \arccos(\frac{1}{b\lambda} - \cos x)$ in the extreme cases $1/(b\lambda) = 1$ and $1/(b\lambda) = 3/2$.

Applying Proposition 3.5 we get either $N = 3$ or $N = 4$, that is: between quasi-regular polygons, only triangles and quadrilaterals can be optimal sets. Indeed for each $N \geqslant 5$ there is no intersection between the curve $\cos x + \cos y = 1/2\lambda b$ and the line $y = \pi - (N-1)x$ as shown in Figure 11 (a). In particular quadrilaterals are not optimal as the (non null) values of x such that there exists a solution to

$$\Big\{ \cos x + \cos y = \frac{1}{b\lambda}, \qquad y = \pi - (N-1)x,$$

for $N = 4$, does not satisfy $\sin x \geqslant (N-1)\sin y$, as shown in Figure 11 (b). Hence the only possible quasi-regular optimal polygons with $\mathscr{A}^a_b = \emptyset$ are triangles of the form $\mathscr{A}_b = \{x, x, y\}$.

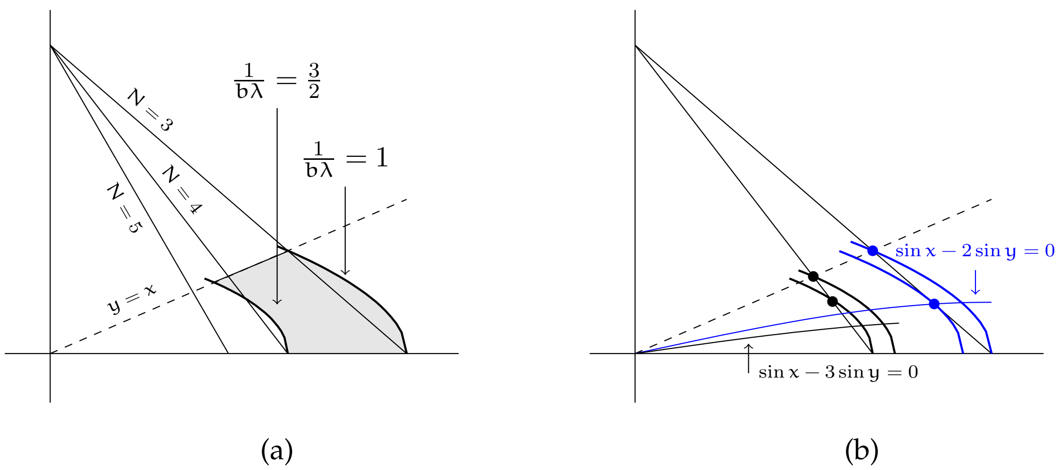


FIGURE 11. Conditions for the existence of a quasi-regular optimal polygon with $\mathscr{A}^a_b = \emptyset$. Case $b \geqslant 2a$, $\frac{1}{a+b} \leqslant \lambda \leqslant \frac{1}{b}$.

Consider now the case of a regular N-gone P_N; it holds

$$J_\lambda(P_N) = 2b\,N \sin\tfrac{\pi}{N}\Big(\lambda\tfrac{b}{2}\cos\tfrac{\pi}{N} - 1\Big).$$

Notice that, as $1/(a+b) \leqslant \lambda \leqslant 1/b$ with $b > 2a$, we have $\lambda b/2 \in (1/3, 1/2)$ and hence Proposition 3.7 guarantees $N = 3$.

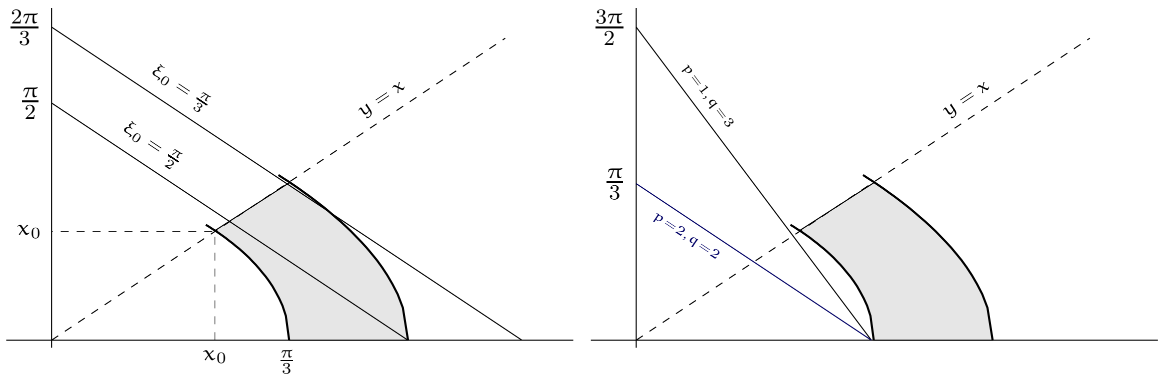
Hence if $\mathscr{A}_b^a$ is empty necessarily Ω_λ is a triangle; either equilateral or isosceles.

Suppose now $\mathscr{A}_b^a$ to be not empty; as $b > 2a$ it holds $|\mathscr{A}_b^a| = p \leqslant 2$ and Proposition 3.5 guarantees that Ω_λ is either a triangle or a quadrilateral. We are going to show that in fact this latter cannot arrive. Assume $\mathscr{A}_b \supseteq \{x, y\}$ with $x > y$ and let $q = |\mathscr{A}_b| \geqslant 2$. By the first order optimality conditions we have

$$\begin{cases} y = \pi - p\xi_0 - (q-1)x, \\ x > y \\ \cos x + \cos y = \frac{1}{\lambda b}, \end{cases} \tag{3.10}$$

where it holds $1 \leqslant \frac{1}{\lambda b} \leqslant \frac{3}{2}$ and $\frac{\pi}{3} < \xi_0 < \frac{\pi}{2}$.

Notice that the constant term and the director coefficient of the line in (3.10) decreases with respect to p and q, respectively. Hence if (3.10) admits no solution for some $\bar{p}, \bar{q}$, then the same will arrive for every $p \geqslant \bar{p}, q \geqslant \bar{q}$.



(a) Case $p = 1, q = 2$. (b) Case $p = 1, q = 3$ and $p = 2, q = 2$.

FIGURE 12. Conditions for the existence of an optimal polygon with $\mathscr{A}_b^a \neq \emptyset$, $\mathscr{A}_b \supseteq \{x, y\}$. Case $b > 2a$, $\frac{1}{a+b} \leqslant \lambda \leqslant \frac{1}{b}$.

Consider the case $p = q = 2$, shown in Figure 12 (b). Notice that, the line $y = \pi - 2\xi_0 - x$ never intersects the curve $\cos x + \cos y = \frac{3}{2}$ for $x > y > 0$ (and hence it never intersects $\cos x + \cos y = 1/\lambda b$ neither). Indeed, thanks to the concavity of the function $\phi_{\frac{2}{3b}}(x) = \arccos(3/2 - \cos x)$, the curve $\cos x + \cos y = 3/2$ for $x > y > 0$ stays above the line through the points $(\pi/3, 0)$, (x_0, x_0), where $x_0 = \arccos 3/4$, and this latter stays above the line $y = \pi - 2\xi_0 - x$ for every $y > 0$. Hence there is no solution to (3.10) for $p = q = 2$. The same arrives for $p = 1, q = 3$ as shown again in Figure 12 (b). This implies that the only possible case is $p = 1, q = 2$, which corresponds to an isosceles triangle whose central angles are $\{\xi_0, z, z\}$, represented in Figure 12 (a).

Assume now that $\mathscr{A}_b$ only contains copies of the same angle x, with $|\mathscr{A}_b| = q \geqslant 2$ and $p\xi_0 + qx = \pi$. By the second order optimality conditions (see Remark 2.11), and the fact that $\lambda \geqslant 1/(a+b) \geqslant 2/3b$, we have

$$\cos x \leqslant \frac{1}{2\lambda b} \leqslant \frac{a+b}{2b} \leqslant \frac{3}{4},$$

that is $x \geqslant u_0 \geqslant x_0$, where u_0 is such that $\cos u_0 = (a+b)/2b$. Hence we have

$$3 - p \leqslant q \leqslant \frac{\pi}{u_0} - p\frac{\xi_0}{u_0} \leqslant \frac{\pi(3-p)}{3\,u_0}, \tag{3.11}$$

where $p = 1, 2$ by construction, as $b > 2a$ and $\xi_0 \geqslant \pi/3$. Let us analyse these cases separately; notice that $u_0 \geqslant x_0 = \arccos(3/4) \geqslant 0.72$.

For $p = 1$ we obtain $2 \leqslant q \leqslant 2.9$, which implies that the only possible polygon with $\mathscr{A}_b^a = \{\xi_0\}$ is the triangle with $\mathscr{A}_b = \{x, x\}$. For $p = 2$ condition (3.11) reads as $1 \leqslant q \leqslant 1.44$ which gives $q = 1$ and hence again the only possibility is a triangle, which can be identified by its central angles as $\{\xi_0, \xi_0, z\}$.

Hence the optimal polygons are triangles, in particular they are of the form:

$$T = \{\pi/3, \pi/3, \pi/3\}, \quad T' = \{x, x, y\}, \qquad T'' = \{\xi_0, z, z\}, \quad T''' = \{\xi_0, u, v\} \quad T'^{v} = \{\xi_0, \xi_0, w\},$$

where the polygons are indicated using their central angles and $z = \frac{\pi - \xi_0}{2}$, $w = \pi - 2\xi_0$ are fixed. The values of x, y and u, v are given accordingly to Theorem 2.9. It is possible to simply compare these five kind of triangles by splitting them in two (non disjoint!) classes:

- the class of triangles with at least one central angle ξ_0;
- the class of isosceles triangle.

Let us consider first the class of triangles with at least one central angle ξ_0. All of them can be represented as triangles whose central angles are $\{\xi_0, u, \pi - \xi_0 - u\}$ with $u \in (\frac{\pi}{2} - \frac{\xi_0}{2}, \xi_0)$. Notice that the limit cases $u = \frac{\pi}{2} - \frac{\xi_0}{2}$ and $u = \xi_0$ correspond to the triangles T'' and T'^{v} respectively. Writing down the functional J_λ as a function of u, we get three different optimal triangles depending on the value of λ:

$$T'' = \{\xi_0, \tfrac{\pi}{2} - \tfrac{\xi_0}{2}, \tfrac{\pi}{2} - \tfrac{\xi_0}{2}\} \qquad \text{for} \quad \tfrac{1}{a+b} \leqslant \lambda \leqslant \tfrac{2b^2}{(b-a)(b+2a)},$$

$$T''' = \{\xi_0, \bar{u}, \pi - \xi_0 - \bar{u}\} \qquad \text{for} \quad \tfrac{1}{\sqrt{2b}\sqrt{b-a}} \leqslant \lambda \leqslant \tfrac{2b^2}{(b-a)(b+2a)},$$

$$T'^{v} = \{\xi_0, \xi_0, \pi - 2\xi_0\} \qquad \text{for} \quad \tfrac{2b^2}{(b-a)(b+2a)} \leqslant \lambda \leqslant \tfrac{1}{b}.$$

where $\bar{u}$ is such that $\sin(\bar{u} + \frac{\xi_0}{2})(2\lambda b \sin\frac{\xi_0}{2}) = 1$. Hence there exists only one possible optimal triangle of the type T''' corresponding to $u = \bar{u}$.

On the other hand, in the class of isosceles triangles determined by their central angles $\{x, x, y\}$, with $x \in [\frac{\pi}{3}, \xi_0]$, we have seen that there exists at most one triangle of type T' which can be optimal. More precisely the functional J_λ is an increasing function of x if $\lambda \leqslant \frac{8}{9b}$ or if $2a \leqslant b \leqslant 4a$ for every λ and hence in these cases the only possible optimal isosceles triangles is T'^{v}. In the case $b > 4a$ with $\frac{8}{9b} < \lambda \leqslant \frac{1}{b}$ there could exist a unique optimal triangle T', corresponding to the unique possible point $\bar{x}$ of local minimum for J_λ:

$$\cos\bar{x} = \frac{1 + \sqrt{9 - 8\lambda b}}{4}.$$

Hence for each $\frac{1}{a+b} \leqslant \lambda \leqslant \frac{1}{b}$ the solution to Problem (1.2) is a triangle and the comparison between the two above classes yields the precise optimal one. Let us remark that, using a straightforward but tedious calculation, it is possible to prove that the optimal triangle is always one of the following: T, T' with $x = \bar{x}$ or T'^{v}. □

4. An example

Let us consider in detail an example to explain how the previous results allow us to easily get any solution of the problem for any value of the parameter λ. We choose here to fix $a = 1, b = 3$. Then $\xi_0 = \arccos(a/b) \simeq 1.2310$.

The cases $\lambda > \frac{2}{a} = 2$ and $\lambda \leqslant \frac{1}{2b} = \frac{1}{6}$ are covered by Theorem 2.13 and the solutions are respectively D_a and D_b.

For $\lambda = 2$, as explained in Remark 2.10, any polygon circumscribed to D_a and any combination of sides tangent to D_a and arcs of the circle D_a solves the problem.

Let us consider the case $1/2b < \lambda < 2/a$. First we want to apply Theorem 3.1. Since $\xi_0 \simeq 1.2310$, we have $p = 2$ and $x = \pi - 2\xi_0 \simeq 0.6797$. The critical value of λ which is equal to $2/(b\cos x + a)$ equals

$$\tilde{\lambda} = \frac{2}{1 - 3\cos 2\xi_0} = \frac{2b}{b^2 + ab - 2a^2} = 0.6\,.$$

Therefore, for $\lambda \geqslant 0.6$, the optimal solution is the isosceles triangle circumscribed to D_a while for $1/3 < \lambda \leqslant 0.6$ the optimal solution is the isosceles triangle inscribed into D_b, see Table 1.

Now for λ between $0.25 = 1/(a+b)$ and $1/b$, we use the analysis done in Proposition 3.8 and the comparison between all triangles. This shows that the isosceles triangle inscribed into D_b (and defined by its three angles $\xi_0, \xi_0, \pi - 2\xi_0$ remains the optimal domain for $\lambda \in (0.308, 1/3)$ while the equilateral triangle becomes the optimal domain for $\lambda \in (0.25, 0.3080)$.

For $\lambda < 1/(a+b) = 0.25$, according to Proposition 3.4, we know that the optimal domain is inscribed in D_b (and does not touch D_a). Moreover, by Proposition 3.7, we are able to compare all regular polygons. More precisely, the following table shows the values of λ for which we switch from the regular N-gone to the regular $(N+1)$-gone (e.g. we switch from the equilateral triangle to the square for $\lambda \leqslant 0.2191$).

λ	0.2191	0.1951	0.1847	0.1792	0.1759	0.1738	0.1723	0.1713
N	3	4	5	6	7	8	9	10

Now we have seen in Theorem 2.9 that the only other possible optimal domain is a quasi-regular polygon with $N-1$ angles x and one angle $y = \pi - (N-1)x$. Moreover, Proposition 3.6 shows that there exists at most one possible value of N for such a quasi-regular polygon (and we have explicit bounds for this N), therefore the numerical study is easy. In our case, it turns out that we are able to find such quasi-regular polygons only twice (for two small intervals):

- If $\lambda \in (0.21874; 0.22222)$ the optimal domain is a quasi-regular quadrilateral.
- If $\lambda \in (0.19506; 0.19525)$ the optimal domain is a quasi-regular pentagon (with a very small angle y, thus it is not easy to recognize a pentagon in the corresponding Figure of Table 1).

For the other values of λ, the optimal domain is the regular N-gone and we just have to follow the Table in the Appendix (Section 6). Thus, we have represented the solutions in Table 1 only up to the regular hexagon. Let us remark that, in this table, the values of the angles x and y for the quasi-regular polygons are given as an example for one choice of λ.

5. Some related problems

In this section we begin by investigating the same problem when we remove one of the unilateral constraint $D_a \subset \Omega$ or $\Omega \subset D_b$. We show that the previous study allows to handle also these cases. Then, choosing particular values for the parameter λ, we are able to recover a classical inequality due to Bonnesen and Fenchel involving the area, the perimeter and the inradius. Then we recover another one due to J. Favard which involves the area, the perimeter and the circumradius. We are also able to find a refinement of such inequality for large perimeter. We close this section with a discussion about the problem of maximizing perimeter with a volume constraint in the class $\mathscr{C}_{a,b}$.

5.1. **Variation of constraints.**

It is interesting to investigate Problem (1.2) with different constraints. In particular it is often useful to consider convex sets which either contain a common fixed ball or which are contained in it. This corresponds to consider the class of convex sets with not too small inradius, or on the opposite side, the class of not too large convex sets.

5.1.1. *Analysis of convex sets with not too small inradius.* For a fixed positive real number a we define the class $\mathscr{I}_a$ as the class of convex sets which contain the ball D_a and we consider the problem

$$\min_{\Omega \in \mathscr{I}_a} J_\lambda(\Omega), \tag{5.1}$$

where J_λ is defined as in (1.1).

Notice that not for every values of λ a solution exists. Indeed for small values of λ the perimeter has in fact the heaviest weight, and it is not bounded. More precisely, solutions to (5.1) can be seen as limit of solutions to Problem (1.2) in the class $\mathscr{C}_{a,b}$ for b which tends to infinity. Hence for

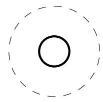
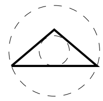
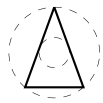
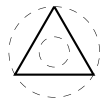
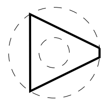
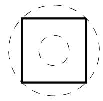
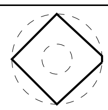
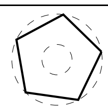
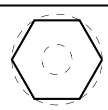
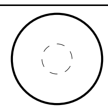
Interval for λ	Optimal Solution	Class of Angles			Figure	Area
		$\mathscr{A}_b^a$	$\mathscr{A}^a$	$\mathscr{A}_b$		
$(2;+\infty)$	disk D_a	$\emptyset$	$\emptyset$	$\emptyset$		π
$(0.6;2)$	isosceles triangle	1.2310×2	0.6797	$\emptyset$		6.4650
$(0.3080;0.6)$	isosceles triangle	1.2310×2	$\emptyset$	0.6797		10.0566
$(0.2222;0.3080)$	equilateral triangle	$\emptyset$	$\emptyset$	$\frac{\pi}{3} \times 3$		11.6913
$(0.2187;0.2222)$	quasi-regular quadrilater	$\emptyset$	$\emptyset$	$3 \times x = 1.0135$ $y = 0.1012$		13.0245
$(0.19525;0.2187)$	square	$\emptyset$	$\emptyset$	$\frac{\pi}{4} \times 4$		18
$(0.19506;0.19525)$	quasi-regular pentagone	$\emptyset$	$\emptyset$	$4 \times x = 0.7829$ $y = 0.0098$		18.0879
$(0.1847;0.19506)$	regular pentagone	$\emptyset$	$\emptyset$	$\frac{\pi}{5} \times 5$		21.3988
$(0.1792;0.1847)$	regular hexagone	$\emptyset$	$\emptyset$	$\frac{\pi}{6} \times 6$		23.3827
$(2\hat{\beta}_N/3;2\hat{\beta}_{N-1}/3)$	regular N-gone	$\emptyset$	$\emptyset$	$\frac{\pi}{N} \times N$	$\vdots$	$\vdots$
$(0;1/6)$	disk D_b	$\emptyset$	$\emptyset$	$\emptyset$		9π

TABLE 1. Optimal sets for $a = 1, b = 3, 0 \leqslant \lambda \leqslant +\infty$.

$0 \leqslant \lambda < \frac{2}{a}$ a possible solution should be the limit of the triangle T'' in Figures 7 (c). However $\lim_{b\to\infty} J_\lambda(T'') = -\infty$ and hence a minimum does not exists.

More generally, as for values of $\lambda \geqslant \frac{2}{a}$ solutions to (1.2) do not depend on the exterior ball D_b, they solve Problem (5.1) as well. Indeed let Ω_λ be a solution to (5.1); then either Ω_λ is contained in a ball D_b or it is a limit of a sequence $\{\Omega_n\}$ with $\Omega_n \subseteq D_{b_n}$ for some b_n, since otherwise the functional could not be defined. Hence we can apply the analysis of Problem (1.2) and we get the following.

Proposition 5.1. *For $\lambda < \frac{2}{a}$ there is no solution to Problem (5.1) while for $\lambda \geqslant 2/a$ solutions exist and they coincide with the corresponding solutions to Problem (1.2). More precisely for $\lambda = 2/a$ there exist an infinite number of solutions, which are circumscribed figures composed by arcs of D_a and tangent segment, while for $\lambda > \frac{2}{a}$ the ball D_a is the unique solution.*

5.1.2. *Analysis of not too large convex sets.* For $b > 0$ we define the class $\mathscr{O}_b$ as the class of convex sets contained in the ball D_b and we consider the problem

$$\min_{\Omega \in \mathscr{O}_b} J_\lambda(\Omega), \tag{5.2}$$

where J_λ is defined as in (1.1).

Since for every fixed $b > 0$ the class $\mathscr{O}_b$ is compact for the Hausdorff distance, the existence of a solution to Problem (5.2) is guaranteed for every $\lambda \geqslant 0$. We would like to solve Problem (5.2) passing to the limit $a \to 0$ in Problem (1.2), but this cannot be done directly since we cannot assume that an optimal set Ω_λ to (5.2) contains the ball D_a, even for very small $a > 0$. However we can circumvent this difficulty by considering a "translated" problem.

Let $\Omega \in \mathscr{O}_b$ be given. If the origin is in the exterior of Ω, it means that Ω lies in an open half-disc and we can translate it (without changing the value of the functional) to assume either that the origin is in the interior of Ω or that it is on its boundary . If the origin is in the interior of Ω there exists $\varepsilon > 0$ such that $\Omega \in \mathscr{C}_{\varepsilon,b}$ which entails

$$J_\lambda(\Omega) \geqslant J_\lambda(\Omega_\lambda^\varepsilon), \tag{5.3}$$

where $\Omega_\lambda^\varepsilon$ is an optimal set for the Problem (1.2) in the class $\mathscr{C}_{\varepsilon,b}$. We can now use the analysis done for Problem (1.2). Hence for $\lambda \leqslant 1/2b$ the set $\Omega_\lambda^\varepsilon$ is the ball D_b, while for $\frac{1}{2b} < \lambda \leqslant \frac{1}{b+\varepsilon}$ the set $\Omega_\lambda^\varepsilon$ is strictly inscribed into D_b and it is either regular or quasi-regular. For $\frac{1}{b} \leqslant \lambda \leqslant \frac{2b}{(b-\varepsilon)(b+2\varepsilon)}$ we have $\Omega_\lambda^\varepsilon = T_b'$ the set in Figure 7 (b) whose circumradius is b and inradius is ε, while for $\frac{2b}{(b-\varepsilon)(b+2\varepsilon)} \leqslant \lambda \leqslant \frac{2}{\varepsilon}$ the set $\Omega_\lambda^\varepsilon$ is the triangle T_b'' in Figure 7 (c), with circumradius b and inradius ε. Passing to the limit for ε which tends to zero we get inequality (5.3) with Ω_λ equal to the optimal set of Problem (1.2) for $0 \leqslant \lambda \leqslant 1/b$, while for $\lambda \geqslant 1/b$ we obtain as optimal set a double diameter.

If Ω contains the origin on its boundary then we consider a translation of the origin such that $O_\varepsilon = O - \varepsilon$, $\Omega_\varepsilon = \Omega - \varepsilon$. Hence $\Omega_\varepsilon \in \mathscr{C}_{\varepsilon,b_\varepsilon}$ for sufficiently small ε and $b_\varepsilon = b + \varepsilon$. As $|\Omega_\varepsilon| = |\Omega|, P(\Omega_\varepsilon) = P(\Omega_\varepsilon)$, inequality (5.3) still holds true, with $\Omega_\lambda^\varepsilon$ an optimal set for the Problem (1.2) in the class $\mathscr{C}_{\varepsilon,b_\varepsilon}$. The same argument as before (passing to the limit when $\varepsilon \to 0$) leads to the following result.

Proposition 5.2. *For every $\lambda \geqslant 0$ there exists a solution Ω_λ to the problem (5.2). In particular Ω_λ coincides with the optimal set in Problem (1.2) for $\lambda < 1/b$, while Ω_λ is a double diameter for $\lambda \geqslant 1/b$.*

5.2. **Inequalities for convex sets.** In the study of the theory of convex sets, geometric inequalities play a crucial rule as they allow to connect important geometric quantities (as the area and the perimeter) and to have an estimate of them. We refer to [10] for a summary of the most famous.

5.2.1. *Area, perimeter and inradius.* A well known inequality involving the area $|\Omega|$, the perimeter $P(\Omega)$ and the inradius $r(\Omega)$ of a convex set Ω is due to Bonnesen and Fenchel (see [2]). They proved that for every planar convex set Ω,

$$P(\Omega) \leqslant 2\frac{|\Omega|}{r(\Omega)}. \tag{5.4}$$

Notice that Theorem 2.13 offers a new proof of this result. Indeed: let Ω be a planar convex set, up to translation of the origin we can assume $D_r \subset \Omega$, where $r = r(\Omega)$; moreover there exists $R > r$ such that $\Omega \subset D_R$ and hence $\Omega \in \mathscr{C}_{r,R}$. Then Remark 2.10 entails

$$\frac{2}{r}|\Omega| - P(\Omega) \geqslant \frac{2}{r}|D_r| - P(D_r) = 0,$$

which corresponds to Bonnesen-Fenchel inequality (5.4) and in particular equality holds in (5.4) for every polygon circumscribed to D_a as well as for every convex set Ω whose boundary is composed by arcs of D_r and tangent sides to it.

5.2.2. *Area, perimeter and circumradius.* Another interesting inequality regards the area, the perimeter and the circumradius $R(\Omega)$. In [5] it is proved that for every planar convex set Ω it holds

$$|\Omega| \geqslant R(\Omega)(P(\Omega) - 4R(\Omega)), \tag{5.5}$$

with equality for linear segments.

Using Theorem 3.1 for $\lambda = 1/b$, we can recover this result. Indeed, let Ω be a planar convex set and let $R = R(\Omega)$ be its circumradius; up to translation of the origin we can assume $\Omega \subseteq D_R$. If Ω contains the origin in its interior, then there exists $\varepsilon > 0$ such that $D_\varepsilon \subset \Omega$ and hence $\Omega \in \mathscr{C}_{\varepsilon,R}$, which implies

$$\frac{1}{R}|\Omega| - P(\Omega) \geqslant \frac{1}{R}|T'_\varepsilon| - P(T'_\varepsilon) = 4\sqrt{R^2-\varepsilon^2}\Big(\frac{\varepsilon^3}{R^3} - \frac{\varepsilon}{R} - 1\Big), \tag{5.6}$$

where T'_ε is the triangle in Figure 7 (b), whose inradius is ε. Passing to the limit for ε which tends to zero, we obtain

$$\frac{1}{R}|\Omega| - P(\Omega) \geqslant -4R,$$

with equality for segments, which are in fact obtained as limits of triangles T'_ε. If the origin is on the boundary of Ω then using the same argument than in Section 5.1.2 we have $\Omega_\varepsilon = \Omega - \varepsilon \in \mathscr{C}_{\varepsilon,R+\varepsilon}$. Applying Theorem 3.1 for $\lambda = 1/(R+\varepsilon)$, we get inequality (5.6) for $R_\varepsilon = R + \varepsilon$,

$$\frac{1}{R_\varepsilon}|\Omega| - P(\Omega) \geqslant 4\sqrt{R_\varepsilon^2-\varepsilon^2}\Big(\frac{\varepsilon^3}{R_\varepsilon^3} - \frac{\varepsilon}{R_\varepsilon} - 1\Big),$$

and passing to the limit for ε which tends to zero, we get (5.5), with equality for diameters of the ball D_R.

Actually, we can get another similar inequality which improves the previous one for "large" perimeter. Indeed if we choose now $\lambda = 1/2b$ in Proposition 5.2, the optimal domain is the ball D_b. Thus, for any domain included in the ball D_b, the following inequality holds

$$\frac{1}{2b}|\Omega| - P(\Omega) \geqslant \frac{\pi b^2}{2b} - 2\pi b = -\frac{3\pi b}{2}.$$

In particular, replacing b by the circumradius yields the following proposition:

Proposition 5.3. *For a convex set Ω the following inequality holds*

$$|\Omega| \geqslant R(\Omega)(2P(\Omega) - 3\pi R(\Omega)) \tag{5.7}$$

with equality for a ball. Moreover inequality (5.7) improves inequality (5.5) when $P(\Omega) \geqslant (3\pi - 4)R(\Omega)$.

5.3. **Maximum for the perimeter.** Let us consider the following problem

$$\max_{\substack{\Omega \in \mathscr{C}_{a,b},\\ |\Omega| \leqslant c}} P(\Omega), \tag{5.8}$$

where $c > 0$ is a given constant. If $\pi a^2 \leqslant c \leqslant \pi b^2$ then a solution exists by the compactness of the class $\mathscr{C}_{a,b} \cap \{|\Omega| \leqslant c\}$ and the continuity of $P(\cdot)$ (for the Hausdorff distance). In particular using the formulation of the perimeter in terms of the so called *gauge function*, Theorem 2.1 of [9] guarantees that all the possible solutions are locally polygons in the interior of the annulus $D_b \setminus \overline{D_a}$.

Notice that each solution Ω_c to (5.8) in fact saturates the constraint on the volume, that is $|\Omega_c| = c$. Indeed, for every set $\Omega \in \mathscr{C}_{a,b}$ with volume strictly smaller than c, there exists $\Omega' \in \mathscr{C}_{a,b}$, with $|\Omega'| = c$ and $\Omega' \supset \Omega$; as Ω, Ω' are planar convex sets, it holds $P(\Omega') > P(\Omega)$.

Let Ω_c be a solution to (5.8) for some fixed c; hence Ω_c is a critical point for the functional J_λ with λ corresponding to a Lagrange multiplier associated to the area constraint. However Ω_c is not necessarily a minimum for it. In particular, as shown in the graph below (see Figure 13), there are many values of $c \in (\pi a^2, \pi b^2)$ for which there is no solution to (1.2) of volume c, and hence an optimal set to (5.8) for those values of c cannot be a solution to (1.2).

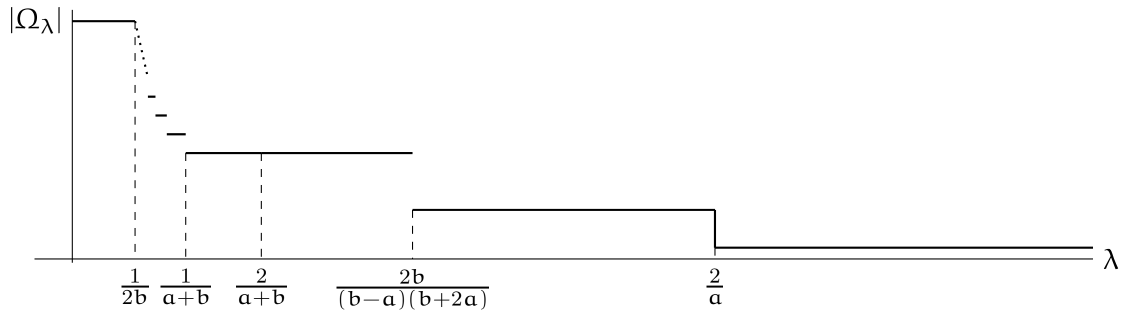


FIGURE 13. Graph of the possible values of the volume of solutions to (1.2), as λ varies.

The main difference between the two problems is that in Problem (5.8) solutions are not necessarily polygons and hence they could contain parts of arcs of D_b and D_a, as explained below. Notice that, in fact, the proof of Theorem 2.1 does not work for Problem (5.8) as the considered perturbations do not preserve the volume.

As an example, consider the case of a fixed volume closed to that of the ball D_b: $c = \pi b^2 - \varepsilon$, for some positive small ε. The class of sets belonging to $\mathscr{C}_{a,b}$ with volume equal to c only contains sets closed to the ball D_b and hence each possible side is not tangent to the interior ball D_a. This allows us to assume that each side of the boundary is a chord of D_b since otherwise a technique of parallel chord movements would increase the perimeter. Hence if a polygon is a critical point for Problem (5.8), the first order conditions (2.8) hold and they imply that the polygon has at most two different values for its central angles: x, y with $x > y$. In particular, following Remark 2.11, we can check that the second order optimality conditions guarantee that there are at most two copies of the angle y (we have here two equality constraints, thus the critical cone is of codimension 2). Hence a possible critical polygon for (5.8) is determined by its central angles as q copies of an angle x with either zero, one or two copies of an angle $y < x$; the value of the central angles are established using the volume constraint.

However direct computations show that all the possible critical polygons have a perimeter less than the set Ω_c whose boundary is composed by an arc of the circle D_b and a chord of D_b and hence for values of c closed to πb^2, solutions to Problem (5.8) are not polygons.

6. APPENDIX

A list of values for the constants $\hat{\beta}_N$ of Proposition 3.7.

N	$\hat{\beta}_N$	N	$\hat{\beta}_N$	N	$\hat{\beta}_N$	N	$\hat{\beta}_N$	N	$\hat{\beta}_N$
3	0.32862	13	0.25413	23	0.25135	33	0.25066	43	0.25039
4	0.29260	14	0.25357	24	0.25124	34	0.25062	44	0.25037
5	0.27706	15	0.25312	25	0.25114	35	0.25059	45	0.25036
6	0.26881	16	0.25275	26	0.25106	36	0.25056	46	0.25034
7	0.26388	17	0.25244	27	0.25098	37	0.25053	47	0.25033
8	0.26068	18	0.25218	28	0.25091	38	0.25050	48	0.25032
9	0.25848	19	0.25196	29	0.25085	39	0.25048	49	0.25030
10	0.25690	20	0.25177	30	0.25080	40	0.25045	50	0.25029
11	0.25572	21	0.25161	31	0.25075	41	0.25043	51	0.25028
12	0.25483	22	0.25147	32	0.25070	42	0.25041	52	0.25027

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