Morita Theory in Deformation Quantization

Dedicated to the memory of Nikolai Neumaier

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Abstract

Various aspects of Morita theory of deformed algebras and in particular of star product algebras on general Poisson manifolds are discussed. We relate the three flavours ring-theoretic Morita equivalence, *-Morita equivalence, and strong Morita equivalence and exemplify their properties for star product algebras. The complete classification of Morita equivalent star products on general Poisson manifolds is discussed as well as the complete classification of covariantly Morita equivalent star products on a symplectic manifold with respect to some Lie algebra action preserving a connection.

1 Introduction

Morita theory is a classical topic on ring theory having a more analytic cousin called strong Morita theory in the context of C^*-algebras. The underlying idea is that one wants to learn something about the categories of modules over a given ring by comparing it with the corresponding category for some other ring: even if the category of left modules might be a very complicated object, it can still be possible to state that for two (non-isomorphic) rings the corresponding categories are equivalent. In the realm of unital rings this is precisely Morita equivalence. In the C^*-algebraic framework one is interested not just in modules but in *-representations on Hilbert spaces or, more generally, on Hilbert modules over some auxiliary C^*-algebra.

In deformation quantization one is interested in the representation theories of the deformed algebras. But now the star product algebras are more specific than "just a ring" and hence a purely ring-theoretic treatment would not seem to be appropriate. It simply will not capture all interesting properties of the star products. One can achieve Hermitian star products yielding *-algebras and using the ring ordering of $\mathbb{R}[[\hbar]]$ one has a natural notion of positivity at hand. Thus from this and many other aspects the star product algebras behave much more like C^*-algebras. Hence one requires a more refined notion of representation theory leading to *-representations on pre Hilbert spaces over $\mathbb{C}[[\hbar]]$ as well as on pre Hilbert modules over auxiliary *-algebras over $\mathbb{C}[[\hbar]]$.

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It turns out that one can transfer the notions of strong Morita theory from the C^*-algebraic theory into this entirely algebraic framework, thereby extending the previous notions tremendously. This way, star product algebras and C^*-algebras can be treated almost on the same footing even though the star product algebras are not at all C^*-algebras.

In this review, we will focus on the recent developments in the understanding of the Morita theory of deformed algebras in general and of the star product algebras as major class of examples. Since recently in [9] the final classification of Morita equivalent star products in the general case of Poisson manifolds was obtained, it seems to be a good point to give such an overview.

In Section 2, we start with some elementary presentation of the ring-theoretic aspects of Morita theory as it can be found in any algebra textbook, see e.g. [28]. Then we pass to the notion of *-Morita equivalence as it was developed by Ara [1] and to the notion of strong Morita equivalence which was first established by Rieffel for C^*-algebras in [30] and then for general *-algebras over ordered rings in [11, 14]. Here we stress the functorial aspects of Morita theory and discuss in particular the functoriality of the classical limit. Section 3 contains a brief introduction to the existence and classification results in deformation theory based on Kontsevich's formality theorem [27]. Our focus is on the notions of equivalence of star products and of formal Poisson structures. In Section 4 we explain the main result of [9] by first establishing the gauge action of formal series of closed two-forms on formal Poisson structures. On the level of equivalence classes this provides precisely the description of Morita equivalent star products in terms of classical data *provided* the two-form is integral. The particular case of symplectic star products is easier and was discussed earlier in [12]. In Section 5 we recall the basic notions of Morita theory in presence of symmetries which will be modelled by a Hopf algebra action. Here the equivalence bimodules are required to carry an action of the Hopf algebra, too, such that all structure maps have nice covariance properties. One can now study the relations between the various notions of Morita equivalence. Finally, in Section 6 we outline the main results of [24] where the classification of invariant star products on a symplectic manifold up to covariant Morita equivalence with respect to a Lie algebra action was obtained.

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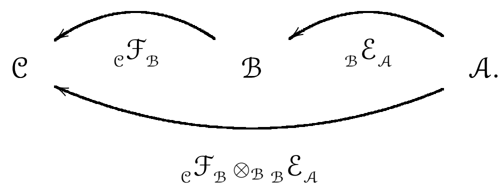
2 Morita Equivalence

In this section we first recall some basic notions of ring-theoretic Morita equivalence and specialize this to *-algebras over ordered rings to establish the notions of *-Morita equivalence and strong Morita equivalence.

We consider unital algebras over some fixed commutative unital ring C. Later on, in deformation quantization the cases $\mathsf{C} = \mathbb{C}$ or $\mathbb{C}[[\hbar]]$ will be used. One can abandon the condition of having unital algebras by imposing some slightly weaker requirements (non-degeneracy and idempotency) but we do not need this more general framework here.

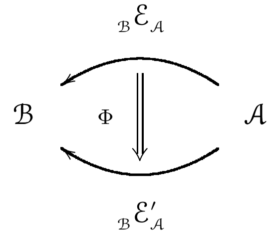
The idea of Morita theory is to replace ordinary algebra homomorphisms $\Phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ by something more general in order to have more freedom when comparing algebras. The new arrows between algebras will now be *bimodules*. Consider a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$: this notation says that $\mathcal{B}$ acts from the left while $\mathcal{A}$ acts from the right. By convention, all bimodules will have a compatible C-module structure and all structure maps will be C-(multi-)linear in the following. Moreover, the units of $\mathcal{A}$ and $\mathcal{B}$ will act as identity on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We will view ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ now as an arrow from $\mathcal{A}$ to $\mathcal{B}$. If ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ is another bimodule for some additional algebra $\mathcal{C}$ then the composition of bimodules will be the tensor product over the algebra in the middle, i.e.



(1)

As a unit morphism one uses the canonical bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ where $\mathcal{A}$ acts on itself by left and right multiplications. Now $\otimes$ is not directly associative but only associative up to a natural isomorphism. Also, ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is not directly a unit element for $\otimes$ but again only up to a natural isomorphism. Thus we have to identify isomorphism classes of bimodules: this results in an honest category which we denote by Bimod. Strictly speaking, we should confine ourselves to bimodules and algebras from some Grothendieck universe in order to get honest sets of morphisms. But this is a technical issue which will not affect the notion of Morita equivalence at all.

While passing to isomorphism classes of bimodules has the advantage to yield a category we can also stay with the bimodules directly and taking into account that $\otimes$ is not really associative. This leads to a *bicategory* where the 1-morphisms are the bimodules and the 2-morphisms, i.e. the morphisms between the 1-morphisms, are the bimodule homomorphisms. Such 2-morphisms can then be depicted by



(2)

for a bimodule homomorphism $\Phi : {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow {}_{\mathcal{B}}\mathcal{E}'_{\mathcal{A}}$. Note that we require Φ to be C-linear as well. It is then a classical result that the natural isomorphisms implementing the associativity for $\otimes$ and the unit properties of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ satisfy the necessary coherence properties to yield a bicategory (weak 2-category), see [3]. This bicategory will then be denoted by $\underline{\mathsf{Bimod}}$.

We are now in the position to state the definition of Morita equivalence:

Definition 2.1 (Morita equivalence and Picard group) *Two unital* C*-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are Morita equivalent if they are isomorphic in* Bimod*. A bimodule* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *representing an invertible arrow in* Bimod *is called an equivalence bimodule. The groupoid of invertible arrows in* Bimod *is called the Picard groupoid, denoted by* Pic*. The isotropy group of it at* $\mathcal{A}$ *is called the Picard group* $\mathsf{Pic}(\mathcal{A})$ *of* $\mathcal{A}$*.*

Alternatively, we can also use isomorphisms in $\underline{\mathsf{Bimod}}$ in the sense of bicategories, i.e. two objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if there are 1-morphisms in both directions such that their compositions are isomorphic (via 2-morphisms) to the identity morphisms ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ and ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$, respectively. This gives then the *Picard bigroupoid* $\underline{\mathsf{Pic}}$ as well as the *Picard bigroup* $\underline{\mathsf{Pic}}(\mathcal{A})$ at $\mathcal{A}$.

The main task of Morita theory is then twofold: first one would like to know which algebras are Morita equivalent, this is described by the *orbits* of the Picard groupoid. Second, one would like to understand in how many different ways two algebras can be Morita equivalent. Thanks to the groupoid structure this is equivalent to determine the isotropy groups, i.e. the Picard groups: they encode how many self-equivalences an algebra has. The classical theorem of Morita determines now the structure of equivalence bimodules:

Theorem 2.2 (Morita) *${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is an equivalence bimodule for two unital C-algebras $\mathcal{A}$ and $\mathcal{B}$ iff there is an idempotent $e = e^2 \in M_n(\mathcal{A})$ in some matrix algebra over $\mathcal{A}$ such that the right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ is isomorphic to $e\mathcal{A}^n$ and one has $\mathcal{A}e\mathcal{A} = \mathcal{A}$ as well as $\mathcal{B} \cong \mathsf{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}) \cong eM_n(\mathcal{A})e$.*

Here $\mathcal{A}e\mathcal{A}$ denotes the two-sided ideal generated by the n^2 components of e and the isomorphism in the last statement are those induced by the action of $\mathcal{B}$ and $eM_n(\mathcal{A})e$ on $\mathcal{E}_{\mathcal{A}} \cong e\mathcal{A}^n$. In particular, an equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a finitely generated and projective module over $\mathcal{A}$. This shows already that we have to look for equivalence bimodules inside the K_0-theory of $\mathcal{A}$. Since Morita equivalence is a symmetric relation by the very definition, we can equivalently formulate things in terms of $\mathcal{B}$ instead.

Example 2.3 As a first example relevant for the following we consider the smooth complex-valued functions $\mathcal{A} = C^\infty(M)$ for a smooth manifold M. Then it is well-known that any finitely generated projective module over $C^\infty(M)$ is isomorphic as a right module to the sections $\Gamma^\infty(E)$ of a complex vector bundle $E \longrightarrow M$. Moreover, this gives an equivalence bimodule with $\mathsf{End}_{C^\infty(M)}(\Gamma^\infty(E)) \cong \Gamma^\infty(\mathsf{End}(E))$ iff the fiber dimension of E is not zero. Thus the Morita equivalent algebras to $C^\infty(M)$ are isomorphic to the sections $\Gamma^\infty(\mathsf{End}(E))$ of the endomorphism bundle $\mathsf{End}(E)$ for arbitrary non-zero vector bundles $E \longrightarrow M$.

This example also allows to determine the Picard group of $C^\infty(M)$: Since the only way to get the endomorphism algebra $\Gamma^\infty(\mathsf{End}(E))$ to be isomorphic to $C^\infty(M)$ is by a *line bundle* $L \longrightarrow M$ we have the following result: Implementing the isomorphism by choosing an appropriate automorphism of $C^\infty(M)$ we can arrange to get a *symmetric* bimodule where $C^\infty(M)$ acts on $\Gamma^\infty(L)$ from left and right in the same way. Since $\Gamma^\infty(L)$ determines L completely, we are left with the classification of line bundles, which is done via the Chern class. Since the automorphisms of $C^\infty(M)$ are just the pull-backs with diffeomorphism, we arrive at the result that

$$\mathsf{Pic}(C^\infty(M)) = \mathrm{Diffeo}(M) \ltimes \mathrm{H}^2(M, \mathbb{Z}) \tag{3}$$

as group where the semidirect product comes from the usual action of the diffeomorphism on the integral cohomology classes by pull-backs.

Let us now pass to the more specific case where the underlying scalars are of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with an ordered ring R and $\mathrm{i}^2 = -1$. In this case we have on one hand the complex conjugation in C and on the other hand, inherited from the ordering of R, the notion of positivity. We want to transfer this now to algebras over C as well: instead of general unital algebras we consider now *-algebras, i.e. algebras equipped with an anti-linear, involutive anti-automorphism, the *-*involution* denoted by $a \mapsto a^*$. Then we can speak of positivity in the following way: a linear functional $\mathcal{A} \longrightarrow \mathsf{C}$ is *positive* if $\omega(a^* a) \geq 0$ for all $a \in \mathcal{A}$. Using this, we say that an algebra element $a \in \mathcal{A}$ is *positive* if $\omega(a) \geq 0$ for all positive linear functionals. We denote the positive algebra elements by $\mathcal{A}^+$. Clearly, $a^* a$ and any convex combination of such elements are in $\mathcal{A}^+$ but there might be more. Note also that there are other scenarios where one employs a more sophisticated version of positivity for the price of additional structures like for O^*-algebras, see e.g. [31].

As before we want now to replace the obvious notion of *-homomorphism by some bimodule version. Here we rely on the particular case of C^*-algebras where this theory was studied first. However, the essence is entirely algebraic and thus works in our general setting as well. We consider again a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, now together with a *inner product*

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \times {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{A}, \tag{4}$$

such that $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is C-linear in the second argument and satisfies $\langle x, y \cdot a \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}}\, a$ as well as $\langle x, y \rangle_{\mathcal{A}} = (\langle y, x \rangle_{\mathcal{A}})^*$, and $\langle b \cdot x, y \rangle_{\mathcal{A}} = \langle x, b^* \cdot y \rangle_{\mathcal{A}}$ for all $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$. Finally, we require $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ to be non-degenerate. In this case we will call ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ together with $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ an *inner product* $(\mathcal{B}, \mathcal{A})$*-bimodule*. Note that the definition is not symmetric in $\mathcal{A}$ and $\mathcal{B}$. As a second variant we consider a *completely positive* inner product where we require in addition

$$\left(\langle x_i, x_j \rangle_{\mathcal{A}} \right) \in M_n(\mathcal{A})^+. \tag{5}$$

Here $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and the *-algebra structure on $M_n(\mathcal{A})$ is the usual one induced from the one on $\mathcal{A}$. In general it will be quite difficult to determine whether an inner product is completely positive because we first have to determine all positive linear functionals of $M_n(\mathcal{A})$ in order to determine the positivity of a matrix. However, there are many examples and more particular situations where things simplify. A bimodule with such a completely positive inner product is then called a *pre Hilbert bimodule*.

The inner product bimodules as well as pre Hilbert bimodules are now used to define (bi-) categories Bimod^* and $\mathsf{Bimod}^{\mathrm{str}}$ ($\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively) by adapting the notion of the tensor product appropriately. In fact, it is rather easy to see that the definition

$$\langle \phi \otimes x, \psi \otimes y \rangle_{\mathcal{A}}^{\mathcal{F} \otimes \mathcal{E}} = \left\langle x, \langle \phi, \psi \rangle_{\mathcal{B}}^{\mathcal{F}} \cdot y \right\rangle_{\mathcal{A}}^{\mathcal{E}} \tag{6}$$

for $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and $\phi, \psi \in {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ extends to a well-defined inner product on ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *except* that it might still be degenerate. However, a further quotient by the degeneracy space will yield a well-defined inner product. The quotient together with this new inner product then defines a new composition, the tensor product $\widehat{\otimes}_{\mathcal{B}}$. It is slightly more tricky to see that also complete positivity is preserved by $\widehat{\otimes}_{\mathcal{B}}$. In both cases, with this new tensor product we get honest categories Bimod^* and $\mathsf{Bimod}^{\mathrm{str}}$ after passing to isometric isomorphism classes or, without identifying, bicategories $\underline{\mathsf{Bimod}}^*$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}$, respectively. In the bicategory case we have to require that the 2-morphisms are not just bimodule morphisms but *adjointable* bimodule morphisms, i.e. there exists an adjoint with respect to the inner product. In analogy to Definition 2.1 we can now state the following:

Definition 2.4 (*-Morita and strong Morita equivalence) *Two unital *-algebras $\mathcal{A}$ and $\mathcal{B}$ are *-Morita equivalent or strongly Morita equivalent if they are isomorphic in* Bimod^* *or in* $\mathsf{Bimod}^{\mathrm{str}}$*, respectively. A bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ representing an invertible arrow in* Bimod^* *or* $\mathsf{Bimod}^{\mathrm{str}}$ *is called a *-equivalence or strong equivalence bimodule, respectively. The groupoid of invertible arrows in* Bimod^* *and in* $\mathsf{Bimod}^{\mathrm{str}}$ *is called the *-Picard groupoid* Pic^* *and the strong Picard groupoid* $\mathsf{Pic}^{\mathrm{str}}$*. The isotropy groups of them at $\mathcal{A}$ are called the *-Picard group* $\mathsf{Pic}^*(\mathcal{A})$ *and the strong Picard group* $\mathsf{Pic}^{\mathrm{str}}(\mathcal{A})$ *of $\mathcal{A}$.*

For unital *-algebras one can prove that the tensor product $\widehat{\otimes}$ of *-equivalence or strong equivalence bimodules does not require the additional quotient procedure. Moreover, it is easy to see that after forgetting about the inner product one obtains an equivalence bimodule in the ring-theoretic sense. This yields well-defined groupoid morphisms

$$\begin{array}{ccc} \mathsf{Pic}^{\mathrm{str}} & \longrightarrow & \mathsf{Pic}^* \\ & \searrow \quad \swarrow & \\ & \mathsf{Pic} & \end{array} \tag{7}$$

by successively forgetting structure. On the level of the Picard groups many properties of these forgetful morphisms have been discussed in [14]. In particular, even for a C^*-algebra $\mathcal{A}$ over $\mathbb{C}$ the group morphism $\mathsf{Pic}^{\mathrm{str}}(\mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{A})$ is in general nor surjective, though always injective.

In a last step we consider now formal deformations. Here we will consider a unital algebra $\mathcal{A}$ over C together with a formal associative deformation $\star$ in the sense of Gerstenhaber [20]. This means we have C-bilinear maps $C_r \colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$a \star b = \sum_{r=0}^{\infty} \hbar^r C_r(a, b) \tag{8}$$

defines a $\mathsf{C}[[\hbar]]$-bilinear associative product for $\mathcal{A}[[\hbar]]$ where $C_0(a, b) = ab$ is the original product of $\mathcal{A}$. Since we work in the unital setting, we require that $\mathbb{1}_{\mathcal{A}}$ is still a unit with respect to $\star$. We will abbreviate this by $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\hbar]], \star)$.

Passing to formal power series gives rise to various classical limit maps, where we set $\hbar = 0$. On the level of algebra elements we write

$$\operatorname{cl}(a) = a_0 \quad \text{for} \quad a = \sum_{r=0}^{\infty} \hbar^r a_r \in \mathcal{A}[[\hbar]], \tag{9}$$

which is a C-linear algebra morphism $\operatorname{cl} \colon (\mathcal{A}[[\hbar]], \star) \longrightarrow \mathcal{A}$. If we have two deformations $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{A}}$ of $\mathcal{B}$ and $\mathcal{A}$, respectively, then for a $(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}})$-bimodule ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}}$ we define the classical limit as the quotient

$$\operatorname{cl} \colon {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}} \longrightarrow \operatorname{cl}({}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}}) = {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}} \big/ \hbar\, {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}}. \tag{10}$$

The result $\operatorname{cl}({}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}})$ is viewed as a $(\mathcal{B}, \mathcal{A})$-bimodule. It is now easy to see that $\operatorname{cl}({}_{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{A}}}) \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ and that the tensor product is compatible with cl up to natural isomorphisms. This simple observation can be summarized as follows. To keep track of the ring we denote the category of bimodule over algebras over C and $\mathsf{C}[[\hbar]]$ by $\mathsf{Bimod}_{\mathsf{C}}$ and $\mathsf{Bimod}_{\mathsf{C}[[\hbar]]}$, respectively. Then one has the sub-category $\mathbf{Bimod} \subseteq \mathsf{Bimod}_{\mathsf{C}[[\hbar]]}$ of those algebras over $\mathsf{C}[[\hbar]]$ which are formal deformations of algebras over C as above. For the morphisms in $\mathbf{Bimod}$ we allow *all* bimodules and not just those of the form $\boldsymbol{\mathcal{E}} = \mathcal{E}[[\hbar]]$. Then cl induces a functor

$$\operatorname{cl} \colon \mathbf{Bimod} \longrightarrow \mathsf{Bimod}_{\mathsf{C}}, \tag{11}$$

called the classical limit functor. Hence we also get immediately a groupoid morphism

$$\operatorname{cl} \colon \mathbf{Pic} \longrightarrow \mathsf{Pic}_{\mathsf{C}}, \tag{12}$$

which ultimately results in a group morphism

$$\operatorname{cl} \colon \mathsf{Pic}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathsf{Pic}(\mathcal{A}) \tag{13}$$

for every deformation $\boldsymbol{\mathcal{A}}$ of $\mathcal{A}$. Again, one is interested in understanding the properties of this classical limit morphism (12) and in particular the behaviour of the Picard groups under deformation (13). Many results on this have been obtained in [13].

Also for *-algebras one can define a classical limit functor similar to (11): here one considers *Hermitian* deformations which are formal deformations $\star$ such that the original *-involution is still a *-involution also with respect to the deformed product $\star$. Then the quotient procedure for the bimodules has to be modified as the resulting inner product on the naive quotient ${}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}} / \hbar\, {}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{A}}}$ will in general be degenerate. Thus we have to divide by the degeneracy space and get a functor

$$\operatorname{cl} \colon \mathbf{Bimod}^{*} \longrightarrow \mathsf{Bimod}^{*}{}_{\mathsf{C}}, \tag{14}$$

which also gives groupoid and group morphisms for the *-Picard groupoid and *-Picard groups, respectively. Finally, for the strong version one has to take care once more: the complete positivity of the inner product on the classical limit may fail. The way out is to allow only those Hermitian deformations which are *completely positive deformations*, see e.g. [15] for examples.

3 Deformation Quantization

After the algebraic preliminaries on Morita theory we pass now to the geometric situation of deformation quantization of Poisson manifolds.

Let (M, π_1) be a Poisson manifold with Poisson tensor $\pi_1 \in \Gamma^\infty(\Lambda^2 TM)$. Then a *formal deformation* of π_1 is a formal series

$$\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]], \tag{15}$$

such that we still have the Jacobi identity $[\![\pi, \pi]\!] = 0$. Such a formal series is also called a formal Poisson tensor. The set of all formal Poisson tensors is denoted by $\underline{\mathrm{FPoiss}}(M)$ and those with fixed first order π_1 are denoted by $\underline{\mathrm{FPoiss}}(M, \pi_1)$. A *formal vector field* is a formal series $X = \hbar X_1 + \hbar^2 X_2 + \cdots \in \hbar\Gamma^\infty(TM)[[\hbar]]$. For a formal vector field X the exponential series $\exp(\mathscr{L}_X)$ of the Lie derivative is a well-defined operator on formal series with coefficients in some type of tensor fields on M. We call $\exp(\mathscr{L}_X)$ the *formal diffeomorphism* induced by X. The Baker-Campbell-Hausdorff theorem shows that the composition of two formal diffeomorphisms $\exp(\mathscr{L}_X)$ and $\exp(\mathscr{L}_Y)$ is again a formal diffeomorphism $\exp(\mathscr{L}_{\mathrm{BCH}(X,Y)})$ since both X and Y start in first order of $\hbar$ making the BCH series $\mathrm{BCH}(X, Y)$ a well-defined formal vector field again. This way, we obtain the *group of formal diffeomorphisms* $\mathrm{FDiffeo}(M)$ acting on various tensor fields. Indeed, one can think of formal vector fields, formal Poisson structures, formal diffeomorphisms, etc. as the ∞-jet around $\hbar = 0$ of vector fields, Poisson structures, diffeomorphisms, etc., depending smoothly on the parameter $\hbar$. However, we will not need this point of view here.

Since the Schouten bracket $[\![\cdot, \cdot]\!]$ is natural with respect to the Lie derivative, it is clear that a formal diffeomorphism $\exp(\mathscr{L}_X)$ maps a formal Poisson structure π to a formal Poisson structure $\exp(\mathscr{L}_X)(\pi)$ again. Moreover, the first order term π_1 is preserved by this action. This motives the definition that two formal Poisson structures π and π' are called *equivalent* if they are in the same $\mathrm{FDiffeo}(M)$-orbit, i.e. if there is a formal vector field such that

$$\mathrm{e}^{\mathscr{L}_X}(\pi) = \pi'. \tag{16}$$

In this case we necessarily have $\pi_1 = \pi_1'$ and we write $\pi \sim \pi'$. The equivalence classes of this equivalence relation are then denoted by

$$\mathrm{FPoiss}(M) = \underline{\mathrm{FPoiss}}(M) \big/ \mathrm{FDiffeo}(M) \quad \text{and} \quad \mathrm{FPoiss}(M, \pi_1) = \underline{\mathrm{FPoiss}}(M, \pi_1) \big/ \mathrm{FDiffeo}(M). \tag{17}$$

They are the moduli space for the inequivalent formal deformations of a given Poisson structure π_1. In general, it will be very complicated to determine the set $\mathrm{FPoiss}(M, \pi_1)$ for a given Poisson structure π_1. By abstract deformation theory one can say that $[\pi_2]$ is a well-defined class in the second Poisson cohomology of π_1 but it is not clear which such infinitesimal deformations can actually be lifted to formal deformations of all order in $\hbar$. If however, π_1 is symplectic and comes from a symplectic form ω_1 then the moduli space $\mathrm{FPoiss}(M, \pi_1)$ is easily be described by the inequivalent formal deformations of ω. Here the result is

$$\mathrm{FPoiss}(M, \pi_1) = [\omega] + \hbar\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]], \tag{18}$$

where we have (artificially) put an affine space modeled on $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]]$ instead of $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})[[\hbar]]$ itself, just to keep track of the symplectic form we started with. The proofs of these facts are well-known and can e.g. be found in the textbook [33, Sect. 4.2.4].

Let us now recall the basic notions from deformation quantization [2], see also the textbook [33] for a detailed exposition. On a Poisson manifold (M, π_1) a *star product* $\star$ is a formal associative deformation of $C^\infty(M)$ as in (8) with the additional requirements that the first order commutator

$$C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\} \tag{19}$$

gives the Poisson bracket coming from π_1. Moreover, one requires that the C_r are bidifferential operators. The set of all star products on M is sometimes denoted by $\underline{\mathrm{Def}}(M)$ and the star products quantizing the Poisson structure π_1 are then $\underline{\mathrm{Def}}(M, \pi_1)$.

If $D = \hbar D_1 + \hbar^2 D_2 + \cdots \in \hbar\,\mathrm{DiffOp}(M)[[\hbar]]$ is a formal series of differential operators we construct analogously to (16) an action on star products: the exponential series $T = \exp(D)$ is a well-defined formal series of differential operators, now starting with the identity, i.e. $T = \mathsf{id} + \hbar T_1 + \cdots$. Conversely, every such formal series is of this form as we can always build $D = \log(T)$ as a well-defined formal power series. Note that D vanishes on constants iff T is the identity on constants. Now if $\star$ is a star product for π_1 then also

$$f \star' g = T^{-1}(Tf \star Tg) \tag{20}$$

is easily shown to be a star product quantizing the same Poisson structure π_1. Here we need that $D = \log(T)$ vanishes on constants in order to have again $1 \star' f = f = f \star' 1$. This allows to interpret the operators D as quantum analogs of formal vector fields while the operators T are the quantum analogs of formal diffeomorphisms. We call such an operator T an *equivalence transformation*. Clearly, we get a group structure by multiplying equivalence transformations which corresponds to the Lie algebra structure of the operators D coming from the commutator. We end up with an action of the group of equivalence transformations on $\underline{\mathrm{Def}}(M)$ which preserves each $\underline{\mathrm{Def}}(M, \pi_1)$. This allows to define two star products $\star$ and $\star'$ to be *equivalent*, denoted by $\star \sim \star'$, if they are in the same orbit under the action (20) of the equivalence transformations. This gives us the analog of (17) and we set

$$\mathrm{Def}(M) = \underline{\mathrm{Def}}(M)\big/\sim \quad \text{and} \quad \mathrm{Def}(M,\pi_1) = \underline{\mathrm{Def}}(M,\pi_1)\big/\sim. \tag{21}$$

One of the major achievements in deformation quantization is now the famous statement of Kontsevich that the two moduli spaces $\mathrm{FPoiss}(M, \pi_1)$ and $\mathrm{Def}(M, \pi_1)$ are in bijection for every Poisson structure π_1. More precisely, the formality map $\mathcal{K}$ of Kontsevich gives a construction where a formal Poisson structure $\pi = \hbar\pi_1 + \hbar^2\pi_2 + \cdots$ is used to build a formal star product $\star_\pi$ in such a way that $\star_\pi \sim \star_{\pi'}$ iff $\pi \sim \pi'$. The precise construction

$$\pi \;\mapsto\; \star_\pi \tag{22}$$

requires the formality map $\mathcal{K}$ and is involved, both from the conceptual point of view as well as technically, see [27] for further details. Thus the choice of a formality map results in a bijection

$$\mathcal{K}_*\colon\ \mathrm{FPoiss}(M,\pi_1) \longrightarrow \mathrm{Def}(M,\pi_1). \tag{23}$$

As a remark we note that for a real formal Poisson structure $\pi = \overline{\pi}$ Kontsevich's formality on $\mathbb{R}^n$ produces a Hermitian star product, i.e. $\overline{f \star_\pi g} = \overline{g} \star_\pi \overline{f}$. Also the global formality map of Dolgushev has this property [18]. Finally, one can show that a Hermitian star product is always a completely positive deformation [15].

4 Morita Equivalence of Star Products

The main question concerning Morita theory in deformation quantization is now which star products are Morita equivalent. Since we have a good understanding of the equivalence classes of star products in terms of the equivalence classes of formal Poisson tensors one can refine the task as follows: describe the Morita equivalence of $\star_\pi$ and $\star_{\pi'}$ in terms of the equivalence classes of π and π'. Indeed, since isomorphic algebras are Morita equivalence the Morita equivalence of $\star_\pi$ and $\star_{\pi'}$ will only depend on the *classes* of π and π'. Moreover, since every star product is equivalent (and hence isomorphic) to a $\star_\pi$, it suffices to consider those.

There are now two reasons for star products to be Morita equivalent which we would like to discuss separately. First it is clear that any Poisson diffeomorphism $\Phi\colon (M, \pi_1) \longrightarrow (M, \pi_1)$ maps $\star_\pi$ to an isomorphic star product

$$\star_\pi \;\mapsto\; \Phi^*(\star_\pi) \sim \star_{\Phi^*\pi}, \tag{24}$$

where we use in the second step that the global formalities have a good covariance property up to equivalence. Since $\star_\pi$ and $\Phi^*(\star_\pi)$ are isomorphic (via Φ^*) they are Morita equivalent in a trivial way. This is the simple part of the description. Note that for star products quantizing diffeomorphic but different Poisson structures we still can have isomorphism via general diffeomorphisms. However, we study Morita equivalence of star products for a fixed Poisson structure π_1 on M.

The non-trivial part of Morita equivalence comes from the non-trivial classical equivalence bimodules, the line bundles. We know from the classical limit morphism (12) that the classical limit of an equivalence bimodule has to be a classical equivalence bimodule. Conversely, given a line bundle $L \longrightarrow M$ one can show that there is a unique way up to equivalence to deform the right module structure of $\Gamma^\infty(L)$ into a right module structure $\bullet$ for $\Gamma^\infty(L)[[\hbar]]$ with respect to the star product algebra $(C^\infty(M)[[\hbar]], \star)$. Moreover, it turns out that the module endomorphisms of this new, deformed right module are in bijection to $\Gamma^\infty(\mathsf{End}(L))[[\hbar]] = C^\infty(M)[[\hbar]]$. Hence we get an induced deformed product for $C^\infty(M)[[\hbar]]$ which turns out to be a star product $\star'$. Being isomorphic to the module endomorphisms this induces also a left module structure $\bullet'$ for $\star'$ such that we get a bimodule in the end. Finally, $\star'$ is uniquely determined by L and $\star$ up to equivalence since $\bullet$ was unique up to equivalence. Thus we get, on the level of equivalence classes, a well-defined map

$$L\colon [\star] \mapsto [\star']. \tag{25}$$

It is now easy to see that the deformed bimodule is still a Morita equivalence bimodule and all Morita equivalences arise this way. These results have been obtained very early in [8, 10].

The remaining task is now to compute $\star'$ for a given $\star = \star_\pi$ and determine the corresponding π' such that $\star' \sim \star_{\pi'}$.

The main idea how this is achieved is to use local transition functions to describe L. Then these transition functions allow for a suitable quantum analog obeying a cocycle identity with respect to the star product. This gives a local description of the deformed right module structure and hence also a local description of $\star'$. Moreover, locally the two star products $\star'$ and $\star$ are even equivalent and the difference between equivalence and Morita equivalence is a global effect. Next one uses a two-form B representing $2\pi i c_1(L)$, e.g. the curvature of a connection on L. Then the idea is to pass from the deformed transition functions to local expressions involving B. Here comes now the following construction into the game. Recall that closed two-forms act on Poisson structures, at least on the formal level, as follows: for $B \in \Gamma^\infty(\Lambda^2 T^*M)[[\hbar]]$ and a formal Poisson structure $\pi \in \hbar\Gamma^\infty(\Lambda^2 TM)[[\hbar]]$ we consider the corresponding (formal) bundle maps $B^\sharp$ and $\pi^\sharp$. Then the *gauge transformation* of π by B is defined to be the formal bivector field $\tau_B(\pi)$ characterized by

$$\tau_B(\pi)^\sharp = \pi^\sharp \circ \frac{1}{\mathsf{id} + B^\sharp \circ \pi^\sharp}. \tag{26}$$

Since π starts in first order of $\hbar$ the inverse is well-defined indeed.

If $B = \mathrm{d}\, A$ is an exact two-form then we can build a formal vector field out of the potential A via π and it is a straightforward computation that the corresponding formal diffeomorphism maps

$\tau_B(\pi)$ to π. Thus in the exact case the gauged bivector field is equivalent to π. In particular, it is again a formal Poisson structure. Since the Jacobi identity is a local property and since closed two-forms are locally exact, $\tau_B(\pi)$ is always a Poisson structure if $\mathrm{d}\,B = 0$. However, in general it will no longer be equivalent to π. Note however, that the first order of $\tau_B(\pi)$ coincides with π_1.

Another simple computation shows that the formal closed two-forms *act* on formal Poisson structures via (26) where we view the closed formal two-forms as abelian group with respect to the usual addition. Then the above results show that we get a well-defined action on the level of deRham cohomology classes on one hand and on equivalence classes of formal Poisson structures on the other hand, i.e.

$$\mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})[[\hbar]] \circlearrowleft \mathrm{FPoiss}(M) \quad \text{and} \quad \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})[[\hbar]] \circlearrowleft \mathrm{FPoiss}(M,\pi_1). \tag{27}$$

The following statement is now the first main result of [9]. The formula was already found in [26] on more heuristic arguments:

Theorem 4.1 *Let (M,π_1) be a Poisson manifold and π a formal Poisson structure with first order π_1. Let $L \longrightarrow M$ be a line bundle with $B \in \Gamma^\infty(\Lambda^2 T^*M)$ representing $2\pi\mathrm{i}c_1(L)$. Then the star product $\star'$ obtained from* (25) *out of $\star_\pi$ is equivalent to $\star_{\pi'}$ with*

$$\pi' = \tau_B(\pi). \tag{28}$$

This way, one has the full classification of star products up to ring-theoretic Morita equivalence: it only remains to take into account the simpler part, i.e. the Poisson diffeomorphisms as discussed above. It should be noted that the classification of star products by formal deformations of the Poisson structure π_1 requires the *choice* of a global formality. It should also be noted that in the proof one needs a formality which is differential and vanishes on constants. The global formality constructed in [18] fulfills all these requirements.

Remark 4.2 In the case of symplectic manifolds one has an alternative classification of star products by means of their *characteristic class* $c(\star) \in \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{H}^2_{\mathrm{dR}}(M,\mathbb{C})[[\hbar]]$, see e.g. [21]. Here $\star$ and $\star'$ are equivalent iff $c(\star) = c(\star')$. The important feature is that the class c can be defined intrinsically without reference to any construction method for the star products. In particular, it does not rely on the choice of a formality. The second main result of [9] is that the characteristic class $c(\star_\pi)$ of $\star_\pi$ constructed from a global formality for a formal Poisson structure deforming the symplectic π_1 is the "inverse" of the class of π. This matches the earlier result from [12] that in the symplectic case one has

$$c(\star') = c(\star) + 2\pi\mathrm{i}c_1(L) \tag{29}$$

where again $\star'$ is the star product from (25).

Remark 4.3 As a last remark here we note that the *-Morita and the strong Morita theory of star products is now fairly easy: from general results in [14] one knows that Hermitian star products are Morita equivalent iff they are strongly Morita equivalent. Moreover, the kernel and image of the groupoid morphisms (7) are very well understood for the case of star product algebras. Thus the additional requirements of having (completely positive) inner products on the equivalence bimodules do not cause any further difficulties.

5 Incorporating Symmetries

In many applications one is not just interested in Morita equivalence but in Morita equivalence compatible with some additional symmetry: Here several possibilities arise like a symmetry of a Lie algebra $\mathfrak{g}$ acting on all algebras by derivations or a symmetry of a group G acting by automorphisms. To combine these notions it is convenient to consider a Hopf algebra symmetry.

Let H be a Hopf algebra over C which should encode the type of symmetry which we want to study. Then we consider H-module algebras, i.e. algebras $\mathcal{A}$ over C which carry an action of H: there is a left module structure of H on $\mathcal{A}$ denoted by

$$\triangleright \colon H \times \mathcal{A} \longrightarrow \mathcal{A}, \tag{30}$$

such that in addition one has $g \triangleright (ab) = (g_{(1)} \triangleright a)(g_{(2)} \triangleright b)$ for all $g \in H$ and $a, b \in \mathcal{A}$. Here we use the Sweedler notation for the coproduct $\Delta(g) = g_{(1)} \otimes g_{(2)}$ as usual. In the case of *-algebras we require a Hopf *-algebra and the action should fulfill $(g \triangleright a)^* = S(g)^* \triangleright a^*$ where S is the antipode of H. If $H = \mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra then $\triangleright$ corresponds just to a Lie algebra action by derivations and if $H = \mathsf{C}[G]$ is the group algebra of some group G, then $\triangleright$ reduces to a group representation by automorphisms. Thus we cover the two cases mentioned above.

In a next step we consider bimodules. On a $(\mathcal{B}, \mathcal{A})$-bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ we want to implement also an action of H. We require that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a left H-module such that in addition

$$g \triangleright (b \cdot x) = (g_{(1)} \triangleright b) \cdot (g_{(2)} \triangleright x) \quad \text{and} \quad g \triangleright (x \cdot a) = (g_{(1)} \triangleright x) \cdot (g_{(2)} \triangleright a) \tag{31}$$

for all $g \in H$, $b \in \mathcal{B}$, $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, and $a \in \mathcal{A}$. In this case we call the bimodule H-covariant (or H-equivariant).

It is now a simple check that ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ with its induced left H-module structure is a H-covariant $(\mathcal{A}, \mathcal{A})$-bimodule. Moreover, the tensor product $\otimes$ of H-covariant bimodules gives again a H-covariant bimodule by defining the H-action on the tensor product according to

$$g \triangleright (\phi \otimes x) = (g_{(1)} \triangleright \phi) \otimes (g_{(2)} \triangleright x) \tag{32}$$

for $g \in H$, $\phi \in {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$, and $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. Using now only H-equivariant bimodule morphisms to relate H-covariant bimodules we obtain a category Bimod_H of unital C-algebras with H-action as objects and isomorphism classes of H-covariant bimodules as morphisms. Not yet identifying bimodules up to isomorphisms yields again a bicategory, denoted by $\underline{\mathsf{Bimod}}^*_H$.

There is also a way to incorporate inner products. If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a inner product $(\mathcal{B}, \mathcal{A})$-bimodule with an H-action then the compatibility with the inner product we need is

$$g \triangleright \langle x, y \rangle_{\mathcal{A}} = \langle S(g_{(1)})^* \triangleright x, g_{(2)} \triangleright y \rangle_{\mathcal{A}} \tag{33}$$

for all $g \in H$ and $x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. In this case we call ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ a H-covariant inner product bimodule. For the morphisms between H-covariant inner product bimodules we take now the H-equivariant adjointable bimodule morphisms. Again, it is a routine check that $\widehat{\otimes}$ is compatible with this additional symmetry. This gives the category Bimod^*_H with objects being the unital *-algebras over C with a *-action of H and isometric H-equivariant isomorphism classes of H-covariant inner product bimodules as morphisms. If in addition we take completely positive inner product, no further compatibility is needed. We obtain the category $\mathsf{Bimod}^{\mathrm{str}}_H$. Again, we also have bicategories $\underline{\mathsf{Bimod}}^*_H$ and $\underline{\mathsf{Bimod}}^{\mathrm{str}}_H$ in this case, the check of the needed coherences is slightly more involved but still straightforward. Details on this can be found in [23, 25] as well as in [16].

It is now clear how to define Morita equivalence in the presence of symmetries: we use isomorphism in the categories Bimod_H, Bimod^*_H, and $\mathsf{Bimod}^{\mathrm{str}}_H$ to define *H-covariant Morita equivalence*, *H-covariant *-Morita equivalence* and *H-covariant strong Morita equivalence*, respectively. The corresponding groupoids of invertible arrows in these categories are then the *H-covariant Picard groupoid* Pic_H, the H-covariant *-Picard groupoid Pic^*_H, and the H-covariant strong Picard groupoid $\mathsf{Pic}^{\mathrm{str}}_H$, respectively.

In [16, 24, 25] many general statements about these notions of H-covariant Morita equivalence have been established. We mention just three aspects relevant for deformation quantization:

First one can define again a classical limit functor. Here one can even allow for a formal Hopf algebra deformation $\boldsymbol{H}$ of H with deformed structure maps as usual. Then we get again a subcategory $\boldsymbol{\mathsf{Bimod}}_{\boldsymbol{H}}$ of $\mathsf{Bimod}_{\boldsymbol{H}}$ (defined over $\mathsf{C}[[\hbar]]$ as before) with objects being those unital algebras $\boldsymbol{\mathcal{A}}$ over $\mathsf{C}[[\hbar]]$ with $\boldsymbol{H}$-action which are deformations of unital algebras $\mathcal{A}$ over C with H-action. Then we get a classical limit functor

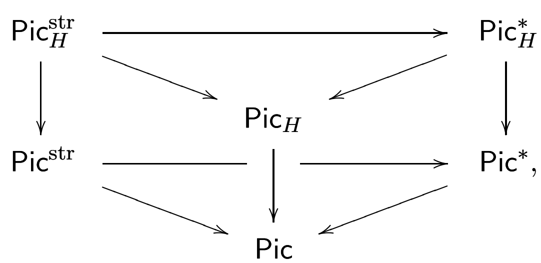
$$\mathrm{cl}\colon \boldsymbol{\mathsf{Bimod}}_{\boldsymbol{H}} \longrightarrow \mathsf{Bimod}_H, \tag{34}$$

restricting to a groupoid morphism on the level of the corresponding covariant Picard groupoids, eventually leading to a group morphism

$$\mathrm{cl}\colon \mathsf{Pic}_{\boldsymbol{H}}(\boldsymbol{\mathcal{A}}) \longrightarrow \mathsf{Pic}_H(\mathcal{A}). \tag{35}$$

Analogously, there is a *-version for Hermitian deformations and a strong version for completely positive deformations as well. We do not spell out the detail which should be clear. First steps in understanding the kernel of (35) are available in [16]. It should also be mentioned that the classical limit is already available on the level of the bicategories and gives a homomorphism of bicategories there (a weak form of a 2-functor).

Second, one can successively forget information. This gives groupoid morphisms leading to the commuting diagram



(36)

between the various types of Picard groupoids. Additionally, the classical limit fits into this diagram nicely and gives yet some more groupoid morphisms making the doubled diagram also commutative. Again, kernels and images of these forgetful morphisms have been studied in various contexts.

Third, one can study the Morita equivalence of crossed products using H-covariant Morita equivalence. Having an H-action on $\mathcal{A}$ gives us a crossed product algebra structure on the tensor product $\mathcal{A} \otimes H$ which we shall denote by $\mathcal{A} \rtimes H$. Moreover, for a H-covariant bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ one can construct a $(\mathcal{B} \rtimes H, \mathcal{A} \rtimes H)$-bimodule structure on the tensor product $\mathcal{E} \otimes H$, the resulting bimodule will then be denoted by $\mathcal{E} \rtimes H$. This results in a functor

$$\mathsf{Bimod}_H \longrightarrow \mathsf{Bimod} \tag{37}$$

restricting to a groupoid morphism $\mathsf{Pic}_H \longrightarrow \mathsf{Pic}$ and ultimately to a group morphism

$$\mathsf{Pic}_H(\mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{A} \rtimes H). \tag{38}$$

The same construction goes through in the case of inner product bimodules and pre Hilbert bimodules, see [25] for further details and examples.

6 $\mathfrak{g}$-Actions on Symplectic Manifolds

In this last section we apply our considerations on symmetries to the particular case of star products on symplectic manifolds which are invariant under a Lie algebra action of some finite-dimensional real Lie algebra $\mathfrak{g}$.

Thus let (M, ω) be a symplectic manifold endowed with an action of $\mathfrak{g}$, e.g. coming from an action of a Lie group G with $\mathfrak{g}$ as its Lie algebra. For technical reasons we require the action to preserve a connection ∇. This is in fact a rather mild requirement: if a Lie group G acts *properly* it preserves a connection, but there are other examples of non-proper actions which still preserve a connection as e.g. the linear action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathbb{R}^{2n}$ which preserves the canonical flat connection. Without restriction we can assume that ∇ is torsion-free and symplectic in addition.

It is now a well-known fact that Fedosov's construction of a star product, see e.g. [19], gives an invariant star product provided one has invariant geometric entrance data. More precisely, in the situation where one has an invariant connection the moduli space $\mathrm{Def}^{\mathfrak{g}}(M, \omega)$ of invariant star products $\underline{\mathrm{Def}}^{\mathfrak{g}}(M, \omega)$ modulo invariant equivalences is in bijection to the second *invariant* deRham cohomology $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}[[\hbar]]$. In fact, there is a $\mathfrak{g}$-invariant characteristic class

$$c^{\mathfrak{g}} \colon \underline{\mathrm{Def}}^{\mathfrak{g}}(M, \omega) \ni \star \mapsto c^{\mathfrak{g}}(\star) \in \frac{[\omega]}{\mathrm{i}\hbar} + \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}[[\hbar]], \tag{39}$$

inducing this bijection. Note that under the canonical forgetful map $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})$ the class $c^{\mathfrak{g}}(\star)$ is mapped to $c(\star)$. The proofs of this result and also the case of Lie group actions can be found in [4].

We are now interested in the $\mathfrak{g}$-covariant Morita theory of invariant symplectic star products. In more detail, we would like to have a refined statement analogously to (29) using the invariant characteristic class $c^{\mathfrak{g}}(\cdot)$ instead.

First we study the classical limit of a $\mathfrak{g}$-covariant equivalence bimodule $\mathcal{L}$ between two $\mathfrak{g}$-invariant star products $\star'$ acting from the left and $\star$ acting from the right. We know that there is a unique symplectomorphism Ψ such that the twisted equivalence bimodule ${}^{\Phi}\mathcal{L}$ for $\Phi^*(\star')$ and $\star$ has a symmetric equivalence bimodule as classical limit, i.e. the smooth sections of a line bundle L, unique up to isomorphism, with the same action of $C^\infty(M)$ from the left and the right. One can show that Ψ is necessarily $\mathfrak{g}$-equivariant and thus $\Psi^*(\star')$ is again a $\mathfrak{g}$-invariant star product. In a next step one shows that on the line bundle L we have a lift of the $\mathfrak{g}$-action on M together with an invariant connection ∇^L: the connection is obtained as the difference of the left and right multiplications in the first order of $\hbar$. In fact, one can show by this construction that the original $\mathcal{L}$ was isomorphic to the Ψ-twist of a $\mathfrak{g}$-covariant *deformation* of L with respect to two $\mathfrak{g}$-invariant star products $\star''$ and $\star$ such that $\star''$ is $\mathfrak{g}$-equivariantly equivalent to $\Psi^*(\star')$. The usage of $\star''$ allows to achieve that $\star''$ and $\star$ have the *same* first order term (and not just the same first order commutator).

In the situation of Hermitian star products and a covariant *-equivalence bimodule one has also a $\mathfrak{g}$-invariant pseudo Hermitian fiber metric h_0 on L and ∇^L is metric such that h_0 is the classical limit of the inner product on $\mathcal{L}$. Finally, in the strong equivalence case, h_0 is positive, i.e. a Hermitian fiber metric.

Conversely, given a line bundle L over M which allows for a lift of the $\mathfrak{g}$-action and given a $\mathfrak{g}$-invariant connection, we can always deform the classical bimodule structure into a quantum bimodule structure *preserving* the $\mathfrak{g}$-invariance. This is a simple application of Fedosov's construction adapted to vector bundles, see [32]. In the Hermitian case we can also deform a classical and $\mathfrak{g}$-invariant fiber metric, preserving its positivity. Since in the Fedosov construction it is very easy to keep track of the (invariant) characteristic classes $c^{\mathfrak{g}}(\cdot)$, one arrives at the following theorem [24]:

Theorem 6.1 *Let M be a symplectic manifold with $\mathfrak{g}$-action such that a $\mathfrak{g}$-invariant (symplectic torsion-free) connection of M exists. Moreover, let $\star'$ and $\star$ be two $\mathfrak{g}$-invariant (Hermitian) star products on M. Then $\star$ and $\star'$ are $\mathfrak{g}$-covariantly (strongly) Morita equivalent iff there is a $\mathfrak{g}$-equivariant symplectomorphism Ψ such that $\Psi^* c^{\mathfrak{g}}(\star') - c^{\mathfrak{g}}(\star)$ is in the image of the first map in*

$$\mathrm{H}^2_{\mathfrak{g}}(M, \mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}} \longrightarrow \mathrm{H}_{\mathrm{dR}}(M, \mathbb{C})^{\mathfrak{g}}, \tag{40}$$

and maps to a $2\pi\mathrm{i}$-integral class under the second map.

Here $\mathrm{H}^2_{\mathfrak{g}}(M, \mathbb{C})$ denotes the $\mathfrak{g}$-equivariant deRham cohomology where we use the Cartan model to define this cohomology. In particular, we do neither require a Lie group action integrating the Lie algebra action of $\mathfrak{g}$ nor any compactness assumptions.

In fact, one even knows that the equivalence bimodules are obtained from $\mathfrak{g}$-equivariant symplectomorphism Ψ on one hand and from $\mathfrak{g}$-invariant deformations of line bundles L with a lifted action on the other hand. Note also, that with the usual arguments it is now easy to lift from the infinitesimal symmetry by $\mathfrak{g}$ to the integrated symmetry by the connected and simply-connected Lie group G integrating $\mathfrak{g}$. Finally, much of the above arguments can be carried through also in the case of a discrete group symmetry, only the existence of invariant classical objects is more involved in this case.

Going beyond the symplectic case to the general Poisson case should be possible by using the equivariant formality theorem of Dolgushev [18] which are based on the existence of an invariant connection on M: it is again the same classical data one has to invest and whose existence is well-studied under various circumstances.

Remark 6.2 It is clear that the question of $\mathfrak{g}$-covariant Morita theory ultimately should result in an understanding of the behaviour of Morita theory under *reduction* of star products. Concerning the reduction aspect, one has by now a rather good understanding, starting from the BRST approach in [7], see also [5, 17]. In [22] the representation theory of the reduced algebras was studied in detail, including some aspects of Morita theory. The usage of invariant star products (and even better: invariant star products with a quantum momentum map) will hopefully allow also to treat the Morita theory of star products on singular quotients, see e.g. [6, 29].

References

[1] Ara, P.: *Morita equivalence for rings with involution.* Alg. Rep. Theo. **2** (1999), 227–247.

[2] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151.

[3] Bénabou, J.: *Introduction to Bicategories.* In: *Reports of the Midwest Category Seminar*, 1–77. Springer-Verlag, 1967.

[4] Bertelson, M., Bieliavsky, P., Gutt, S.: *Parametrizing Equivalence Classes of Invariant Star Products.* Lett. Math. Phys. **46** (1998), 339–345.

[5] Bordemann, M.: *(Bi)Modules, morphisms, and reduction of star-products: the symplectic case, foliations, and obstructions.* Trav. Math. **16** (2005), 9–40.

[6] Bordemann, M., Herbig, H.-C., Pflaum, M. J.: *A homological approach to singular reduction in deformation quantization.* In: Chéniot, D., Dutertre, N., Murolo, C., Trotman, D., Pichon, A. (eds.): *Singularity theory*, 443–461. World Scientific Publishing, Hackensack, 2007. Proceedings of the Singularity School and Conference held in Marseille, January 24–February 25, 2005. Dedicated to Jean-Paul Brasselet on his 60th birthday.

[7] Bordemann, M., Herbig, H.-C., Waldmann, S.: *BRST Cohomology and Phase Space Reduction in Deformation Quantization.* Commun. Math. Phys. **210** (2000), 107–144.

[8] Bursztyn, H.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products.* Int. Math. Res. Not. **2002**.16 (2002), 821–846.

[9] Bursztyn, H., Dolgushev, V., Waldmann, S.: *Morita equivalence and characteristic classes of star products.* Preprint **arXiv:0909:4259** (September 2009), 57 pages. To appear in Crelle's J. reine angew. Math.

[10] Bursztyn, H., Waldmann, S.: *Deformation Quantization of Hermitian Vector Bundles.* Lett. Math. Phys. **53** (2000), 349–365.

[11] Bursztyn, H., Waldmann, S.: *Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization.* J. Geom. Phys. **37** (2001), 307–364.

[12] Bursztyn, H., Waldmann, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds.* Commun. Math. Phys. **228** (2002), 103–121.

[13] Bursztyn, H., Waldmann, S.: *Bimodule deformations, Picard groups and contravariant connections.* K-Theory **31** (2004), 1–37.

[14] Bursztyn, H., Waldmann, S.: *Completely positive inner products and strong Morita equivalence.* Pacific J. Math. **222** (2005), 201–236.

[15] Bursztyn, H., Waldmann, S.: *Hermitian star products are completely positive deformations.* Lett. Math. Phys. **72** (2005), 143–152.

[16] Calon, P.: *Klassischer Limes für H-kovariante Darstellungstheorie.* master thesis, Fakultät für Mathematik und Physik, Physikalisches Institut, Albert-Ludwigs-Universität, Freiburg, 2010.

[17] Cattaneo, A. S., Felder, G.: *Relative formality theorem and quantisation of coisotropic submanifolds.* Adv. Math. **208** (2007), 521–548.

[18] Dolgushev, V. A.: *Covariant and equivariant formality theorems.* Adv. Math. **191** (2005), 147–177.

[19] Fedosov, B. V.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[20] Gerstenhaber, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964), 59–103.

[21] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J. Geom. Phys. **29** (1999), 347–392.

[22] Gutt, S., Waldmann, S.: *Involutions and Representations for Reduced Quantum Algebras.* Adv. Math. **224** (2010), 2583–2644.

[23] Jansen, S.: *H-Äquivariante Morita-Äquivalenz und Deformationsquantisierung.* PhD thesis, Fakultät für Mathematik und Physik, Physikalisches Institut, Albert-Ludwigs-Universität, Freiburg, November 2006.

[24] Jansen, S., Neumaier, N., Waldmann, S.: *Covariant Morita Equivalence of Star Products.* In preparation.

[25] Jansen, S., Waldmann, S.: *The H-covariant strong Picard groupoid.* J. Pure Appl. Alg. **205** (2006), 542–598.

[26] Jurčo, B., Schupp, P., Wess, J.: *Noncommutative Line Bundles and Morita Equivalence.* Lett. Math. Phys. **61** (2002), 171–186.

[27] Kontsevich, M.: *Deformation Quantization of Poisson manifolds.* Lett. Math. Phys. **66** (2003), 157–216.

[28] Lam, T. Y.: *Lectures on Modules and Rings*, vol. 189 in *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[29] Pflaum, M. J.: *On the deformation quantization of symplectic orbispaces.* Diff. Geom. Appl. **19** (2003), 343–368.

[30] Rieffel, M. A.: *Morita equivalence for C^*-algebras and W^*-algebras.* J. Pure. Appl. Math. **5** (1974), 51–96.

[31] Schmüdgen, K.: *Unbounded Operator Algebras and Representation Theory*, vol. 37 in *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, Boston, Berlin, 1990.

[32] Waldmann, S.: *Morita equivalence of Fedosov star products and deformed Hermitian vector bundles.* Lett. Math. Phys. **60** (2002), 157–170.

[33] Waldmann, S.: *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung.* Springer-Verlag, Heidelberg, Berlin, New York, 2007.