PRYM VARIETIES OF SPECTRAL COVERS

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Abstract. Given a possibly reducible and non-reduced spectral cover $\pi : X \to C$ over a smooth projective complex curve C we determine the group of connected components of the Prym variety $\mathrm{Prym}(X/C)$. We also describe the sublocus of characteristics $a \in \mathcal{A}$ for which the Prym variety $\mathrm{Prym}(X_a/C)$ is connected. These results extend special cases of work of Ngô who considered integral spectral curves.

1. Introduction

Recently there have been renewed interest in the topology of the Hitchin fibration. The Hitchin fibration is an integrable system associated to a complex reductive group G and smooth complex projective curve C. It was introduced by Hitchin [H] in 1987, originating in his study of a 2-dimensional reduction of the Yang-Mills equations. In 2006, Kapustin and Witten [KW] highlighted the importance of the Hitchin fibration for S-duality and the Geometric Langlands program. While the work of Ngô [N2] in 2008 showed that the topology of the Hitchin fibration is responsible for the fundamental lemma in the Langlands program. In Ngô's work and later in the work of Frenkel and Witten [FW] a certain symmetry of the Hitchin fibration plays an important role.

In the case of $\mathrm{G} = \mathrm{SL}_n$ this symmetry group is the Prym variety of a spectral cover. For the topological applications the determination of its group of components is the first step. Ngô works with integral, that is irreducible and reduced, spectral curves; but it is interesting to extend his results to non-integral curves. For reducible but reduced spectral curves it was done by Chaudouard and Laumon [CL], who proved the weighted fundamental lemma by generalizing Ngô's results to reduced spectral curves. In this note we determine the group of components of the Prym variety for non-reduced spectral curves as well.

Let C be a complex smooth projective curve of genus g. We associate to any spectral cover $\pi : X \to C$ a finite group K as follows: let $X = \bigcup_{i \in I} X_i$ be its decomposition into irreducible components X_i, let X_i^{red} be the underlying reduced curve of X_i, m_i the multiplicity of X_i^{red} in X_i and $\widetilde{X}_i^{red}$ the normalization of X_i. We denote by $\widetilde{\pi}_i : \widetilde{X}_i^{red} \to C$ the projection onto C and introduce the finite subgroups

$$K_i = \ker\left(\widetilde{\pi}_i^* : \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0(\widetilde{X}_i^{red})\right) \subset \mathrm{Pic}^0(C),$$

as well as the subgroups $(K_i)_{m_i} = [m_i]^{-1}(K_i)$, where $[m_i]$ denotes multiplication by m_i in the Picard variety $\mathrm{Pic}^0(C)$ parameterizing degree 0 line bundles over C. Finally, we put

$$K = \bigcap_{i \in I} (K_i)_{m_i} \subset \mathrm{Pic}^0(C). \tag{1}$$

We denote by C_n the multiple curve with trivial nilpotent structure of order n having underlying reduced curve C.

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We consider the norm map $\mathrm{Nm}_{X/C} : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(C)$ between the connected components of the identity elements of the Picard schemes of the curves X and C and define the Prym variety

$$\mathrm{Prym}(X/C) := \ker(\mathrm{Nm}_{X/C}).$$

Our main result is the following

Theorem 1.1. *Let $\pi : X \to C$ be a spectral cover of degree $n \geq 2$. With the above notation we have the following results:*

(1) *The group of connected components $\pi_0(\mathrm{Prym}(X/C))$ of the Prym variety $\mathrm{Prym}(X/C)$ equals*

$$\pi_0(\mathrm{Prym}(X/C)) = \widehat{K},$$

where $\widehat{K} = \mathrm{Hom}(K, \mathbb{C}^*)$ is the group of characters of K.

(2) *The natural homomorphism from the group of n-torsion line bundles $\mathrm{Pic}^0(C)[n]$ to $\pi_0(\mathrm{Prym}(X/C))$ given by*

$$\Phi : \mathrm{Pic}^0(C)[n] \longrightarrow \pi_0(\mathrm{Prym}(X/C)), \qquad \gamma \mapsto [\pi^*\gamma],$$

where $[\pi^*\gamma]$ denotes the class of $\pi^*\gamma \in \mathrm{Pic}^0(X)$ in $\pi_0(\mathrm{Prym}(X/C))$ is surjective. In particular, we obtain an upper bound for the order

$$|\pi_0(\mathrm{Prym}(X/C))| \leq n^{2g}.$$

(3) *The map Φ is an isomorphism if and only if X equals the non-reduced curve C_n with trivial nilpotent structure of order n.*

Similar descriptions of $\pi_0(\mathrm{Prym}(X/C)$ were given in [N1] in the case of integral spectral curves and by [CL] in the case of reducible but reduced spectral curves. Also [dCHM] use special cases for SL_2.

We fix a line bundle M over C and introduce the SL_n-Hitchin space

$$\mathcal{A}_n^0 = \bigoplus_{j=2}^{n} H^0(C, M^j).$$

For a characteristic $a \in \mathcal{A}_n^0$ we denote by $\pi : X_a \to C$ the associated spectral cover of degree n (see section 2.2) and by K_a the subgroup of $\mathrm{Pic}^0(C)$ defined in (1) and corresponding to the cover X_a. Let $\Gamma \subset \mathrm{Pic}^0(C)[n]$ be a cyclic subgroup of order d and let $\mathcal{A}_\Gamma^0 \subset \mathcal{A}_n^0$ denote the endoscopic sublocus of characteristics a such that the associated degree n spectral cover $\pi : X_a \to C$ comes from a degree $\frac{n}{d}$ spectral cover over the étale Galois cover of C with Galois group Γ (for the precise definition see section 5.1). With this notation we have the following

Theorem 1.2. *We have an equivalence*

$$\Gamma \subset K_a \qquad \Longleftrightarrow \qquad a \in \mathcal{A}_\Gamma^0.$$

This gives a description of the locus of characteristics $a \in \mathcal{A}_n^0$ such that the Prym variety $\mathrm{Prym}(X_a/C)$ is non-connected, because clearly $\mathcal{A}_{\Gamma_2}^0 \subset \mathcal{A}_{\Gamma_1}^0$ if $\Gamma_1 \subset \Gamma_2$.

Corollary 1.3. *The sublocus of characteristics $a \in \mathcal{A}_n^0$ such that the Prym variety* $\mathrm{Prym}(X_a/C)$ *is not connected equals the union*

$$\bigcup \mathcal{A}_\Gamma^0,$$

where Γ varies over all cyclic subgroups of prime order of $\mathrm{Pic}^0(C)[n]$.

The paper is organized as follows: in sections 2 and 3 we recall basic results on spectral covers and on the norm map $\mathrm{Nm}_{X/C}$. In sections 4 and 5 we prove the two main theorems.

Notation: Given a sheaf $\mathcal{F}$ over a scheme X and a subset $U \subset X$ we denote by $\mathcal{F}(U)$ or by $\Gamma(U, \mathcal{F})$ the space of sections of $\mathcal{F}$ over U.

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2. Preliminaries

2.1. **Two lemmas on abelian varieties.** Given an abelian variety A and a positive integer n we denote by $[n] : A \to A$ the multiplication by n, by $A[n] = \ker[n]$ its subgroup of n-torsion points and by $\hat{A} = \mathrm{Pic}^0(A)$ its dual abelian variety. We consider

$$f : A \longrightarrow B$$

a homomorphism between abelian varieties with kernel $K = \ker(f)$ which we assume to be finite. We let $\hat{f} : \hat{B} \to \hat{A}$ denote the dual map induced by f. We introduce the quotient abelian variety $A' = A/K$, so that we can write the homomorphism f as a composite map

$$f = j \circ \mu : A \xrightarrow{\mu} A' \xrightarrow{j} B,$$

where μ is an isogeny with kernel K and j is injective.

Lemma 2.1. *The group of connected components of the abelian subvariety* $\ker(\hat{f}) \subset \hat{B}$ *equals*

$$\pi_0(\ker(\hat{f})) = \widehat{K},$$

where $\widehat{K} = \mathrm{Hom}(K, \mathbb{C}^*)$ *is the group of characters of K.*

Proof. We consider the dual map

$$\hat{f} : \hat{B} \xrightarrow{\hat{j}} \hat{A}' \xrightarrow{\hat{\mu}} \hat{A},$$

and observe that $\hat{\mu} : \hat{A}' \to \hat{A}$ is an isogeny with kernel $\widehat{K}$ (see e.g. [BL] Proposition 2.4.3) and $\hat{j}$ has connected fibers (see e.g. [BL] Proposition 2.4.2). The lemma then follows. □

We also suppose that A and B are principally polarized abelian varieties, i.e. the polarizations induce isomorphisms $A \cong \hat{A}$ and $B \cong \hat{B}$.

Lemma 2.2. *We assume that there exists a homomorphism $g : B \to A$ such that $g \circ f = [n]$ for some integer n. Then the dual of the canonical inclusion $i : K \hookrightarrow A[n]$ is a surjective map*

$$\hat{i} : A[n] = \hat{A}[n] \longrightarrow \widehat{K},$$

which coincides with the restriction to $A[n]$ of the composite map $\hat{j} \circ \hat{g} : A \to B \to \hat{A}'$.

Proof. It suffices to observe that the isogeny $\hat{f} \circ \hat{g} = \widehat{[n]} = [n]$ factorizes as

$$[n] : A \xrightarrow{\hat{j}\circ\hat{g}} \hat{A}' \xrightarrow{\hat{\mu}} A,$$

that $\widehat{K} = \ker(\hat{\mu})$, and that $\hat{j} \circ \hat{g}$ is surjective. Hence a canonical surjection $A[n] \to \widehat{K}$, which is dual to the inclusion $i : K \hookrightarrow A[n]$, since $\widehat{[n]} = [n]$. □

2.2. **Spectral covers.** In this section we review some elementary facts on spectral covers.

Let C be a complex smooth projective curve and let M be a line bundle over C with $\deg M > 0$. We denote by $|M|$ the total space of M and by

$$\pi : |M| \longrightarrow C$$

the projection onto C. There is a canonical coordinate $t \in H^0(|M|, \pi^*M)$ on the total space $|M|$. The direct image decomposes as follows

$$\pi_*\mathcal{O}_{|M|} = \mathrm{Sym}^\bullet(M^{-1}) = \bigoplus_{i=0}^{\infty} M^{-i}.$$

Definition 2.3. *A spectral cover X of degree n over the curve C and associated to the line bundle M is the zero divisor in $|M|$ of a non-zero section $s \in H^0(|M|, \pi^*M^n)$.*

Since a spectral cover X is a subscheme of $|M|$, it is naturally equipped with a projection onto C, which we also denote by π. The decomposition of the section s according to the direct sum

$$\begin{aligned} H^0(|M|, \pi^*M^n) &= H^0(C, M^n \otimes \bigoplus_{i=0}^{\infty} M^{-i}) \quad \text{(projection formula)} \\ &= H^0(C, M^n) \oplus \cdots \oplus H^0(C, M) \oplus H^0(C, \mathcal{O}_C) \end{aligned}$$

gives an expression $s = s_0 + ts_1 + \cdots + t^{n-1}s_{n-1} + t^n s_n$ with $s_j \in H^0(C, M^{n-j})$. Here we also denote by s_j its pull-back to $|M|$. We note that there is an isomorphism $\pi^* : \mathrm{Pic}(C) \xrightarrow{\sim} \mathrm{Pic}(|M|)$, hence any line bundle over $|M|$ is of the form π^*L for some line bundle $L \in \mathrm{Pic}(C)$. More generally, we have a decomposition $H^0(|M|, \pi^*L) = H^0(C, L) \oplus H^0(C, LM^{-1}) \oplus \cdots \oplus H^0(C, LM^{-d})$ for some integer d and any section $s \in H^0(|M|, \pi^*L)$ can be written in the form

$$s = s_0 + ts_1 + \cdots + t^{d-1}s_{d-1} + t^d s_d, \qquad s_j \in H^0(C, LM^{-j}). \tag{2}$$

Lemma 2.4. *Let $\pi : X \to C$ be a spectral cover. Then the underlying reduced curve of each irreducible component of X is again a spectral cover associated to the line bundle M.*

Proof. It suffices to show that if the section $s \in H^0(|M|, \pi^*M^n)$ decomposes as $s = s^{(1)} \cdot s^{(2)}$ with $s^{(i)} \in H^0(|M|, \pi^*L_i)$ for $i = 1, 2$ and $L_1L_2 = M^n$, then $L_i = M^{n_i}$ and $n_1 + n_2 = n$. By (2) the section $s^{(i)}$ can be written as

$$s^{(i)} = s_0^{(i)} + ts_1^{(i)} + \cdots + t^{n_i}s_{n_i}^{(i)}, \tag{3}$$

with $s_j^{(i)} \in H^0(C, L_iM^{-j})$ and $s_{n_i}^{(i)} \neq 0$. Moreover $n_i = \deg(X^{(i)}/C)$ with $X^{(i)} = \mathrm{Zeros}(s^{(i)})$. By considering the highest order terms of (3) we obtain the relations $n_1 + n_2 = n$ and $s_{n_1}^{(1)} \cdot s_{n_2}^{(2)} = s_n \in H^0(C, \mathcal{O}_C)$. Since s_n is a non-zero constant section, we conclude that $L_i = M^{n_i}$. □

We introduce the SL_n- and GL_n-Hitchin space for the line bundle M over the curve C

$$\mathcal{A}_n^0(C,M) = \bigoplus_{j=2}^{n} H^0(C, M^j) \quad \text{and} \quad \mathcal{A}_n(C,M) = \bigoplus_{j=1}^{n} H^0(C, M^j).$$

If no confusion arises, we simply denote these vector spaces by $\mathcal{A}_n^0$ and $\mathcal{A}_n$. Note that $\mathcal{A}_n^0 \subset \mathcal{A}_n$. Given an element $a = (a_1, \ldots, a_n) \in \mathcal{A}_n$ with $a_j \in H^0(C, M^j)$, called a characteristic, we associate to a a spectral cover of degree n

$$\pi_a : X_a \longrightarrow C, \qquad X_a \subset |M|,$$

such that $X_a = \mathrm{Zeros}(s_a)$ and $s_a = t^n + a_1 t^{n-1} + \cdots a_{n-1} t + a_n \in H^0(|M|, \pi^* M^n)$.

Remark 2.5. Given $a \in \mathcal{A}_n$ we observe that the pull-back of the spectral cover $X_a \subset |M|$ by the automorphism of $|M|$ given by translation with the section $-\frac{a_1}{n}$, i.e. $(x,y) \mapsto (x, y - \frac{1}{n} a_1(x))$, equals the spectral cover $X_{a'}$ for some $a' \in \mathcal{A}_n^0$; equivalently do the change of variables $t \mapsto t - \frac{a_1}{n}$. Hence $X_a \cong X_{a'}$. It therefore suffices to restrict our study to spectral covers X_a for $a \in \mathcal{A}_n^0$.

2.3. **Non-reduced curves.** Let X be an irreducible curve contained in a smooth surface and let X^{red} denote its underlying reduced curve. Then there exists a global section s of a line bundle such that $X^{red} = \mathrm{Zeros}(s)$ and an integer k such that $X = \mathrm{Zeros}(s^k)$. We introduce the subschemes $X_i = \mathrm{Zeros}(s^i)$ for $i = 1, \ldots, k$, so that we have a filtration of X by closed subschemes

$$X^{red} = X_1 \subset X_2 \subset \cdots \subset X_k = X.$$

In that case we say that X has a nilpotent structure of order k. For any integer i we denote by $\mathcal{O}_{X_i}$ the structure sheaf of the subscheme $X_i \subset X$. Note that $\mathcal{O}_{X_i}$ is naturally a $\mathcal{O}_X$-module.

We need to recall a result on the local structure of coherent sheaves on non-reduced curves.

Theorem 2.6 ([D] Théorème 3.4.1)**.** *Let X be a curve with nilpotent structure of order k and let $\mathcal{E}$ be a coherent sheaf over X. Then there exists an open subset $V \subset X$ depending on $\mathcal{E}$ and integers m_i such that*

$$\mathcal{E}_{|V} \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathcal{O}_{X_i}^{\oplus m_i}{}_{|V}.$$

The sheaf on the right is called a quasi-free sheaf.

3. The norm map

In this section we recall the definition of the norm map and prove some of its properties. The standard references are [G1] section 6.5 and [G2] section 21.5.

3.1. **Definition.** Let C be a smooth projective curve and let

$$\pi : X \longrightarrow C$$

be any finite degree n covering of C. The $\mathcal{O}_C$-algebra $\pi_* \mathcal{O}_C$ will be denoted $\mathcal{B}$ and the group of invertible elements in $\mathcal{B}$ by $\mathcal{B}^*$. Note that $\mathcal{B}$ is a locally free sheaf of rank n. Let $U \subset C$ be an open subset and let $s \in \Gamma(U, \mathcal{B}) = \Gamma(\pi^{-1}(U), \mathcal{O}_X)$ be a local section. One defines ([G1] section 6.5.1)

$$N_{X/C}(s) := \det(\mu_s) \in \Gamma(U, \mathcal{O}_C)$$

where $\mu_s : \mathcal{B}_{|U} \to \mathcal{B}_{|U}$ is the multiplication with the section s. Moreover s is invertible in $\Gamma(U, \mathcal{B})$ if and only if $N_{X/C}(s)$ is invertible in $\Gamma(U, \mathcal{O}_X)$. We have the following obvious relations

$$N_{X/C}(s \cdot s') = N_{X/C}(s) \cdot N_{X/C}(s'), \qquad N_{X/C}(\lambda s) = \lambda^n N_{X/C}(s) \tag{4}$$

for any local sections s and s' of $\mathcal{B}$ and any local section λ of $\mathcal{O}_C$.

Let $\mathcal{L}$ be an invertible $\mathcal{B}$-module. We can choose a covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of C by open subsets and trivializations $\eta_\lambda : \mathcal{L}_{|U_\lambda} \xrightarrow{\sim} \mathcal{B}_{|U_\lambda}$. Then $(\omega_{\lambda,\mu})_{\lambda,\mu \in \Lambda}$ with

$$\omega_{\lambda,\mu} = \eta_\lambda \circ \eta^{-1}_{\mu | U_\lambda \cap U_\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathcal{B})$$

is a 1-cocycle with values in $\mathcal{B}^*$ and $(N_{X/C}(\omega_{\lambda,\mu}))_{\lambda,\mu \in \Lambda}$ is a 1-cocycle with values in $\mathcal{O}_C^*$, the sheaf of invertible elements of $\mathcal{O}_C$. This 1-cocycle determines an invertible sheaf over C, which we denote by $\mathrm{Nm}_{X/C}(\mathcal{L})$. The following properties easily follow from (4)

$$\mathrm{Nm}_{X/C}(\mathcal{L} \otimes \mathcal{L}') = \mathrm{Nm}_{X/C}(\mathcal{L}) \otimes \mathrm{Nm}_{X/C}(\mathcal{L}'), \qquad \mathrm{Nm}_{X/C}(\pi^* \mathcal{M}) = \mathcal{M}^{\otimes n},$$

for any two invertible sheaves $\mathcal{L}$ and $\mathcal{L}'$ over X and for any invertible sheaf $\mathcal{M}$ over C. We therefore obtain a group homomorphism between the Picard groups of the curves X and C called the norm map

$$\mathrm{Nm}_{X/C} : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(C), \qquad \mathcal{L} \mapsto \mathrm{Nm}_{X/C}(\mathcal{L}).$$

3.2. **Properties.** In the case X is smooth, the norm map $\mathrm{Nm}_{X/C}$ has a more explicit description in terms of divisors associated to line bundles.

Proposition 3.1 ([G2] section 21.5)**.** *Assume that X is a smooth curve. The above defined norm map coincides with the map*

$$\mathcal{L} = \mathcal{O}_X(\sum_{i \in I} n_i p_i) \mapsto \mathrm{Nm}_{X/C}(\mathcal{L}) = \mathcal{O}_C(\sum_{i \in I} n_i \pi(p_i)),$$

where $n_i \in \mathbb{Z}$ and $p_i \in X$. Note that this map is well-defined, i.e. $\mathrm{Nm}_{X/C}(\mathcal{L})$ only depends on the linear equivalence class of the divisor $\sum_{i \in I} n_i p_i$.

From now on the curve X is again an arbitrary cover of C.

Lemma 3.2. *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{T} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. We assume that $\mathcal{E}$ and $\mathcal{F}$ are torsion-free and that $\mathcal{T}$ is a torsion sheaf. Let $\varphi_\bullet$ be a local morphism over $\pi^{-1}(U)$ for some open subset $U \subset C$ between exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\ & & \downarrow^{\varphi_{\mathcal{E}}} & & \downarrow^{\varphi_{\mathcal{F}}} & & \downarrow^{\varphi_{\mathcal{T}}} & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{T} & \longrightarrow & 0. \end{array} \tag{5}$$

We consider the $\mathcal{O}_C$-linear maps induced by $\varphi_{\mathcal{E}}$ and $\varphi_{\mathcal{F}}$ in the direct image sheaves $\pi_*\mathcal{E}$ and $\pi_*\mathcal{F}$. Then we have the equality

$$\det(\varphi_{\mathcal{E}}) = \det(\varphi_{\mathcal{F}}) \in \Gamma(U, \mathcal{O}_U).$$

Proof. It is enough to show that the two local sections $\det(\varphi_{\mathcal{E}})$ and $\det(\varphi_{\mathcal{F}})$ coincide in the local rings $\mathcal{O}_{C,p}$ for every point $p \in U$. We put $A = \mathcal{O}_{C,p}$ and $K = Fr(A)$ and denote by E, F and T the corresponding A-modules of sheaves $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{T}$. Then E and F are free A-modules, hence we have injections $E \hookrightarrow E \otimes_A K$ and $F \hookrightarrow F \otimes_A K$. Since T is a torsion module, we have $T \otimes_A K = 0$. Then after localizing (5) at $p \in C$ and taking tensor product with K, we obtain the commutative diagram

$$\begin{array}{ccc}
E \otimes_A K & \xrightarrow{\cong} & F \otimes_A K \\
\Big\downarrow{\scriptstyle \varphi_E \otimes id} & & \Big\downarrow{\scriptstyle \varphi_F \otimes id} \\
E \otimes_A K & \xrightarrow{\cong} & F \otimes_A K,
\end{array}$$

where the horizontal maps are isomorphisms. So $\varphi_E \otimes id$ and $\varphi_F \otimes id$ are conjugate, hence $\det(\varphi_E \otimes id) = \det(\varphi_F \otimes id) \in K$. On the other hand $\det(\varphi_E \otimes id)$ and $\det(\varphi_F \otimes id)$ are elements in $A \subset K$, hence we obtain the desired equality. $\square$

In the sequel we will use the following properties of the norm map:

Corollary 3.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be two torsion-free $\mathcal{O}_X$-modules such that*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $\mathcal{T}$ is a torsion $\mathcal{O}_X$-module. Let $s \in \Gamma(U, \mathcal{B}) = \Gamma(\pi^{-1}(U), \mathcal{O}_X)$ be a local section of $\mathcal{B}$ over the open subset $U \subset C$. We consider the maps induced by the multiplication with the section s in the direct image sheaves $\pi_*\mathcal{E}$ and $\pi_*\mathcal{F}$, which we denote by $\mu_s^{\mathcal{E}} \in \mathrm{Hom}_{\mathcal{O}_C(U)}(\pi_*\mathcal{E}(U), \pi_*\mathcal{E}(U))$ and $\mu_s^{\mathcal{F}} \in \mathrm{Hom}_{\mathcal{O}_C(U)}(\pi_*\mathcal{F}(U), \pi_*\mathcal{F}(U))$. Then we have the equality

$$\det(\mu_s^{\mathcal{E}}) = \det(\mu_s^{\mathcal{F}}) \in \Gamma(U, \mathcal{O}_C).$$

Lemma 3.4. *Let $p : \widetilde{X} \to X$ be a covering such that the cokernel of the canonical inclusion $\mathcal{O}_X \hookrightarrow p_*\mathcal{O}_{\widetilde{X}}$ is a torsion $\mathcal{O}_X$-module. Then, for any invertible sheaf $\mathcal{L}$ over X we have*

$$\mathrm{Nm}_{\widetilde{X}/C}(p^*\mathcal{L}) = \mathrm{Nm}_{X/C}(\mathcal{L}).$$

Proof. We consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow p_*\mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{T} \longrightarrow 0, \tag{6}$$

where $\mathcal{T}$ is a torsion $\mathcal{O}_X$-module. Note that the direct image $p_*\mathcal{O}_{\widetilde{X}}$ is torsion-free. We denote the $\mathcal{O}_C$-algebra $\pi_*p_*\mathcal{O}_{\widetilde{X}}$ by $\widetilde{\mathcal{B}}$. Note that $\widetilde{\mathcal{B}}$ is a $\mathcal{B}$-module. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, $\eta_\lambda : \mathcal{L}_{|U_\lambda} \xrightarrow{\sim} \mathcal{B}_{|U_\lambda}$ be a set of trivializations of $\mathcal{L}$ as $\mathcal{B}$-module, and $(\omega_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ be the corresponding 1-cocycle with values in $\mathcal{B}^*$. Then the pull-back $p^*\mathcal{L}$ corresponds to a 1-cocycle $(p^*\omega_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ with values in $\widetilde{\mathcal{B}}^*$ obtained from $(\omega_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ under the canonical inclusion $\mathcal{B} \hookrightarrow \widetilde{\mathcal{B}}$. We now apply Corollary 3.3 to the exact sequence (6) and conclude that $N_{\widetilde{X}/C}(p^*\omega_{\lambda,\mu}) = N_{X/C}(\omega_{\lambda,\mu}) \in \Gamma(U_\lambda \cap U_\mu, \mathcal{O}_C)$. This proves the lemma. $\square$

Lemma 3.5. *Let $X = \bigcup_{i=1}^r X_i$ be the decomposition of X into irreducible components X_i. For an invertible sheaf $\mathcal{L}$, we denote by $\mathcal{L}_i = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}$ its restriction to X_i. Then, we have the equality*

$$\mathrm{Nm}_{X/C}(\mathcal{L}) = \bigotimes_{i=1}^{r} \mathrm{Nm}_{X_i/C}(\mathcal{L}_i).$$

Proof. We apply the previous lemma to the covering $p : \widetilde{X} = \coprod_{i=1}^r X_i \to X$ given by the disjoint union of the curves X_i. □

Lemma 3.6. *Let X be an irreducible curve and let $j : X^{red} \hookrightarrow X$ be its underlying reduced curve. Let m be the multiplicity of X^{red} in X. Then, for any invertible sheaf $\mathcal{L}$ over X we have*

$$\mathrm{Nm}_{X/C}(\mathcal{L}) = \mathrm{Nm}_{X^{red}/C}(j^*\mathcal{L})^{\otimes m}.$$

Proof. The $\mathcal{O}_C$-algebra $\mathcal{B} = \pi_*\mathcal{O}_X$ comes equipped with a nilpotent ideal sheaf $\mathcal{J} \subset \mathcal{B}$ such that $\mathcal{B}_{red} = \mathcal{B}/\mathcal{J} = \pi_*\mathcal{O}_{X^{red}}$. We choose a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of C by open subsets which trivialize the invertible sheaf $\mathcal{L}$, i.e. there exists isomorphisms $\eta_\lambda : \mathcal{L}_{|U_\lambda} \xrightarrow{\sim} \mathcal{B}_{|U_\lambda}$ and such that $\mathcal{J}_{|U_\lambda}$ is generated by an element $t \in \mathcal{B}_{|U_\lambda}$. Then multiplication with the invertible element $\omega_{\lambda,\mu} = \eta_\lambda \circ \eta^{-1}_{\mu|U_\lambda\cap U_\mu}$ preserves the filtration $t^{m-1}\mathcal{B}_{|U_\lambda} \subset \cdots \subset t\mathcal{B}_{|U_\lambda} \subset \mathcal{B}_{|U_\lambda}$ and acts on the quotients as multiplication with $\omega^{red}_{\lambda,\mu} \in \mathcal{B}^{red}_{|U_\lambda\cap U_\mu}$. It follows that $N_{X/C}(\omega_{\lambda,\mu}) = N_{X^{red}/C}(\omega^{red}_{\lambda,\mu})^m$, which proves the lemma. □

3.3. **The Prym variety** $\mathrm{Prym}(X/C)$**.** Given a spectral cover $\pi : X \to C$ we denote by $\mathrm{Pic}^0(X)$ the connected component of the identity element of the Picard group of X (see e.g. [K]). We then define the Prym variety $\mathrm{Prym}(X/C)$ to be the kernel of the Norm map $\mathrm{Nm}_{X/C}$

$$\mathrm{Prym}(X/C) := \ker\left(\mathrm{Nm}_{X/C} : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(C)\right).$$

We recall that n denotes the degree of the cover $\pi : X \to C$.

Definition 3.7. *Let $\mathcal{E}$ be a torsion-free $\mathcal{O}_X$-module. The rank of $\mathcal{E}$ is the rational number $r = \mathrm{rk}(\mathcal{E})$ satisfying*

$$\mathrm{rk}(\pi_*\mathcal{E}) = r \cdot n,$$

where $\mathrm{rk}(\pi_*\mathcal{E})$ is the rank of the vector bundle $\pi_*\mathcal{E}$ over C.

Proposition 3.8. *Let $\mathcal{E}$ be a torsion-free $\mathcal{O}_X$-module of integral rank r and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then we have the relation*

$$\det(\pi_*(\mathcal{E} \otimes \mathcal{L})) = \det(\pi_*\mathcal{E}) \otimes \mathrm{Nm}_{X/C}(\mathcal{L})^{\otimes r}.$$

Proof. We shall use the notation of section 3.1. Since $\mathcal{E}$ is torsion-free, the direct image $\pi_*\mathcal{E}$ is a locally free $\mathcal{O}_C$-module. We choose a covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of C for which both $\mathcal{L}$ and $\pi_*\mathcal{E}$ are trivialized, i.e., such that there exists local isomorphisms

$$\alpha_\lambda : \pi_*\mathcal{E}_{|U_\lambda} \xrightarrow{\sim} \mathcal{O}_{U_\lambda}^{\oplus rn}, \qquad \tau_\lambda : \mathcal{L}_{|U_\lambda} \xrightarrow{\sim} \mathcal{B}_{U_\lambda}.$$

Since $\mathcal{L}$ is trivial on U_λ we have an isomorphism

$$\mathrm{id}_{\mathcal{E}} \otimes \tau_\lambda : \mathcal{E} \otimes \mathcal{L}_{|U_\lambda} \longrightarrow \mathcal{E} \otimes \mathcal{B}_{|U_\lambda},$$

which we can consider as an isomorphism between $\mathcal{O}_C$-modules

$$\mathrm{id}_{\mathcal{E}} \otimes \tau_\lambda : \pi_*(\mathcal{E} \otimes \mathcal{L})_{|U_\lambda} \longrightarrow \pi_*\mathcal{E}_{|U_\lambda}.$$

We compose with α_λ to obtain a trivialization of $\pi_*(\mathcal{E}\otimes\mathcal{L})_{|U_\lambda}$

$$\beta_\lambda : \pi_*(\mathcal{E}\otimes\mathcal{L})_{|U_\lambda} \xrightarrow{\mathrm{id}_{\mathcal{E}}\otimes\tau_\lambda} \pi_*\mathcal{E}_{|U_\lambda} \xrightarrow{\alpha_\lambda} \mathcal{O}_{U_\lambda}^{\oplus rn}.$$

Given $\lambda,\mu\in\Lambda$ we can now write the transition functions $f_{\lambda,\mu}=\beta_\lambda\circ\beta_\mu^{-1}$ of the vector bundle $\pi_*(\mathcal{E}\otimes\mathcal{L})$ as

$$f_{\lambda,\mu} : \mathcal{O}_{U_\lambda}^{\oplus rn} \xrightarrow{\alpha_\mu^{-1}} (\pi_*\mathcal{E})_{|U_{\lambda,\mu}} \xrightarrow{\mathrm{id}_{\mathcal{E}}\otimes\omega_{\lambda,\mu}} (\pi_*\mathcal{E})_{|U_{\lambda,\mu}} \xrightarrow{\alpha_\lambda} \mathcal{O}_{U_\lambda}^{\oplus rn},$$

where we denote by $\omega_{\lambda,\mu}=\tau_\lambda\circ\tau_\mu^{-1}$ the $\mathcal{B}^*$-valued transition functions of the line bundle $\mathcal{L}$. We deduce from this expression the relation

$$\det(f_{\lambda,\mu})=\det(g_{\lambda,\mu})\cdot\det(\mathrm{id}_{\mathcal{E}}\otimes\omega_{\lambda,\mu}),$$

where $g_{\lambda,\mu}=\alpha_\lambda\circ\alpha_\mu^{-1}$ denotes the transition functions of the vector bundle $\pi_*\mathcal{E}$. Hence the proposition follows if we show the relation $\det(\mathrm{id}_{\mathcal{E}}\otimes\omega_{\lambda,\mu})=\det(\omega_{\lambda,\mu})^r$, which is proved in the next Lemma. $\square$

Lemma 3.9. *Let $\mathcal{E}$ be a torsion-free $\mathcal{O}_X$-module and let $s\in\Gamma(U,\mathcal{B})=\Gamma(\pi^{-1}(U),\mathcal{O}_X)$ be a local section of $\mathcal{B}$ over the open subset $U\subset C$. We denote by $\mu_s^{\mathcal{E}}\in\mathrm{Hom}_{\mathcal{O}_C(U)}(\pi_*\mathcal{E}(U),\pi_*\mathcal{E}(U))$ the map induced by multiplication with the section s. Then we have an equality*

$$\det(\mu_s^{\mathcal{E}})=\det(\mu_s)^r\in\Gamma(U,\mathcal{O}_C).$$

Proof. By Lemma 2.6 there exists an open subset $j:V\hookrightarrow X$ such that $j^*\mathcal{E}$ is isomorphic to $j^*\mathcal{Q}$ where $\mathcal{Q}$ is a quasi-free sheaf of the form $\oplus_{i=1}^k\mathcal{O}_{X_i}^{\oplus m_i}$. We then apply Corollary 3.3 to the two exact sequences

$$0\longrightarrow\mathcal{E}\longrightarrow j_*j^*\mathcal{E}\longrightarrow\mathcal{T}_1\longrightarrow 0,\qquad\text{and}\qquad 0\longrightarrow\mathcal{Q}\longrightarrow j_*j^*\mathcal{Q}\longrightarrow\mathcal{T}_2\longrightarrow 0,$$

where $\mathcal{T}_i$ are torsion sheaves. This leads to the equality $\det(\mu_s^{\mathcal{E}})=\det(\mu_s^{\mathcal{Q}})$. It therefore suffices to compute $\det(\mu_s^{\mathcal{Q}})$ in terms of $\det(\mu_s)$. We put $n=k\cdot l$ with $l=\deg(X^{red}/C)$. Then we have

$$r=\mathrm{rk}(\mathcal{E})=\mathrm{rk}(\mathcal{Q})=\frac{1}{n}\sum_{i=1}^k m_i\mathrm{rk}(\pi_*\mathcal{O}_{X_i})=\frac{1}{n}\sum_{i=1}^k m_i il=\frac{1}{k}\sum_{i=1}^k m_i i.$$

Let $A=\mathcal{O}_{C,p}$ denote the local ring at the point $p\in C$ and let B denote the localization of $\pi_*\mathcal{O}_X$ at the point $p\in C$. Thus B is a projective A-module of rank n equipped with a filtration

$$t^{k-1}B\subset\cdots\subset tB\subset B,\qquad t\in B\text{ with }t^k=0.$$

We put $B_1=B/tB$, the localization of $\pi_*\mathcal{O}_{X^{red}}$ at the point $p\in C$. Since B is projective we can choose a splitting

$$B=B_1\oplus tB_1\oplus\cdots\oplus t^{k-1}B_1.$$

Using this decomposition we can write a section $s\in B$ as $s=s_0+ts_1+\cdots+t^{k-1}s_{k-1}$ with $s_j\in B_1$. Moreover, the localization of $\pi_*\mathcal{O}_{X_i}$ at the point $p\in C$ is given by $B_i:=B_1\oplus tB_1\oplus\cdots\oplus t^{i-1}B_1$ and the matrix of the multiplication with s in B_i is with respect to this decomposition lower block-triangular and has determinant $\det(\mu_s^{B_i})=\det(\mu_{s_0}^{B_1})^i$. Therefore

$$\det(\mu_s^{\mathcal{Q}})=\prod_{i=1}^k\det(\mu_s^{B_i})^{m_i}=\det(\mu_{s_0}^{B_1})^{\sum_{i=1}^k im_i}.$$

On the other hand $\det(\mu_s)=\det(\mu_s^{\mathcal{O}_X})=\det(\mu_s^{B_k})=\det(\mu_{s_0}^{B_1})^k$, which leads to the desired equality. $\square$

Taking the trivial sheaf $\mathcal{E} = \mathcal{O}_X$ in Proposition 3.8 we obtain the following description of the norm map:

Corollary 3.10. *For any invertible $\mathcal{O}_X$-module, we have*

$$\mathrm{Nm}_{X/C}(\mathcal{L}) = \det(\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_X)^{-1}.$$

Remark 3.11. Note that Proposition 3.8 implies that the group $\mathrm{Prym}(X/C)$ acts on the fibers of the SL_n-Hitchin fibration.

4. The group of connected components of $\mathrm{Prym}(X/C)$

In this section we give the proof of Theorem 1.1.

4.1. **Part (1).** Given a spectral cover X we will associate a covering

$$p : \widetilde{X} \longrightarrow X$$

as follows: let $X = \bigcup_{i=1}^r X_i$ be its decomposition into irreducible components X_i, let X_i^{red} be the underlying reduced curve of X_i, let m_i be the multiplicity of X_i^{red} in X_i and let $\widetilde{X}_i^{red}$ be the normalization of X_i^{red}. Since X_i^{red} is embedded in the smooth surface $|M|$, there exists a sequence of blowing-ups $bl : \widetilde{|M|} \to |M|$ of the surface $|M|$ at reduced points (depending on the curve X_i^{red}) such that the proper transform of X_i^{red} equals its normalization $\widetilde{X}_i^{red}$. We define $\widetilde{X}_i \subset \widetilde{|M|}$ to be the proper transform of the non-reduced curve $X_i \subset |M|$. We take

$$\widetilde{X} = \coprod_{i=1}^r \widetilde{X}_i$$

to be the disjoint union of the curves $\widetilde{X}_i$ together with the natural map p onto X. Note that the multiplicity of $\widetilde{X}_i^{red}$ in $\widetilde{X}_i$ also equals m_i.

Lemma 4.1. *The above contructed covering $p : \widetilde{X} \to X$ has the following properties:*

1. *the cokernel of the canonical inclusion $\mathcal{O}_X \hookrightarrow p_*\mathcal{O}_{\widetilde{X}}$ is a torsion $\mathcal{O}_X$-module,*
2. *the underlying reduced curve $\widetilde{X}^{red}$ of $\widetilde{X}$ is smooth.*
3. *the map induced by pull-back under p*
$$\mathrm{Pic}^0(X) \xrightarrow{p^*} \mathrm{Pic}^0(\widetilde{X})$$
is surjective and has connected kernel.
4. *we have an equality*
$$\pi_0(\mathrm{Prym}(X/C)) = \pi_0(\mathrm{Prym}(\widetilde{X}/C)).$$

Proof. (1) This is clear since $p : \widetilde{X} \to X$ is an isomorphism outside a finite set of points.

(2) We clearly have $\widetilde{X}^{red} = \coprod_{i=1}^r \widetilde{X}_i^{red}$ and the curves $\widetilde{X}_i^{red}$ are smooth by constuction.

(3) We consider the two exact sequences obtained by restricting invertible sheaves to the underlying reduced curve

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X^{red}) & \longrightarrow & 0 \\ & & \downarrow^{\alpha} & & \downarrow^{p^*} & & \downarrow^{p^*_{red}} & & \\ 0 & \longrightarrow & U_2 & \longrightarrow & \mathrm{Pic}(\widetilde{X}) & \longrightarrow & \mathrm{Pic}(\widetilde{X}^{red}) & \longrightarrow & 0, \end{array}$$

which are surjective with unipotent kernels U_1 and U_2 by [L] Lemma 7.5.11. Then by the snake lemma the kernel $\ker(p^*)$ fits into the exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(p^*) \longrightarrow \ker(p^*_{red}) \longrightarrow \mathrm{coker}(\alpha) \longrightarrow 0.$$

Note that $\ker(\alpha)$ and $\mathrm{coker}(\alpha)$ are unipotent groups. We shall denote by V the kernel of the last map. By [L] Lemma 7.5.13 the kernel $\ker(p^*_{red})$ is an extension of a toric group by an unipotent group. The same holds for V, since there are no non-zero maps from a toric group to an unipotent group. Hence V and $\ker(\alpha)$ are connected, so $\ker(p^*)$ is connected. Hence $\ker(p^*)$ is contained in the connected component $\mathrm{Pic}^0(X)$ and we obtain that $p^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\widetilde{X})$ is surjective.

(4) Because of Lemma 3.4 we have an exact sequence

$$0 \longrightarrow \ker(p^*) \longrightarrow \mathrm{Prym}(X/C) \xrightarrow{p^*} \mathrm{Prym}(\widetilde{X}/C) \longrightarrow 0.$$

The equality between the groups of connected components now follows since $\ker(p^*)$ is connected.

□

The previous lemma implies that it is enough to show the equality $\pi_0(\mathrm{Prym}(\widetilde{X}/C)) = \widehat{K}$. By Lemma 3.5 and Lemma 3.6 the Norm map $\mathrm{Nm}_{\widetilde{X}/C}$ factorizes as follows

$$\mathrm{Nm}_{\widetilde{X}/C} : \mathrm{Pic}^0(\widetilde{X}) \xrightarrow{j^*} \mathrm{Pic}^0(\widetilde{X}^{red}) = \prod_{i=1}^{r} \mathrm{Pic}^0(\widetilde{X}_i^{red}) \xrightarrow{\prod[m_i]} \prod_{i=1}^{r} \mathrm{Pic}^0(\widetilde{X}_i^{red}) \xrightarrow{\prod \mathrm{Nm}} \mathrm{Pic}^0(C).$$

Moreover j^* is surjective and $\ker(j^*)$ is connected (see e.g.[L] Lemma 7.5.11). It suffices therefore to compute $\pi_0(\ker(h))$, where $h : \mathrm{Pic}^0(\widetilde{X}^{red}) \to \mathrm{Pic}^0(C)$ denotes the composite of the last two maps. We also consider the composite homomorphism

$$f : \mathrm{Pic}^0(C) \xrightarrow{\Delta} \mathrm{Pic}^0(C)^r \xrightarrow{\prod[m_i]} \mathrm{Pic}^0(C)^r \xrightarrow{\prod \widetilde{\pi}_i^*} \prod_{i=1}^{r} \mathrm{Pic}^0(\widetilde{X}_i^{red}) = \mathrm{Pic}^0(\widetilde{X}^{red}),$$

where $\Delta(L) = (L, \ldots, L)$ is the diagonal map. We note that the duals $\widehat{\widetilde{\pi}_i^*}$ and $\widehat{[m_i]}$ coincide with $\mathrm{Nm}_{\widetilde{X}_i^{red}/C}$ and $[m_i]$ under the identifications $\widehat{\mathrm{Pic}^0(C)} \cong \mathrm{Pic}^0(C)$ and $\widehat{\mathrm{Pic}^0(\widetilde{X}_i^{red})} \cong \mathrm{Pic}^0(\widetilde{X}_i^{red})$ given by the principal polarizations on the Jacobians (see [BL] section 11.4), and that the dual $\widehat{\Delta}$ of Δ is the multiplication map on $\mathrm{Pic}^0(C)$ (see e.g. [BL] exercise 2.6 (12)). Hence we obtain that $\hat{f} = h$. Thus $\pi_0(\mathrm{Prym}(\widetilde{X}/C)) = \pi_0(\ker(\hat{f}))$. Now we apply Lemma 2.1 to f and we obtain the desired result since $\ker(f) = \bigcap_{i=1}^{r} (K_i)_{m_i}$.

4.2. **Part (2).** We consider the morphism $f : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{X}^{red})$ introduced in the previous section. Moreover the morphism $g : \mathrm{Pic}^0(\widetilde{X}^{red}) \to \mathrm{Pic}^0(C)$ defined by

$$g(\mathcal{L}_1, \ldots, \mathcal{L}_r) = \bigotimes_{i=1}^{r} \mathrm{Nm}_{\widetilde{X}_i^{red}/C}(\mathcal{L}_i)$$

satisfies the relation $g \circ f = [n]$. We are therefore in a position to apply Lemma 2.2 to the morphism f. This proves part (2) for the Prym variety $\mathrm{Prym}(\widetilde{X}/C)$. Since by Lemma 4.1 the natural map $p^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\widetilde{X})$ induces an isomorphism $\pi_0(\mathrm{Prym}(X/C)) = \pi_0(\mathrm{Prym}(\widetilde{X}/C))$, we are done.

4.3. **Part (3).** The if part follows immediately from the formula proved in part (1). Suppose now that $K = \mathrm{Pic}^0(C)[n]$. With the above notation we have $n = \sum_{i=1}^r m_i \deg(X_i^{red}/C)$ and $K = \bigcap_{i=1}^r (K_i)_{m_i}$, from which we deduce that $r = 1$. On the other hand $K = (K_1)_{m_1} = \mathrm{Pic}^0(C)[n]$ implies that $K_1 = \mathrm{Pic}^0(C)[d_1]$ with $d_1 = \deg(X_1^{red}/C)$. But this can only happen if $d_1 = 1$. Hence $m_1 = n$ and we are done.

5. Endoscopic subloci of $\mathcal{A}_n$

5.1. **Cyclic Galois covers.** We consider a smooth projective curve C and a line bundle $M \in \mathrm{Pic}(C)$. Let Γ be a cyclic subgroup of order d of the group of n-torsion line bundles $\mathrm{Pic}^0(C)[n]$ and let

$$\varphi : D \longrightarrow C$$

be the étale Galois covering of C associated to $\Gamma \subset \mathrm{Pic}^0(C)[n]$. By definition $D = \mathrm{Spec}(\mathcal{E}_\Gamma)$ where $\mathcal{E}_\Gamma = \oplus_{L \in \Gamma} L$ is the direct sum of all line bundles L in Γ with the natural $\mathcal{O}_C$-algebra structure. Note that the Galois group of the covering $\varphi : D \to C$ equals $\Gamma \cong \mathrm{Aut}(D/C)$. We introduce the line bundle $N = \varphi^* M$. Then the line bundle N has a canonical Γ-linearization, hence we obtain a canonical action of Γ on the total space $|N|$. We notice that the canonical coordinate $t \in H^0(|N|, \pi^* N)$ is invariant under this Γ-action.

We consider a spectral cover of degree m over D with associated line bundle N given by a global section $s \in H^0(|N|, \pi^* N^m)$. We can apply a Galois automorphism $\sigma \in \Gamma$ to s and denote its image by s^σ. We introduce

$$\widehat{s} = \prod_{\sigma \in \Gamma} s^\sigma \in H^0(|N|, \pi^* N^n), \qquad \text{with } \; n = d \cdot m.$$

We observe that $\widehat{s}$ is Γ-invariant, hence $\widehat{s}$ descends to a section over $|M|$, which we also denote by $\widehat{s}$. Hence we obtain a map

$$\Phi_\Gamma : \mathcal{A}_m(D, N) \longrightarrow \mathcal{A}_n := \mathcal{A}_n(C, M), \qquad b \mapsto \Phi_\Gamma(b).$$

with $a = \Phi_\Gamma(b)$ defined by the relation $\widehat{s}_b = s_a$, where $s_b \in H^0(|N|, \pi^* N^m)$ is the global section $s_b = t^m + b_1 t^{m-1} + b_2 t^{m-2} + \cdots + b_m$ associated to $b = (b_1, \ldots, b_m)$ with $b_j \in H^0(D, N^j)$. We also introduce the subspace

$$\mathcal{A}_m^\Gamma(D, N) = H^0(D, N)_{var} \oplus \bigoplus_{j=2}^m H^0(D, N^j) \subset \mathcal{A}_m(D, N),$$

where $H^0(D, N)_{var}$ denotes the Γ-variant subspace of $H^0(D, N)$, i.e. the direct sum of the character spaces $H^0(D, N)_\chi$ for non-trivial characters χ of the group Γ.

Lemma 5.1. *We have the inclusion*

$$\Phi_\Gamma(\mathcal{A}_m^\Gamma(D, N)) \subset \mathcal{A}_n^0.$$

Proof. It suffices to compute the coefficient of t^{n-1} in $\widehat{s}_b$, which equals $\sum_{\sigma \in \Gamma} \sigma^* b_1$. We immediately see that the relation $\sum_{\sigma \in \Gamma} \sigma^* b_1 = 0$ is equivalent to $b_1^{(0)} = 0$, where $b_1^{(0)}$ denotes the Γ-invariant component of b_1. □

We denote the images of Φ_Γ by

$$\mathcal{A}^0_\Gamma \subset \mathcal{A}^0_n \qquad \text{and} \qquad \mathcal{A}_\Gamma \subset \mathcal{A}_n.$$

The subvariety $\mathcal{A}_\Gamma$ admits the following characterization: for $a \in \mathcal{A}_n$ we denote by

$$Y_a = X_a \times_C D$$

the fiber product of X_a and D over C. Then Y_a is a spectral cover over D of degree n associated to the line bundle N. The following lemma follows immediately from the definition of $\mathcal{A}_\Gamma$.

Lemma 5.2. *The characteristic a lies in $\mathcal{A}_\Gamma$ if ond only if the fiber product Y_a decomposes as*

$$Y_a = \bigcup_{\sigma \in \Gamma} Z^\sigma,$$

where Z is a spectral cover of degree $m = \frac{n}{d}$ over D and Z^σ is its image under the Galois automorphism $\sigma \in \Gamma$.

We also need to introduce some natural subvarieties of the Hitchin spaces $\mathcal{A}^0_n$ and $\mathcal{A}_n$, which will be used in the proof of Theorem 1.2.

For any divisor $l \neq 1$ of n, with $n = k \cdot l$, we consider the natural k-th power map

$$\Phi_k : \mathcal{A}_l \longrightarrow \mathcal{A}_n,$$

where $\Phi_k(b) = a$ is defined by the relation

$$s_a = (t^l + b_1 t^{l-1} + \cdots + b_l)^k \in H^0(|M|, \pi^* M^n), \qquad \text{for } b = (b_1, \ldots, b_l) in \mathcal{A}_l.$$

We shall abuse notation and will also denote by $\mathcal{A}_l$ its image $\Phi_k(\mathcal{A}_l) \subset \mathcal{A}_n$. Note that $\Phi_k(\mathcal{A}^0_l) \subset \mathcal{A}^0_n$ and we also denote this image by $\mathcal{A}^0_l$.

Given two positive integers n_1, n_2 such that $n_1 + n_2 = n$, we introduce the map

$$\Phi_{n_1,n_2} : \mathcal{A}_{n_1} \times \mathcal{A}_{n_2} \longrightarrow \mathcal{A}_n,$$

with $a = \Phi_{n_1,n_2}(b, c)$ defined by the relation $s_a = s_b \cdot s_c$, where $s_b = t^{n_1} + b_1 t^{n_1 - 1} + \cdots + b_{n_1}$ and $s_c = t^{n_2} + c_1 t^{n_2-1} + \cdots + c_{n_2}$ for $b = (b_1, \ldots, b_{n_1}) \in \mathcal{A}_{n_1}$ and $c = (c_1, \ldots, c_{n_2}) \in \mathcal{A}_{n_2}$. We define $(\mathcal{A}_{n_1} \times \mathcal{A}_{n_2})_0 \subset \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$ to be the subset of pairs (b, c) satisfying the relation $b_1 + c_1 = 0$. We shall denote by $\mathcal{A}_{n_1,n_2} \subset \mathcal{A}_n$ the image of Φ_{n_1,n_2} and by $\mathcal{A}^0_{n_1,n_2}$ the subset $\mathcal{A}_{n_1,n_2} \cap \mathcal{A}^0_n = \Phi_{n_1,n_2}[(\mathcal{A}_{n_1} \times \mathcal{A}_{n_2})_0]$.

5.2. **Proof of Theorem 1.2.** We show here the analogue of Theorem 1.2 for the $\mathrm{GL}(n)$-Hitchin space $\mathcal{A}_n$. Note that both statements are equivalent by Remark 2.5. Given a spectral cover $\pi : X_a \to C$ with $a \in \mathcal{A}_n$, we denote the subgroup of $\mathrm{Pic}^0(C)$ defined in (1) by K_a. Let $\Gamma \subset \mathrm{Pic}^0(C)[n]$ be a cyclic subgroup of order d.

Theorem 5.3. *We have an equivalence*

$$\Gamma \subset K_a \qquad \Longleftrightarrow \qquad a \in \mathcal{A}_\Gamma.$$

Proof. We first show the equivalence in the case the spectral cover X_a is integral. In that case we can consider its normalization $\widetilde{X}_a$, which comes with a natural projection

$$\widetilde{\pi}_a : \widetilde{X}_a \longrightarrow C.$$

By [BL] Proposition 11.4.3 we have $\Gamma \subset K_a$ if and only if $\widetilde{\pi}_a$ factors through the map φ, i.e. there exists a map $u : \widetilde{X}_a \to D$ such that $\widetilde{\pi}_a = \varphi \circ u$. By the universal property of the fiber product there exists a map $\delta : \widetilde{X}_a \to Y_a$ into the fiber product Y_a of X_a with D over C. We denote by $Z = \mathrm{im}(\delta) \subset Y_a$ the image of the smooth irreducible curve $\widetilde{X}_a$. Then Z is irreducible too. Moreover, since X_a is reduced and φ is étale, the curve Y_a is also reduced, hence Z is integral. The group Γ acts on Y_a, hence permutes its irreducible components. Since Γ acts transitively on the fibers of $Y_a \to X_a$, all irreducible components are of the form Z^σ for some $\sigma \in \Gamma$. We therefore obtain a factorization $\widetilde{X}_a \to Z \to X_a$. Since this composite map is birational, we deduce that $\deg(Z/X_a) = 1$. Hence, since $\deg(Y_a/X_a) = d$, we conclude that

$$Y_a = \bigcup_{\sigma \in \Gamma} Z^\sigma \qquad \text{and} \qquad Z^\sigma \neq Z^{\sigma'} \ \text{ if } \sigma \neq \sigma'.$$

By Lemma 2.4 the curve Z is a spectral cover of degree m over D and by Lemma 5.2 we obtain that $a \in \mathcal{A}_\Gamma$.

Conversely, for $a \in \mathcal{A}_\Gamma$ the map $Z \to X_a$ given by Lemma 5.2 is birational. Hence the normalization of Z equals $\widetilde{X}_a$ and we obtain a factorization $\widetilde{X}_a \to Z \to D \to C$, which implies that $\Gamma \subset K_a$ by [BL] Proposition 11.4.3.

Now we will prove the equivalence for more general characteristics $a \in \mathcal{A}_n$. We start with $a \in \mathcal{A}_n$ such that the spectral cover X_a is irreducible, but not reduced. Let X_a^{red} be the underlying reduced curve of X_a and let k be the multiplicity of X_a^{red} in X_a. We put $n = k \cdot l$. By Lemma 2.4 we have $X_a^{red} = X_{a_{red}}$ for some characteristic $a_{red} \in \mathcal{A}_l \subset \mathcal{A}_n$ and $a = \Phi_k(a_{red})$ — see section 2.2. Then by formula (1) we have $K_a = [k]^{-1}(K_{a_{red}})$. We introduce $\Gamma_{red} = [k](\Gamma) \subset \mathrm{Pic}^0(C)[l]$. Then Γ_{red} is a cyclic subgroup of order $d_{red} = \frac{d}{\gcd(k,d)}$. With this notation we easily obtain the equivalence

$$\Gamma \subset K_a \qquad \Longleftrightarrow \qquad \Gamma_{red} \subset K_{a_{red}}.$$

We combine this equivalence with the statement of the Theorem written for the integral characteristic a_{red}, which was proved above:

$$\Gamma_{red} \subset K_{a_{red}} \qquad \Longleftrightarrow \qquad a_{red} \in \mathcal{A}_{\Gamma_{red}}.$$

Therefore it remains to show the following equivalence

$$a_{red} \in \mathcal{A}_{\Gamma_{red}} \qquad \Longleftrightarrow \qquad a = \Phi_k(a_{red}) \in \mathcal{A}_\Gamma.$$

In order to show this equivalence we introduce the subgroup $S = \ker(\Gamma \to \Gamma_{red})$. By Galois theory there exists an intermediate cover $D \to \overline{D} \to C$ with $\mathrm{Aut}(\overline{D}/C) = \Gamma_{red}$ and $\mathrm{Aut}(D/\overline{D}) = S$.

Consider a characteristic $a_{red} \in \mathcal{A}_{\Gamma_{red}}$. By Lemma 5.2 applied to $a_{red} \in \mathcal{A}_{\Gamma_{red}}$ we obtain that the fiber product $Y_{a_{red}} = X_{a_{red}} \times_C \overline{D}$ decomposes as $\bigcup_{\sigma \in \Gamma_{red}} W^\sigma$, where W is a spectral cover of degree $\frac{l}{d_{red}}$ over $\overline{D}$. Now, observing that $Y_a = k(Y_{a_{red}} \times_{\overline{D}} D)$ as divisors in $|N|$, we can write

$$Y_a = k \bigcup_{\sigma \in \Gamma_{red}} (W \times_{\overline{D}} D)^\sigma = \bigcup_{\sigma \in \Gamma} Z^\sigma,$$

where we have put $Z = \frac{k}{\gcd(k,d)}(W \times_{\overline{D}} D) \subset |N|$. Note that $Z^\sigma = Z$ for $\sigma \in S$ and that Z is a spectral cover of degree $\frac{n}{d}$. This proves that $a = \Phi_k(a_{red}) \in \mathcal{A}_\Gamma$.

Conversely, we consider a characteristic $a_{red} \in \mathcal{A}_l$ with $\Phi_k(a_{red}) \in \mathcal{A}_\Gamma$. We assume that the spectral cover $X_{a_{red}}$ is integral. This assumption implies that the fiber product $X_{a_{red}} \times_C D$ is reduced. Let $\mathcal{I}$ denote an irreducible component of $X_{a_{red}} \times_C D$, let $\mathrm{Stab}(\mathcal{I})$ denote its stabilizer, i.e.,

$$\mathrm{Stab}(\mathcal{I}) = \{\sigma \in \Gamma \mid \mathcal{I}^\sigma = \mathcal{I}\},$$

and let $\delta = |\mathrm{Stab}(\mathcal{I})|$ be the order. Since Γ acts transitively on the fibers of $X_{a_{red}} \times_C D \to X_{a_{red}}$ we obtain the decomposition into irreducible components

$$X_{a_{red}} \times_C D = \bigcup_{\sigma \in \Gamma/\mathrm{Stab}(\mathcal{I})} \mathcal{I}^\sigma.$$

Let us denote by s the global section over $|N|$ with $\mathrm{Zeros}(s) = \mathcal{I}$. Then the spectral cover $Y_a = k(X_{a_{red}} \times_C D)$ is the zero set of the section

$$\prod_{\sigma \in \Gamma/\mathrm{Stab}(\mathcal{I})} (s^\sigma)^k,$$

which has $k\frac{d}{\delta}$ irreducible factors of the same degree. The assumption $\Phi_k(a_{red}) \in \mathcal{A}_\Gamma$ implies that this product can be written as a product of d factors of the same degree, hence $k\frac{d}{\delta}$ is divisible by d, i.e., δ divides k, so δ divides $\gcd(k,d)$. Since $\delta = |\mathrm{Stab}(\mathcal{I})|$ and $\gcd(k,d) = |S|$, we conclude that $\mathrm{Stab}(\mathcal{I}) \subset S$. We then introduce the section

$$t = \prod_{\sigma \in S/\mathrm{Stab}(\mathcal{I})} s^\sigma.$$

Since t is S-invariant, its zero divisor descends as a spectral cover W over $\overline{D} = D/S$. Moreover we have the equality

$$\prod_{\sigma \in \Gamma/\mathrm{Stab}(\mathcal{I})} s^\sigma = \prod_{\tau \in \Gamma_{red} = \Gamma/S} t^\tau,$$

which proves that $Y_{a_{red}} = X_{a_{red}} \times_C \overline{D}$ splits into d_{red} spectral covers W^τ for $\tau \in \Gamma_{red}$, and we conclude by Lemma 5.2 that $a_{red} \in \mathcal{A}_{\Gamma_{red}}$.

Finally, we will show the equivalence for a characteristic $a \in \mathcal{A}_{n_1,n_2} \subset \mathcal{A}_n$, i.e. the spectral cover X_a equals the union $X_{a_1} \cup X_{a_2}$ for two spectral covers X_{a_i} with $a_i \in \mathcal{A}_{n_i}$, which we assume to be irreducible. Then by (1) we have

$$\Gamma \subset K_a \qquad \Longleftrightarrow \qquad \Gamma \subset K_{a_1} \text{ and } \Gamma \subset K_{a_2}.$$

On the other hand since the curves X_{a_i} are irreducible, we can apply what we have proved above, i.e., for $i = 1, 2$

$$\Gamma \subset K_{a_i} \qquad \Longleftrightarrow \qquad a_i \in \mathcal{A}_\Gamma^{n_i}.$$

Note that $\Gamma \subset \mathrm{Pic}^0(C)[n_i]$ for $i = 1, 2$. Here $\mathcal{A}_\Gamma^{n_i}$ denotes the corresponding subspace of $\mathcal{A}_{n_i}$. Hence it remains to show that

$$a_1 \in \mathcal{A}_\Gamma^{n_1} \text{ and } a_2 \in \mathcal{A}_\Gamma^{n_2} \qquad \Longleftrightarrow \qquad a = \Phi_{n_1,n_2}(a_1, a_2) \in \mathcal{A}_\Gamma.$$

The implication $\Rightarrow$ is trivial. In order to show the implication $\Leftarrow$ we note that Lemma 5.2 gives the decomposition $Y_a = \bigcup_{\sigma \in \Gamma} Z^\sigma$ for some spectral cover Z. We then put $Z_i = Z \cap Y_{a_i}$, which gives the desired decomposition for the fiber product Y_{a_i}.

Now the statement follows for arbitrary characteristic $a \in \mathcal{A}_n$ by induction on the number of irreducible components of X_a.

□

References

[BL] Ch. Birkenhake, H. Lange: Complex Abelian Varieties (Second edition), Grundlehren der mathematischen Wissenschaften Vol. 302, Springer (2004)

[dCHM] M.A. de Cataldo, T. Hausel, L. Migliorini: Topology of Hitchin systems and Hodge theory of character varieties: the case A_1, (preprint arXiv:104.1420)

[CL] P.-H. Chaudouard, G. Laumon: Le lemme fondamental pondéré. II. Énoncés cohomologiques, (preprint arXiv:0912.4512)

[D] J.-M. Drézet: Faisceaux sans torsion et faisceaux quasi localement libres sur les courbes multiples primitives, Mathematische Nachrichten, Vol. 282, No.7 (2009), 919-952

[FW] E. Frenkel, E. Witten: Geometric endoscopy and mirror symmetry. Commun. Number Theory Phys. 2 (2008), no. 1, 113283.

[G1] A. Grothendieck: Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): II. Étude globale élémentaire de quelques classes de morphismes. Publications Mathématiques de l'IHÉS, 8 (1961), 5-222

[G2] A. Grothendieck: Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. Publications Mathématiques de l'IHÉS, 32 (1967), 5-361

[H] N. J. Hitchin Stable bundles and integrable systems. Duke Math. J., 54(1):91–114, 1987.

[KW] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric Langlands program. Commun. Number Theory Phys. 1 (2007), no. 1, 1–236.

[K] S. Kleiman: The Picard scheme, Fundamental algebraic geometry, Math. Surveys Monogr., 123, Amer. Math. Soc., Providence, RI (2005), 235-321

[L] Q. Liu: Algebraic geometry and arithmetic curves, Oxford Graduate Texts in Mathematics, No. 6 (2002)

[N1] B.C. Ngô: Fibration de Hitchin et endoscopie, Inventiones mathematicae, Vol. 164 (2006), 399-453

[N2] B.C. Ngô: Le lemme fondamental pour les algèbres de Lie, Publ. Math. Inst. Hautes Études Sci. No. 111 (2010), 1169.

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