THE INCIDENCE HOPF ALGEBRA OF GRAPHS

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ABSTRACT. The *graph algebra* is a commutative, cocommutative, graded, connected incidence Hopf algebra, whose basis elements correspond to finite simple graphs and whose Hopf product and coproduct admit simple combinatorial descriptions. We give a new formula for the antipode in the graph algebra in terms of acyclic orientations; our formula contains many fewer terms than Takeuchi's and Schmitt's more general formulas for the antipode in an incidence Hopf algebra. Applications include several formulas (some old and some new) for evaluations of the Tutte polynomial.

1. INTRODUCTION

The *graph algebra* $\mathcal{G}$ is a commutative, cocommutative, graded, connected Hopf algebra, whose basis elements correspond to finite simple graphs G, and whose Hopf product and coproduct admit simple combinatorial descriptions. The graph algebra was first considered by Schmitt in the context of incidence Hopf algebras [Sch94, §12] and furnishes an important example in the work of Aguiar, Bergeron and Sottile [ABS06, Example 4.5].

In this paper, we derive a nonrecursive formula (Theorem 3.1) for the Hopf antipode in $\mathcal{G}$. Our formula is specific to the graph algebra in that it involves acyclic orientations. Therefore, it is not a consequence of the antipode formulas of Takeuchi [Tak71] and Schmitt [Sch94] in the more general settings of, respectively, connected bialgebras and incidence Hopf algebras. Aguiar and Ardila [Agu] have independently discovered a more general antipode formula than ours, in the context of Hopf monoids; their work will appear in a forthcoming paper.

Our formula turns out to be well suited for studying graph invariants, including the Tutte polynomial $T_G(x, y)$ and various specializations of it. The idea is to make $\mathcal{G}$ into a combinatorial Hopf algebra in the sense of Aguiar, Bergeron and Sottile [ABS06] by defining a character on it, then to define a graph invariant by means of a Hopf morphism to a polynomial ring. The antipode formula leads to combinatorial interpretations of the convolution inverses of several natural characters, as we discuss in Section 3.1.

The Tutte polynomial can itself be viewed as a character. We prove that its k-th convolution power itself is a Tutte evaluation at rational functions in x, y, k (Theorem 4.1). This implies several well-known formulas such as

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Stanley's formula for acyclic orientations in terms of the chromatic polynomial [Sta73], as well as some interpretations of less familiar specializations of the Tutte polynomial, and an unusual-looking reciprocity relation between complete graphs of different sizes (Eqns. (22) and (23)).

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2. Hopf algebras

2.1. **Basic definitions.** We briefly review the basic facts about Hopf algebras, omitting the proofs. Good sources for the full details include Sweedler [Swe69] and (for combinatorial Hopf algebras) Aguiar, Bergeron and Sottile [ABS06]. For the more general setting of Hopf monoids, see Aguiar and Mahajan [AM10].

Fix a field $\mathbb{F}$ (typically $\mathbb{C}$). A *bialgebra* $\mathcal{H}$ is a vector space over $\mathbb{F}$ equipped with linear maps

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \qquad u : \mathbb{F} \to \mathcal{H}, \qquad \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad \epsilon : \mathcal{H} \to \mathbb{F},$$

respectively the *multiplication*, *unit*, *comultiplication*, and *counit*, such that the following properties are satisfied:

(1) $m \circ (m \otimes I) = m \circ (I \otimes m)$ (associativity);
(2) $m \circ (u \otimes I) = m \circ (I \otimes u) = I$ (where I is the identity map on $\mathcal{H}$);
(3) $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$ (coassociativity);
(4) $(\epsilon \otimes I) \circ \Delta = (I \otimes \epsilon) \circ D = I$; and
(5) Δ and ϵ are multiplicative (equivalently, m and u are comultiplicative).

If there exists a bialgebra automorphism $S : \mathcal{H} \to \mathcal{H}$ such that $m \circ (S \otimes I) \circ \Delta = m \circ (I \otimes S) \circ \Delta = u \circ \epsilon$, then $\mathcal{H}$ is a *Hopf algebra* and S is its *antipode*.[1]

A Hopf algebra $\mathcal{H}$ is *graded* if $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ as vector spaces, and multiplication and comultiplication respect this decomposition, i.e.,

$$m(\mathcal{H}_i \otimes \mathcal{H}_j) \subseteq \mathcal{H}_{i+j} \quad \text{and} \quad \Delta(\mathcal{H}_k) \subseteq \sum_{i+j=k} \mathcal{H}_i \otimes \mathcal{H}_j.$$

Meanwhile, $\mathcal{H}$ is *connected* if $\dim(\mathcal{H}_0) = 1$. If $\mathcal{H}$ is a graded and connected bialgebra, then its antipode can be defined inductively as follows: $S(h) = h$ for $h \in \mathcal{H}_0$, and, then $(m \circ (S \otimes I) \circ \Delta)(h) = 0$ for $h \in \mathcal{H}_i$, $i > 0$. Most (if not all) of the Hopf algebras arising naturally in combinatorics are graded and connected, and every algebra we consider henceforth will be assumed to have these properties.

A *character* of a Hopf algebra $\mathcal{H}$ is a multiplicative linear map $\phi : \mathcal{H} \to \mathbb{F}$. The *convolution product* of two characters is $\phi * \psi = (\phi \otimes \psi) \circ \Delta$. That is, if $\Delta h = \sum_i h_1^{(i)} \otimes h_2^{(i)}$, then $(\phi * \psi)(h) = \sum_i \phi(h_1^{(i)}) \psi(h_2^{(i)})$. (This formula can be writen more concisely in Sweedler notation: if $\Delta h = \sum h_1 \otimes h_2$, then

[1] It can be shown that S is the unique automorphism of $\mathcal{H}$ with this property.

$(\phi * \psi)(h) = \sum \phi(h_1)\psi(h_2)$.) Convolution makes the set of characters $\mathbb{X}(\mathcal{H})$ into a group, with identity ϵ and inverse given by

$$\phi^{-1} = \phi \circ S. \tag{1}$$

There is a natural involutive automorphism $\phi \mapsto \overline{\phi}$ of $\mathbb{X}(\mathcal{H})$, given by $\overline{\phi}(h) = (-1)^n\phi(h)$ for $h \in \mathcal{H}_n$. If $\mathcal{H}$ is a graded connected Hopf algebra and $\zeta \in \mathbb{X}(H)$, then the pair $(\mathcal{H}, \zeta)$ is called a *combinatorial Hopf algebra*, or CHA for short. A *morphism* of CHAs $\Phi : (\mathcal{H}, \zeta) \to (\mathcal{H}', \zeta')$ is a linear transformation $\mathcal{H} \to \mathcal{H}'$ that is a morphism of Hopf algebras (i.e., a linear transformation that preserves the operations of a bialgebra) such that $\zeta \circ \Phi = \zeta'$.

2.2. **The binomial and graph Hopf algebras.**

The *binomial Hopf algebra* is the ring of polynomials $\mathbb{F}[k]$ in one variable k, with the usual multiplicative structure; comultiplication $\Delta(f(k)) = f(k \otimes 1 + 1 \otimes k)$; counit $\epsilon(f(k)) = \epsilon_0(f(k)) = f(0)$; and character $\epsilon_1(f(k)) = f(1)$. The following proposition is a consequence of work of Aguiar, Bergeron, and Sottile [ABS06, Thm. 4.1].

Proposition 2.1 (Polynomiality)**.** *Every combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ has a unique CHA morphism to $(\mathbb{F}[k], \epsilon_1)$.*

We regard this Hopf morphism as a way to associate a polynomial invariant $P_{\zeta,h}(k) = \zeta^k(h) \in \mathbb{F}[k]$ with each element $h \in \mathcal{H}$. In fact, Aguiar, Bergeron, and Sottile proved something much stronger: the algebra Q of quasisymmetric functions is a terminal object in the category of CHAs, i.e., every CHA has a unique morphism to Q. Composing this morphism with the principal specialization[2] gives the morphism of Proposition 2.1. We will not use the full power of the Aguiar–Bergeron–Sottile theorem (which can be viewed as a way to associate a quasisymmetric-function invariant to each element of $\mathcal{H}$). Note that for $k \in \mathbb{Z}$, the identity $\zeta^k(h) = P_{\zeta,h}(k)$ follows from the definition of a CHA morphism; therefore, it is actually an identity of polynomials in k.

The *graph algebra*[3] is the $\mathbb{F}$-vector space $\mathcal{G} = \bigoplus_{n\geq 0} \mathcal{G}_n$, where $\mathcal{G}_n$ is the linear span of isomorphism classes of simple graphs on n vertices. This is a graded connected Hopf algebra, with multiplication $m(G \otimes H) = G \cdot H = G \uplus H$ (where $\uplus$ denotes disjoint union); unit $u(1) = \emptyset$ (the graph with no vertices); comultiplication

$$\Delta(G) = \sum_{T \subseteq V(G)} G|_T \otimes G|_{\overline{T}}$$

[2] If $F(x_1, x_2, \dots)$ is a formal power series, then its principal specialization is obtained by setting $x_i = 1$ and $x_i = 0$ for all $i > 1$.

[3] The literature contains many other instances of "Hopf algebras of graphs"; for example, the algebra $\mathcal{G}$ is not the same as that studied by Novelli, Thibon and Thiéry [NTT04].

(where $G|_T$ denotes the induced subgraph on vertex set T, and $\overline{T} = V(G) \setminus T$); and counit

$$\epsilon(G) = \begin{cases} 1 & \text{if } G = \emptyset, \\ 0 & \text{if } G \neq \emptyset. \end{cases}$$

This Hopf algebra is commutative and cocommutative; in particular, its character group $\mathbb{X}(G)$ is abelian. As proved by Schmitt [Sch94, eq. (12.1)], the antipode in $\mathcal{G}$ is given combinatorially by

$$S(G) = \sum_{\pi} (-1)^{|\pi|} |\pi|! G_\pi$$

where the sum runs over all ordered partitions π of $V(G)$ into nonempty sets (or "blocks"), and G_π is the disjoint union of the induced subgraphs on the blocks. This is a consequence of Takeuchi's more general formula for connected Hopf algebras [Tak71, Lemma 14]; see also [AM10, §2.3.3 and §8.4], [AS05, §5], [Mon93].

The graph algebra admits two canonical involutions on characters:

$$\overline{\phi}(G) = (-1)^{n(G)} \phi(G), \qquad \tilde{\phi}(G) = (-1)^{\mathrm{rk}(G))} \phi(G),$$

where $\mathrm{rk}(G)$ denotes the graph rank of G (that is, the number of edges in a spanning tree). As always, $\phi \mapsto \overline{\phi}$ is an automorphism of $\mathbb{X}(G)$; on the other hand, $\phi \mapsto \tilde{\phi}$ is not. The graph algebra was studied by Schmitt [Sch94] and appears as the *chromatic algebra* in the work of Aguiar, Bergeron and Sottile [ABS06], where it is equipped with the character

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges}, \\ 0 & \text{if } G \text{ has an edge}. \end{cases}$$

We will study several characters on $\mathcal{G}$ other than ζ.

3. A combinatorial antipode formula

In this section, we prove a new combinatorial formula for the Hopf antipode in $\mathcal{G}$. Unlike Takeuchi's and Schmitt's formulas, our formula applies only to $\mathcal{G}$ and and does not generalize to other incidence algebras. On the other hand, our formula involves many fewer summands, which makes it useful for enumerative formulas involving characters. As noted in the introduction, Aguiar and Ardila have independently discovered a more general antipode formula in the context of Hopf monoids.

First, we set up some standard graph-theoretic notation and terminology. The notation $G = (V, E)$ means that G is a finite, simple, undirected graph with vertex set V and edge set E; we may then write $G_{V',E'}$ for the subgraph with vertex set V' and edge set E'. (We could also write simply (V', E'), but we often wish to emphasize that this is a subgraph of G.) The sets of vertices and edges of a graph G will be denoted $V(G)$ and $E(G)$ respectively; no confusion should arise from this apparent abuse of notation. The induced subgraph on a vertex set $T \subseteq V$ will be denoted $G|_T$. The complement of T will be denoted $\overline{T}$.

The *rank* $\mathrm{rk}(F)$ of a subset $F \subseteq E$ is the size of any maximal acyclic subset of F. Meanwhile, the set F is called a *flat* if, whenever the endpoints of an edge e are connected by a path in F, then $e \in A$. (The are precisely the flats of the graphic matroid of G.) An equivalent condition is that $\mathrm{rk}(F') > \mathrm{rk}(F)$ for every $F' \supsetneq F$. Finally, an *acyclic orientation* of G is a choice of orientation of all the edges that admits no directed cycles. Let

$$\begin{aligned}
\mathcal{F}(G) &= \{\text{flats of } G\},\\
\mathcal{A}(G) &= \{\text{acyclic orientations of } G\},\\
a(G) &= |\mathcal{A}(G)|.
\end{aligned}$$

Theorem 3.1. *Let $G = (V, E)$ be a graph with $n = |V|$. Then*

$$S(G) = \sum_{F \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F)} a(G/F) G_{V,F}.$$

Proof. We proceed by induction on n. If G has no vertices, then $\mathcal{F}(G) = \{\emptyset\}$, and our formula then gives $S(\emptyset) = \emptyset$ as desired. On the other hand, if G has at least one vertex, then

$$\begin{aligned}
S(G) &= -\sum_{\emptyset \neq T \subseteq V} G|_T \cdot S(G|_{\overline{T}})\\
&= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{F \in \mathcal{F}(G|_{\overline{T}})} (-1)^{n-|T|-\mathrm{rk}(F)} a(G|_{\overline{T}}/F) G_{\overline{T},F}\\
&= -\sum_{\emptyset \neq T \subseteq V} G|_T \sum_{F \in \mathcal{F}(G|_{\overline{T}})} \sum_{\mathcal{O} \in \mathcal{A}(G|_{\overline{T}}/F)} (-1)^{n-|T|-\mathrm{rk}(F)} G_{\overline{T},F}. \qquad (2)
\end{aligned}$$

Now we establish a bijection which will allow us to interchange the order of summation.

First, suppose we are given a nonempty vertex set $T \subseteq V$, a flat F of $G|_{\overline{T}}$, and an acyclic orientation $\mathcal{O}$ of $G|_{\overline{T}}/F$. Let $F' = E(G|_T) \cup F$; this is a flat of G. Moreover, we can construct an acyclic orientation $\mathcal{O}'$ of G/F' by orienting all edges in $[\overline{T}, \overline{T}]$ as in $\mathcal{O}$, and orienting all edges in $[T, \overline{T}]$ towards $\overline{T}$. Let $S_{\mathcal{O}'}$ be the set of sources of $\mathcal{O}'$ (that is, vertices with no in-edges); then the image T' of T under the contraction of F' is a nonempty subset of $S_{\mathcal{O}'}$.

Second, suppose we are given a flat F' of G, an acyclic orientation $\mathcal{O}'$ of G/F', and a set T' such that $\emptyset \neq T' \subseteq S_{\mathcal{O}'}$. Let T be the inverse image of T' under contraction of F'. Then $F = F' \setminus E(G|_T)$ is a flat of $G|_{\overline{T}}$, and we can construct an acyclic orientation $\mathcal{O}$ of $G|_{\overline{T}}/F$ by orienting all edges as in $\mathcal{O}'$.

It is straightforward to check that these constructions are inverses. Therefore, we have a bijection

$$\begin{array}{rcl} \mathcal{F}(G|_{\overline{T}}) \times \mathcal{A}(G|_{\overline{T}}/F) & \to & \mathcal{F}(G) \times \mathcal{A}(G/F') \\ (F, \mathcal{O}) & \mapsto & (F', \mathcal{O}') \end{array}$$

with the following properties:

- $|T'|$ is the number of components of $G|_T$.
- $|T| - |T'| = \mathrm{rk}(G|_T) = \mathrm{rk}(F') - \mathrm{rk}(F)$, so $|T| + \mathrm{rk}(F) = |T'| + \mathrm{rk}(F')$.
- $G|_T \cdot G_{\overline{T},F} = G_{V,F'}$ in the graph algebra $\mathcal{G}$.

Therefore, (2) gives

$$\begin{aligned} S(G) &= -\sum_{F' \in \mathcal{F}(G)} \sum_{\mathcal{O}' \in \mathcal{A}(G/F')} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{n-|T'|-\mathrm{rk}(F')} G_{V,F'} \\ &= -\sum_{F' \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F')} G_{V,F'} \sum_{\mathcal{O}' \in \mathcal{A}(G/F')} \sum_{\emptyset \neq T' \subseteq S_{\mathcal{O}'}} (-1)^{|T'|} \\ &= \sum_{F' \in \mathcal{F}(G)} (-1)^{n-\mathrm{rk}(F')} a(G/F') G_{V,F'}. \end{aligned}$$ ☐

3.1. **Inversion of characters.** We now apply the antipode formula to give combinatorial interpretations of several instances of inversion in the group of characters.

Proposition 3.2. *Let P be any family of graphs such that $G \uplus H \in P$ if and only if $G \in P$ and $H \in P$; equivalently, such that the function*

$$\psi_P(G) = \begin{cases} 1 & \text{if } G \in P, \\ 0 & \text{if } G \notin P \end{cases}$$

is a character. Then

$$\psi_P^{-1}(G) = \sum_{F \in \mathcal{F}(G):\ G_{V,F} \in P} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

Proof. From Equation 1 and Theorem 3.1, we have

$$\begin{aligned} \chi_P^{-1}(G) = \chi(SG) &= \sum_F (-1)^{n-\mathrm{rk}(F)} a(G/F) \chi(G_{V,F}) \\ &= \sum_{F \in P} (-1)^{n-\mathrm{rk}(F)} a(G/F). \end{aligned}$$ ☐

Example 3.3. Let P be the family of graphs with no edges. Then $\psi_P = \zeta$ and $\chi_P^{-1}(G)(-1)^n a(G)$, which is Stanley's well-known formula [Sta73].

Example 3.4. Let P be the family of acyclic graphs, and let $\alpha = \psi_P$. Then

$$\alpha^{-1}(G) = \sum_{\text{acyclic flats } F} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

First, consider $G = C_n$, the cycle of length n. The acyclic flats of G are just the sets of $n-2$ or fewer edges, so an elementary calculation (which we omit) gives $\alpha^{-1}(C_n) = (-1)^n + 1$, the Euler characteristic of an n-sphere.

For many other families P, the P-free flats of C_n are just its flats, i.e., the edge sets of cardinality $\neq n-1$. In such cases, the same omitted calculation gives $\psi_P(C_n) = (-1)^n$.

Second, consider $G = K_n$. Now the acyclic flats of G are matchings; for $0 \leq k \leq \lfloor n/2 \rfloor$, the number of k-edge matchings is $n!/(2^k(n-2k)!k!)$, and contracting such a matching yields a graph whose underlying simple graph is K_{n-k}. Therefore

$$\alpha^{-1}(K_n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \frac{n!}{2^k(n-2k)!k!}(n-k)!.$$

Starting at $n = 1$, these numbers are as follows:

$$-1,\ 1,\ 0,\ -6,\ 30,\ -90,\ 0,\ 2520,\ -22680,\ 113400,\ 0,\ -7484400,\ \ldots.$$

This is sequence A009775 in [Slo10], for which the generating function is $-\tanh(\ln(1+x))$.

Example 3.5. Fix any connected graph S. Say that G is *S-free* if it has no subgraph isomorphic to S. (This is a stronger condition than saying that G has no *induced* subgraph isomorphic to S.) The corresponding *avoidance character η_S* is defined by

$$\eta_S(G) = \begin{cases} 1 & \text{if } G \text{ is } S\text{-free}, \\ 0 & \text{otherwise}. \end{cases}$$

For instance, $\eta_{K_1} = \epsilon$ and $\eta_{K_2} = \zeta$. More generally, the character $\eta_{K_{m,1}}$ detects whether or not G has maximum degree $< m$. For an avoidance character, the sum in Proposition 3.2 is taken over all S-free flats F. For example, we have

$$\eta^{-1}_{K_m}(K_n) \ = \ \sum_{j=0}^{m-1} \binom{n}{j} (-1)^{n-j-1}(n-j)!\ .$$

Another consequence is that if T is a tree with $r = n-1$ edges, then

$$\eta_S^{-1}(G) \ = \ \sum_F (-1)^{r+1-|F|} 2^{r-|F|} \ = \ -\sum_F (-2)^{r-|F|}.$$

with both sums over all S-free forests $F \subseteq T$. Moreover, $P_{\eta_T}(T;k) = k^{n(T)} - k$.

Example 3.6. Let S be a connected graph and η_S the corresponding avoidance character. Then $P_{\eta_S}(G;k)$ equals the number of k-colorings such that every color-induced subgraph is S-free. For instance, if S is the star $K_{m,1}$, then $P_{\eta_S}(G;k)$ is the number of k-colorings such that no vertex belongs to m or more monochromatic edges. This "degree-chromatic polynomial" counts colorings of G in which no color-induced subgraph has a vertex of degree $\geq m$; if $m = 1$, then we recover the usual chromatic polynomial. In general, two trees with the same number of vertices need not have the same degree-chromatic polynomials. For example, if G is the three-edge path on four vertices, H is the three-edge star, and S is the two-edge path, then $P_{\eta_S}(G;k) = k^4 - 2k^2 + k$ and $P_{\eta_S}(H;k) = k^4 - 3k^2 + 2k$. Based on experimental evidence, we conjecture that if T is any tree on n vertices and $m < n$, then

$$P_{\eta_{K_{m-1}}}(T;k) = k^n - \sum_{v \in V(T)} \binom{d_T(v)}{m} k^{n-m} + \text{(lower order terms)}.$$

4. Tutte characters

The *Tutte polynomial* $T_G(x,y)$ is a powerful graph invariant with many important properties (for a comprehensive survey, see [BO92]). It is defined in closed form by the formula

$$T_G(x,y) \;=\; \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}(G)-\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}$$

where $\mathrm{rk}(A)$ is the graph rank of A, and $\mathrm{null}(A) = |A| - \mathrm{rk}(A)$ (the *nullity* of A). The Tutte polynomial is a universal deletion-contraction invariant in the sense that every graph invariant satisfying a deletion-contraction recurrence can be obtained from $T_G(x,y)$ via a standard "recipe" [Bol98, p. 340]. In particular, $T_G(x,y)$ is multiplicative on connected components, so we can regard it as a character on the graph algebra:

$$\tau_{x,y}(G) = T_G(x,y).$$

We may regard x, y either as indeterminates or as (typically integer-valued) parameters. It is often more convenient to work with the *rank-nullity polynomial*

$$R_G(x,y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)} = (x-1)^{\mathrm{rk}(G)} T_G(x/(x-1), y) \quad (3)$$

which carries the same information as $T_G(x,y)$, and is also multiplicative on connected components, hence is a character on $\mathcal{G}$. Note that $R_G(1,y) = 1$, and that

$$T_G(x,y) = (x-1)^{\mathrm{rk}(G)} R_G(x/(x-1), y). \quad (4)$$

Let $\rho_{x,y}$ denote the function $G \mapsto R_G(x,y)$, viewed as a character of the graph algebra $\mathcal{G}$. Let $P_{x,y}(G;k) = \rho^k_{x,y}(G)$ be the image of G under the CHA morphism $(\mathcal{G}, \rho_{x,y}) \to \mathbb{F}(x,y)[k]$ (see Proposition 2.1); note that $P_{x,y}(G;k)$ is a polynomial function of k.

For later use, we record the relationship between ρ and τ:

$$\tau_{x,y} = (x-1)^{\mathrm{rk}(G)} \rho_{x/(x-1),y}, \qquad \rho_{x,y} = (x-1)^{\mathrm{rk}(G)} \tau_{x/(x-1),y}. \quad (5)$$

In particular,

$$\tau_{0,y} = \widetilde{\rho_{0,y}} \quad \text{and} \quad \rho_{2,y} = \tau_{2,y}. \quad (6)$$

4.1. **The main theorem on Tutte characters.** We now prove that $P_{x,y}(G;k)$ is itself a Tutte polynomial evaluation.

Theorem 4.1. *We have*

$$\rho^k_{x,y}(G) = P_{x,y}(G;k) = k^{c(G)}(x-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+x-1}{x-1}, y\right).$$

Proof. We have

$$P_{x,y}(G;k) = \rho^k_{x,y}(G) = \sum_{V_1 \uplus \cdots \uplus V_k} \prod_{i=1}^{k} \rho_{x,y}(G|_{V_i}) \tag{7a}$$

$$= \sum_{V_1 \uplus \cdots \uplus V_k} \prod_{i=1}^{k} \sum_{A_i \subseteq E(G|_{V_i})} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{7b}$$

$$= \sum_{f:V\to[k]} \prod_{i=1}^{k} \sum_{A_i \subseteq f^{-1}(i)} (x-1)^{\mathrm{rk}(A_i)}(y-1)^{\mathrm{null}(A_i)} \tag{7c}$$

$$= \sum_{f:V\to[k]} \sum_{A \subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)} \tag{7d}$$

where $M(f,G)$ denotes the set of edges $e = uv \in E(G)$ such that $f(u) = f(v)$ (including, in particular, all loops). Here the sum is over all ordered partitions of $V(G)$ into pairwise disjoint subsets (possibly empty). In order to find a recipe for $P_{x,y}(G;k)$ as a Tutte specialization, we need to know its value on edgeless graphs, and how it behaves with respect to deleting a loop, deleting a cut-edge, or deletion and contraction of an "ordinary" edge.

Step 1: Edgeless graphs. If $E(G) = \emptyset$, then $R_H(x,y) = 1$ for every subgraph $H \subseteq G$. Therefore, every summand in (7a) is 1, so $P_{x,y}(G;k)$ is just the number of ordered partitions with $n = |V(G)|$ parts, that is:

$$P_{x,y}(\overline{K_n};k) = k^n. \tag{8}$$

Step 2: Loops. Suppose G has a loop ℓ. For every ordered partition $V_1 \uplus \cdots \uplus V_k$, let V_i be the part that contains the endpoint of ℓ. Then $\rho_{2,y}(G|_{V_i}) = y\rho_{2,y}((G-\ell)|_{V_i})$, and we conclude that

$$P_{x,y}(G;k) = y \cdot P_{x,y}(G-\ell;k). \tag{9}$$

Step 3: Nonloop edges. Suppose G has a nonloop edge e (possibly a cut-edge) with endpoints u,v. For a function $f : V \to [k]$, if $f(u) \neq f(v)$ then $M(f, G-e) = M(f,G)$, while if $f(u) = f(v)$ then $M(f,G-e) = M(f,G) \setminus \{e\}$. For every edge set $A \subseteq M(f,G)$ containing e, the edge set $B = A \setminus \{e\} \subseteq M(f, G/e)$ satisfies $\mathrm{null}(B) = \mathrm{null}(A)$ and $\mathrm{rk}(B) = \mathrm{rk}(A) - 1$;

moreover, the correspondence between A and B is a bijection. Therefore,

$$\begin{aligned}
P_{x,y}(G;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)},\\
P_{x,y}(G-e;k) &= \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)},\\
P_{x,y}(G;k)-P_{x,y}(G-e;k) &= \sum_{\substack{f:V\to[k]\\ e\in M(f,G)}} \sum_{\substack{A\subseteq M(f,G)\\ e\in A}} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}\\
&= \sum_{\substack{f:V\to[k]\\ f(u)=f(v)}} \sum_{B\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(B)+1}(y-1)^{\mathrm{null}(B)}\\
&= (x-1)P_{x,y}(G/e;k).
\end{aligned}$$

To put this recurrence in a more familiar form,

$$P_{x,y}(G;k) = P_{x,y}(G-e;k) + (x-1)P_{x,y}(G/e;k). \tag{10}$$

Step 4: Cut-edges. Now suppose that $e = uv$ is a cut-edge. We have

$$P_{x,y}(G-e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G-e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}$$

and

$$P_{x,y}(G/e;k) = \sum_{f:V\to[k]} \sum_{A\subseteq M(f,G/e)} (x-1)^{\mathrm{rk}(A)}(y-1)^{\mathrm{null}(A)}.$$

Let H be the connected component of $G-e$ containing u, and let $H' = G-H$. Then we have $E(G-e) = E(H)\cup E(H')$. The cyclic group $\mathbb{Z}_k$ acts on colorings f by cycling the colors of vertices in H and fixing the colors of vertices in H'; i.e., if ζ is a generator of $\mathbb{Z}_k$, then $(\zeta f)(w)f(w)+\zeta$ for $w\in V(H)$, while $(\zeta f)(w) = f(w)$ for $w\in V(H')$. Then the set $M(f,G-e)$ is invariant under the action of $\mathbb{Z}_k$; moreover, each orbit has size k and has exactly one coloring for which $f(u) = f(v)$. In that case, contracting the edge e does not change the nullity or rank. Therefore, $P_{x,y}(G/e;k) = k^{-1}P_{x,y}(G-e;k)$, which when combined with (10) yields

$$\begin{aligned}
P_{x,y}(G;k) &= P_{x,y}(G-e;k) + (x-1)P_{x,y}(G-e;k)/k\\
&= \left(\frac{k+x-1}{k}\right) P_{x,y}(G-e;k).
\end{aligned} \tag{11}$$

Now combining (8), (9), (10), and (11) with the "recipe theorem" [Bol98, p. 340] (replacing Bollobás' $x, y, \alpha, \sigma, \tau$ with $(k+x-1)/k,\ y,\ k,\ 1,\ x-1$ respectively) gives the desired result. □

4.2. **Applications to Tutte polynomial evaluations.** Theorem 4.1 has many enumerative consequences, some familiar and some less so. Many of the formulas we obtain resemble those in the work of Ardila [Ard07]; the precise connections remain to be investigated.

First, observe that setting $x = y = t$ in Theorem 4.1 yields

$$\rho^k_{t,t}(G) \;=\; P_{t,t}(G;k) \;=\; k^{c(G)}(t-1)^{\mathrm{rk}(G)}T_G\left(\frac{k+t-1}{t-1}, t\right)$$
$$= k^{c(G)}\overline{\chi}C_G(k;t) \tag{12}$$

where $\overline{\chi}$ denotes Crapo's coboundary polynomial[4]; see [MR05, p. 236] and [BO92, §6.3.F].

Corollary 4.2. *For $k \in \mathbb{Z}$ and y arbitrary, the Tutte characters $\tau_{2,y}$ and $\tau_{0,y}$ satisfy the identities*

$$(\tau_{2,y})^k(G) = k^{c(G)}T_G(k+1, y), \tag{13}$$
$$(\widetilde{\tau_{0,y}})^k(G) = k^{c(G)}(-1)^{\mathrm{rk}(G)}T_G(1-k, y). \tag{14}$$

In particular, $(\widetilde{\tau_{0,y}})^{-1} = \overline{\tau_{2,y}}$.

Proof. Recall from (6) that $\rho_{2,y} = \tau_{2,y}$ and $\rho_{0,y} = \widetilde{\tau_{0,y}}$. Setting $x = 2$ or $x = 0$ in Theorem 4.1 yields respectively

$$(\tau_{2,y})^k(G) = (\rho_{2,y})^k(G) = P_{2,y}(G;k) = k^{c(G)}T_G(k+1,y),$$
$$(\widetilde{\tau_{0,y}})^k(G) = (\rho_{0,y})^k(G) = P_{0,y}(G;k) = k^{c(G)}(-1)^{\mathrm{rk}(G)}T_G(1-k,y).$$

In particular, setting $k = -1$ in (14) gives

$$(\widetilde{\tau_{0,y}})^{-1}(G) = (-1)^{c(G)}(-1)^{\mathrm{rk}(G)}T_G(2,y) = (-1)^{n(G)}\tau_{2,y}(G) = \overline{\tau_{2,y}}(G). \qquad \square$$

Similarly, we can find combinatorial interpretations of convolution powers of the characters $\tau_{2,2}$, $\tau_{2,0}$, $\widetilde{\tau_{0,2}}$, and $\widetilde{\tau_{0,0}}$. In the last case, we recover the standard formula for the chromatic polynomial as a specialization of the Tutte polynomial. Note that $\widetilde{\tau_{0,0}} = \tau_{0,0}$, because these characters are both zero on any graph with one or more edges.

One can deduce combinatorial interpretations of other evaluations of the Tutte polynomial. If G is connected, then substituting $y = 2$ and $k = 2$ into (13) yields

$$T(G;3,2) \;=\; \frac{P_{2,2}(G;2)}{2} \;=\; \frac{(\tau_{2,2} * \tau_{2,2})(G)}{2} \;=\; \sum_{U \subseteq V(G)} 2^{e(G|_U)+e(G|_{\overline{U}})-1}. \tag{15}$$

That is, $T(G;3,2)$ counts the pairs (f, A), where f is a 2-coloring of G and A is a set of monochromatic edges.

In order to interpret more general powers of Tutte characters, we use (5) to rewrite the left-hand side of Theorem 4.1 as

$$k^{c(G)}(x-1)^{\mathrm{rk}(G)}T_G\left(\frac{k+x-1}{x-1}, y\right) \;=\; \sum_{V_1 \uplus \cdots \uplus V_k = V(G)} \;\prod_{i=1}^{k}(x-1)^{\mathrm{rk}(G_i)}\tau_{x/(x-1),y}(G_i)$$

[4]The bar in the notation $\overline{\chi}$ has no relation to the involution $\phi \mapsto \overline{\phi}$ on $\mathbb{X}(\mathcal{G})$.

where $G_i = G|_{V_i}$. Note that in the special case $G = K_n$, we have $G_i \cong K_{|V_i|}$ and $\mathrm{rk}(G_i) = |V_i| - 1$ for all i, so the equation simplifies to

$$(x-1)^{n-1} T_{K_n}\left(\frac{k+x-1}{x-1}, y\right) = k^{-1} (\tau_{x/(x-1),y})^k (K_n). \qquad (16)$$

This equation has further enumerative consequences: setting $x = 2$ gives

$$T_{K_n}(k+1, y) = \frac{1}{k} \sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1! a_2! \ldots a_k!} \tau_{2,y}(K_{a_1}) \ldots \tau_{2,y}(K_{a_k}). \qquad (17)$$

Setting $y = 0$ in (17), and observing that $\tau_{2,0}(K_a) = a!$ gives $T_{K_n}(k+1, 0) = (n+k-1)!/k!$. This is not a new formula; it follows from the standard specialization of the Tutte polynomial to the chromatic polynomial [BO92, Prop. 6.3.1], together with the well-known formula $k(k-1)\cdots(k-n+1)$ for the chromatic polynomial of K_n. On the other hand, setting $y = 2$ in (17), and recalling that $\tau_{2,2}(K_a) = 2^{|E(K_a)|} = 2^{\binom{a}{2}}$, gives

$$T_{K_n}(k+1, 2) \ = \ k^{-1} \sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1! a_2! \ldots a_k!} 2^{\binom{a_1}{2}+\cdots+\binom{a_k}{2}}. \qquad (18)$$

This formula may be obtainable from the generating function for the coboundary polynomials of complete graphs, as computed by Ardila [Ard07, Thm. 4.1]; see also sequence A143543 in [Slo10]. Notice that setting $k = 2$ in (18) recovers (15).

It is natural to ask what happens when we set $x = 1$, since this specialization of the Tutte polynomial has well-known combinatorial interpretations in terms of, e.g., the chip-firing game [ML97] and parking functions [GS96]. The equations (3) and (4) degenerate upon direct substitution, but we can instead take the limit of both sides of Theorem 4.1 as $x \to 1$, obtaining (after some calculation, which we omit)

$$\rho^k_{1,y}(G) = k^{n(G)}.$$

What can be said about Tutte characters in light of Proposition 2.1? Replacing x with $(k + x - 1)/(x - 1)$ in Theorem 4.1, we get

$$\begin{aligned} P_{(k+x-1)/(x-1),y}(G; k) \ &= \ k^{c(G)} (k/(x-1))^{\mathrm{rk}(G)} T(G; x, y) \\ &= \ k^{n(G)} (x-1)^{-\mathrm{rk}(G)} T(G; x, y). \end{aligned} \qquad (19)$$

One consequence is a formula for the Tutte polynomial in terms of P:

$$T(G; x, y) = k^{-n(G)} (x-1)^{\mathrm{rk}(G)} P_{(k+x-1)/(x-1),y}(G; k). \qquad (20)$$

In addition, the left-hand-side of (19) — which is an element of $\mathbb{F}(x, y)[k]$ — is actually just $k^{n(G)}$ times a rational function in x and y. Setting $k = x-1$ or $k = 1-x$, we can write down simpler formulas for the Tutte polynomial in terms of P:

$$\begin{aligned} T(G; x, y) &= (x-1)^{-c(G)} P_{2,y}(G; x-1), \\ T(G; x, y) &= (-1)^{n(G)} (x-1)^{c(G)} P_{0,y}(G; 1-x). \end{aligned}$$

5. A reciprocity relation between K_n and K_m

For each scalar $c \in \mathbb{C}$, there is a character ξ_C on $\mathcal{G}$ defined by $\xi_c(G) = c^{n(G)}$. It is not hard to see that

$$(\xi_c * \zeta)(G) = \sum_{\text{cocliques } Q} c^{n-|Q|}.$$

In particular, $(\xi_1 * \zeta)(G)$ is the number of cocliques in G, and $-(\xi_{-1} * \zeta)(G)$ is the reduced Euler characteristic of its clique complex.

Define a *k-near-coloring* of a graph $G = (V, E)$ to be a function $f : V \to [0, k]$, not necessarily surjective, with the following property: each of the color classes $V_1 = f^{-1}(1), \ldots, V_k = f^{-1}(k)$, but not necessarily $V_0 = f^{-1}(0)$, is a coclique. Let $\chi^\natural(G; k)$ denote the number of k-near colorings of G. Then

$$(\xi_c * \zeta)^k(G) \;=\; \sum_f (ck)^{|V_0|} \;=\; \sum_{V_0 \subseteq V(G)} (ck)^{|V_0|}\, \chi^\natural(G - V_0; k). \tag{21}$$

(To see the first equality in (21), consider a partition of V into $2k$ subsets. The union of the first k blocks is V_0, and the last k blocks are $V_1, \ldots, V_k$. Since V_0 is arbitrarily divided into k blocks, each k-near-coloring is counted $k^{|V_0|}$ times.) Now, equation (21) implies that

$$\begin{aligned}(\xi_1 * \zeta^n)(K_m) = \sum_{W \subseteq [m]} \zeta^n(K_W) \;&=\; \sum_{j=0}^{m} \binom{m}{j} \chi^\natural(K_m; n)\\ &=\; \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} \frac{n!}{(n-j)!}.\end{aligned}$$

But this expression is symmetric in n and m, which yields a surprising (to us, at least) reciprocity relation:

$$(\xi_1 * \zeta^n)(K_m) = (\xi_1 * \zeta^m)(K_n). \tag{22}$$

If we apply the bar involution to both sides of (22) (or, equivalently, redo the calculation, replacing ξ_1 with ξ_{-1}) we obtain

$$(\overline{\xi}_1 * \zeta^n)(K_m) = (-1)^{n+m} (\overline{\xi}_1 * \zeta^m)(K_n). \tag{23}$$

Experimental evidence indicates that

$$(\xi_1 * \zeta^{-1})(K_n) = (-1)^n D_n, \qquad (\xi_{-1} * \zeta^{-1})(K_n) = (-1)^n A_n,$$

where D_n is the number of derangements of $\{1, 2, \ldots, n\}$ and B_n is the number of arrangements (sequences A000166 and A000522 of [Slo10], respectively). More generally, we conjecture that for all scalars k and c, the exponential generating function for $\xi_k * \zeta^c$ is

$$\sum_{n \geq 0} (\xi_k * \zeta^c)(K_n) \frac{x^n}{n!} \;=\; e^{-kx}(1-x)^{-c}$$

(see [Sta99, Example 5.1.2]).

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