Preprint

ON CONVOLUTIONS OF EULER NUMBERS

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Abstract. We show that if p is an odd prime then

$$\sum_{k=0}^{p-1}E_kE_{p-1-k}\equiv 1 \pmod p$$

and

$$\sum_{k=0}^{p-3}E_kE_{p-3-k}\equiv(-1)^{(p-1)/2}2E_{p-3} \pmod p,$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers. Moreover, we prove that for any positive integer n and prime number $p>2n+1$ we have

$$\sum_{k=0}^{p-1+2n}E_kE_{p-1+2n-k}\equiv s(n) \pmod p$$

where $s(n)$ is an integer only depending on n.

1. Introduction

The Euler numbers E_n $(n\in\mathbb{N}=\{0,1,2,\ldots\})$ are integers defined by

$$E_0=1 \text{ and } \sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}E_{n-k}=0 \quad \text{for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

It is well known that $E_{2n+1}=0$ for all $n\in\mathbb{N}$ and

$$\sec x=\sum_{n=0}^{\infty}(-1)^nE_{2n}\frac{x^{2n}}{(2n)!} \quad \left(|x|<\frac{\pi}{2}\right).$$

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The exponential generating function for Euler numbers is given by

$$\frac{2e^x}{e^{2x}+1}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\qquad\left(|x|<\frac{\pi}{2}\right).$$

Thus

$$\left(\frac{2e^x}{e^{2x}+1}\right)^2=\sum_{k=0}^{\infty}E_k\frac{x^k}{k!}\sum_{l=0}^{\infty}E_l\frac{x^l}{l!}=\sum_{n=0}^{\infty}f(n)\frac{x^n}{n!},$$

where

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}E_kE_{n-k}.$$

In this paper we are interested in the usual convolution of Euler numbers given by $\sum_{k=0}^{n}E_kE_{n-k}$. The reader may consult [PS], [SP] and [S11] for related background.

Now we present our main results.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-3}E_kE_{p-3-k}\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\pmod p,\tag{1.1}$$

where $(-)$ denotes the Jacobi symbol. Moreover, for any $n=0,1,2,\ldots$ we have

$$\sum_{k=0}^{p-1+2n}E_kE_{p-1+2n-k}\equiv s(n)+\delta(p,n)\pmod p,\tag{1.2}$$

where

$$s(n)=\sum_{k=0}^{n}E_{2k}E_{2n-2k}\tag{1.3}$$

and

$$\delta(p,n)=\begin{cases}1 & \text{if } n>0\ \&\ p-1\mid 2n,\\ 0 & \text{otherwise.}\end{cases}$$

Example 1.1. Here are the values of $s(n)$ with $n\in\{0,1,2,3,4,5\}$:

$$s(0)=1,\ s(1)=-2,\ s(2)=11,\ s(3)=-132,\ s(4)=2917,\ s(5)=-104422.$$

Thus, for any odd prime p we have

$$\sum_{k=0}^{p-1}E_kE_{p-1-k}\equiv 1\pmod p,\tag{1.4}$$

$$\sum_{k=0}^{p+1}E_kE_{p+1-k}\equiv -2\pmod p\quad\text{if } p>3,\tag{1.5}$$

$$\sum_{k=0}^{p+3}E_kE_{p+3-k}\equiv 11\pmod p\quad\text{if } p>5.\tag{1.6}$$

Applying (1.2) again and again we immediately obtain the following consequence.

Corollary 1.1. *Let* $n = \frac{p-1}{2}q + r$ *with* $q \in \{1, 2, 3, \dots\}$ *and* $r \in \{0, \dots (p-3)/2\}$. *Then we have*

$$s(n) \equiv s(r) + (q-1)\delta_{r,0} \pmod{p}. \tag{1.7}$$

By a further refinement of our method to prove Theorem 1.1 and some complicated discussions, we can deduce the following theorem though we will not give the details of the proof since it is similar to that of Theorem 1.1.

Theorem 1.2. *For any odd prime* p, *we have*

$$\sum_{i+j+k=p-3} E_i E_j E_k \equiv -2E_{p-3} \pmod{p}. \tag{1.8}$$

Also, for each $n \in \mathbb{N}$ *there is a unique integer* $t(n)$ *such that if* $p > 2n+1$ *is a prime then*

$$\sum_{i+j+k=p-1+2n} E_i E_j E_k \equiv t(n) \pmod{p}. \tag{1.9}$$

In particular,

$$t(0) = 3,\ t(1) = -9,\ t(2) = 68,\ t(3) = -1068.$$

Theorems 1.1 and 1.2 should have their q-analogues. We leave this to those who are interested in such things.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* p *be an odd prime and let* $k \in \mathbb{N}$ *be even. Then*

$$E_k \equiv 2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1} \left(\frac{-1}{j}\right) j^k + \delta_{k,0}\left(\frac{-1}{p}\right) \pmod{p}. \tag{2.1}$$

Proof. By [S05, (1.1)],

$$E_k \equiv \sum_{i=0}^{p-1} (-1)^i (2i+1)^k \pmod{p}.$$

Observe that

$$\begin{aligned}&\sum_{i=0}^{p-1}(-1)^i(2i+1)^k-(-1)^{(p-1)/2}p^k\\ =&\sum_{i=0}^{(p-3)/2}\big((-1)^i(2i+1)^k+(-1)^{p-1-i}(2(p-1-i)+1)^k\big)\\ \equiv&2\sum_{i=0}^{(p-3)/2}(-1)^i(2i+1)^k=2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1}\left(\frac{-1}{j}\right)j^k \pmod p.\end{aligned}$$

So (2.1) follows. □

Proof of Theorem 1.1. (i) In view of Lemma 2.1,

$$\begin{aligned}\sum_{k=0}^{p-3}E_kE_{p-3-k}\equiv&2\left(\frac{-1}{p}\right)\times 2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1}\left(\frac{-1}{j}\right)j^{p-3}\\ &+2\sum_{k=0}^{(p-3)/2}\sum_{\substack{i=1\\ 2\nmid i}}^{p-1}\left(\frac{-1}{i}\right)i^{2k}2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1}\left(\frac{-1}{j}\right)j^{p-3-2k}\\ \equiv&2\left(\frac{-1}{p}\right)E_{p-3}+4\sum_{\substack{j=1\\ 2\nmid j}}^{p-1}\left(\frac{-1}{j}\right)^2j^{p-3}\\ &+8\sum_{\substack{1\leqslant i<j<p\\ 2\nmid ij}}\left(\frac{-1}{ij}\right)j^{p-3}\frac{(i^2/j^2)^{(p-1)/2}-1}{i^2/j^2-1}\\ \equiv&2\left(\frac{-1}{p}\right)E_{p-3}+4\sum_{j=1}^{p-1}\frac{1}{j^2}-4\sum_{k=1}^{(p-1)/2}\frac{1}{(2k)^2}\\ \equiv&2\left(\frac{-1}{p}\right)E_{p-3}\pmod p.\end{aligned}$$

In the last step we noted that

$$2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p$$

by the Wolstenhomle congruence. Thus (1.1) holds.

(ii) Observe that

$$\sum_{k=0}^{p-1+2n} E_k E_{p-1+2n-k} = \sum_{k=0}^{(p-3)/2} E_{2k} E_{p-1+2n-2k} + \sum_{k=0}^{n} E_{p-1+2k} E_{2n-2k}. \tag{2.2}$$

By Lemma 2.1,

$$\begin{aligned} E_{p-1} \equiv& 2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1} \left(\frac{-1}{j}\right) = 2\left(1 - 1 + \cdots + (-1)^{(p-3)/2}(p-2)\right) \\ =& 1 - \left(\frac{-1}{p}\right) = E_0 - \left(\frac{-1}{p}\right) \pmod p \end{aligned}$$

and also

$$E_{p-1+2k} \equiv E_{2k} \pmod p \quad \text{for } k = 1, 2, 3, \ldots .$$

Therefore

$$\sum_{k=0}^{n} E_{p-1+2k} E_{2n-2k} \equiv \sum_{k=0}^{n} E_{2k} E_{2n-2k} - \left(\frac{-1}{p}\right) E_{2n} \pmod p. \tag{2.3}$$

In view of Lemma 2.1, we also have

$$\begin{aligned} &\sum_{k=0}^{(p-3)/2} E_{2k} E_{p-1+2n-2k} \\ \equiv& \left(\frac{-1}{p}\right) 2\sum_{\substack{j=1\\ 2\nmid j}}^{p-1} \left(\frac{-1}{j}\right) j^{p-1+2n} \\ &+ \sum_{k=0}^{(p-3)/2} 2\sum_{\substack{0<i<p\\ 2\nmid i}} \left(\frac{-1}{i}\right) i^{2k} 2\sum_{\substack{0<j<p\\ 2\nmid j}} \left(\frac{-1}{j}\right) j^{p-1+2n-2k} \\ \equiv& \left(\frac{-1}{p}\right)\left(E_{2n} - \delta_{n,0}\left(\frac{-1}{p}\right)\right) + 4\sum_{\substack{0<i,j<p\\ 2\nmid ij}} \left(\frac{-1}{ij}\right) j^{2n} \sum_{k=0}^{(p-3)/2} \frac{i^{2k}}{j^{2k}} \\ \equiv& \left(\frac{-1}{p}\right) E_{2n} - \delta_{n,0} + 4\sum_{\substack{0<j<p\\ 2\nmid j}} \left(\frac{-1}{j^2}\right) j^{2n} \frac{p-1}{2} \\ &+ 8\sum_{\substack{0<i<j<p\\ 2\nmid ij}} \left(\frac{-1}{ij}\right) j^{2n} \frac{(i^2/j^2)^{(p-1)/2} - 1}{i^2/j^2 - 1} \end{aligned}$$

Note that if $0 < i, j < p$ and $2 \nmid ij$ then $i \not\equiv -j \pmod p$ since $i + j$ is even while p is odd. Applying Fermat's little theorem we obtain from the above

$$\sum_{k=0}^{(p-3)/2} E_{2k}E_{p-1+2n-2k} \equiv \left(\frac{-1}{p}\right) E_{2n} - \delta_{n,0} - 2 \sum_{\substack{0<j<p \\ 2\nmid j}} j^{2n} \pmod p.$$

If $p - 1$ divides $2n$, then

$$\sum_{\substack{0<j<p \\ 2\nmid j}} j^{2n} \equiv |\{0 < j < p :\ 2 \nmid j\}| = \frac{p-1}{2} \pmod p.$$

When $p - 1 \nmid 2n$, we have

$$2 \sum_{j=1}^{(p-1)/2} j^{2n} \equiv \sum_{j=1}^{(p-1)/2} \left(j^{2n} + (p-j)^{2n}\right) = \sum_{j=1}^{p-1} j^{2n} \equiv 0 \pmod p$$

(cf. [IR, p. 235]) and hence

$$\sum_{\substack{0<j<p \\ 2\nmid j}} j^{2n} = \sum_{j=1}^{p-1} j^{2n} - \sum_{j=1}^{(p-1)/2} (2j)^{2n} \equiv 0 \pmod p.$$

Thus

$$\sum_{k=0}^{(p-3)/2} E_{2k}E_{p-1+2n-2k} \equiv \left(\frac{-1}{p}\right) E_{2n} - \delta_{n,0} + [p-1 \mid 2n] \pmod p, \quad (2.4)$$

where $[p - 1 \mid 2n]$ takes 1 or 0 according as $p - 1 \mid 2n$ or not.

Combining (2.2)-(2.4) we get

$$\sum_{k=0}^{p-1+2n} E_k E_{p-1+2n-k} \equiv \sum_{k=0}^{n} E_{2k}E_{2n-2k} - \delta_{n,0} + [p-1 \mid 2n] \pmod p.$$

This proves (1.2).

So far we have completed the proof of Theorem 1.1. □

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